Invariant Ideal Axiom

Michael Hrušák
joint with Alexander Shibakov

Centro de Ciencias Matemáticas
Universidad Nacional Autónoma de México
michael@matmor.unam.mx

Wien
December 2020

Fréchet and sequential spaces

- A topological space X is Fréchet if whenever $x \in \bar{A}$ then $x = \lim x_n$ for some $\{x_n : n \in \omega\} \subseteq A$.
- A space X is sequential if $A \subseteq X$ is not closed then there is $x \notin A$ such that $x = \lim x_n$ for some $\{x_n : n \in \omega\} \subseteq A$.
- Arens space - $S_2 = [\omega]^{\leqslant 2}$ where $U \subseteq S_2$ is open if and only if for every $s \in U$ the set $\{ s \cup \{n\} \in S_2 : s \cup \{n\} \notin U\}$ is finite.
- A sequential space X contains a copy of S_2 if and only if it is not Fréchet.
- Sequential fan - the quotient $S(\omega) = S_2/[\omega]^{\leqslant 1}$
- (van Douwen) The product of the sequential fan and the convergent sequence of closed discrete sets is not sequential
($\emptyset \times fin$ and $fin \times \emptyset$ generate $fin \times fin$).

Fréchet and sequential groups

- (Nyikos) Fréchet groups do not contain a copy of the sequential fan.
- Sequential groups that are not Fréchet contain a copy of the sequential fan.
- (Todorčević) There are X, Y such that $C_p(X)$ and $C_p(Y)$ are Fréchet but $C_p(X) \times C_p(Y)$ is not countably tight.

General question:

What is the structure and behavior under products of separable (countable) sequential (Fréchet) groups?

Fréchet groups/Malykhin's problem

Problem (Malykhin 1978)

Is there a separable (equivalently, countable) Fréchet group which is not metrizable?

Partial positive solutions:

- $\mathfrak{p} > \omega_1 \ldots$ Yes
- (Gerlits-Nagy 1982) There is an uncountable γ-set ... Yes
- (Nyikos 1989) $\mathfrak{p} = \mathfrak{b} \ldots$ Yes

Theorem (H.-Ramos García 2014)

It is consistent with ZFC that every separable Fréchet group is metrizable.

Sequential groups/Nyikos's problem

Problem (Nyikos 1981)

Is there a sequential group of intermedate sequential order?

Partial positive solutions (all due to Shibakov)

- (1996) Consistently . . . Yes
- (1998) CH . . . Yes

Theorem (Shibakov 2017)

It is consistent with ZFC that every sequential group is metrizable or has sequential order ω_1.

Invariant Ideal Axiom

IIA : *For every countable* groomed *topological group* $\mathbb{G}$ *and every* tame, invariant *ideal* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ *one of the following holds:*

1. *there is a countable* $\mathcal{S} \subseteq \mathcal{I}$ *such that for every infinite sequence* C *convergent in* $\mathbb{G}$ *there is an* $I \in \mathcal{S}$ *such that* $C \cap I$ *is infinite, (*$=$ *sequence capture)*
2. *there is a countable* $\mathcal{H} \subseteq \mathcal{I}^+$ *such that for every non-empty open* $U \subseteq \mathbb{G}$ *there is an* $H \in \mathcal{H}$ *such that* $H \setminus U \in \mathcal{I}$ *(*$=$ *almost* π*-net.).*

ω-hitting and tame invariant ideals

- An ideal $\mathcal{I}$ on a set X is ω-hitting if for every collection $\{X_n : n \in \omega\}$ of infinite subsets of X there is an element I of $\mathcal{I}$ having infinite intersection with all X_n.
- An ideal $\mathcal{I}$ is tame if for every $X \in \mathcal{I}^+$ and every $f : X \to \omega$ there is a partition $\{P_n : n \in \omega\}$ of ω into infinite pieces such that for every $I \in \mathcal{I} \restriction X$ there is an $n \in \omega$ such that $P_n \cap f[I] = \emptyset$,

  i.e. if no ideal Katětov below a restriction of $\mathcal{I}$ is ω-hitting.
- An ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ is invariant if both $g \cdot I = \{g \cdot h : h \in I\}$ and $I \cdot g = \{h \cdot g : h \in I\}$, and $I^{-1} = \{h^{-1} : h \in I\}$ are in $\mathcal{I}$ for every $I \in \mathcal{I}$ and $g \in \mathbb{G}$.

Groomed spaces

- Given a topological space X and $x \in X$ let

$$\mathcal{I}_x = \{A \subseteq X : x \notin \overline{A}\}.$$

- A subset Y of a topological space X is entangled if $\mathcal{I}_x \restriction Y$ is ω-hitting for every $x \in X$.
- A topological space X is groomed if it does not contain a dense entangled set.
- The class of groomed spaces includes all Fréchet and sequential spaces (also all subspaces of sequential spaces).

