Can You Take Komjath's Inaccessible Away?

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This is a joint work with Stevo Todorcevic.

Introduction

Definition

An Aronszajn tree is an ω_1-tree with no uncountable chain. A Kurepa tree is an ω_1-tree which has at least $\aleph_2$ many branches.

Fact

Aronszajn trees exist in any model of ZFC. If there is an inaccessible cardinal then it is consistent that there is no Kurepa tree.

Comparing Aronszajn trees and Kurepa trees.

Theorem (Jensen)

Assume $\mathrm{V} = \mathrm{L}$. Then there is a Kurepa tree with no Aronszajn subtree.

Theorem (Todorcevic)

There is a σ-closed forcing which adds a Kurepa tree with no Aronszajn subtree.

Theorem (Komjath)

Assume there are two inaccessible cardinals. Then it is consistent with ZFC that there is a Kurepa tree and every Kurepa tree has an Aronszajn subtree.

Goal

Theorem

If there is an inaccessible cardinal, then it is consistent with ZFC that there is a Kurepa tree and every Kurepa tree has a Souslin subtree.

Theorem

If every Kurepa tree has an Aronszajn subtree then ω_2 is inaccessible in L.

Walks on ordinals in ω_2

$\square_{\omega_1}$

We fix a sequence $\langle C_\alpha : \alpha \in \omega_2 \rangle$ such that

- C_α is a closed unbounded subset of α,
- for all α, $\mathrm{otp}(C_\alpha) \leq \omega_1$ and if $\mathrm{cf}(\alpha) = \omega$ then $\mathrm{otp}(C_\alpha) < \omega_1$,
- for all $\beta \in \omega_2$ if $\alpha \in C_\beta$ then $\mathrm{cf}(\alpha) \leq \omega$,
- if α is a limit point of C_β then $C_\beta \cap \alpha = C_\alpha$.

Fact

Assume λ is the first inaccessible cardinal in L. Let $G \subset coll(\omega_1, < \lambda)$ be L-generic. Then $\square_{\omega_1}$ holds in $\mathrm{L}[G]$.

Definition (Todorcevic)

The function $\rho : [\omega_2]^2 \longrightarrow \omega_1$ is defined recursively as follows: for $\alpha < \beta$,

$$\rho(\alpha,\beta) = \max\{\mathrm{otp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi,\alpha) : \xi \in C_\beta \cap [\lambda(\alpha,\beta),\alpha)\}.$$

We define $\rho(\alpha,\alpha) = 0$ for all $\alpha \in \omega_2$.

Lemma (Todorcevic)

Assume $\xi \in \alpha$ and α is a limit point of C_β. Then $\rho(\xi,\alpha) = \rho(\xi,\beta)$.

Lemma (Todorcevic)

If $\alpha < \beta$, α is a limit ordinal such that there is a cofinal sequence of $\xi \in \alpha$, with $\rho(\xi,\beta) \leq \nu$ then $\rho(\alpha,\beta) \leq \nu$.

Lemma (Todorcevic)

For all $\nu \in \omega_1$ and $\alpha \in \omega_2$, the set $\{\xi \in \alpha : \rho(\xi,\alpha) \leq \nu\}$ is countable.

Lemma (Todorcevic)

Assume $\alpha \leq \beta \leq \gamma$. Then

- $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$,
- $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$.

Lemma (Todorcevic)

Assume $\alpha < \beta < \gamma$. We have $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$, if $\rho(\beta, \gamma) < \max\{\rho(\alpha, \beta), \rho(\alpha, \gamma)\}$.

Lemma (Todorcevic)

Assume A is an uncountable family of finite subsets of ω_2 and $\nu \in \omega_1$. Then there is an uncountable $B \subset A$ such that B forms a Δ-system with root r and for all a, b in B:

- $a \setminus r < b \setminus r$ *implies that for all* $\alpha \in a \setminus r$ *and* $\beta \in b \setminus r$, $\rho(\alpha, \beta) > \nu$,
- $r < a \setminus r < b \setminus r$ *implies that for all* $\alpha \in a \setminus r$, $\beta \in b \setminus r$, *and* $\gamma \in r$, $\rho(\alpha, \beta) \geq \min\{\rho(\gamma, \alpha), \rho(\gamma, \beta)\}$.

Definition

Q is the poset consisting of all finite functions p such that the following holds.

