Partition forcing

Vera Fischer, Jaroslav Šupina[1 2]

[1]UPJŠ Košice

Institute of Mathematics
Faculty of Science
P.J. Šafárik University in Košice
Slovakia

[2]The results obtained during research stay at the University of Vienna,
Kurt Gödel Research Center for Mathematical Logic,
scholarship OeAD-GmbH ICM-2020-00442,
Aktion Österreich-Slowakei.

KGRC 2021

$\mathtt{MAD}$ number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

MAD number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

$\mathcal{A}$ is an almost disjoint family, shortly **AD** family, if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$.

MAD number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

$\mathcal{A}$ is an almost disjoint family, shortly **AD** family, if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$.

$\mathcal{A}$ is maximal almost disjoint family of $\mathcal{E}$, shortly a **MAD** family of $\mathcal{E}$, if $\mathcal{A}$ is maximal **AD** family below $\mathcal{E}$ with respect to inclusion.

MAD number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

$\mathcal{A}$ is an almost disjoint family, shortly **AD** family, if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$.

$\mathcal{A}$ is maximal almost disjoint family of $\mathcal{E}$, shortly a **MAD** family of $\mathcal{E}$, if $\mathcal{A}$ is maximal **AD** family below $\mathcal{E}$ with respect to inclusion.

Definition

$\mathfrak{a}$ is the minimal size of a **MAD** family of subsets of a countable set.

MAD number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

$\mathcal{A}$ is an almost disjoint family, shortly **AD** family, if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$.

$\mathcal{A}$ is maximal almost disjoint family of $\mathcal{E}$, shortly a **MAD** family of $\mathcal{E}$, if $\mathcal{A}$ is maximal **AD** family below $\mathcal{E}$ with respect to inclusion.

Definition

$\mathfrak{a}$ is the minimal size of a **MAD** family of subsets of a countable set.

The family of sets $\{f \restriction i \colon i \in \omega\}$ for all sequences $f \in {}^{\omega}2$ is an **AD** family of size $\mathfrak{c}$.

MAD number

M a set, $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{P}(M)$

$\mathcal{A}$ is an almost disjoint family, shortly **AD** family, if $|A \cap B| < \omega$ for each $A, B \in \mathcal{A}$.

$\mathcal{A}$ is maximal almost disjoint family of $\mathcal{E}$, shortly a **MAD** family of $\mathcal{E}$, if $\mathcal{A}$ is maximal **AD** family below $\mathcal{E}$ with respect to inclusion.

Definition

$\mathfrak{a}$ is the minimal size of a **MAD** family of subsets of a countable set.

The family of sets $\{f \restriction i \colon i \in \omega\}$ for all sequences $f \in {}^{\omega}2$ is an **AD** family of size $\mathfrak{c}$.

It can be extended to a **MAD** family of subsets of $2^{<\omega}$.

Tree **MAD** number

Definition

$\mathfrak{a}_T$ is the minimal size of a **MAD** family of subtrees of $2^{<\omega}$.

Compact partitions of ω^ω

$\mathfrak{a}_T$ is the minimal size of a partition of ω^ω into compact sets.

Compact partitions of ω^ω

$\mathfrak{a}_T$ is the minimal size of a partition of ω^ω into compact sets.

$$\boxed{\mathfrak{d} \leq \mathfrak{a}_T \leq \mathfrak{c}}$$

$\mathfrak{a}_T$ is really (topologically) invariant

Theorem (A. Miller 1980, O. Spinas 1997)

Let X be an uncountable Polish space. $\mathfrak{a}_T$ is the minimal size of an uncountable partition of X into closed sets.

Some history

J. Stern 1977, K. Kunen

- $\mathfrak{a}_T = \omega_1$ in the random real model ($\mathfrak{i} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$)

Some history

J. Stern 1977, K. Kunen

- $\mathfrak{a}_T = \omega_1$ in the random real model ($\mathfrak{i} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$)

A. Miller 1980

- $\mathfrak{a}_T$ is invariant
- it is consistent that $\mathfrak{a}_T = \omega_2$ and $\mathrm{cov}(\mathcal{M}) = \omega_1$

Some history

J. Stern 1977, K. Kunen

- $\mathfrak{a}_T = \omega_1$ in the random real model ($\mathfrak{i} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$)

A. Miller 1980

- $\mathfrak{a}_T$ is invariant
- it is consistent that $\mathfrak{a}_T = \omega_2$ and $\mathrm{cov}(\mathcal{M}) = \omega_1$

L. Newelski 1987

- $\mathfrak{d} \leq \mathfrak{a}_T$
- $\mathfrak{a}_T = \omega_1$ in the Sacks model

Some history

J. Stern 1977, K. Kunen

- $\mathfrak{a}_T = \omega_1$ in the random real model ($\mathfrak{i} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$)

A. Miller 1980

- $\mathfrak{a}_T$ is invariant
- it is consistent that $\mathfrak{a}_T = \omega_2$ and $\mathrm{cov}(\mathcal{M}) = \omega_1$

L. Newelski 1987

- $\mathfrak{d} \leq \mathfrak{a}_T$
- $\mathfrak{a}_T = \omega_1$ in the Sacks model

O. Spinas 1997

- $\mathfrak{d} \leq \mathfrak{a}_T$, $\mathfrak{a}_T$ is invariant
- it is consistent that $\mathfrak{a}_T = \omega_2$ and $\mathrm{cof}(\mathcal{N}) = \omega_1$

Some history

M. Hrušák 2000

- notation $\mathfrak{a}_T$
- $\mathfrak{d}, \mathfrak{ra} \leq \mathfrak{a}_T$, where $\mathfrak{ra}$ is the minimal size of **AD** family without certain Ramsey-like properties

Some history

M. Hrušák 2000

- notation $\mathfrak{a}_T$
- $\mathfrak{d}, \mathfrak{ra} \leq \mathfrak{a}_T$, where $\mathfrak{ra}$ is the minimal size of **AD** family without certain Ramsey-like properties

