On Continuous Tree-Like Scales and Freeness properties of Internally Approachable Structures

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We will present here results from a

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Plan

Part I Free Sets and Tree-Like Scales

Part II Some Ideas and Insights to the Proofs

Part III The Approachable Bounded Subset Property and Open Problems

Part I

Free Sets and Tree-Like Scales

Conventions I

A **Structure** on H_θ will mean a model $\mathfrak{A} = \langle H_\theta, \in, f_n \rangle_{n<\omega}$ with countably finitary functions f_n.

We will assume that the functions f_n contain Skolem functions and a well-ordering of $\mathfrak{A}$

A **substructure** means an elementary substructure $N \prec \mathfrak{A}$

The **characteristic function** χ_N, of a substructure N is defined by $\chi_N(\tau) = \sup(N \cap \tau)$ for a cardinal $\tau \in H_\theta$

Free Sets

Definition

A set x is **free** with respect to a structure $\mathfrak{A} = \langle H_\theta, \in, f_n \rangle_{n<\omega}$ if for every $n < \omega$ and $\delta \in x$,

$$\delta \notin f_n''[x \setminus \{\delta\}]^{<\omega}$$

We say x is **free over** $N \prec \mathfrak{A}$ if $\delta \notin f_n''[N \cup (x \setminus \{\delta\})]^{<\omega}$,

Equivalently, if for every function $f \in N$ of some arity $k < \omega$,and $\delta \in x$, $\delta \notin f''[x \setminus \{\delta\}]^k$

Cardinal Arithmetic

Shelah's celebrated bound in cardinal arithmetic asserts that if $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \aleph_{\omega_4}, \aleph_{(2^{\aleph_0})^+}$$

It is open whether $2^{\aleph_\omega} \geq \aleph_{\omega_1}$ is consistent

Cardinal Arithmetic and Free Sets

Theorem (Shelah)

If $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} \geq \aleph_{\omega_1}$ then for every structure $\mathfrak{A}$ and a substructure N of size $|N| < \aleph_\omega$ there is a cofinal set $x \subseteq \aleph_\omega$ which is free over N

More generally,

Theorem (Shelah)

If $\langle \tau_n \rangle_n$ is an interval of regular cardinals and $|PCF(\langle \tau_n \rangle_n)| \geq \aleph_1$, then for every structure $\mathfrak{A}$ and a substructure N of size $|N| < \lambda = \sup_n \tau_n$, there is a cofinal set $x \subseteq \lambda$ which is free over N

Internally Approachable Structures

Let $\mathfrak{A}$ be a structure and $N \prec \mathfrak{A}$. We say that N is **Internally Approachable** if $N = \bigcup_{i<\mu} N_i$ is a union of a $\subseteq$-increasing chain of substructures $\langle N_i \mid i < \mu \rangle$, $\mathrm{cof}(\mu) > \aleph_0$ so that $N_i \in N$ for all $i < \mu$

Approachable Free Subset Property

Let $\langle \tau_n \rangle_n$ be an increasing sequence of regular cardinals, and $\lambda = \sup_n \tau_n$.

Definition (Pereira)

The **Approachable Free Subset Property (AFSP)** with respect to $\langle \tau_n \rangle_n$ asserts that for every internally approachable $N \in (H_\theta, \langle \tau_n \rangle_n)$ there is an infinite subset $x \subseteq \{\chi_N(\tau_n) \mid n < \omega\}$ is free over N

Theorem (Shelah)

If $|PCF(\langle \tau_n \rangle_n)| > \aleph_0$ then AFSP holds with respect to $\langle \tau_n \rangle_n$.

Improving the Bound for $2^{\aleph_\omega}$?

Conclusion:
If $\aleph_\omega$ is strong limit cardinal and $2^{\aleph_\omega} \geq \aleph_{\omega_1}$, then AFSP holds with respect to $\langle \omega_n \rangle_n$.

Therefore, a strategy for proving (in ZFC) that $2^{\aleph_\omega} < \aleph_{\omega_1}$ would be to show that AFSP must fail at $\langle \omega_n \rangle_n$. For this, Pereira introduced the notion of a **tree-like scale**.

Tree-Like Scales

Let $\langle \tau_n \rangle_n$ be an increasing sequence of regular cardinals. For two functions $f, g \in \prod_n \tau_n$.
$f <^* g$ means $f(n) < g(n)$ for all but finitely many n's.

Let $\vec{f} = \langle f_\alpha \mid \alpha < \eta \rangle \subseteq \prod_n \tau_n$, be a sequence of regular length η.

