Forcing the Σ^1_3-Separation property

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The separation property is an old concept, introduced in the 1920s. Given two disjoint sets of reals A_1 and A_2, we say that a set C *separates* A_1 and A_2 iff $A_1 \subset C$ and $A_2 \subset C^c$. The problem becomes interesting when considered through the lens of definability.

Definition

We say that the $\mathbf{\Sigma^1_3}$-separation property holds iff every pair A_1 and A_2 of disjoint $\mathbf{\Sigma^1_3}$-sets has a separating set C such that C is both $\mathbf{\Sigma^1_3}$ and $\mathbf{\Pi^1_3}$.

The separation property is connected to two notions, the reduction- and the uniformization property.

Definition

A projective pointclass Γ has the reduction property if for every pair $A, B \in \Gamma$ there are $A' \subset A, A' \in \Gamma$ and $B' \subset B, B \in \Gamma$ such that $A' \cap B' = \emptyset$ and $A \cup B = A' \cup B'$.

Definition

A projective pointclass Γ has the uniformization property if for every set A in the plane, there exists a partial function $f \in \Gamma$ which uniformizes A, i.e. whenever $(x, y) \in A$, then f is defined and $(x, f(x)) \in A$.

Theorem

Let Γ be some projective pointclass, and let $\neg\Gamma$ denote its dual class. Then

1. *If Γ has the uniformization property then Γ has the reduction property.*
2. *If Γ has the reduction property then $\neg\Gamma$ has the separation property.*

Classical work of M. Kondo shows that the $\mathbf{\Sigma^1_2}$, $\mathbf{\Pi^1_1}$-uniformization property and hence the $\mathbf{\Pi^1_2}$, $\mathbf{\Sigma^1_1}$-separation property are true in ZFC. This is as much as ZFC can prove about the separation property in the projective hierarchy.

In Gödels constructible universe L, the reals have a good Σ^1_2-well-order, which in particular implies that the $\mathbf{\Sigma^1_3}$, in fact the $\mathbf{\Sigma^1_n}$-uniformization property is true, and so $\mathbf{\Pi^1_n}$-separation holds in L.

On the other hand $\mathbf{\Delta^1_2}$-determinacy implies that $\mathbf{\Sigma^1_3}$-separation is true. Note however that the determinacy assumption has considerable large cardinal strength, as it implies an inner model with a Woodin cardinal.

Question (Mathias)

Is it possible to force the $\mathbf{\Sigma}^1_3$-separation property over a model of ZFC*?*

The answer is yes and its proof will be the topic of this talk.

Theorem

There is a generic extension of L in which the $\mathbf{\Sigma}^1_3$-separation property is true.

We outline first how this talk is organized.

1. We introduce first a generic extension of L called W which will be the suitable ground model for our needs.
2. We use an iteration over W to prove an easier result, stated on the next slide, in a way which will introduce some key ideas.
3. Then extend the proof to give a proof of the theorem.

Theorem

There is a generic extension of L which satisfies that there is a countable ordinal α_0, such that any two disjoint (lightface) Σ^1_3-sets A_m and A_k can be separated by a set which is $\Delta^1_3(\alpha_0)$.

We start to define our suitable ground model W which will be a generic extension of L. Recall that an (ω_1-) Suslin tree S is a tree of height ω_1 such that every antichain of S is countable.

Definition

A sequence $(S_\alpha : \alpha < \omega_1)$ of Suslin trees is independent if for every finite set $e \subset \omega_1$, the product tree $\prod_{n \in e} S_n$ is a Suslin tree again.

Independent sequences can be generated via forcing.

Fact

The countably supported product of ω_1-Cohen forcing will introduce an independent sequence of length ω_1 of Suslin trees.

Fact

Let $\vec{S} = (S_\alpha : \alpha < \omega_1)$ be an independent sequence of Suslin trees in V, and let $A \in P(\omega_1)^V$ be an arbitrary subset. Let $\prod_{i \in A} S_i$ be the finitely supported product, G the generic filter, then for any $\alpha \notin A$, S_α remains a Suslin tree in $V[G]$.

So adding branches through trees from the independent sequence can be used to code up information, as long as we do have the the sequence $\vec{S}$ available as a parameter. As we are aiming for Σ^1_3-predicates we have to add further forcings which will turn membership in $\vec{S}$ into a Σ^1_3-property.
So we use an additional coding forcing, taking advantage of club shooting forcing, to make $\vec{S}$ nicely definable.