J.T. Moore, M. Hrušák, M. Džamonja 2004

- if $\diamondsuit_{\mathfrak{d}}$ holds then $\mathfrak{a}_T = \omega_1$ (also $\mathfrak{a} = \omega_1$, M. Hrušák 2001)

Some history

M. Hrušák 2000

- notation $\mathfrak{a}_T$
- $\mathfrak{d}, \mathfrak{ra} \leq \mathfrak{a}_T$, where $\mathfrak{ra}$ is the minimal size of **AD** family without certain Ramsey-like properties

J.T. Moore, M. Hrušák, M. Džamonja 2004

- if $\diamondsuit_{\mathfrak{d}}$ holds then $\mathfrak{a}_T = \omega_1$ (also $\mathfrak{a} = \omega_1$, M. Hrušák 2001)

K. Ciesielski, J. Pawlikowski 2004

- $\mathfrak{a}_T = \omega_1$ is the consequence (C7) of Covering Property Axiom

Some history

M. Hrušák 2000

- notation $\mathfrak{a}_T$
- $\mathfrak{d}, \mathfrak{ra} \leq \mathfrak{a}_T$, where $\mathfrak{ra}$ is the minimal size of **AD** family without certain Ramsey-like properties

J.T. Moore, M. Hrušák, M. Džamonja 2004

- if $\diamondsuit_{\mathfrak{d}}$ holds then $\mathfrak{a}_T = \omega_1$ (also $\mathfrak{a} = \omega_1$, M. Hrušák 2001)

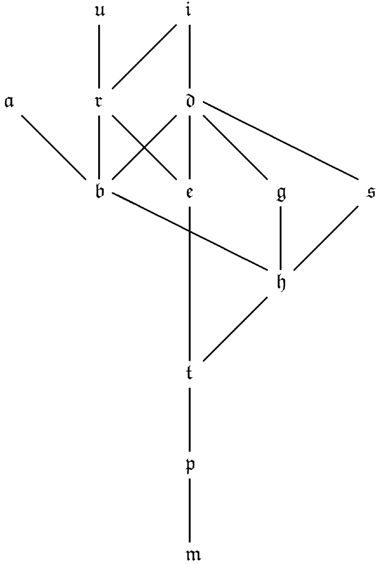
K. Ciesielski, J. Pawlikowski 2004

- $\mathfrak{a}_T = \omega_1$ is the consequence (C7) of Covering Property Axiom

O. Guzmán, M. Hrušák, O. Téllez 2020

- it is consistent that $\mathfrak{a}_T = \omega_2$ and $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \omega_1$

Cardinal invariants of the continuum



Main result

Theorem (V. Fischer–J.Š.)

There is a cardinals preserving generic extension in which

$$\operatorname{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \omega_2.$$

Question

Is any of the inequalities $\mathfrak{a} \leq \mathfrak{a}_T$ or $\mathrm{non}(\mathcal{N}) \leq \mathfrak{a}_T$ provable in **ZFC**?

Proof of the main result.

The plan

(1) The overall model.

(2) Partition forcing.

(3) Fusion arguments.

(4) Indestructibility - ultrafilter base.

(5) Indestructibility - independent family.

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.
- Bookkeeping device such that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}_T = \omega_2$.

The overall model.

- V is the ground model with $V \vDash \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.
- Bookkeeping device such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \omega_2$.
- There is $\mathbb{P}_{\omega_2}$-indestructible independent family in V, so $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{i} = \omega_1$.

The overall model.

- V is the ground model with $V \vDash \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.
- Bookkeeping device such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \omega_2$.
- There is $\mathbb{P}_{\omega_2}$-indestructible independent family in V, so $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{i} = \omega_1$.
- There is $\mathbb{P}_{\omega_2}$-indestructible ultrafilter in V, therefore $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{u} = \omega_1$.

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.
- Bookkeeping device such that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}_T = \omega_2$.
- There is $\mathbb{P}_{\omega_2}$-indestructible independent family in V, so $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{i} = \omega_1$.
- There is $\mathbb{P}_{\omega_2}$-indestructible ultrafilter in V, therefore $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{u} = \omega_1$.
- There is $\mathbb{P}_{\omega_2}$-indestructible **MAD** family in V, so $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \omega_1$ (O. Guzmán, M. Hrušák, O. Téllez 2020).

The overall model.

- V is the ground model with $V \models \mathrm{ZFC} + \mathrm{CH}$.
- Countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets.
- $\mathbb{Q}_\alpha$ destroys a partition of 2^ω.
- Bookkeeping device such that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}_T = \omega_2$.
- There is $\mathbb{P}_{\omega_2}$-indestructible independent family in V, so $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{i} = \omega_1$.
- There is $\mathbb{P}_{\omega_2}$-indestructible ultrafilter in V, therefore $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{u} = \omega_1$.
- There is $\mathbb{P}_{\omega_2}$-indestructible **MAD** family in V, so $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \omega_1$ (O. Guzmán, M. Hrušák, O. Téllez 2020).
- $\mathbb{P}_{\omega_2}$ has Sacks property, therefore $\mathrm{cof}(\mathcal{N}) = \omega_1$ (O. Spinas 1997).

(1) The overall model.

(2) **Partition forcing.**

(3) Fusion arguments.

(4) Indestructibility - ultrafilter base.

(5) Indestructibility - independent family.

Partition forcing

Definition (A. Miller 1980, partition forcing)

Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in \omega_1}$ be an uncountable partition of 2^ω into closed sets.

(1) $\mathbb{Q}(\mathcal{C})$ is the set of perfect trees $p \subseteq 2^{<\omega}$ such that each C_α is nowhere dense in $[p]$.

(2) The order of $\mathbb{Q}(\mathcal{C})$ is inclusion.

Let us recall that a set A which is contained in $[p]$ for some perfect subtree p of $2^{<\omega}$ is nowhere dense in $[p]$ if for every $s \in p$ there is $t \in p$ extending s and

$$\{f \in [p] \colon t \subseteq f\} \cap A = \emptyset.$$

Some history on partition forcing

A. Miller 1980

- The poset $\mathbb{Q}(\mathcal{C})$ is proper.
- $\mathbb{Q}(\mathcal{C})$ has the Laver property.
- In the $\mathbb{Q}(\mathcal{C})$-generic extension, $\mathcal{C}$ is no longer a partition of 2^ω.

