Tree forcings, sharps and absoluteness

Philipp Schlicht, University of Vienna

22 October 2020

Overview

- We study interactions of tree forcings, determinacy and absoluteness.
- Tree forcings are often used in iterated forcing constructions to control cardinal characteristics.
- Determinacy of infinite games is important in descriptive set theory.
- Our aims:
  - Prove preservation of analytic (and higher) determinacy under tree forcings
  - Study the impact of tree forcings on definable equivalence relations
- The results are from a joint project with Fabiana Castiblanco from 2018.

  The project left open some natural problems about tree forcings that I want to discuss.

Preserving large cardinals

A cardinal κ is measurable if the following equivalent conditions hold:

- There is an elementary embedding $j : V \to N$ into a transitive model N with $\mathrm{crit}(j) = \kappa$.
- There is a non-principal $<\kappa$-complete ultrafilter on κ.

Theorem (Levy-Solovay)

If κ is measurable and $\mathbb{P}$ is a forcing of size $|\mathbb{P}| < \kappa$, then κ remains measurable in any $\mathbb{P}$-generic extension $V[G]$.

Proof sketch.

Lift $j\colon V \to N$ to $k\colon V[G] \to N[G]$ by letting $k(\sigma^G) = j(\sigma)^G$. □

There are variants to this theorem stating that supercompact, strong and Woodin cardinals are preserved after doing "small" forcing.

Are consequences of large cardinals preserved by sufficiently nice forcings?

Sharps

Definition

$0^\#$ exists if each (at least one) of the following objects exist:

1. An uncountable set of ordinals which are order-indiscernible over L.
2. A non-trivial elementary $j : L \to L$.
3. A countable structure $(L_\alpha, \in, U)$ such that
   - $(L_\alpha, \in)$ is a model of ZFC^- with a largest cardinal κ,
   - $(L_\alpha, \in, U)$ is a model of Σ_0-separation,
   - U is a $<\kappa$-complete ultrafilter on $P(\kappa)^{L_\alpha}$, and
   - All iterated ultrapowers of $(L_\alpha, \in, U)$ are wellfounded.

More generally, $x^\#$ is defined for any set x of ordinals by replacing L with $L[x]$ and letting $\mathrm{crit}(j) > \sup(x)$.

The existence of $x^\#$ follows from the existence of a measurable cardinal above $\sup(x)$.

Theorem (folklore)

The statement “For every set of ordinals x, $x^\#$ exists” is preserved by any forcing.

Are consequences of large cardinals in $H(\omega_1)$ preserved by sufficiently nice forcings?

Background: sharps for reals and determinacy

Let A be a subset of 2^ω. In the game $G(A)$, two players play $n_i \in \{0,1\}$ in turn. Player I wins if $\vec{n} = \langle n_i \mid i \in \omega \rangle \in A$.

Table: $G(A)$

	Round 0		Round 1	
Player I	n_0		n_2	
Player II		n_1		n_3

Theorem (Harrington, Martin)

The following statements are equivalent:

- *Every real has a sharp.*
- *For every analytic subset A of 2^ω, the game $G(A)$ is determined.*

Preserving sharps

Which forcings preserve the statement “For every real x, $x^{\#}$ exists”?

Theorem (David)

It is consistent that every real has a sharp and there is a c.c.c. Σ^1_3 forcing that forces $V = L[x]$ for some real x.

However, it is easy to show that absolutely c.c.c. Σ^1_2 forcings preserve the statement “Every real has a sharp”.

Question (Ikegami 2010) Does every provably Δ^1_2 proper tree forcing preserve the statement “Every real has a sharp”?

Results

We consider the tree forcings Sacks, Mathias, Miller, Laver and Silver forcing.

Theorem (Castiblanco-S.)

Suppose that $\boldsymbol{\Pi}^1_n$-determinacy holds in V. Let $V[G]$ be a $\mathbb{P}$-generic extension of V, for some $\mathbb{P}$ as above. Then:

1. *$\boldsymbol{\Pi}^1_n$-determinacy holds in $V[G]$.*
2. *$V \prec_{\Sigma^1_{n+2}} V[G]$.*
3. *$V[G]$ has no new equivalence classes of thin absolutely $\boldsymbol{\Delta}^1_{n+2}$ equivalence relations.*

Some of the results hold for more general classes of forcings, for instance absolutely Axiom A forcings.

