Actions of automorphism groups of Fraïssé limits on the space of linear orderings

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joint work with Todor Tsankov

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Toy model

The random graph R.
Construction : Take $\mathbb{N}$ as a domain (vertices) and put an edge between two points with probability $1/2$. Almost surely you obtain one structure (up to isomorphism), call it R.

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If you denote $\mathrm{LO}(\mathbb{N})$ the space of linear orderings on $\mathbb{N}$, then there is an action $\mathrm{Aut}(R) \curvearrowright \mathrm{LO}(\mathbb{N})$ in the following way :

$$a(g\cdot <)b \Leftrightarrow g^{-1}a < g^{-1}b$$

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$$\mu(g \cdot A) = \mu(A).$$

Here the invariant measure is the one such that

$$\mu(x_1 < \cdots < x_n) = \frac{1}{n!}.$$

More on Fraïssé limits

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Theorem (Fraïssé '54)

A Fraïssé class $\mathcal{F}$ admits a Fraïssé limit $\mathbb{F}$, i.e. a countable homogeneous structure such that $\mathrm{Age}(\mathbb{F})$, the class of finite structures embeddable in $\mathbb{F}$, is exactly $\mathcal{F}$.

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Examples :

Fraïssé class	Fraïssé limit	Aut. group
finite graphs	Random graph	$\mathrm{Aut}(R)$
finite sets	$\mathbb{N}$	S_∞
finite linear orderings	$(\mathbb{Q}, <)$	$\mathrm{Aut}(\mathbb{Q})$
finite partial orderings	The generic poset $\mathcal{PO}$	$\mathrm{Aut}(\mathcal{PO})$
finite complete partite graphs	ω-partite graph	$\mathrm{Aut}(\mathrm{Part})$

Some dynamics

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Examples : If $G = \mathrm{Aut}(\mathbb{F})$ for some Fraïssé limit $\mathbb{F}$, then there are two remarkable G-flows.

1) $G \curvearrowright [0,1]^{\mathbb{F}}$ by permuting the coordinates.
   This flow always admits some invariant measures of the form $\nu^{\mathbb{F}}$ for some ν measure on $[0,1]$.
2) $G \curvearrowright \mathrm{LO}(\mathbb{F})$ as before. The invariant measure mentioned before is also an invariant measure for G.

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Remark : There can be more invariant measures than these.

Some Dynamics

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A G- flow is minimal if it admits no proper subflow.

Theorem (Ellis '69)

There exists a unique universal minimal flow (UMF) $\mathrm{M}(G)$.

This means that for any minimal G-flow $G \curvearrowright X$, there is a surjective G-map from $\mathrm{M}(G)$ to X.

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- S_∞, $\mathrm{Aut}(\mathbb{Q})$ (and all extremely amenable groups).
- Angel, Kechris and Lyons prove that $\mathrm{Aut}(R)$ is uniquely ergodic.

Question (Angel, Kechris, Lyons '12)

If G is amenable with metrizable UMF, is G uniquely ergodic?

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If G is amenable with metrizable UMF, is G uniquely ergodic ?

Problem : Finding G with interesting $\mathrm{M}(G)$.

Computing UMFs - The Kechris-Pestov-Todorcevic correspondence

Theorem (Kechris-Pestov-Todorcevic, '05)

Let $\mathbb{F}$ be a Fraïssé limit, $\mathrm{Aut}(\mathbb{F})$ is extremely amenable iff $\mathrm{Age}(\mathbb{F})$ has the Ramsey property.

If G admits a "nice enough" extremely amenable subgroup G^*, then

$$M(G) = \widehat{G/G^*}.$$

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Theorem (Ben Yaacov-Melleray-Nguyen Van Thé-Tsankov '14-'17, Zucker '14)

G has metrizable UMF iff there exists $G^* \leq G$ extremely amenable such that

$$M(G) = \widehat{G/G^*}.$$

The class of finite linear orderings has the Ramsey property, therefore (Pestov) $\mathrm{Aut}((\mathbb{Q},<))$ is extremely amenable.
If $G = S_\infty$ then $G^* = \mathrm{Aut}(\mathbb{Q})$ and $\mathrm{M}(S_\infty) = \mathrm{LO}(\mathbb{N})$.

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For the limit of the class of partite complete graphs, the UMF of the automorphism group is the space of linear orderings for which each part is an interval.
For the limit of the class of partial orderings, the UMF of the automorphism group is the space of linear orderings extending the generic poset.

Main result

Theorem (J.)