Some history on partition forcing

A. Miller 1980

- The poset $\mathbb{Q}(\mathcal{C})$ is proper.
- $\mathbb{Q}(\mathcal{C})$ has the Laver property.
- In the $\mathbb{Q}(\mathcal{C})$-generic extension, $\mathcal{C}$ is no longer a partition of 2^ω.

O. Spinas 1997

- $\mathbb{Q}(\mathcal{C})$ has the ${}^\omega\omega$-bounding property.
- $\mathbb{Q}(\mathcal{C})$ has the Sacks property.

Some history on partition forcing

A. Miller 1980

- The poset $\mathbb{Q}(\mathcal{C})$ is proper.
- $\mathbb{Q}(\mathcal{C})$ has the Laver property.
- In the $\mathbb{Q}(\mathcal{C})$-generic extension, $\mathcal{C}$ is no longer a partition of 2^ω.

O. Spinas 1997

- $\mathbb{Q}(\mathcal{C})$ has the ${}^\omega\omega$-bounding property.
- $\mathbb{Q}(\mathcal{C})$ has the Sacks property.

J. Zapletal 2008

- $\mathbb{Q}(\mathcal{C})$ is isomorphic to a dense subset of P_I (I a σ-ideal on 2^ω generated by $\mathcal{C}$, P_I are I-positive Borel subsets of 2^ω ordered by inclusion).

Some history on partition forcing

A. Miller 1980

- The poset $\mathbb{Q}(\mathcal{C})$ is proper.
- $\mathbb{Q}(\mathcal{C})$ has the Laver property.
- In the $\mathbb{Q}(\mathcal{C})$-generic extension, $\mathcal{C}$ is no longer a partition of 2^ω.

O. Spinas 1997

- $\mathbb{Q}(\mathcal{C})$ has the ${}^\omega\omega$-bounding property.
- $\mathbb{Q}(\mathcal{C})$ has the Sacks property.

J. Zapletal 2008

- $\mathbb{Q}(\mathcal{C})$ is isomorphic to a dense subset of P_I (I a σ-ideal on 2^ω generated by $\mathcal{C}$, P_I are I-positive Borel subsets of 2^ω ordered by inclusion).

O. Guzmán, M. Hrušák, O. Téllez 2020

- $\mathbb{Q}(\mathcal{C})$ strongly preserves the tightness of a tight **MAD** family.

Some history on partition forcing

L.J. Halbeisen 2012, Notes in the Chapter on Miller forcing:

Notes

All non-trivial results presented in this chapter are essentially due to Miller and can be found in [14]. In that paper, he introduced what is now called *Miller forcing*, but which he called *rational perfect set forcing*. Miller thought about this forcing notion when he worked on his paper [13], where he used a fusion argument which involved preserving a dynamically chosen countable set of points (see [13, Lemmata 8 & 9]). This led him to perfect sets in which the rationals in them are dense, and shortly after, he realised that this is equivalent to forcing with superperfect trees. Even though superperfect trees appeared first in papers of Kechris [10] and Louveau [12], Miller was the first who investigated the corresponding forcing notion.

(1) The overall model.

(2) Partition forcing.

(3) **Fusion arguments.**

(4) Indestructibility - ultrafilter base.

(5) Indestructibility - independent family.

Fusion arguments

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \upharpoonright (m+1) = \check{f}_t.$$

$\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \upharpoonright (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding.

$\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding.

Proof.

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.

$\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding.

Proof.

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- There is $q \leq p$ such that for all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ with $q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$.

$\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding.

Proof.

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- There is $q \leq p$ such that for all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ with $q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$.
- $g(n) = \max\{f_s(n) + 1\colon s \in \text{split}_n(q)\}$.

$\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^{\omega}\omega$-bounding.

Proof.

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- There is $q \leq p$ such that for all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ with $q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$.
- $g(n) = \max\{f_s(n) + 1 \colon s \in \mathrm{split}_n(q)\}$.
- The set $\{q(s) \colon s \in \mathrm{split}_n(q)\}$ is pre-dense in q.

$\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Corollary (O. Spinas 1997)

The poset $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding.

Proof.

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- There is $q \leq p$ such that for all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ with $q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$.
- $g(n) = \max\{f_s(n) + 1 \colon s \in \mathrm{split}_n(q)\}$.
- The set $\{q(s) \colon s \in \mathrm{split}_n(q)\}$ is pre-dense in q.
- $q \Vdash \forall n(\dot{f}(n) < g(n))$.

$\square$

Fusion arguments used before

Definition (A. Miller 1980)

Let p, q be conditions in $\mathbb{Q}(\mathcal{C})$. Then $p \leq^n q$ if and only if

(1) $p \leq q$ and $\mathrm{split}_n(p) = \mathrm{split}_n(q)$,

(2) for all $t \in \mathrm{split}_n(q)$ the left most branch x_t^q of q through t belongs to $[p]$,

(3) for each $t \in \mathrm{split}_n(q)$ if $x_t^q \in C_\alpha$ then there is $s \supseteq t$ such that $s \in \mathrm{split}_{n+1}(p)$ such that $[p(s)] \cap C_\alpha = \emptyset$.

If $p_{n+1} \leq^n p_n$ for each n then the $\bigcap\{p_n : n \in \omega\}$ is a fusion of $\{p_n\}_{n\in\omega}$.

Fusion arguments used before

Definition (O. Spinas 1997, O. Guzmán, M. Hrušák, O. Téllez 2020)
A family of reals $X = \{x_s : s \in \omega^{<\omega}\}$ is said to be nice if the following conditions hold:

(1) for every $s \in \omega^{<\omega}$ the sequence $\langle x_{s^\frown n}\rangle_{n\in\omega}$ has the property that $\Delta(x_s, x_{s^\frown n}) < \Delta(x_s, x_{s^\frown (n+1)})$,

(2) for every $s, t, z \in \omega^{<\omega}$ if $s \subseteq t \subseteq z$ then $\Delta(x_s, x_z) < \Delta(x_t, x_z)$, and

(3) if for every $s \in \omega^{<\omega}$, $\alpha_s \in \omega_1$ is such that $x_s \in C_{\alpha_s}$ then whenever $s \subseteq t$ then $\alpha_s \neq \alpha_t$.