Tree forcings

"Organic" illustrations of binary trees.

Tree forcings

Definition

A partial order $\mathbb{P}$ is called a tree forcing if its conditions are perfect trees on ω or 2 such that for all $T \in \mathbb{P}$:

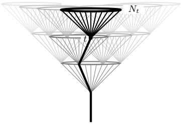
- If $t \in T$, then $T_t = \{s \in T : \text{either } s \subseteq t \text{ or } t \subseteq s\} \in \mathbb{P}$.

$\mathbb{P}$ is ordered by inclusion.

If G is $\mathbb{P}$-generic over V, then

$$x_G = \bigcup\{\text{stem}(T) : T \in G\} = \bigcap\{[T] : T \in G\}$$

is a real and $G = \{T \in \mathbb{P} : x_G \in [T]\}$.



Proper forcing and names for reals

Fact

Suppose that $\mathbb{P} \subseteq \mathbb{R}$ is a proper forcing and σ is a $\mathbb{P}$-name for a real. The for densely many $p \in \mathbb{P}$, there is a nice $\mathbb{P}$-name $\tau \in H_{\omega_1}$ with $p \Vdash \sigma = \tau$.

Capturing

Definition

Let $\mathbb{P}$ and $\mathbb{Q}$ be tree forcings. Let $M : x \mapsto M(x)$ be an operator that sends every real to an inner model, e.g. to $L[x]$.

We say that $\mathbb{Q}$ *captures* $\mathbb{P}$ over M if for any $\mathbb{P}$-generic filter G over V and any $a \in \mathbb{R} \cap V[G]$ there is some $x \in \mathbb{R} \cap V$ such that:

1. $V[G] \models$ "x_G generates a $\mathbb{Q}$-generic extension of $M(x)$" and
2. $a \in \mathbb{M}(x)[x_G]$.

If $\mathbb{P}$ is captured by a small forcing $\mathbb{Q}$ over L, then forcing with $\mathbb{Q}$ preserves the statement "Every real has a sharp".

To see this, take a nontrivial elementary embedding $j : L[x] \to L[x]$. Lift this to:

$$\begin{aligned} k : L[x][x_G] &\to L[x][x_G] \\ \tau^{x_G} &\to (j(\tau))^{x_G} \end{aligned}$$

Since $a \in L[x][x_G]$, k witnesses the existence of $a^{\#}$ in $V[G]$.

Sacks forcing

$\mathbb{S} = \{T : T \text{ is a perfect tree on } 2\}$ denotes Sacks forcing.

Definition

Suppose that $S \in \mathbb{S}$. We define:

$$\mathbb{A}_{\mathbb{S},S} = \{t \subseteq S : t \text{ is a finite subtree of } S\}$$

ordered by end-extension, i.e. $t \leq s$ if and only if t extends s, but only at end nodes of s.

Given $S \in \mathbb{S}$, let $\pi_S : \mathrm{Split}(S) \to {}^{<\omega}2$ be the natural order isomorphism.

Lemma

Suppose that G is $\mathbb{A}_{\mathbb{S},S}$-generic over V. Then:

- $T_G = \bigcup G$ is a perfect subtree of S.
- For every $x \in [T_G]$, $\pi_S(x) := \bigcup_{n<\omega} \pi_S(x{\restriction}n)$ is Cohen-generic over V.

Cohen forcing captures Sacks forcing

Lemma

Suppose that $\forall x \in \mathbb{R}(x^{\#}$ exists) and let $\sigma \in \mathrm{H}(\omega_1)$. Let $\dot{x}$ a name for the $\mathbb{S}$-generic real. For every $S \in \mathbb{S}$, there is some $T \leq S$ such that

$$T \Vdash_{\mathbb{S}} \dot{x} \text{ is } \mathbb{C}\text{-generic over } L[\sigma, S] \text{ modulo } \pi_S$$

Proof

Since $(\sigma, S)^{\#}$ exists, we have that $|P(\mathbb{A}_{\mathbb{S},S})^{L[\sigma,S]}| < \omega_1$ so there is a $\mathbb{A}_{\mathbb{S},S}$-generic T in V over $L[\sigma, S]$. By the lemma above, every branch in T is $\mathbb{C}$-generic over $L[\sigma, S]$ modulo π_S and $T \leq S$.