Let $\mathbb{F}$ be a transitive, ω-categorical Fraïssé limit with no algebraicity that admits weak elimination of imaginaries. Denote $G = \mathrm{Aut}(\mathbb{F})$ and consider the action $G \curvearrowright \mathrm{LO}(\mathbb{F})$. Then exactly one of the following holds :

1. *The action $G \curvearrowright \mathrm{LO}(\mathbb{F})$ has a fixed point (i.e., there is a definable linear order on $\mathbb{F}$);*
2. *The action $G \curvearrowright \mathrm{LO}(\mathbb{F})$ is uniquely ergodic.*

Discussing the hypothesis

a) ω-categoricity : for any $n \in \mathbb{N}$ there are finitely many n-types. This is the only hypothesis we are not sure is necessary. Allows us to use a theorem of Tsankov on group representations (Tsankov '12).

Discussing the hypothesis

b) No algebraicity : fixing finitely many points in the structure fixes no other point.

Counterexample : Take $\mathbb{F}$ the countable-dimensional vector space over $\mathbf{F}_2$, the $\mathrm{M}(Aut(\mathbb{F}))$ is a proper subflow of $LO(\mathbb{F})$ (KPT) and the group is uniquely ergodic (AKL).
The group therefore admits at least two invariant measures on $\mathrm{LO}(\mathbb{F})$: the uniform and the one supported on a proper subflow. There is also no definable ordering on $\mathbb{F}$.

Discussing the hypothesis

c) Weak elimination of imaginaries : for every proper, open subgroup $V < G$, there exists k and a tuple $\bar{a} \in M^k$ such that $G_{\bar{a}} \leq V$ and $[V : G_{\bar{a}}] < \infty$.

Counterexample : ω-partite complete graph : we saw that again the UMF of its automorphism group is a proper subflow of $\mathrm{LO}(\mathbb{F})$ and it also is the support for a measure.

Discussing the hypothesis

d) Transitivity : for any $a, b \in \mathbb{F}$, there is $g \in G$ such that $g(a) = b$.

Counterexample : Take $\mathbb{N}$ with two unary predicates P, Q. Consider the measure that orders elements of P above elements of Q and orders each part uniformly.

Consequences of the result

a) Recovers a lot of known unique ergodicity results.

The random graph, the homogenenous K_n-free graph, the generic tournament...

Consequences of the result

b) Since there is always one invariant fully supported measure on $\mathrm{LO}(\mathbb{F})$, this allows us to prove non-amenability results.

Corollary

Suppose that $\mathbb{F}$ satisfies the assumptions of the Theorem and let $G = \mathrm{Aut}(\mathbb{F})$. If the action $G \curvearrowright \mathrm{LO}(\mathbb{F})$ is not minimal and has no fixed points, then G is not amenable.

Applies for instance for the generic poset (Kechris and Sokić).

Consequences of the result

c) Allows us to get combinatorial results

Corollary

Suppose that $\mathbb{F}$ satisfies the assumptions of the Theorem. If $\mathbb{F}$ has the Hrushovski property, then it has the ordering property, i.e. for every $A \in \mathrm{Age}(\mathbb{F})$, there exists $B \in \mathrm{Age}(\mathbb{F})$ such that for any two linear orders $<$ and $<'$ on A and B respectively, there is an embedding of $(A, <)$ into $(B, <')$.

A very important ingredient

The proof of this relies on

Theorem (Tsankov)

Let $\mathbb{F}$ be an ω-categorical structure with no algebraicity and weak elimation of imaginaries. Then the only $\mathrm{Aut}(\mathbb{F})$-ergodic invariant measures on $[0,1]^{\mathbb{F}}$ are of the type $\nu^{\mathbb{F}}$, where ν is a Borel measure on $[0,1]$.

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Counterexample : ω-partite complete graph. Denote E the equivalence relation being in the same part. There is a map from $[0,1]^{\mathbb{F}/E}$ to $[0,1]^{\mathbb{F}}$. The pushfoward of $\nu^{\mathbb{F}/E}$ to $[0,1]^{\mathbb{F}}$ is not of the form $\nu^{\mathbb{F}}$.

(Sketch of) proof

$G = \mathrm{Aut}(\mathbb{F})$.
Step 1 : An efficient way to produce measures on $\mathrm{LO}(\mathbb{F})$.

Consider the map $\rho\colon [0,1]^{\mathbb{F}} \to \mathrm{LO}(\mathbb{F})$ where $a <_{\rho(x)} b \Leftrightarrow x(a) < x(b)$.
For any atomless measure λ on $[0,1]$, ρ is $\lambda^{\mathbb{F}}$-a.s. well-defined. We therefore have a measure $\mu_\lambda = \rho_* \lambda^{\mathbb{F}}$.

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For any atomless measure λ on $[0,1]$, ρ is $\lambda^{\mathbb{F}}$-a.s. well-defined. We therefore have a measure $\mu_\lambda = \rho_* \lambda^{\mathbb{F}}$.
Moreover μ_λ is S_∞-invariant, so

$$\mu_\lambda(x_1 < \ldots < x_n) = \frac{1}{n!}.$$

Therefore μ_λ does not depend on λ and we really produced just one measure.