If p is a Sacks tree and there is a family $X \subseteq [p]$ which is nice with respect to $\mathcal{C}$ and dense in $[p]$, then $p \in \mathbb{Q}(\mathcal{C})$.

Our fusion arguments

We say that $x, y \in {}^{\omega}2$ are $\mathcal{C}$-different if x, y belong to different elements of $\mathcal{C}$.

A tree $p \subseteq 2^{<\omega}$ is said to be $\mathcal{C}$-branching if for any $s \in p$ there are $\mathcal{C}$-different branches in $[p]$ extending s.

Our fusion arguments

We say that $x, y \in {}^{\omega}2$ are $\mathcal{C}$-different if x, y belong to different elements of $\mathcal{C}$.

A tree $p \subseteq 2^{<\omega}$ is said to be $\mathcal{C}$-branching if for any $s \in p$ there are $\mathcal{C}$-different branches in $[p]$ extending s.

Lemma

Let $p \subseteq 2^{<\omega}$ be a tree. The following are equivalent:

(a) $p \in \mathbb{Q}(\mathcal{C})$.

(b) p is $\mathcal{C}$-branching.

(c) p is perfect and $[p]$ contains a countable dense subset with $\mathcal{C}$-different branches.

Our fusion arguments

We say that a set $X \subseteq {}^{\omega}2$ is a p-witness for the n-th level if

(a) $X \subseteq [p]$,

(b) X has $\mathcal{C}$-different elements,

(c) for each $s \in \mathrm{split}_n(p)$ there is $x \in X$ extending s.

If X is a p-witness for the $(n+1)$-st level then each node from n-th splitting level of p is contained in $\mathcal{C}$-different branches.

Our fusion arguments

Definition (Fusion sequence with witnesses)

(1) $(p, X) \leq^n (q, Y)$ if

- p, q are conditions in $\mathbb{Q}(\mathcal{C})$ such that $p \leq q$,
- $X \supseteq Y$ are p-witness for the $(n+1)$-st level, q-witness for the n-th level, respectively.

(2) A sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses if for each n,

$$(p_{n+1}, X_{n+1}) \leq^n (p_n, X_n).$$

Note that if $(p, X) \leq^n (q, Y)$ then $\mathrm{split}_{<n}(p) = \mathrm{split}_{<n}(q)$.

Lemma

If a sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses then the fusion $\bigcap\{p_n : n \in \omega\}$ is a condition in $\mathbb{Q}(\mathcal{C})$.

Pre-processed tree

Lemma

Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$:

For all $m \in \omega$, for all $t \in \text{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.

One branch

Claim

For each function h in ${}^{\omega}\omega \cap V$, the set of all conditions r satisfying the following property is dense below p:

There is a real $x \in [r]$ and a sequence $\{f_s\}_{s \in x \upharpoonright \mathrm{split}(r)}$ of functions in ${}^{<\omega}\omega$ such that
$r(s) \Vdash \dot{f} \upharpoonright h(n) = f_s$ for any $s = x \upharpoonright \mathrm{split}_n(r)$.

One branch

Claim

For each function h in ${}^{\omega}\omega \cap V$, the set of all conditions r satisfying the following property is dense below p:

There is a real $x \in [r]$ and a sequence $\{f_s\}_{s \in x \restriction \mathrm{split}(r)}$ of functions in ${}^{<\omega}\omega$ such that
$$r(s) \Vdash \dot{f} \restriction h(n) = f_s \text{ for any } s = x \restriction \mathrm{split}_n(r).$$

Proof.

- Construct a decreasing sequence $\{r_i\}_{i \in \omega}$ of extensions of a condition below p with strictly increasing stems such that $r_n \Vdash \dot{f} \restriction h(n) = f_n$ for some $f_n \in {}^{h(n)}\omega$.

One branch

Claim

For each function h in ${}^{\omega}\omega \cap V$, the set of all conditions r satisfying the following property is dense below p:

There is a real $x \in [r]$ and a sequence $\{f_s\}_{s \in x \restriction \mathrm{split}(r)}$ of functions in ${}^{<\omega}\omega$ such that
$$r(s) \Vdash \dot{f} \restriction h(n) = f_s \text{ for any } s = x \restriction \mathrm{split}_n(r).$$

Proof.

- Construct a decreasing sequence $\{r_i\}_{i \in \omega}$ of extensions of a condition below p with strictly increasing stems such that $r_n \Vdash \dot{f} \restriction h(n) = f_n$ for some $f_n \in {}^{h(n)}\omega$.
- Denote $s_n = \mathrm{stem}\, r_n$ and set $x = \bigcup_{i \in \omega} s_i$.

One branch

Claim

For each function h in ${}^{\omega}\omega \cap V$, the set of all conditions r satisfying the following property is dense below p:

There is a real $x \in [r]$ and a sequence $\{f_s\}_{s \in x \restriction \mathrm{split}(r)}$ of functions in ${}^{<\omega}\omega$ such that

$$r(s) \Vdash \dot{f} \restriction h(n) = f_s \text{ for any } s = x \restriction \mathrm{split}_n(r).$$

Proof.

- Construct a decreasing sequence $\{r_i\}_{i\in\omega}$ of extensions of a condition below p with strictly increasing stems such that $r_n \Vdash \dot{f} \restriction h(n) = f_n$ for some $f_n \in {}^{h(n)}\omega$.
- Denote $s_n = \mathrm{stem}\, r_n$ and set $x = \bigcup_{i\in\omega} s_i$.
- Take the amalgamation $r = \bigcup_{i\in\omega} r_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$.