Definition

For a stationary $a \subset \omega_1$ the club-shooting forcing for a, denoted by $\mathbb{P}_a$ consists of conditions which are closed subsets of ω_1, which are also subsets of a, ordered by end-extension.

Fact

The club-shooting forcing $\mathbb{P}_a$ generically adds a club through the stationary set $a \subset \omega_1$, while being ω-distributive and hence ω_1-preserving. Moreover Suslin trees remain Suslin in the generic extension.

We proceed now as follows. We work in $L[\vec{S}]$.

1. Use $\diamondsuit$ which holds in L to find a $\Sigma_1(\omega_1)$-definable sequence of stationary subsets $\vec{a} = (a_\alpha : \alpha < \omega_1 \cdot \omega_1)$ of ω_1.
2. Then use club shooting forcing to code up the independent Suslin trees $\vec{S}$ into the stationary sets $\vec{a}$.

The resulting universe is our desired W.

In W, if $\alpha < \omega_1$ is arbitrary then there is a set $X_\alpha \subset \omega_1$ such that membership in the tree S_α can be written in a $\Sigma_1(X_\alpha, \omega_1)$-way. The set X_α just consists of codes for the relevant clubs through the a_β's:

$$\forall \gamma < \omega_1 (\gamma \in S_\alpha \Leftrightarrow \exists M (M \models \mathsf{ZF}^- \wedge |M| = \aleph_1 \wedge X_\alpha \in M \wedge \\ M \text{ is transitive} \\ \wedge M \models (a_{\omega_1 \cdot \alpha + \gamma \cdot 2})^L \text{ is nonstationary}))$$

Likewise

$$\forall \gamma < \omega_1 (\gamma \notin S_\alpha \Leftrightarrow \exists M (M \models \mathsf{ZF}^- \wedge |M| = \aleph_1 \wedge M \text{ is transitive} \\ \wedge X_\alpha \in M \wedge \\ M \models (a_{\omega_1 \cdot \alpha + \gamma \cdot 2 + 1})^L \text{ is nonstationary}))$$

This description will play a role later .

We start to prove the auxiliary theorem.

Theorem

There is a generic extension of L which satisfies that there is a countable ordinal α_0, such that any two disjoint (lightface) Σ^1_3-sets A_m and A_k can be separated by a set which is $\Delta^1_3(\alpha_0)$.

We outline first a rough description of the proof strategy.

1. We split the definable sequence of Suslin trees $\vec{S}$ from W into two sequences $\vec{S}^1$ and $\vec{S}^2$.
2. We will use a forcing iteration of length ω_1 which is guided by some bookkeeping function F. We list all the Σ^1_3-formulas with one free variable $(\varphi_n(v_0) \, : \, n \in \omega)$, and let A_i denote the set of reals which is defined using the according formula φ_i. Whenever $F(\alpha)$ yields a triple (x, m, k) where $x \in 2^\omega, m, k \in \omega$, then we decide where to "put" x, i.e. we decide whether to force the characteristic function of (a real coding) (x, m, k) into and ω-block of elements of $\vec{S}^1$ or $\vec{S}^2$.

3. Use an almost disjoint coding forcing to turn the property "(x, m, k) is coded into an ω-block of $\vec{S}^i$" into a $\Sigma^1_3(x, m, k)$-statement $\Phi_i(x, m, k)$.
4. After ω_1-many stages, for every triple (x, m, k), either $\Phi_1(x, m, k)$ is true or $\Phi_2(x, m, k)$ and we let $D^1_{m,k} := \{x \in 2^\omega \ : \ \Phi_1(x, m, k)$ holds$\}$ and $D^2_{m,k} := \{x \in 2^\omega \ : \ \Phi_2(x, m, k)$ holds$\}$ which are both Σ^1_3-sets. The main task and difficulty is of course that we decided at every stage of the iteration correctly such that $A_m \subset D^1_{m,k}$ and $A_k \subset D^2_{m,k}$.

We briefly mention the difficulties with the above strategy.

- Of course Σ^1_3-sets change during our iteration, they grow. Thus, the following pathological situation could occur: at some stage we decide for some x to be put in $D^1_{m,k}$, yet in the course of our iteration $\phi_k(x)$ becomes true. In that case $D^1_{m,k}$ and $D^2_{m,k}$ will not separate A_k and A_m.
- As the separating sets are Σ^1_3 themselves, the set up of the proof contains some amount of self-reference. If not handled this can lead to all sorts of diagonalizations we know from logic.

