Forcing against bounded arithmetic

Moritz Müller

joint with Albert Atserias

The problem

Given

M nonstandard model of arithmetic.

R new binary relation symbol.

Goal

expansion (M, R^M) such that

(a) R^M does something prohibitive

e.g. R^M is a bijection from $[n+1]$ onto $[n]$ for some $n \in M$

(b) much of arithmetic is preserved

i.e. (M, R^M) satisfies LNP for a large class of formulas

Results

(M, R^M) with R^M a bijection from $[n+1]$ onto $[n]$

Paris, Wilkie 1985

... and LNP for existential formulas.

Riis 1994

... and LNP for $\exists\Delta_0^{b_0}(R)$ any $b_0 < n^{o(1)}$.

$\Delta_0^{b_0}(R)$: formulas in language with R, only b_0-bounded quantifiers $\exists x{<}b_0, \forall x{<}b_0$

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Some background on these results

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$T^1_2(R) \nvdash \forall x PHP(R,x)$.

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> *we are not satisfied with current methods of proving independence results. The main reason is that, except for Gödel's theorem which gives only special formulas, no general method is known to prove independence of Π_1 sentences.* Pudlák 1996

Some comments on Ajtai's proof

Ajtai's argument is

> *"done according to the general ideas of Cohen's method of forcing"*
> (Ajtai)
>
> *"mostly combinatorial or probabilistic"*
> (Ajtai)
>
> *"similar to the terminology of forcing but we actually do not use any result from it"*
> (Ajtai)
>
> *"extremely diffcult to understand and explain"*
> (Ben-Sasson, Harsha)
>
> *"[the start of] contemporary research in lower bounds for propositional proofs"*
> (Krajíček)

Set theoretic forcing

M a countable transitive model of ZFC, $(P, \leq) \in M$

with generic filter $G \subseteq P$ associate $M[G]$

Principal Theorem $M[G] \models$ ZFC.

Forcing (semantic)

$p \Vdash \varphi$ iff for every generic filter G with $p \in G$: $M[G] \models \varphi$.

forcing language: $\in$ plus constants M.

Extension if $q \leq p \Vdash \varphi$, then $q \Vdash \varphi$.

Stability if $p \Vdash \neg\neg\varphi$, then $p \Vdash \varphi$.

Truth $M[G] \models \varphi$ iff $p \Vdash \varphi$ for some $p \in G$.

Definability $\Vdash$ is in a certain sense definable in M.

Set theoretic forcing

Definability for every $\varphi(\bar{x})$ the set $\{p\bar{a} \mid p \Vdash \varphi(\bar{a})\}$ is definable in M.

Forcing (syntactic) by universal recurrence:

$$\begin{aligned} p \Vdash \forall x \varphi(x) &\iff \forall a \in M : p \Vdash \varphi(a) \\ p \Vdash (\varphi \wedge \psi) &\iff p \Vdash \varphi \ \& \ p \Vdash \psi \\ p \Vdash \neg\varphi &\iff \forall q \le p : q \nVdash \varphi \\ p \Vdash \text{atom} &\iff ? \end{aligned}$$

Forcing Completeness

The syntactic and semantic definitions of forcing are equivalent.

“A simple forcing argument” Paris, Wilkie 1985

Given M countable model of arithmetic, $n \in M \setminus \mathbb{N}$.

Want bijection R^M from $[n+1]$ onto $[n]$ such that

$$(M, R^M) \models \text{LNP for existential formulas.}$$

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"forcing frame": finite partial bijections from $[n+1]$ onto $[n]$

construct $\emptyset = p_0 \subseteq p_1 \subseteq \cdots \subseteq R^M := \bigcup_i p_i$

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choice of p_{2i}: some $q \supseteq p_{2i-1}$ that has

- "ith" element of $[n+1]$ in domain,
- "ith" element of $[n]$ in range

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choice of p_{2i+1}:

- let $\varphi(x)$ be the ith existential $L_{\mathsf{PA}}(M) \cup \{R\}$-formula
- choose M-minimal b such that

$$(M, R^M) \models \varphi(b) \text{ for some bijection } R^M \supseteq p_{2i}$$

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$$\Longleftrightarrow$$

$$\text{some } q \supseteq p_{2i} \text{ forces } \varphi(b)$$

$$\text{i.e. } (M, R^M) \models \varphi(b) \text{ for all bijections } R^M \supseteq q.$$

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p is compatible with $\varphi(b)$

Forcing in general: universal forcings

M a countable L-structure

$(P, \leq, D_0, D_1, \ldots)$ countable forcing frame

Forcing language $L^* \supseteq L$ plus constants M

Forcing syntactic definition via universal recurrence

$p \Vdash$ atom $\iff$?

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Lemma Extension and Stability hold for all φ.

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Lemma Extension and Stability hold for all φ.

conservative $\quad p \Vdash L(M)$-atom $\iff M \models L(M)$-atom.

Forcing in general: genericity

Want intersect many dense subsets P.

e.g. in set theory: all dense sets from M

e.g. Feferman, Robinson. . . : all sets of the form $[\varphi] \cup [\neg\varphi]$ where

$$[\varphi] := \{p \mid p \Vdash \varphi\}.$$

Need e.g. $\bigcap_{a \in M} \bigcup_{b \in M} [\varphi(a, b)]$.

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The Stern formalism (Stern 1975, Knight 1973).