□

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:
  - For $x \in X_n$ extending s do nothing: set $q(s, i) = q_n(r)$ for $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and $i = x(|s|)$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:
  - For $x \in X_n$ extending s do nothing: set $q(s, i) = q_n(r)$ for $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and $i = x(|s|)$.
  - Otherwise take $t \supseteq s^\frown\langle 1 - i\rangle$ such that $[q_n(t)]$ avoids all already considered sets in $\mathcal{C}$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:
  - For $x \in X_n$ extending s do nothing: set $q(s, i) = q_n(r)$ for $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and $i = x(|s|)$.
  - Otherwise take $t \supseteq s^\frown\langle 1 - i\rangle$ such that $[q_n(t)]$ avoids all already considered sets in $\mathcal{C}$.
  - Apply previous observation to obtain condition $q(s, 1 - i) \leq q_n(t)$, branch x and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_n)}$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:
  - For $x \in X_n$ extending s do nothing: set $q(s, i) = q_n(r)$ for $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and $i = x(|s|)$.
  - Otherwise take $t \supseteq s^\frown\langle 1 - i\rangle$ such that $[q_n(t)]$ avoids all already considered sets in $\mathcal{C}$.
  - Apply previous observation to obtain condition $q(s, 1 - i) \leq q_n(t)$, branch x and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_n)}$.
  - Amalgamate $q_{n+1} = \bigcup\{q(s,i)\colon s \in \mathrm{split}_n(q_n), i \in \{0,1\}\}$.

Proof

- $\dot{f}$ a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^{\omega}\omega$, $p \in \mathbb{Q}(\mathcal{C})$.
- Build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \leq p$ such that its fusion q has the required property.
- Apply claim to p and $h(n) = n + 1$ to obtain condition q_0, branch x, and sequence $\{f_s\}_{s \in x \restriction \mathrm{split}(q_0)}$.
- If $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$ are defined then consider each $s \in \mathrm{split}_n(q_n)$:
    - For $x \in X_n$ extending s do nothing: set $q(s, i) = q_n(r)$ for $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and $i = x(|s|)$.
    - Otherwise take $t \supseteq s^\frown\langle 1 - i\rangle$ such that $[q_n(t)]$ avoids all already considered sets in $\mathcal{C}$.
    - Apply previous observation to obtain condition $q(s, 1 - i) \leq q_n(t)$, branch x and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_n)}$.
    - Amalgamate $q_{n+1} = \bigcup\{q(s, i) \colon s \in \mathrm{split}_n(q_n), i \in \{0, 1\}\}$.
    - X_{n+1} is the set of all considered branches in this step.

(1) The overall model.

(2) Partition forcing.

(3) Fusion arguments.

(4) **Indestructibility - ultrafilter base.**

(5) Indestructibility - independent family.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P*-points and Ramsey ultrafilters.*

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \text{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t : t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t \colon t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).
- $Y_t = \bigcup\{u_s \colon s \subseteq x_t\}$.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t : t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).
- $Y_t = \bigcup\{u_s : s \subseteq x_t\}$.
- $\{Y_s : s \in p\} \subseteq \mathcal{U}$ or $\{\omega \setminus Y_s : s \in p\} \subseteq \mathcal{U}$.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t : t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).
- $Y_t = \bigcup \{u_s : s \subseteq x_t\}$.
- $\{Y_s : s \in p\} \subseteq \mathcal{U}$ or $\{\omega \setminus Y_s : s \in p\} \subseteq \mathcal{U}$.
- There is a pseudointersection $Z \in \mathcal{U}$ of one of the above sets.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \text{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t : t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).
- $Y_t = \bigcup\{u_s : s \subseteq x_t\}$.
- $\{Y_s : s \in p\} \subseteq \mathcal{U}$ or $\{\omega \setminus Y_s : s \in p\} \subseteq \mathcal{U}$.
- There is a pseudointersection $Z \in \mathcal{U}$ of one of the above sets.
- Fusion argument: $q_0, q_1 \leq p$, and a partition of Z into two sets Z_0, Z_1 such that $q_0 \Vdash \check{Z}_0 \subseteq \dot{Y}$ and $q_1 \Vdash \check{Z}_1 \subseteq \dot{Y}$. Similarly for the second case.

Indestructibility - ultrafilter base

Theorem

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters.

Proof.

- $\mathbb{Q}(\mathcal{C})$ preserves P-points and is ${}^{\omega}\omega$-bounding - $\mathbb{Q}(\mathcal{C})$ preserves Ramsey ultrafilters.
- $\mathcal{U}$ a P-point in V, $p \Vdash \dot{Y} \subseteq \omega$.
- For all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that $p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$.
- $\{x_t : t \in p\} \subseteq [p]$ a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$).
- $Y_t = \bigcup\{u_s : s \subseteq x_t\}$.
- $\{Y_s : s \in p\} \subseteq \mathcal{U}$ or $\{\omega \setminus Y_s : s \in p\} \subseteq \mathcal{U}$.
- There is a pseudointersection $Z \in \mathcal{U}$ of one of the above sets.
- Fusion argument: $q_0, q_1 \leq p$, and a partition of Z into two sets Z_0, Z_1 such that $q_0 \Vdash \check{Z}_0 \subseteq \dot{Y}$ and $q_1 \Vdash \check{Z}_1 \subseteq \dot{Y}$. Similarly for the second case.
- $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{P}(\omega) = \langle \mathcal{U} \rangle_{\mathrm{up}} \cup \langle \mathcal{U}^* \rangle_{\mathrm{dn}}$.

□

(1) The overall model.

(2) Partition forcing.

(3) Fusion arguments.

(4) Indestructibility - ultrafilter base.

(5) **Indestructibility - independent family.**

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

$\boxed{\mathcal{A} \subseteq \mathcal{P}(\omega)}$

$\mathcal{A}^h = \bigcap\{A^{h(A)} \colon A \in \operatorname{dom} h\}$ for $h \in \mathrm{FF}(\mathcal{A})$

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

$\boxed{\mathcal{A} \subseteq \mathcal{P}(\omega)}$ $\qquad$ $\mathcal{A}^h = \bigcap\{A^{h(A)} \colon A \in \operatorname{dom} h\}$ for $h \in \mathrm{FF}(\mathcal{A})$

$\mathcal{A}$ is an independent family if $\mathcal{A}^h$ is infinite for each $h \in \mathrm{FF}(\mathcal{A})$.