To deal with the first problem we have to find a way of ensuring that, given a real r which is not in A_m, it will not become an element of A_m in all future extensions we are about to define. This sounds like a circular definition, which is bad. To deal with the second problem, we allow ourselves to be greedy, meaning that whenever we can force a real x into both A_m and A_k, we will do so. As A_m and A_k now have non-emtpy intersection, we can neglect them.

We only force (with finite support) with partial orders of the following form. We first use a fresh ω-block $(S^i_{\omega\cdot\alpha+n} \,:\, n \in \omega)$ of Suslin trees from either $\vec{S}^1$ or $\vec{S}^2$ and shoot a branch though either $S^i_{\omega\cdot\alpha+2n}$ or $S^i_{\omega\cdot\alpha+2n+1}$. In a second step we collect the added branches in some set X and code them into one real r using almost disjoint coding forcing $\mathbb{A}_h(X)$ relative to a fixed, L-definable, almost disjoint family of reals $h = \{h_\alpha \,:\, \alpha < \aleph_1\}$. Recall that if $X \subset \aleph_1$ be a set of ordinal, then there is a ccc forcing, the almost disjoint coding $\mathbb{A}_h(X)$ which adds a new real x which codes X relative to the family h in the following way

$$\alpha \in X \text{ if and only if } x \cap h_\alpha \text{ is finite.}$$

Definition

The almost disjoint coding $\mathbb{A}_h(X)$ relative to an almost disjoint family h consists of conditions $(r, R) \in \omega^{<\omega} \times h^{<\omega}$ and $(s, S) < (r, R)$ holds if and only if

1. $r \subset s$ and $R \subset S$.
2. If $\alpha \in X$ and $h_\alpha \in R$ then $r \cap h_\alpha = s \cap h_\alpha$.

We can repeat this if desired. Forcings of this form will be called *legal*. Recall that the almost disjoint coding forcing is Knaster, thus it will never destroy any Suslin tree. As an easy consequence we obtain:

Fact

Legal forcings $\mathbb{P}$ preserve all the Suslin trees from W which are not explicitly used in $\mathbb{P}$.

We use an iteration of length ω_1 of legal forcings over W guided by some bookkeeping F. Assume we are at some stage $\alpha < \omega_1$ and $F(\alpha) = (x, m, k), x \in 2^\omega, m, k \in \omega$. By induction we assume that we have defined already $\mathbb{P}_\alpha$ and $W[G_\alpha]$. Goal is to define the next forcing $\mathbb{P}(\alpha)$. We split into three cases.
Assume that $x \in A_m$. In that case we code a real w coding the triple (x, m, k) into $\vec{S}^1$, i.e. we let $\mathbb{P}(\alpha) = \mathbb{Q}_0 * \mathbb{Q}_1$, where $\mathbb{Q}_0 = \prod_{i \in w} S^1_{\omega \cdot \xi + 2i} \times \prod_{i \notin w} S^1_{\omega \cdot \xi + 2i+1}$. We let $X_{x,m,k} \subset \omega_1$ be a code for the ω-many branches we just added and the ω_1-many clubs we need to define $\{S_{\omega \cdot \xi + n} \,:\, n \in \omega\}$ then we rewrite $X_{x,m,k}$ into a set $Y_{x,m,k} \subset \omega_1$ (David's trick) and code, using $\mathbb{Q}_1 := \mathbb{A}_h(Y_{x,m,k})$ into one real $r_{x,m,k}$ which will satisfy:

$(*)$ For any countable, transitive model M of ZF$^-$ such that $\omega_1^M = (\omega_1^L)^M$ and $r_{x,m,k} \in M$, M can construct its version of $L[r_{x,m,k}]$ which in turn thinks that there is an ordinal $\xi < \omega_1$ such that for any $i \in \omega$, $S^L_{\xi+2i}$ is Suslin iff $i \in w$ and $S^L_{\xi+2i+1}$ is Suslin iff $i \notin w$.