Invariant Ideal Axiom (repeated)

IIA: For every countable groomed topological group $\mathbb{G}$ and every tame, invariant ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ one of the following holds:

1. there is a countable $\mathcal{S} \subseteq \mathcal{I}$ such that for every infinite sequence C convergent in $\mathbb{G}$ there is an $I \in \mathcal{S}$ such that $C \subseteq^* I$, (= sequence capture)
2. there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every non-empty open $U \subseteq \mathbb{G}$ there is an $H \in \mathcal{H}$ such that $H \setminus U \in \mathcal{I}$ (= almost π-net.).

IIA for nice ideals

- For nice ideals such as nwd and scattered we can do better:

IIA for nice ideals : *For every countable groomed topological group $\mathbb{G}$ and every nice tame invariant ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ one of the following holds:*

1. *there is an $I \in \mathcal{I}$ such that for every $C \to 1_{\mathbb{G}}$ $C \subseteq^* I$ (= sequence capture)*
2. *there is a countable $\mathcal{H} \subseteq \mathcal{I}^+$ such that for every open U nbhd of $1_{\mathbb{G}}$ there is an $H \in \mathcal{H}$ such that $H \subseteq U$ (= local π-network).*

For others, i.e. compact we can get local π-network in (2) but need countably many I in (1).

nwd in Fréchet groups

Lemma (Barman-Dow)

Given a point x in a Fréchet space X and a family $\{N_i : i \in \omega\} \subseteq nwd$ there is a $C \to x$ such that $C \cap N_i$ is finite for every $i \in \omega$.

Proof.

- Pick $\{x_i : i \in \omega\} \to x$.
- For $i \in \omega$ pick $C_i \to x_i$ such that $C_i \cap \bigcup_{j \leqslant i} N_j = \emptyset$.
- $C \subseteq \bigcup_{i \in \omega} C_i$ such that $C \to x$. □

Corollary

In a Fréchet group, nwd is a tame invariant ideal.

Proof.

- Let $f : X \to \omega$, $X \in nwd^+$. WLOG $f^{-1}(n) \in nwd$ for all $i \in \omega$.
- Fix for $g \in \mathbb{X}$, $C_g \subseteq X$, $C_g \to g$ such that $f \restriction C_g$ is finite-to-one.
- $\{P_n : n \in \omega\}$ disjoint refinement of $\{f[C_g] : g \in \mathbb{G}\}$. □

Simple consequences of IIA

Theorem

Assuming IIA, *every countable (separable) Fréchet group is metrizable.*

Proof.

As *nwd* is nice, and every dense open set contains a sequence convergent to $1_{\mathbb{G}}$, (1) of IIA fails, so there is a countable family $\mathcal{X}$ of somewhere dense subsets of $\mathbb{G}$ such that every open set contains an element of $\mathcal{X}$. Then

$$\{\mathrm{int}(\overline{X}) : X \in \mathcal{X}\},$$

form a ctble π-base, and as π-weight and weight coincide in topological groups, the group $\mathbb{H}$ is first countable hence metrizable. □

Corollary

Assuming IIA, $\mathfrak{p} = \omega_1$ *and* $\mathfrak{b} > \omega_1$.

Some restrictions are necessary

Proposition

There is a countable topological group $\mathbb{G}$ and a tame invariant ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that IIA *fails for $\mathbb{G}$ and $\mathcal{I}$.*

$\{A_\alpha, B_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$, $A_\alpha \subseteq^* A_\beta \subseteq^* B_\beta \subseteq^* B_\alpha$ - a Hausdorff gap

$\mathbb{G} = [\omega]^{<\omega}$ with $[F]^{<\omega}$ open nbhds of $\emptyset$, with $B_\alpha \subseteq^* F$ for some $\alpha < \omega_1$, and let $\mathcal{I} = \{A \subseteq \mathbb{G} : \ \forall \alpha < \omega_1 \ \bigcup A \subseteq^* B_\alpha\}$.

- $\mathcal{I}$ is a tame invariant ideal: No restriction of $\mathcal{I}$ to a positive set is tall, and $\mathcal{I}$ is invariant as $\bigcup a \triangle I =^* \bigcup I$ for every $I \in \mathcal{I}$ and $a \in \mathbb{G}$.
- (1) of IIA fails for $\mathcal{I}$: $C \to 0$ if and only if C is a point-finite family of finite sets and $C \in \mathcal{I}$. ...
- (2) fails: $X \in \mathcal{I}^+$ iff $\bigcup X \setminus B_\alpha$ is infinite for some $\alpha < \omega_1$, and having ctbly many such X, there is an α which is a witness for all of them, hence none of them is mod $\mathcal{I}$ contained in $[B_\alpha]^{<\omega}$.