If G is $\mathbb{S}$-generic over V, we have

$$V[G] \models \text{every branch in } T \text{ is } \mathbb{C}\text{-generic over } L[\sigma, S] \text{ modulo } \pi_S$$

In particular, as $\dot{x}$ is a Sacks real, if $T \in G$ we have

$$V[G] \models \dot{x} \text{ is } \mathbb{C}\text{-generic over } L[\sigma, S] \text{ modulo } \pi_S$$

i.e., $T \Vdash \dot{x}$ is $\mathbb{C}$-generic over $L[\sigma, S]$ modulo π_S.

Thus, for every $\mathbb{S}$-generic real x over V, there exists some real $y \in V$ such that x is equivalent to a $\mathbb{C}$-generic over $L[r, y]$.

Other forcings

Cohen forcing captures Sacks and Silver forcing.

Mathias forcing captures Mathias, Laver and Miller and Silver forcing.

Therefore these forcings preserve sharps for reals.

Grounds for tree forcings

The above results suggest to ask about the nature of reals added by tree forcings.

How much information does a real contain about the forcing for which it is generic?

Let $\mathbb{P}$, $\mathbb{Q}$ denote simply definable tree forcings.

Question

For which such forcings $\mathbb{P}$, $\mathbb{Q}$ are there inner models (transitive class models of ZFC*) $M \subseteq N$ and a real x such that:*

1. *x generates a $\mathbb{P}$-generic extension $M[x]$ of M, and*
2. *x generates a $\mathbb{Q}$-generic extension $N[x]$ of N?*

Note that x itself is not necessarily $\mathbb{P}$-generic over M.

Question

Can a Mathias real x over V generate a Cohen extension $L[x]$ of L?

Note that the usual proof that Mathias forcing does not add Cohen reals (via the Laver property) does not show this.

Amoebas $\leftrightsquigarrow$ Master conditions

An Amoeba for $\mathbb{P}$ is a forcing that adds a tree $T \in \mathbb{P}$ all of whose branches are $\mathbb{P}$-generic.

Assuming the genericity of branches persists to outer models, an Amoeba thus adds a master condition over a countable M. For a master condition, we only need the weaker property that "many" branches of T are $\mathbb{P}$-generic over M.

Mathias forcing is its own "weak Amoeba".

Products of tree forcings

We also need to work with products of the above forcings. Spinas (2009) showed the finite powers of Laver and Miller forcing are proper (and in fact, satisfy Axiom A), answering questions of Brendle and Zapletal.

Our arguments avoid using properness of the products, by working with capturing.

Question (Spinas 2009): Find a reasonably nice class of simply definable proper forcings that is closed under products.

Axiom A forcings

A forcing $(\mathbb{P}, \leq)$ satisfies Axiom A if there is a sequence $\vec{\leq} = \langle \leq_n \mid n \in \omega \rangle$ with:

- $\leq \, = \, \leq_0 \, \supseteq \, \leq_1 \, \supseteq \ldots$.
- (Fusion) If $p_0 \geq_0 p_1 \geq_2 p_2 \geq_3 \ldots$, then there is some $p \in \mathbb{P}$ with $p \leq_n p_n$ for all $n \in \omega$.
- (Chain condition) For every maximal antichain A and every $p \in \mathbb{P}$, there is some $q \leq_n p$ such that $\{r \in A \mid r \parallel q\}$ is countable.

Many tree forcings satisfy Axiom A. In most cases, $S \leq_n T$ means that $S \leq T$ and $\mathrm{split}(S)$, $\mathrm{split}(T)$ agree up to their n-th level.

Axiom A forcings

Lemma (Castiblanco-S.)

Suppose that $\bar{\mathbb{P}} = \langle \mathbb{P}, \leq, \leq_n \mid n \in \omega \rangle$ *is definable such that:*

- *Its definition is absolute to every inner model, and*
- $\bar{\mathbb{P}}$ *satisfies Axiom A in every inner model.*

Then $\mathbb{P}$ *preserves the statement "every real has a sharp".*

Proof, part 1.

Let $\dot{G}$ be the $\mathbb{P}$-name for the $\mathbb{P}$-generic filter. Let σ be a $\mathbb{P}$-name for a real; we can assume that σ is a nice name in H_{ω_1}.