In the second case we assume that $F(\alpha) = (x, m, k)$, $x \notin A_m \cup A_k$ and there is a legal forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \exists z(z \in A_m \cap A_k)$. As we decided on being greedy, we force with $\mathbb{P}$, ignoring the fact that $\mathbb{P}$ might introduce new codes on $\vec{S}$ which are unwanted.
In case c, $F(\alpha) = (x, m, k)$, $x \notin A_m \cup A_k$ and there is no legal forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \exists z(z \in A_m \cap A_k)$. We distinguish two subcases. First, if there is a legal forcing $\mathbb{Q}$ such that there is a $q \in \mathbb{Q}$ and $q \Vdash_{\mathbb{Q}} x \in A_m$. In that situation we mark all the (ctbly many) blocks b_α of Suslin trees which are used by $\mathbb{Q}$, but do not force with $\mathbb{Q}$, instad we ensure that from now on, none of these blocks must be touched by any future forcing we use. Then we code (x, m, k) into $\vec{S^1}$ into some fresh block of Suslin trees. This has the following consequence:

Lemma

Under the assumption of case c, let $\mathbb{Q}$ be legal such that $\Vdash_{\mathbb{Q}} x \in A_m$ which uses the set b_α of blocks of elements of $\vec{S}$. Let $\mathbb{P}$ be a legal forcing in $W[G_\alpha]$ which does not use any tree $S \in b_\alpha$. Then $\Vdash_{\mathbb{P}} x \notin A_k$.

Proof.

Assume not, then there is a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} x \in A_k$. There is also a $q \in \mathbb{Q}$ such that $q \Vdash_{\mathbb{Q}} x \in A_m$, by assumption.
Now $\mathbb{P} \times \mathbb{Q}$ is a legal forcing, and if $g \times h$ is $\mathbb{P} \times \mathbb{Q}$-generic over $W[G_\alpha]$ which contains the condition (p, q) then $W[G_\alpha * g] \models x \in A_k$, hence because Shoenfield $W[G_\alpha * (g \times h)] \models x \in A_k$. On the other hand $W[G_\alpha * h] \models x \in A_m$ so again by Shoenfield $W[G_\alpha * (h \times g)] \models x \in A_m$, so there has been a legal forcing, which forces $A_m \cap A_k \neq \emptyset$, which is a contradiction. $\square$

In the second subcase of case c, there is no legal forcing $\mathbb{P}$ for which $\Vdash_{\mathbb{P}} \exists z(z \in (A_m \cap A_k))$, and there is no legal $\mathbb{Q}$ such that $\Vdash_{\mathbb{Q}} x \in A_m$. In that situation, we can safely write a code for (x, m, k) into $\vec{S}^2$, and will avoid the pathological situation that later in the iteration we might have $x \in A_m$. This ends the definition of the iteration.

We continue to discuss the resulting universe $W[G_{\omega_1}]$. The following is true in this universe:

- CH holds, as CH is true in W and we used an ω_1-length iteration of ccc forcings of size $\aleph_1$.
- For any pair $(m, k) \in \omega^2$, if we let $D^1_{m,k} := \{x \in 2^\omega \,:\, \exists \alpha < \omega_1((x, m, k)$ is coded into $\vec{S}^1$ at $\alpha)\}$, and $D^2_{m,k} := \{x \,:\, \exists \alpha < \omega_1((x, m, k)$ is coded into $\vec{S}^2$ at $\alpha)\}$, then $D^1_{m,k} \cup D^2_{m,k} = 2^\omega$.

Even more is true, both $D^1_{m,k}$ and $D^2_{m,k}$ are Σ^1_3-sets because of property $(*)$. Recall

$(*)$ For any countable, transitive model M of ZF$^-$ such that $\omega_1^M = (\omega_1^L)^M$ and $r_{x,m,k} \in M$, M can construct its version of $L[r_{x,m,k}]$ which in turn thinks that there is an ordinal $\xi < \omega_1$ such that for any $i \in \omega$, $S^L_{\xi+2i}$ is Suslin iff $i \in w$ and $S^L_{\xi+2i+1}$ is Suslin iff $i \notin w$.

So $x \in D^1_{m,k}$ iff $\exists r$(for any countable, transitive model M of ZF$^-$ such that $\omega_1^M = (\omega_1^L)^M$ and $r, x \in M$, M can construct its version of $L[r]$ which in turn thinks that there is an ordinal $\xi < \omega_1$ such that for any $i \in \omega$, $S^L_{\xi+2i}$ is Suslin iff $i \in (x, m, k)$ and $S^L_{\xi+2i+1}$ is Suslin iff $i \notin (x, m, k)$).