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

$\boxed{\mathcal{A} \subseteq \mathcal{P}(\omega)}$ $\qquad \mathcal{A}^h = \bigcap\{A^{h(A)} \colon A \in \operatorname{dom} h\}$ for $h \in \mathrm{FF}(\mathcal{A})$

$\mathcal{A}$ is an independent family if $\mathcal{A}^h$ is infinite for each $h \in \mathrm{FF}(\mathcal{A})$.

The cardinal invariant $\mathfrak{i}$ is the minimal cardinality of a maximal independent family.

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

$\boxed{\mathcal{A} \subseteq \mathcal{P}(\omega)}$ $\quad$ $\mathcal{A}^h = \bigcap\{A^{h(A)} \colon A \in \operatorname{dom} h\}$ for $h \in \mathrm{FF}(\mathcal{A})$

$\mathcal{A}$ is an independent family if $\mathcal{A}^h$ is infinite for each $h \in \mathrm{FF}(\mathcal{A})$.

The cardinal invariant $\mathfrak{i}$ is the minimal cardinality of a maximal independent family.

$\mathcal{A}$ is selective independent family if

Indestructibility - independent family

The concept developed by S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019, V. Fischer, D.C. Montoya 2019.

$\boxed{\mathcal{A} \subseteq \mathcal{P}(\omega)}$ $\qquad \mathcal{A}^h = \bigcap\{A^{h(A)} \colon A \in \operatorname{dom} h\}$ for $h \in \mathrm{FF}(\mathcal{A})$

$\mathcal{A}$ is an independent family if $\mathcal{A}^h$ is infinite for each $h \in \mathrm{FF}(\mathcal{A})$.

The cardinal invariant $\mathfrak{i}$ is the minimal cardinality of a maximal independent family.

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

(3) $\operatorname{fil}(\mathcal{A})$ is Ramsey.

Indestructibility - independent family

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

(3) $\mathrm{fil}(\mathcal{A})$ is Ramsey.

Indestructibility - independent family

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

- $\operatorname{fil}(\mathcal{A})$ is the density filter of $\mathcal{A}$.

(3) $\operatorname{fil}(\mathcal{A})$ is Ramsey.

Indestructibility - independent family

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

- $\mathrm{fil}(\mathcal{A})$ is the density filter of $\mathcal{A}$.
- $Y \in \mathrm{fil}(\mathcal{A})$ if for each $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq Y$.

(3) $\mathrm{fil}(\mathcal{A})$ is Ramsey.

Indestructibility - independent family

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

- $\mathrm{fil}(\mathcal{A})$ is the density filter of $\mathcal{A}$.
- $Y \in \mathrm{fil}(\mathcal{A})$ if for each $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq Y$.
- $\mathcal{A}$ is densely maximal if $\mathcal{P}(\omega) = \mathrm{fil}(\mathcal{A}) \cup \langle \{\omega \backslash \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\mathsf{dn}}$.

(3) $\mathrm{fil}(\mathcal{A})$ is Ramsey.

Indestructibility - independent family

$\mathcal{A}$ is selective independent family if

(1) $\mathcal{A}$ is an independent family,

(2) $\mathcal{A}$ is densely maximal,

- $\mathrm{fil}(\mathcal{A})$ is the density filter of $\mathcal{A}$.
- $Y \in \mathrm{fil}(\mathcal{A})$ if for each $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq Y$.
- $\mathcal{A}$ is densely maximal if $\mathcal{P}(\omega) = \mathrm{fil}(\mathcal{A}) \cup \langle \{\omega \setminus \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\mathsf{dn}}$.

(3) $\mathrm{fil}(\mathcal{A})$ is Ramsey.

- $\mathrm{fil}(\mathcal{A})$ is a P-filter.
- $\mathrm{fil}(\mathcal{A})$ is a Q-filter, i.e., for each partition of ω into finite sets there is a selector in $\mathrm{fil}(\mathcal{A})$.

Preservation Theorems

Lemma (S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019)

Let $\mathcal{A} \in V$ be an independent family and let $\mathbb{P} \in V$ have Sacks property. Then the filter $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}}$ is generated by $\mathrm{fil}(\mathcal{A}) \cap V$. That is, $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}} = \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\mathsf{up}}$.

Preservation Theorems

Lemma (S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019)

Let $\mathcal{A} \in V$ be an independent family and let $\mathbb{P} \in V$ have Sacks property. Then the filter $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}}$ is generated by $\mathrm{fil}(\mathcal{A}) \cap V$. That is, $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}} = \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\mathrm{up}}$.

Theorem (S. Shelah 1992)

(CH) *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of proper ${}^\omega\omega$-bounding posets. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a Ramsey set and let $\mathcal{H} \subseteq \mathcal{P}(\omega) \backslash \langle \mathcal{F} \rangle_{up}$. Suppose for each $\alpha < \delta$, $V^{\mathbb{P}_\alpha} \vDash \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}$. Then, the same property holds at δ, i.e.*

$$V^{\mathbb{P}_\delta} \vDash \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}.$$

Preservation Theorems

Lemma (S. Shelah 1992, D. Chodounský, V. Fischer, H. Grebík 2019)

Let $\mathcal{A} \in V$ be an independent family and let $\mathbb{P} \in V$ have Sacks property. Then the filter $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}}$ is generated by $\mathrm{fil}(\mathcal{A}) \cap V$. That is, $\mathrm{fil}(\mathcal{A})^{V^{\mathbb{P}}} = \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\mathsf{up}}$.

Theorem (S. Shelah 1992)

(CH) *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of proper ${}^\omega\omega$-bounding posets. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a Ramsey set and let $\mathcal{H} \subseteq \mathcal{P}(\omega) \backslash \langle \mathcal{F} \rangle_{up}$. Suppose for each $\alpha < \delta$, $V^{\mathbb{P}_\alpha} \vDash \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}$. Then, the same property holds at δ, i.e.*

$$V^{\mathbb{P}_\delta} \vDash \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}.$$

Corollary

(CH) Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of proper posets which preserve selective independent families and possess Sacks property. If $\mathcal{A}$ is a selective independent family then ($\mathcal{A}$ is a selective independent family)$^{V^{\mathbb{P}_\alpha}}$.