It suffices to show that

$$D = \{T \mid \exists S \geq T \quad T \Vdash \dot{G} \text{ is } \mathbb{P}\text{-generic over } L[\sigma, S]\}$$

is dense. Then for any $\mathbb{P}$-generic G over V, there is some $S \in G$ such that G is $\mathbb{P}$-generic over $L[\sigma, S]$. The embedding $j : L[\sigma, S] \to N$ then lifts to $L[\sigma, S][G]$.

□

Proof, part 2.

We claim that

$$D = \{T \mid \exists S \geq T \ \exists x \ T \Vdash \dot{G} \text{ is } \mathbb{P}\text{-generic over } L[\sigma, S]\}$$

is dense.

To prove this, take $S \in \mathbb{P}$. Let $\vec{D} = \langle D_n \mid n \in \omega \rangle$ enumerate the dense open subsets $D \in L[\sigma, S]$ of $\mathbb{P}^{L[\sigma,S]}$.

Construct a sequence $S \geq_0 p_0 \geq_1 p_1 \geq_2 \ldots$ of conditions $p_n \in L[\sigma, S]$ with $p_n \in D_n$. This works since the statement "$\bar{\mathbb{P}}$ satisfies Axiom A" holds in both $L[\sigma, S]$ and in V.

By fusion, there is a condition $T \in \mathbb{P}$ with $T \leq p_n$ for all $n \in \omega$. Therefore, for any $\mathbb{P}$-generic filter G over V with $T \in G$, $G \cap L[\sigma, S]$ is $\mathbb{P}^{L[\sigma,S]}$-generic over $L[\sigma, S]$. $\square$

Absoluteness

Theorem

Assume $\mathbf{\Pi}^1_n$-determinacy, where $n \geq 1$. If $\mathbb{P}$ is a tree forcing as above, then for any $\mathbb{P}$-generic extension $V[G]$ of V, we have

$$V \prec_{\Sigma^1_{n+2}} V[G].$$

The proof uses the fact that for any Σ^1_n-formula φ, the formula $\Vdash_{\mathbb{P}} \varphi$ is also Σ^1_n. It further uses:

Theorem (Martin, Harrington, Neeman, Woodin)

Let $n \in \omega$. Then, the following are equivalent:

1. *For every $x \in {}^\omega\omega$, $M_n^\#(x)$ exists.*
2. *Every $\mathbf{\Pi}^1_{n+1}$ set is determined.*

Thin equivalence relations

Theorem (Castiblanco-S.)

Assume $\mathbf{\Pi}^1_{n+1}$-determinacy. Suppose that E is a thin absolutely $\mathbf{\Delta}^1_{n+3}$ equivalence relation. Then forcing with any of the tree forcings above does not add new equivalence classes to E.

The main idea is to take a name τ for a real in a new equivalence class and argue that for densely many conditions $p \in \mathbb{P}$, the condition $(p, p) \in \mathbb{P} \times \mathbb{P}$ forces that the two copies of τ are E-equivalent.

Thin equivalence relations: Examples

Definition

We say an equivalence relation E on 2^ω or ω^ω is thin if there is no perfect set P of pairwise E-inequivalent elements.

Let WO denote the set of all wellorders on ω. Let WO_α denote the set of wellorders on ω with order type α.

Example

Let $(x, y) \in F$ if either $x, y \in WO_\alpha$ for some α, or $x, y \notin WO$.

F is thin, since otherwise the would be a Borel wellorder of 2^ω. Note that F is analytic.

Example

Assume that every real has a sharp. Define

$$xEy \iff (\omega_1^{+L[x]} = \omega_1^{+L[y]})$$

Notice that xEy iff

$$\exists z(x, y \leq_T z \text{ and } z^\# \models \kappa^{+L[x]} = \kappa^{+L[y]})$$

where κ is the critical point of the measure of $z^\#$.

Therefore, in the presence of sharps for reals, E is a Δ^1_3 equivalence relation.

Thin equivalence relations: Examples

Lemma

E is thin.

Proof, part 1.