1. $dom(p) \subset \omega_2$.
2. For all $\alpha \in dom(p)$, $p(\alpha) \in [\omega_1]^{<\omega}$ such that for all $\nu \in \omega_1$, $p(\alpha) \cap [\nu, \nu + \omega)$ has at most one element.
3. For all α, β in $dom(p)$, $p(\alpha) \cap p(\beta)$ is an initial segment of both $p(\alpha)$ and $p(\beta)$.
4. For all $\alpha < \beta$ in $dom(p)$, $\max(p(\alpha) \cap p(\beta)) \leq \rho(\alpha, \beta)$.

We let $q \leq p$ if $dom(p) \subset dom(q)$ and $\forall \alpha \in dom(p)$, $p(\alpha) \subset q(\alpha)$.

Proposition (Todorcevic)

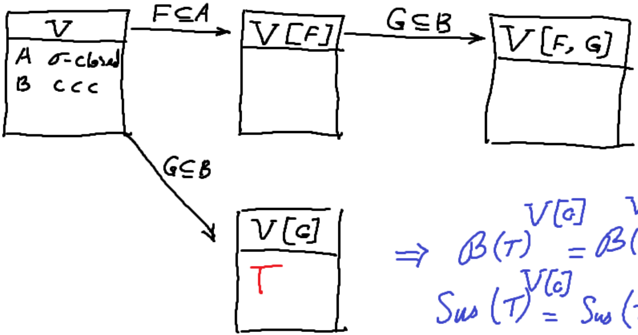
The poset Q satisfies the Knaster condition.

Trees in extensions by σ-closed $\times$ ccc

Theorem (Jensen - Schlechta)

Assume $A \in \mathrm{V}$ is a countably closed poset, $F \subset A$ is V-generic, $B \in \mathrm{V}$ is a ccc poset and $G \subset B$ is $\mathrm{V}[F]$-generic. Let $T \in \mathrm{V}[G]$ be an ω_1-tree.

1. If $b \in \mathrm{V}[F][G]$ is a cofinal branch in T, then $b \in \mathrm{V}[G]$.
2. If $S \in \mathrm{V}[F][G]$ is a downward closed Souslin subtree of T then $S \in \mathrm{V}[G]$.



$$\Rightarrow \mathcal{B}(T)^{V[G]} = \mathcal{B}(T)^{V[F,G]}$$

$$Sus(T)^{V[G]} = Sus(T)^{V[F,G]}$$

Lemma

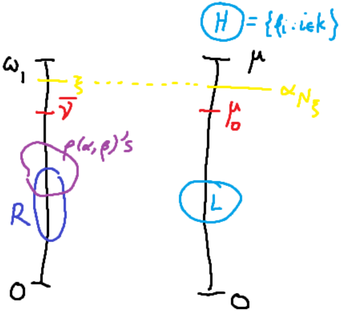
Assume CH. Let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain of countable elementary submodels of H_θ where θ is a regular large enough cardinal, $N_{\omega_1} = \bigcup_{\xi \in \omega_1} N_\xi$, and $\mu = \sup(N_{\omega_1} \cap \omega_2)$. Then Q_μ is a complete suborder of Q.

proof

- We need to show that for all $q \in Q$ there is $p \in Q_\mu$ such that if $r \leq p$ and $r \in Q_\mu$ then r is compatible with q.

Fix the following notation:

- $H = dom(q) \setminus \mu = \{\beta_i : i \in k\}$ such that β_i is increasing, $L = dom(q) \cap \mu$, and $R = \bigcup range(q)$.
- $\bar{\nu} \in \omega_1$ such that $\bar{\nu} > \max(R)$ and for all α, β in $dom(q)$, $\bar{\nu} > \rho(\alpha, \beta)$.
- $\mu_0 \in \mu$ such that $\mu_0 > \max(L)$ and $\forall \gamma \in \mu \setminus \mu_0 \; \forall \beta \in H$, we have $\rho(\gamma, \beta) > \bar{\nu}$.

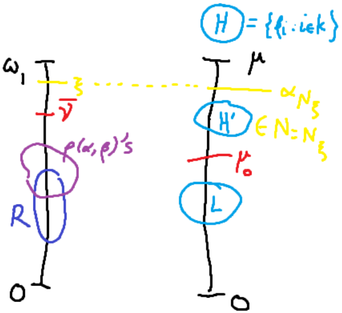


$Dom(q) = L \cup H$

$\forall \gamma \in (\mu \setminus \mu_0) \; \forall \beta \in H \quad \rho(\gamma, \beta) > \bar{\nu}$