Step 2 : Proving that all measures are produced this way or exhibiting a fixed point of the action.

Take μ a G-invariant ergodic measure on $\mathrm{LO}(\mathbb{F})$, i.e. an extreme point of the set of G-invariant measures on $\mathrm{LO}(\mathbb{F})$.
We want a map from $\mathrm{LO}(\mathbb{F})$ to $[0,1]^{\mathbb{F}}$ that reverses ρ and pushes μ to some $\lambda^{\mathbb{F}}$.

We want to associate a number to each $a \in \mathbb{F}$ and each ordering.

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We want to associate a number to each $a \in \mathbb{F}$ and each ordering.
First idea : associate to a, $<_x$ the number

$$\lim_{n\to\infty} \frac{\#\{b \in F_n \ : \ b <_x a\}}{\#F_n}$$

where F_n is an enumeration of $\mathbb{F}$.

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where F_n is an enumeration of $\mathbb{F}$.
Problem : this is not well defined.

Solution :

Consider τ a 2-type and $a \in \mathbb{F}$, we call

$$D_\tau(a) = \{b \in \mathbb{F} \colon \mathrm{tp}(ab) = \tau\}.$$

Lemma

Let $a \in \mathbb{F}$ and τ a 2-type. Take $A \subset D_\tau(a)$ be a definable, infinite set. Then for μ-a.a. x,

$$\lim_{n\to\infty} \frac{\#\{b \in F_n \cap A \ : \ \ b <_x a\}}{\# F_n \cap A}$$

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exists and does not depend on A.

Consequence of Tsankov's Theorem.

We can now define for almost all $x \in \mathrm{LO}(\mathbb{F})$

$$\eta_a^\tau(x) = \lim_{n\to\infty} \frac{\#\{b \in F_n \cap D_\tau(a) \ : \ b <_x a\}}{\# F_n \cap D_\tau(a)}.$$

Lemma

If we denote λ the distribution of η_a^τ, then the family $(\eta_a^\tau)_{a\in\mathbb{F}}$ has distribution $\lambda^{\mathbb{F}}$.

This is again a consequence of Tsankov's Theorem.

We want to prove (if possible) that a.s.

1) λ is atomless.

2) For all $a, b \in \mathbb{F}$, we have

$$a < b \Leftrightarrow \eta_a^\tau < \eta_b^\tau.$$

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1) λ is atomless.

2) For all $a, b \in \mathbb{F}$, we have

$$a < b \Leftrightarrow \eta^{\tau}_{a} < \eta^{\tau}_{b}.$$

1) is not always true, we will have to assume it (for now), and prove 2).

Remark

If $D_\tau(a) \cap D_\tau(b) \neq \varnothing$, then

$$
\begin{aligned}
a < b \Rightarrow & \{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \ : \ c < a\} \\
& \quad \subset \{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \ : \ c < b\} \\
\Rightarrow & \frac{\#\{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \ : \ c < a\}}{\# D_\tau(a) \cap D_\tau(b) \cap F_n} \\
& \quad \leq \frac{\#\{c \in D_\tau(a) \cap D_\tau(b) \cap F_n \ : \ c < b\}}{\# D_\tau(a) \cap D_\tau(b) \cap F_n} \\
\Rightarrow & \eta_a^\tau \leq \eta_b^\tau .
\end{aligned}
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$$
\begin{tikzcd}
& c & \\
a \arrow[ur, "\tau"] & & b \arrow[ul, "\tau"']
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Lemma

If for all $a, b \in \mathbb{F}$ *we have* $\mu(\eta_a^\tau = \eta_b^\tau) = 0$*, then we have a.s. for all* $a, b \in M$ *:*

$$a < b \Leftrightarrow \eta_a^\tau < \eta_b^\tau.$$

Facts :

1) We only have to show that $\eta_a^\tau < \eta_b^\tau \Rightarrow a < b$.
2) The above remark becomes : if $D_\tau(a) \cap D_\tau(b) \neq \varnothing$, then

$$a < b \Leftrightarrow \eta_a^\tau < \eta_b^\tau.$$

If $D_\tau(a) \cap D_\tau(b) = \varnothing$, our hypothesis imply that there are infinitely many "alternating τ-paths" between a and b.

$$
\begin{tikzcd}
 & y_1 & & y_3 & & \cdots & & y_{2n-1} & \\
a \arrow[ur, "\tau"] & & y_2 \arrow[ul, "\tau"'] \arrow[ur, "\tau"] & & y_4 \arrow[ul, "\tau"'] & & y_{2n-2} \arrow[ur, "\tau"] & & b \arrow[ul, "\tau"']
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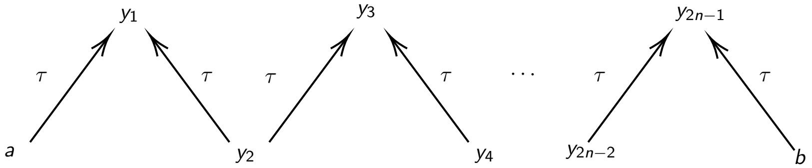