1. $\vec{f}$ is a **scale** in $\prod_n \tau_n$ if it is increasing and cofinal in the $<^*$-ordering

2. $\vec{f}$ is **continuous** if for every limit ordinal $\delta < \eta$. $\mathrm{cof}(\delta) > \aleph_0$, $\vec{f} \restriction \delta = \langle f_\alpha \mid \alpha < \delta \rangle$ is $<^*$-cofinal in $\prod_n f_\delta(n)$

3. $\vec{f}$ is **tree-like** if for every $\alpha, \beta < \eta$ and $n < \omega$, if $f_\alpha(n+1) = f_\beta(n+1)$ then $f_\alpha(n) = f_\beta(n)$

Relations between Properties

Let $\langle \tau_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals, $\lambda = \cup_n \tau_n$.

Fact: Suppose that $\vec{f} = \langle f_\alpha \mid \alpha < \eta \rangle$ is a *continuous* scale on $\prod_n \tau_n$, and $N \prec (H_\theta, \langle \tau_n \rangle_n)$ is an *internally approachable* substructure. Then $\chi_N =^* f_{\delta_N}$ where $\delta_N = \sup(N \cap \eta)$.

Claim: If there exists a continuous tree-like scale on $\prod_n \tau_n$ then AFSP fails with respect to $\langle \tau_n \rangle$

Consistency Results

Theorem (Pereira)

Continuous tree-like scale (TLS) can exists on a product of $\langle \tau_n \rangle_n$ *from an I0 sequence*

Theorem (Cummings)

Continuous tree-like scale (TLS) can exists above a supercompact cardinal

Consistency Results cont.

Theorem (Welch)

If the Approachable Free Subset Property (AFSP) holds w.r.t a sequence $\langle \tau_n \rangle_n$ then there is an ***inner model with*** *a cardinal λ such that $\{o(\mu) \mid \mu < \lambda\}$ is unbounded in λ*

Theorem (Gitik)

It is ***consistent relative*** *to a cardinal κ carriying a (κ, κ^{++})-extender that there is a product $\prod_n \kappa_n^{++}$ which does not carry a continuous TLS.*

Questions:

Question1: Is AFSP consistent with respect to some sequence $\langle \tau_n \rangle_n$?

Answer1: Yes

Recall that AFSP with respect to $\langle \tau_n \rangle$ implies that there cannot be a continuous tree-like scale on $\prod_n \tau_n$.

Question2: Is the consistency strength of AFSP strictly stronger than the inexistence of continuous tree-like scale ?

Answer2: No

Consistency Results cont.

Theorem (Adolf-BN)

AFSP w.r.t $\langle \tau_n \mid n < \omega \rangle$ *is consistent relative to the existence of a cardinal* λ *such that the set of Mitchell orders* $\{o(\mu) \mid \mu < \lambda\}$ *is unbounded in* λ*.*

Moreover, the sequence τ_n *can be a subsequence of the* $\aleph_k$*'s*

Theorem (Adolf-BN)

If there exists a product $\prod_n \tau_n$ *which does not carry a continuous TLS then there is an inner model with a cardinal* λ *such that* $\{o(\mu) \mid \mu < \lambda\}$ *is unbounded in* λ

The machinery developed for the proof of the lower-bound reveals that tree-like scales naturally exists in canonical inner models.

Theorem (Adolf-BN)

Let $\mathcal{M}$ be a premouse such that each countable hull has an $(\omega_1 + 1)$-iteration strategy and suppose that $\langle \tau_n \rangle_n \in \mathcal{M}$ is a sequence of regular cardinals. Then $\prod_{n<\omega} \tau_n$ carries a continuous tree-like scale in $\mathcal{M}$.

Part II
Some Ideas and Insights to the proofs

Diagonal Prikry forcing and Free Sets

Let $\langle \lambda_n \rangle_n$ an increasing sequence of measurable cardinals, and $\vec{U} = \langle U_n \mid n < \omega \rangle$ a sequence of ultrafilters, where each U_n is a λ_n-complete normal measure on λ_n

The Diagonal Prikry forcing $\mathbb{P}_{\vec{U}}$ consists of sequences $p = (p_n \mid n, \omega))$, so that for some $\ell^p < \omega$ the following holds:

1. For $n < \ell^p$, $p_n = \rho_n$ is an ordinal $\rho_n < \lambda_n$
2. For $n \geq \ell^p$, $p_n = A_n$ where $A_n \in U_n$

When extending a condition p, we may choose finitely many new points $\rho_n \in A_n$ (increasing ℓ^p) and shrink the measure one sets A_n to some $A^*_n \in U_n$.
A $\mathbb{P}_{\vec{U}}$ generic filter naturally gives rise to an ω-sequence $\langle \rho_n \mid n < \omega \rangle$

Claim: For every function $F : \lambda \to \lambda$ in V there is a cofinite $x \subseteq \langle \rho_n \rangle_n$ which is free with respect to F.