Indestructibility - independent family

Theorem (V. Fischer–J.Š.)

The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves selective independent families. That is, if $\mathcal{A}$ is a selective independent family then $(\mathcal{A}$ is a selective independent family$)^{V^{\mathbb{Q}(\mathcal{C})}}$.

Proof

- Let $\mathcal{A}$ be a selective independent family.

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \vDash \mathcal{A}$ is densely maximal.

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{A}$ is densely maximal.
- In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \backslash \langle \text{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$.

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{A}$ is densely maximal.
- In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \setminus \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$.
- To the contradiction, suppose $Y \notin \langle \{\omega \setminus \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\text{dn}}$.

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{A}$ is densely maximal.
- In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \backslash \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$.
- To the contradiction, suppose $Y \notin \langle \{\omega \backslash \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\text{dn}}$.
- Thus we can fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ for Y such that for all $h \in \mathrm{FF}(\mathcal{A})$,

$$p \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty.$$

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{A}$ is densely maximal.
- In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \setminus \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$.
- To the contradiction, suppose $Y \notin \langle \{\omega \setminus \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\text{dn}}$.
- Thus we can fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ for Y such that for all $h \in \mathrm{FF}(\mathcal{A})$,
$$p \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty.$$
- We assume that for all $m \in \omega$, for all $t \in \text{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that
$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$

Proof

- Let $\mathcal{A}$ be a selective independent family.
- It is enough to show $V^{\mathbb{Q}(\mathcal{C})} \vDash \mathcal{A}$ is densely maximal.
- In $V^{\mathbb{Q}(\mathcal{C})}$, take any $Y \in \mathcal{P}(\omega) \backslash \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\text{up}}$.
- To the contradiction, suppose $Y \notin \langle \{\omega \backslash \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\} \rangle_{\text{dn}}$.
- Thus we can fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ for Y such that for all $h \in \mathrm{FF}(\mathcal{A})$,
$$p \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty.$$
- We assume that for all $m \in \omega$, for all $t \in \text{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that
$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$
- For each $t \in p$, let $Y_t = \{m \in \omega : p(t) \nVdash \check{m} \notin \dot{Y}\}$.

Outer hull

$p \Vdash \lvert\dot{Y} \cap \mathcal{A}^h\rvert = \infty$ for $h \in \mathrm{FF}(\mathcal{A})$	$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$ for all $t \in \text{split}_m(p)$

- For each $t \in p$, let $Y_t = \{m \in \omega \colon p(t) \nVdash \check{m} \notin \dot{Y}\}$.

Outer hull

$p \Vdash \lvert\dot{Y} \cap \mathcal{A}^h\rvert = \infty$ for $h \in \mathrm{FF}(\mathcal{A})$	$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$ for all $t \in \mathsf{split}_m(p)$

- For each $t \in p$, let $Y_t = \{m \in \omega \colon p(t) \nVdash \check{m} \notin \dot{Y}\}$.
  - $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.

Outer hull

$p \Vdash \lvert\dot{Y} \cap \mathcal{A}^h\rvert = \infty$ for $h \in \mathrm{FF}(\mathcal{A})$	$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$ for all $t \in \mathsf{split}_m(p)$

- For each $t \in p$, let $Y_t = \{m \in \omega \colon p(t) \nVdash \check{m} \notin \dot{Y}\}$.
  - $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.
  - $Y_t \in \mathrm{fil}(\mathcal{A}) \cap V$.

Outer hull

$p \Vdash \lvert\dot{Y} \cap \mathcal{A}^h\rvert = \infty$ for $h \in \mathrm{FF}(\mathcal{A})$	$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t$ for all $t \in \text{split}_m(p)$

- For each $t \in p$, let $Y_t = \{m \in \omega \colon p(t) \nVdash \check{m} \notin \dot{Y}\}$.
  - $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.
  - $Y_t \in \mathrm{fil}(\mathcal{A}) \cap V$.
  - If $m \in Y_s$ for $s \in \text{split}_n(p)$, and $n < m$ then there is $t \in \text{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$.

Claim

We can assume there is a dense set $\{x_s \colon s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s \colon s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Lemma

Let $\mathcal{F}$ be a filter. The following are equivalent:

(a) $\mathcal{F}$ is a Ramsey filter.

(b) For any sequence $\{F_i\}_{i\in\omega}$ in $\mathcal{F}$ there is $a \in \mathcal{F}$ such that

$$a(n+1) \in F_{a(n)}.$$

(c) For any sequence $\{\mathcal{G}_i\}_{i\in\omega}$ of finite subsets of $\mathcal{F}$ there is $a \in \mathcal{F}$ such that

$$a(n+1) \in \bigcap \mathcal{G}_{a(n)}.$$

Claim

We can assume there is a dense set $\{x_s : s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s : s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t : t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.

Claim

We can assume there is a dense set $\{x_s\colon s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s\colon s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t\colon t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.
- For each $x \in [p]$, we set $i(x) = \{i\colon p(t) \Vdash \check{a}(i+1) \in \dot{Y}$ for $t = x \restriction \mathrm{split}_{a(i+1)}(p)\}$.

Claim

We can assume there is a dense set $\{x_s : s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s : s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t : t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.
- For each $x \in [p]$, we set $i(x) = \{i : p(t) \Vdash \check{a}(i+1) \in \dot{Y}$ for $t = x \restriction \mathrm{split}_{a(i+1)}(p)\}$.
- We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite.

Claim

We can assume there is a dense set $\{x_s\colon s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s\colon s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t\colon t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.
- For each $x \in [p]$, we set $i(x) = \{i\colon p(t) \Vdash \check{a}(i+1) \in \dot{Y}$ for $t = x \restriction \mathrm{split}_{a(i+1)}(p)\}$.
- We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite.
- One of properties of Y_t's: There are acceptable branches extending each $s \in p$.