Since $\eta_a^\tau < \eta_b^\tau$ and $\mu(\eta_{y_k}^\tau = \eta_{y_j}^\tau) = 0$ for all $k \neq j$, there must be a path such that

$$\eta_a^\tau < \eta_{y_2}^\tau < \cdots \eta_{y_{2n-2}}^\tau < \eta_b^\tau$$

which implies

$$a < y_2 < \cdots < b.$$

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Denote λ the distribution of η_a^τ and assume it is atomless.
The hypothesis of the Lemma are verified, and the map ϕ

$$\begin{aligned} \mathrm{LO}(\mathbb{F}) &\to [0,1]^{\mathbb{F}} \\ <_x &\mapsto (\eta_a^\tau(x))_{a \in \mathbb{F}} \end{aligned}$$

is the converse of ρ and $\phi_* \mu$ is of the form $\lambda^{\mathbb{F}}$.

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is the converse of ρ and $\phi_*\mu$ is of the form $\lambda^{\mathbb{F}}$.
By step 1, μ is the uniform measure !

There remains the case when $\mu(\eta_a^\tau = \eta_b^\tau = p) > 0$ for some p. This is the case when there will be a definable ordering.

The important remark is that if

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then

$$\mu(a < c < b | \eta_a^\tau = \eta_b^\tau = p) = 0.$$

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Indeed,

$$\begin{aligned} &\mu(a < c < b | \eta_a^\tau = \eta_b^\tau = p) \\ &\qquad = \mathbb{E}\left[\frac{\#\{c' \in F_n \cap (G_{a,b} \cdot c) \ : \ a < c' < b\}}{\# F_n \cap (G_{a,b} \cdot c)} | \eta_a^\tau = \eta_b^\tau = p\right] \\ &\qquad \to \mathbb{E}\left[\eta_b^\tau - \eta_a^\tau | \eta_a^\tau = \eta_b^\tau = p\right] = 0. \end{aligned}$$

In particular :

$$\mu(a < c < b | \eta_a^\tau = \eta_b^\tau = \eta_c^\tau = p) = 0$$

for all $c \in D_\tau(a) \cap D_\tau(b)$.

In particular :

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for all $c \in D_\tau(a) \cap D_\tau(b)$.
We define a new measure ν by taking

$$\nu(x_1 < \cdots < x_n) = \mu(x_1 < \cdots < x_n | \eta_{x_1}^\tau = \ldots = \eta_{x_n}^\tau = p).$$

ν is supported on a proper subflow of $G \curvearrowright \mathrm{LO}(\mathbb{F})$.

Under ν, one can again define $\eta_a^{\tau^{-1}}$ for all $a \in \mathbb{F}$.
Necessarily, this $\nu(\eta_a^{\tau^{-1}} = q) > 0$ for some $q \in [0, 1]$.
We define ν' as

$$\nu'(x_1 < \cdots < x_n) = \nu(x_1 < \cdots < x_n | \eta_{x_1}^{\tau^{-1}} = \ldots = \eta_{x_n}^{\tau^{-1}} = q).$$

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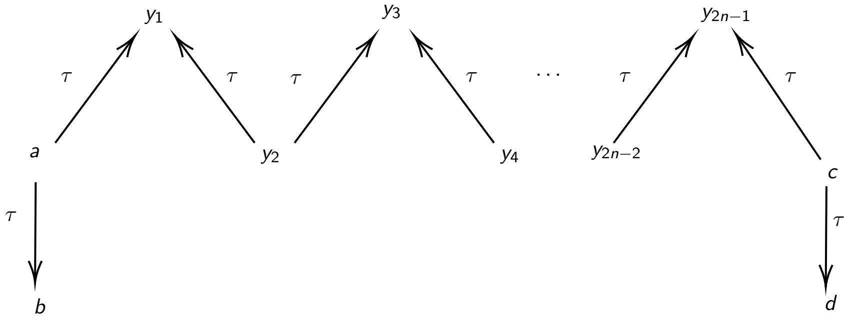
Take $a, b, c, d \in \mathbb{F}$ such that $\mathrm{tp}(ab) = \mathrm{tp}(cd) = \tau$, then ν'-as $a < b$ iff $c < d$.
We say that ν' respects τ.


y_1
y_3
y_{2n-1}
τ
$\cdots$
a
y_2
y_4
y_{2n-2}
c
b
d

By iterating this process for all 2-types, we get a measure that is a Dirac mass. Therefore we have a fixed point !

Thank you !