Ideas for obtaining the Upper-Bound

Theorem (Adolf-BN)

AFSP w.r.t $\langle \tau_n \mid n < \omega \rangle$ is consistent relative to the existence of a cardinal λ such that the set of Mitchell orders $\{o(\mu) \mid \mu < \lambda\}$ is unbounded in λ.

The large cardinal hypothesis allows us to form a model V with a sequence of measurable cardinals $\langle \lambda_n \rangle_n$, each carrying a (baby) extenders $E_n = \{U_{n,\alpha} \mid \alpha < \kappa_n\}$, with κ_n measures, $\lambda_{n-1} < \kappa_n < \lambda_n$.

Force over V with the extender based forcing $\mathbb{P}_{\vec{E}}$. In a generic extension $V[G]$, λ^+-many new sequences $\langle t_\alpha \mid \alpha < \lambda^+ \rangle$ are added ($\lambda = \sup_n \tau_n$)

$$t_\alpha = \langle \alpha_n \mid n < \omega \rangle$$

Ideas for obtaining the Upper-Bound, cont.

Every function F in a generic extension $V[G]$ will belong to an intermediate extension $V[G_\beta]$ in which only $\beta < \lambda^+$ many sequences were added, $\langle t_\alpha \mid \alpha < \beta\rangle$ for some $\beta < \lambda^+$

Taking any $\delta > \beta$, the sequence $t_\delta = \langle \delta_n \mid n < \omega\rangle$ is generic over $V[G_\beta]$ and will satisfy a property similar to $\langle \rho_n\rangle_n$ in the previous claim. Thus, free with respect to $F \in V[G_\beta]$

Ideas for obtaining the Lower-Bound (successor cardinals)

Let $\mathcal{M} \models$ ZFC- be a premouse such that every countable hull of $\mathcal{M}$ has an $(\omega_1 + 1)$ iteration strategy, $\lambda \in M$ a limit cardinal (in $\mathcal{M}$) of V-cofinality ω such that λ^+ exists in $\mathcal{M}$.

Let $\langle \kappa_n \rangle_n$ be a sequence of $\mathcal{M}$-cardinals cofinal in λ, and $\tau_n := (\kappa_n^+)^{\mathcal{M}}$. We would like to define a sequence in $\prod_{n<\omega} \tau_n$ that is increasing, continuous, and **tree-like**.

For $\alpha < \lambda^+$ we define:

1. $\mathcal{M}_\alpha$ to be the collapsing level for α
2. n_α be minimal such that $\rho_{n+1}^{\mathcal{M}_\alpha} = \lambda$
3. $p_\alpha := p_{n_\alpha+1}^{\mathcal{M}_\alpha}$, and $w_\alpha := w_{n_\alpha+1}^{\mathcal{M}_\alpha}$ the associated solidity witness

Ideas for obtaining the Lower-Bound, cont.

By the Condensation Lemma there exists some $\mathcal{M}_\alpha^n \trianglelefteq \mathcal{M}$ which is isomorphic to $\mathrm{Hull}_{n_\alpha+1}^{\mathcal{M}_\alpha}(\kappa_n \cup \{p_\alpha\})$.

$$f_\alpha(n) = \begin{cases} (\kappa_n^+)^{\mathcal{M}_\alpha^n} & \{w_\alpha, \lambda\} \in \mathrm{Hull}_{n_\alpha+1}^{\mathcal{M}_\alpha}(\kappa_n \cup \{p_\alpha\}) \\ 0 & \text{otherwise} \end{cases}$$

Keys: Let $n < m$ finite.

1. $f_m(\alpha)$ determines $\mathcal{M}_\alpha^m$, as $\mathcal{M}_\alpha^m$ is the collapsing level of $f_m(\alpha)$
2. Given $n < m$, $\mathcal{M}_\alpha^m$ determines $\mathcal{M}_\alpha^n$, i.e., by
$$\mathrm{Hull}_{n_\alpha+1}^{\mathcal{M}_\alpha^m}(\kappa_n \cup \{p_{n_\alpha+1}^{\mathcal{M}_\alpha^m}\})$$
3. Hence $f_\alpha(m)$ determines $f_\alpha(n)$. This gives the **tree-like property**

Ideas for obtaining the Lower-Bound, cont.