Two sorted structure (P, M):

- first sort carries $(P, \leq, D_0, D_1, \ldots)$
- second sort carries L-structure M
- for every L^*-atom $\varphi(x_1, \ldots, x_r)$ add relation symbol of sort $P \times M^r$ denoting $\{p\bar{a} \mid p \Vdash \varphi(\bar{a})\}$

generic: intersect all dense sets definable in (P, M).

Forcing in general: generic associates

Given $G \subseteq P$ generic filter.

Want $M[G] \models \mathsf{Th}(G) := \{\varphi \mid \exists p \in G : p \Vdash \varphi\}$.

Need $\mathsf{Th}(G)$ respects equality axioms

for all closed terms t there is $a \in M$ such that $t = a \in \mathsf{Th}(G)$

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Then $M[G]$ is defined and

$$M[G] := \text{Henkin term model of Th}(G)$$

is the generic associate of G.

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- **Truth**
- **Forcing Completeness** assuming $M[G]$ is defined for every generic G.
- if $\Vdash$ is conservative, then $M[G]$ is an expansion of M

(identifying $a \in M$ with its term congruence class)

Principal Theorems

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M satisfies LNP

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$\Vdash$ is definable for $\varphi(\bar{x})$ iff

$\forall p \in P : \{\bar{a} \mid p \| \varphi(\bar{a})\}$ is M-definable.

$\Vdash$ is densely definable for $\varphi(\bar{x})$ up to b_0 iff

$\forall p \in P \exists r \leq p : \{\bar{a} < b_0 \mid r \| \varphi(\bar{a})\}$ is M-definable.

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Principal Theorem

If $\Vdash$ is densely definable for all $\varphi \in \Phi$ up to b_0,

then every generic expansion of M satisfies LNP Φ up to b_0.

Paris Wilkie forcing

M nonstandard model of true arithmetic, $n \in M \setminus \mathbb{N}$,

$L^* := L \cup \{R\}$.

$P :=$ finite partial bijections from $[n+1]$ onto $[n]$ coded in M.

$D_0, D_1, \ldots$ enumerate $\{p \mid a \in \mathsf{dom}(p)\}, \{p \mid b \in \mathsf{im}(p)\}$ for $a \in [n+1], b \in [n]$.

Set $p \Vdash Rst \iff (s^M, t^M) \in p$

This determines a universal conservative forcing.

$M[G] \cong (M, \bigcup G)$ violates PHP for every generic G.

Suffices to show: $\Vdash$ is definable for existential $L^*(M)$-formulas.

The method of definable antichains

X maximal antichain in $[\varphi]$. Then $p||\varphi \iff \exists q \in X : p||q$.

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A refines X iff for every $p \in A$

$$\exists q \in X : p \| q \quad \iff \quad \exists q \in X : p \leq q.$$

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Antichain Lemma

X maximal ac in $[\varphi]$,

A maximal ac refining X,

then $A \setminus (A \downarrow X)$ maximal ac in $[\neg\varphi]$.

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X_a maximal ac in $[\neg\varphi(a)]$ for every $a < b_0$,

A maximal ac refining $\bigcup_{a<b_0} X_a$,

then $A \setminus (A \downarrow \bigcup_{a<b_0} X_a)$ maximal ac in $[\forall x < b_0 \varphi(x)]$.

Ajtai forcing

M nonstandard model of true arithmetic, $b_0 < 2^{n^{o(1)}}$

Ajtai's Theorem

There is a bijection R^M from $[n+1]$ onto $[n]$ st. $(M, R^M) \models$ LNP $\Delta_0^{b_0}(R)$.

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$P(n) :=$ partial bijections from $[n+1]$ onto $[n]$ coded in M.

$P := \bigcup_{\ell \in \mathbb{N}} \{p \in P(n) \mid M \models \mathit{Card}(p) < n - n^{1/\ell}\}$.

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i.e. for every b_0-bounded $\varphi(x)$

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i.e. for every b_0-bounded $\varphi(x)$

$$\forall p \in P \; \exists r \leq p : \{\bar{a} < b_0 \mid r \| \varphi(\bar{a})\} \text{ is } M\text{-definable.}$$

Find in M a sequence $(X_{\bar{a}})_{\bar{a} < b_0}$ of maximal ac in $[\varphi(\bar{a})]{\downarrow} r$

Construct such a sequence by recursion on φ.

Finite combinatorics: Switching Lemma

Given $(X_{\bar{a}})_{\bar{a}<b_0}$ find r and $(A_{\bar{a}})_{\bar{a}<b_0}$ in M such that:

$$A_{\bar{a}} \text{ maximal ac in } P{\downarrow}\, r, \text{ refines } X_{\bar{a}} \cup r := \{p \cup r \mid r \| p \in X_{\bar{a}}\}$$

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Switching Lemma

$X_0, \ldots, X_{N-1} \subseteq P(m)$, elements of X_i have card $\leq k$, $\ell < m$

For some $r \in P(m)$ of card $\leq m - \ell$ there are $A_0, \ldots, A_{N-1}$:

- $A_i \subseteq P(n){\downarrow}\, r$ is an ac refining $X_i \cup r$
- elements of A_i have card $\leq 2k$ larger than r
- A_i is $2k$-predense below r

(every $q \leq r$ of card $\leq m - 2k$ is compatible with some $p \in A_i$)

provided that $\dfrac{(m-\ell)^k}{(\ell+1)^{4k} \cdot k^{3k}} > N$.