Claim

We can assume there is a dense set $\{x_s\colon s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s\colon s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t\colon t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.
- For each $x \in [p]$, we set $i(x) = \{i\colon p(t) \Vdash \check{a}(i+1) \in \dot{Y}$ for $t = x \restriction \mathrm{split}_{a(i+1)}(p)\}$.
- We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite.
- One of properties of Y_t's: There are acceptable branches extending each $s \in p$.
- For each acceptable branch x, $y_x = \{a(i+1)\colon i \in i(x)\} \in \mathrm{fil}(\mathcal{A}) \cap V$.

Claim

We can assume there is a dense set $\{x_s\colon s \in p\} \subseteq [p]$ with $\mathcal{C}$-different elements and the family $\{y_s\colon s \in p\}$ of sets in $\mathrm{fil}(\mathcal{A}) \cap V$ such that:

(1) x_s extends s and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

(2) If $t = x_s \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_s(n) \in \dot{Y}$.

Proof.

- There is $a \in \mathrm{fil}(\mathcal{A}) \cap V$ such that $a(n+1) \in \bigcap\{Y_t\colon t \in \mathrm{split}_{\leq a(n)+2}(p)\}$.
- For each $x \in [p]$, we set $i(x) = \{i\colon p(t) \Vdash \check{a}(i+1) \in \dot{Y}$ for $t = x \restriction \mathrm{split}_{a(i+1)}(p)\}$.
- We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite.
- One of properties of Y_t's: There are acceptable branches extending each $s \in p$.
- For each acceptable branch x, $y_x = \{a(i+1)\colon i \in i(x)\} \in \mathrm{fil}(\mathcal{A}) \cap V$.
- Proceed with fusion argument, and use exclusively acceptable branches to build a dense set of a subtree of p.

□

Proof

- $y_t \in \mathrm{fil}(\mathcal{A}) \cap V$ for each $t \in \mathrm{split}(p)$.

Proof

- $y_t \in \mathrm{fil}(\mathcal{A}) \cap V$ for each $t \in \mathrm{split}(p)$.
- There is $C = \{l(n) \colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that

$$l(n+1) \in \bigcap \{y_t \colon t \in \mathrm{split}_{\leq l(n)+2}(p)\}.$$

Proof

- $y_t \in \mathrm{fil}(\mathcal{A}) \cap V$ for each $t \in \mathrm{split}(p)$.
- There is $C = \{l(n) \colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that
$$l(n+1) \in \bigcap \{y_t \colon t \in \mathrm{split}_{\leq l(n)+2}(p)\}.$$
- Construct a condition $q \leq p$ such that $q \Vdash \check{C} \subseteq \dot{Y}$.

Proof

- $y_t \in \text{fil}(\mathcal{A}) \cap V$ for each $t \in \text{split}(p)$.
- There is $C = \{l(n) \colon n \in \omega\} \in \text{fil}(\mathcal{A})$ such that
$$l(n+1) \in \bigcap \{y_t \colon t \in \text{split}_{\leq l(n)+2}(p)\}.$$
- Construct a condition $q \leq p$ such that $q \Vdash \check{C} \subseteq \dot{Y}$.
- Then $q \Vdash \dot{Y} \in \text{fil}(\mathcal{A})$ which is a contradiction.

$\square$

Question

Does the forcing notion $\mathbb{Q}(\mathcal{C})$ preserve Q-points?

Question

Does the forcing notion $\mathbb{Q}(\mathcal{C})$ preserve Q-points?

Question

Does the forcing notion $\mathbb{Q}(\mathcal{C})$ preserve some Q-point which is not a P-point?

Other work

Theorem (J.A. Cruz-Chapital–V. Fischer–O. Guzmán–J.Š.)

It is relatively consistent that

$$\mathrm{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2.$$

K. Ciesielski, J. Pawlikowski, Covering Property Axiom CPA. A combinatorial core of the iterated perfect set model, Cambridge Tracts in Math. 164, Cambridge Univ. Press, 2004.

D. Chodounský, V. Fischer, H. Grebík, Free sequences in $\mathcal{P}(\omega)/\mathrm{fin}$, Arch. Math. Logic 31 (7-8), 2019, pp. 1035–1051.

V. Fischer, D. C. Montoya, Ideals of Independence, Arch. Math. Logic 58 (5-6), 2019, pp. 767–785.

O. Guzmán, M. Hrušák, O. Téllez, Restricted mad families, J. Symbolic Logic 85 (2020), pp. 149–165.

M. Hrušák, Selectivity of almost disjoint families, Acta Univ. Carolin. Math. Phys. 41 (2000), pp. 13–21.

M. Hrušák, Another $\diamondsuit$-like principle, Fund. Math. 167 (2001), pp. 277–289.

A. Miller, Covering 2^ω with ω_1 disjoint closed sets, In J. Barwise, H. J. Keisler, K. Kunen, editors, The Kleene Symposium, Volume 101, Stud. Logic Found. Math., 1980, pp. 415–421.

J.T. Moore, M. Hrušák, M. Džamonja, Parametrized $\diamondsuit$ principles, Trans. Amer. Math. Soc. 356 (2004), pp. 2281–2306.

L. Newelski, On partitions of the real line into compact sets, J. Symbolic Logic 52 (1987), pp. 353–359.

S. Shelah, $Con(\mathfrak{u} > \mathfrak{i})$, Arch. Math. Logic 58 31 (6), 1992, pp. 433–443.

J. Stern, Partitions of the real line into F_σ or G_δ subsets, C. R. Acad. Sci. Paris Sér. A 284 (1977), pp. 921–922.

O. Spinas, Partition numbers, Ann. Pure Appl. Logic 90 (1-3), 1997, pp. 243–262.

J. Zapletal, Forcing idealized, Cambridge Tracts in Math. 174, Cambridge University Press, Cambridge, 2008.

Thanks for your attention!

jaroslav.supina@upjs.sk

Preprint 1

Preprint 2