The consistency of IIA (very rough sketch)

Theorem

IIA *is relatively consistent with* ZFC.

Proof.

Let $V = L$. Construct a finite support iteration $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ so that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a $\mathbb{L}_{\mathcal{I}^*}$ where $\mathcal{I}$ is some tame invariant ideal on a ctble groomed group $\mathbb{G}$ such that both (1) and (2) of IIA fail (handed to us by a bookkeeping using $\diamondsuit(E^2_1)$).
Then:

- $\mathbb{L}_{\mathcal{I}^*}$ adds a dense entangled $D \subseteq \mathbb{G}$
- D remains dense and entangled by preservation theorems from (H.–Ramos-García)
- All 'reflects' so every candidate is trapped.

□

The main result

Theorem

Assuming IIA, *every countable sequential group is either metrizable or* k_ω*.*

- A topological space X is k_ω if there is a countable family $\mathcal{K}$ of compact subsets of X such that a set U is open if and only if its intersection with every $K \in \mathcal{K}$ is relatively open in K.
- k_ω groups do not contain a convergent sequence of closed discrete sets.
- Countable k_ω groups are definable objects, they have $F_{\sigma\delta}$ topologies, and are completely classified by their compact scatteredness rank defined as the supremum of the Cantor-Bendixson index of their compact subspaces by the theorem of Zelenyuk:

Theorem (Zelenyuk 1995)

Countable k_ω *groups of the same compact scatteredness rank are homeomorphic.*

k_ω-groups

- $\alpha < \omega_1$ let $\mathcal{K}_\alpha$ be a fixed countable family of compact subsets of the rationals $\mathbb{Q}$ closed under translations, inverse and algebraic sums such that $\omega^\alpha = \sup\{\mathrm{rank}_{CB}(K) : K \in \mathcal{K}_\alpha\}$, and let

$$\tau_\alpha = \{U \subseteq \mathbb{Q} : \ \forall K \in \mathcal{K}_\alpha : U \cap K \text{ is open in } K\}.$$

- τ_0 is the discrete topology on $\mathbb{Q}$,
- τ_α is a k_ω sequential group topology on $\mathbb{Q}$ and $\mathbb{Q}_\alpha = (\mathbb{Q}, \tau_\alpha)$.
- $\mathbb{Q}$ is determined by taking into account *all* of its compact subsets, so it makes sense to denote it as $\mathbb{Q}_{\omega_1}$.

Corollary

Assuming IIA, *for every infinite countable sequential group* $\mathbb{G}$ *there is exactly one* $\alpha \leqslant \omega_1$ *such that* $\mathbb{G}$ *is homeomorphic to* $\mathbb{Q}_\alpha$*.*

“ Proof ” of the main result

Theorem

Assuming IIA, *every countable sequential group is either metrizable or* k_ω.

Proof

Assume $\mathbb{G}^2$ sequential (the general case is much harder) and $\mathbb{G}$ not metrizable (hence not Fréchet).

- $\mathbb{G}$ contains a copy of $S(\omega)$.
- Consider IIA with $\mathbb{G}$ and cpt - the ideal generated by compact sets
- Sequence capturing $\Leftrightarrow$ k_ω.
- local π-base $\Rightarrow$ there is a sequence $\{C_n : n \in \omega\}$ of infinite closed discrete sets such that $1_{\mathbb{G}}$ is the only accumulation point of $\bigcup_{n\in\omega} C_n$ and every nbhd of $1_{\mathbb{G}}$ contains one of the C_n's.
- Then $\mathbb{G}^2$ is NOT sequential (the “diagonal product” of $\{C_n : n \in \omega\}$ and $S(\omega)$ has the point $(1_{\mathbb{G}}, 1_{\mathbb{G}})$ in its closure but contains no convergent sequence). □

Products of sequential groups under IIA

Corollary

Assume IIA.

1. *The product of countably many separable Fréchet groups is Fréchet, and*
2. *the product of finitely many countable sequential groups which are either discrete or not Fréchet is sequential.*

Proof.

It suffices to note that

1. The product of two k_ω groups is k_ω.
2. The product of two metrizable groups is metrizable
3. The product of a k_ω group and a discrete group is k_ω
4. The product of a k_ω group and a non-discrete metrizable group is not sequential.

□

Open problems

1. Is it consistent (follows from IIA) that every countable group is either metrizable, k_ω or contains a dense set without a convergent subsequence?
2. Is there (in ZFC) a Fréchet group whose square is not Fréchet?
3. Is there a sequential group whose square is not sequential?
4. Does IIA imply that every countably compact sequential group is metrizable? every sequential group has sequential order 1 or ω_1? If not, how to strengthen IIA to an axiom that does?
5. What about IIA for definable groups?