Suppose that there is a perfect set $P \subset {}^{\omega}\omega$ such that $[P]^2 \subset \mathbb{R}^2 \setminus E$. Since E is Δ^1_3, the formula

$$\forall x, y \in P(x \neq y \implies (x, y) \in \mathbb{R}^2 \setminus E)$$

is Π^1_3.

As every real has a sharp, we have Σ^1_3-absoluteness for any provably Σ^1_2 c.c.c. forcing notion. For any Cohen generic c over V,

$$V[c] \models [P]^2 \subset \mathbb{R}^2 \setminus E$$

Notice that P induces a Δ^1_3 well-ordering of the reals by taking

$$x \prec y \text{ iff } \omega_1^{+L[\varphi(x)]} < \omega_1^{+L[\varphi(y)]}$$

where $\varphi : {}^{\omega}\omega \to P$ is a recursive bijection with parameters in the ground model.

□

Proof, part 2.

Therefore, there exists $a \in {}^\omega\omega \cap V$ and a $\Delta^1_3(a)$ formula $\phi(x, y)$ such that

$$V[c] \models \{(u, v) : \phi(u, v, a)\} \text{ is a well-ordering of } \mathbb{R} \quad (*)$$

Let $f : {}^\omega\omega \to \alpha$, $\alpha \in \mathrm{Ord}$ be an order isomorphism witnessing (*). Note that f is definable from the real $a \in V$.

Thus, c is the only solution to the formula

$$\psi(x, a) : \exists x(f(x) = \gamma)$$

for some $\gamma < \alpha$.

This means that the Cohen generic real c is definable with a formula using parameters from the ground model which is impossible. Thus, E is thin. $\square$

Thin equivalence relations: Examples

For each x, let C_x denote the club of Silver indiscernibles for $L[x]$. The elements of $\bigcap_{x \in 2^\omega} C_x$ are called uniform indiscernibles.

Let $u_1 = \omega_1$ denote the first and u_2 the second uniform indiscernible.

Theorem (Kunen-Martin)

If for every $x \in {}^\omega\omega$, $x^\#$ exists the following are all equal:

- u_2;
- $\sup\{(\omega_1)^{+L[x]} : x \in {}^\omega\omega\}$ where $\omega_1 = \omega_1^V$;
- $\sup\{\alpha : \alpha \text{ is the rank of a } \Pi^1_1 \text{ well-founded relation}\}$;
- $\delta^1_2 = \sup\{\alpha : \exists f : {}^\omega\omega \to \alpha \text{ such that } f \text{ defines a } \Delta^1_2 \text{ well-ordering of } {}^\omega\omega\}$

We thus obtain:

Corollary

Suppose that $x^\#$ exists for every real x and let $\mathbb{P}$ be a forcing as above. Then $\mathbb{P}$ does not change the value of u_2, i.e. $u_2^V = u_2^{V^{\mathbb{P}}}$.

Thin equivalence relations: Examples

Corollary

Suppose that $x^\#$ exists for every real x and let $\mathbb{P}$ be a forcing as above. Then $\mathbb{P}$ does not change the value of u_2, i.e. $u_2^V = u_2^{V^{\mathbb{P}}}$.

u_2 has been studied in the context of large cardinals. For instance, Woodin showed that $u_2 = \omega_2$ assuming the nonstationary ideal on ω_1 is saturated and a measurable cardinal exists.

It cannot by changed by proper forcing assuming sufficient large cardinals, but Claverie and Schindler showed that u_2 can be increased by a stationary set preserving forcing, from strong assumptions.

Some open questions

The arguments for specific tree forcings suggest the following problems:

Question

- *For which tree forcings $\mathbb{P}$, $\mathbb{Q}$ is there a real x and there inner models $M \subseteq N$ such that:*
  1. *x generates a $\mathbb{P}$-generic extension $M[x]$ of M, and*
  2. *x generates a $\mathbb{Q}$-generic extension $N[x]$ of N?*
- *Which tree forcings have Amoebas?*

The results about preserving determinacy and about equivalence relations suggest the following problems:

Question

- *Is there a natural class of forcings for which the above results hold?*
- *Do the results hold for countable support iterations of length ω_2 of the above tree forcings?*
- *Do similar results hold for (not) adding connected components to thin projective graphs?*

References

- Castiblanco, Schlicht, Preserving levels of projective determinacy by tree forcings, 2018, in revision for APAL.

Thank you for listening!