Lemma

In $W[G_{\omega_1}]$, there is an $\alpha_0 < \omega_1$ such that if we let $D^1_{m,k}(\alpha_0)$ be the set of x such that (x, m, k) is coded in $\vec{S}^1$ above α_0, and $D^2_{m,k}(\alpha_0)$ accordingly, then both sets are $\Sigma^1_3(\alpha_0)$ and separate every pair (A_m, A_k) of disjoint Σ^1_3-sets.

Proof.

We let α_0 be the supremum of indices of ordinals which are used in $\mathbb{P}_{\omega_1}$ when in case b. As there are only ctbly many fromulas, $\alpha_0 < \omega_1$. Any code written above α_0 by $\mathbb{P}_{\omega_1}$ will be a correct one, as for them only case a or c is responsible and we ensured that no ”bad” codes are added by these forcings.
Consequentially, for any $m, k \in \omega$, $D^1_{m,k}(\alpha_0) \cap D^2_{m,k}(\alpha_0)$ is empty. $\square$

Now we turn to proving the main theorem. Our old strategy will not work anymore, as there are real parameters in the formulas, so there is no chance to apply case b only countably many times. The right modification of the old proof is to successively narrow down the forcings we allow to use in case b.

Returning briefly to the discussion of case c, we see that under these assumptions we can define an assignment function $g_{m,k}$ which maps a real x (x can even belong to a legal extension of the model we are currently in) to the set $\{m, k\} \times [\omega_1]^\omega$. We think of the value of $g_{m,k}(x)$ as the outcome of the reasoning in case c, thus $g_{m,k}(x) = (m, b)$ is true whenever we code x into $\vec{S^1}$ and ensure that no pathological situation arises as long as we don't touch trees from $\vec{S}$ with indices in b.

So the right modification of the iteration looks as follows: the set up is the same as before, but we inductively assume that we have a notion of β-legal for some $\beta < \omega_1$. Let $F(\alpha) = (x, m, k, y)$, where y now is the parameter for $A_m(\cdot, y)$ and $A_k(\cdot, y)$. Now split again into three cases a,b and c.
Case a is defined exactly as before.
For case b, we ask whether there is a β-legal forcing which forces $\exists z(z \in A_m(y) \cap A_k(y))$. If the answer is yes, we use it.
For case c, no β-legal forcing $\mathbb{P}$ exists such that $\Vdash \exists z(z \in A_m(y) \cap A_k(y))$. We split into the same subcases as before, and define a new assignment function $g_{m,k,y}$ which can be applied to any (even future) real. We use the assignment function to define the new notion $\beta + 1$-legal: A legal forcing is $\beta + 1$-legal if it is β-legal and respects the assignment $g_{m,k,y}$. Then we code (x, m, k) into the according sequence of Suslin trees and continue, ensuring that no blocks of Suslin trees, given by the assignment functions we have defined so far, are touched.

Remark

A legal forcing $(\mathbb{P}(\gamma) : \gamma < \delta)$ *is thus* β*-legal, whenever every factor* $\mathbb{P}(\gamma)$*, if it codes one quadruple* (r, m, k, y) *into* $\vec{S}^1$ *or* $\vec{S}^2$*, and* $g_{m,k,y}$ *is defined, then the placement of* (r, m, k, y) *is in accordance with* $g_{m,k,y}$ *and moreover the set of indices of Suslin trees of* $g_{m,k,y}(r)$ *is not touched by* $(\mathbb{P}(\gamma) : \gamma < \delta)$*. Note that we can always decide whether some legal forcing is* β*-legal, as soon as* β*-legal is defined.*

Lemma

- *For every* $\alpha < \omega_1$*, if at stage* α *of the iteration* $\mathbb{P}_{\omega_1}$*, we have defined the notion of* β_α*-legal, then the tail of the iteration* $\mathbb{P}_{[\alpha,\omega_1)}$ *is* β_α*-legal.*
- *For every* $\alpha < \omega_1$ *and any quadruple* (x, m, k, y)*, if at stage* $\alpha + 1$ *the assignment function* $g_{m,k,y}$ *has already been defined, then the iteration will not add "bad" codes for* (x, m, k, y) *from that stage on. In particular the set of codes in* $\vec{S}^1$ *and* $\vec{S}^2$ *which are created from that stage on which contain* (m, k, y) *form a separating set for* $A_m(y)$ *and* $A_k(y)$*.*

As a last fact we note that,