Definable ideal dichotomy

Theorem

Given a filter $\mathcal{F}$ and a definable ideal $\mathcal{I}$ on ω there is either

1. *an $\mathcal{F}^+$-branching tree with all branches in $\mathcal{I}$, or*
2. *an $\mathcal{F}$-branching tree with all branches in $\mathcal{I}^+$.*

- A tree $T \subseteq \omega^{<\omega}$ is $\mathcal{X}$-branching if $\emptyset \in T$ and

$$succ_T(t) = \{n : t^\frown n \in T\} \in \mathcal{X}$$

for every $t \in T$.

- $[T] = \{f \in \omega^\omega : \ \forall n \ (f \restriction n \in T)\}$.
- All branches in $\mathcal{X}$ abbreviates $\forall f \in [T] \ (\mathrm{rng}(f) \in \mathcal{X})$.

Definable ideal dichotomy

Theorem

Given a filter $\mathcal{F}$ and a definable ideal $\mathcal{I}$ on ω there is either

1. *an $\mathcal{F}^+$-branching tree with all branches in $\mathcal{I}$, or*
2. *an $\mathcal{F}$-branching tree with all branches in $\mathcal{I}^+$.*

Proof.

Consider the following infinite two-player game:

I	$X_0 \in \mathcal{F}^+$		$\dots$	$X_n \in \mathcal{F}^+$			$\dots$
II		$j_0 \in X_0$	$\dots$		$j_n \in X_n$		

I winning if $\{j_n : n \in \omega\} \in \mathcal{I}$

□

Definable ideal trichotomy

Theorem

Given a filter $\mathcal{F}$ and a definable ideal $\mathcal{I}$ on ω then exactly one of the following holds:

1. *There is an $\mathcal{F}$-branching tree with all branches in $\mathcal{I}^+$,*
2. *there is an $\mathcal{F}$-branching tree with all branches in $\mathcal{I}$.*
3. *there is an $\mathcal{F}^+$-branching tree with all branches in $\mathcal{I}$ and an $\mathcal{F}^+$-branching tree with all branches in $\mathcal{I}^+$.*

Strong form of IIA from DID

Theorem (LC)

Let X be a countable definable space, $x \in X$ and $\mathcal{J}$ an ideal on X. Either

1. *there is a ctble $\mathcal{S} \subseteq \mathcal{J}$ such that every sequence convergent to x is contained in an element of $\mathcal{S}$, or*
2. *there is a countable local π-network consisting of $\mathcal{J}$-positive sets.*

Proof.

Consider the Definable Ideal Dichotomy for $\mathcal{F} = \mathcal{J}^*$ and $\mathcal{I}$ - the ideal generated by sequences convergent to x.

- If there is an $\mathcal{F}^+$-branching tree T with all branches in $\mathcal{I}$ then $\{succ_T(t) : t \in T\}$ forms a local π-network.
- If there is an $\mathcal{F}$-branching tree with all branches in $\mathcal{I}^+$ then $\{X \setminus succ_T(t) : t \in T\}$ captures convergent sequences.

□

Consequences of the definable IIA

- (Todorčević-Uzcátegui) Every definable countable Fréchet group is metrizable.
- (Shibakov) Every definable countable sequential group is metrizable or k_ω, i.p. has sequential order 1 or ω_1.
- (Shibakov) Every definable countable Fréchet space has a ctble π-base. (compare to...)
- (Dow) There is a countable Fréchet space with unctble π-base.

Selective ultrafilters - one more application of DID

Lemma

An ultrafilter $\mathcal{U}$ on ω is selective if and only if for every $\mathcal{U}$-branching tree T there is a $U \in \mathcal{U}$ such that every infinite subset of U is a branch of T.

Theorem (Mathias)

An ultrafilter $\mathcal{U}$ on ω is selective if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every tall Borel ideal $\mathcal{I}$.

Proof.

Consider the DID for $\mathcal{F} = \mathcal{F}^+ = \mathcal{U}$ and a Borel ideal $\mathcal{I}$.

- If there is an $\mathcal{F}^+$-branching tree T with all branches in $\mathcal{I}$ then $\mathcal{U} \cap \mathcal{I} \neq \emptyset$.
- If there is an $\mathcal{F}$-branching tree T with all branches in $\mathcal{I}^+$ then CONTRADICTION!.

□

DID and ultrafilters

Proposition

For every ultrafilter $\mathcal{U}$ and every Borel ideal $\mathcal{I}$ there is either

1. *a $\mathcal{U}$-branching tree with all branches in $\mathcal{I}$, or*
2. *a $\mathcal{U}$-branching tree with all branches in $\mathcal{I}^+$.*

Question

- Is there an ultrafilter $\mathcal{U}$ such that for every Borel ideal there is a $\mathcal{U}$-branching tree with all branches in $\mathcal{I}^+$?
- What does the dichotomy say for some well-known ideals/classes of ultrafilters?

That's all!

Thank you for your attention!