Theorem (Adolf-BN)

If there exists a product $\prod_n \tau_n$ which does not carry a continuous TLS then there is an inner model with a cardinal λ such that $\{o(\mu) \mid \mu < \lambda\}$ is unbounded in λ

We derived from $\mathcal{M}$ a tree-like sequence continuous sequence $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$.

Under the smallness assumption of no inner model with the stated large cardinal property, and taking $\mathcal{M}$ to be a suitable initial segment of the core model, a Covering-type argument shows that $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ is a **scale in** V**.**

Part III

The Approachable Bounded Subset Property and Open Problems

Recall AFSP

Definition (Pereira)

The **Approachable Free Subset Property (AFSP)** with respect to $\langle \tau_n \rangle_n$ asserts that for every internally approachable $N \in (H_\theta, \langle \tau_n \rangle_n)$ there is an infinite subset $x \subseteq \{\chi_N(\tau_n) \mid n < \omega\}$ is free over N

This means that for every function $F \in N$ of of finite arity k, and $(k+1)$ distinct values $\chi_N(\tau_{n_0}), \chi_N(\tau_{n_1}), \ldots, \chi_N(\tau_{n_k}) \in x$ we have

$$F\left(\chi_N(\tau_{n_1}), \ldots, \chi_N(\tau_{n_k})\right) \neq \chi_N(\tau_{n_0})$$

Definition (Approachable Bounded Subset Property (ABSP))

ABSP w.r.t $\langle \tau_n \rangle_n$ asserts that for every internally approachable substructure $N \prec (H_\theta, \langle \tau_n \rangle_n)$ there exists some $m < \omega$ so that for every $F \in N$ of some finite arity k, and $(k+1)$-distinct numbers $n_0, n_1, \ldots, n_k > m$, we have that

$$\text{If } F\left(\chi_N(\tau_{n_1}), \ldots, \chi_N(\tau_k n_k)\right) < \tau_{n_0}$$

$$\text{then } F\left(\chi_N(\tau_{n_1}), \ldots, \chi_N(\tau_{n_k})\right) < \chi_N(\tau_{n_0})$$

Equivalently, for the set

$$x = \{\chi_N(\tau_n) \mid n \neq n_0\}$$

we have that

$$\chi_{N[x]}(\tau_{n_0}) = \chi_N(\tau_{n_0})$$

Theorem (Adolf-BN)

ABSP w.r.t $\langle \tau_n \mid n < \omega \rangle$ *is equi-consistent with the existence of a cardinal* λ *such that the set of Mitchell orders* $\{o(\mu) \mid \mu < \lambda\}$ *is unbounded in* λ.

Moreover, the sequence τ_n *can be a subsequence of the* $\aleph_k$*'s*

Remark: In the above theorem, ABSP is shown to hold on a subsequence of the $\aleph_k$'s with gaps.

Question1: Is ABSP (or a cofinite version of AFSP) consistent with respect to a tail of the $\langle \aleph_n \rangle_n$.

Question2: Can the principles AFSP, ABSP remain consistent if we remove the restriction to internally approachable structures?

Pereira proved from the assumption of a cardinal κ which is κ^{++}-supercompact, that it is consistent to have a (long) continuous tree-like scale $\langle f_\alpha \mid \alpha < \kappa^{++} \rangle$ on a product of cardinals $\langle \kappa_n \rangle_n$ cofinal in κ.

Question3: Is it possible to obtain a long continuous tree-like scale from an assumption of a strong cardinal?

A Stronger Principle

Consider the following natural strengthening of ABSP, where for $\chi_N(\tau_{n_0}) < \chi_N(\tau_{n_1}) < \dots \chi_N(\tau_{n_k})$, the requirement-

$$\text{“If } F(\chi_N(\tau_{n_1}), \dots, \chi_N(\tau_k n_k)) < \tau_{n_0}$$

$$\text{then } F(\chi_N(\tau_{n_1}), \dots, \chi_N(\tau_{n_k})) < \chi_N(\tau_{n_0})\text{”}$$

Is replaced with-

$$\text{“If } F(\chi_N(\tau_{n_1}), \dots, \chi_N(\tau_k n_k)) < \tau_{n_0}$$

$$\text{then } F(\chi_N(\tau_{n_1}), \dots, \chi_N(\tau_{n_k})) \in N \cap \tau_{n_0}\text{”}$$

Question4: Is it consistent?