$A_{\nu,\beta} \in [\mu_0]^\omega$

- For each $\beta \in H$, $\nu \in \bar{\nu}$ let $A_{\nu,\beta} = \{\alpha \in \mu_0 : \rho(\alpha, \beta) = \nu\}$.
- Fix $N = N_\xi$ such that $\mu_0, \bar{\nu}, L, R, \langle A_{\nu,\beta} : \beta \in H, \nu \in \bar{\nu}\rangle$ are in N.
- By elementarity, there is $H' = \{\beta'_i : i \in k\}$ which is in N and
  1. β'_i is increasing,
  2. $\min(H') > \mu_0$
  3. for all $i \in k$ and for all $\nu \in \bar{\nu}$, $A_{\nu,\beta_i} = \{\alpha \in \mu_0 : \rho(\alpha, \beta'_i) = \nu\}$, and
  4. for all $i < j$ in k, $\rho(\beta_i, \beta_j) = \rho(\beta'_i, \beta'_j)$.



$Dom(q) = L \cup H$

- $\forall \gamma \in (\mu \setminus \mu_0)\ \forall \beta \in H\ \rho(\gamma, \beta) > \bar{\nu}$

$A_{\nu,\beta} \in [\mu_0]^\omega$

Let p be the condition that $dom(p) = L \cup H'$, for all $\xi \in L$, $p(\xi) = q(\xi)$ and for all $i \in k$, $p(\beta'_i) = q(\beta_i)$.

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Suppose $r \leq p$ is in Q_μ. We will find $s \in Q$ which is a common extension of r, q. Pick s such that $dom(s) = dom(r) \cup H$, $s \restriction dom(r) = r$, and for all $i \in k$ $s(\beta_i) = r(\beta'_i) \cap (\max(q(\beta_i)) + 1)$.
We need to show that s is a condition in Q. All of the conditions in Definition of Q obviously hold, except for condition 4. If $\alpha < \beta$ are in H, by the last requirement for H' and the fact that r is a condition, $\max(s(\alpha) \cap s(\beta)) \leq \rho(\alpha, \beta)$.
Now assume that $\alpha \in dom(r)$ and $\beta = \beta_i \in H$. If $\rho(\alpha, \beta) \geq \bar{\nu}$, everything is obvious because $\max(s(\beta)) < \bar{\nu}$. Assume $\rho(\alpha, \beta) = \nu < \bar{\nu}$. So $\alpha \in A_{\nu,\beta}$. Since $r \in Q_\mu$, we have $\max(s(\alpha) \cap s(\beta_i)) < \max(r(\alpha), r(\beta'_i)) \leq \nu$.

Taking Komjath's inaccessible away

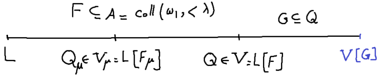
Now we are ready to show that if there is an inaccessible cardinal in L then there is a model of ZFC in which every Kurepa tree has a Souslin subtree.

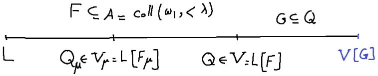
Our model

We start from L. Assume $\lambda \in \mathrm{L}$ is the first inaccessible cardinal. First we collapse every $\kappa \in \lambda$ to ω_1 using the standard Levy collapse. Then we force with $Q = Q_\lambda$.

Some notation

λ is the first inaccessible cardinal in L. $A_\mu = \mathrm{coll}(\omega_1, < \mu)$ for every uncountable cardinal $\mu \in \lambda$, F is L-generic for $A = A_\lambda$ and $F_\mu = F \cap A_\mu$. $\mathrm{V}_\mu = \mathrm{L}[F_\mu]$, and $\mathrm{V} = \mathrm{L}[F]$. Let $G \subset Q$ be V-generic and for each $\mu \in \lambda$, let $G_\mu = G \cap Q_\mu$.



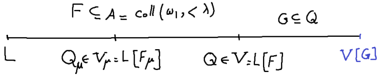


Claim

Every Kurepa tree in $\mathrm{V}[G]$, has a Souslin subtree.

proof

Assume K is an ω_1-tree with no Souslin subtree. We will show that it has at most $\aleph_1$ many branches.



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Every Kurepa tree in $\mathrm{V}[G]$, *has a Souslin subtree.*

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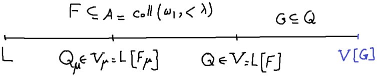
Assume K is an ω_1-tree with no Souslin subtree. We will show that it has at most $\aleph_1$ many branches.
Let H be the set of all $\mu \in \lambda$ such that Q_μ is a complete suborder of Q. Obviously, $H \in \mathrm{V}$ and it is stationary in $\mathrm{V}[G]$.



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Let H be the set of all $\mu \in \lambda$ such that Q_μ is a complete suborder of Q. Obviously, $H \in \mathrm{V}$ and it is stationary in $\mathrm{V}[G]$.
For each $\alpha \in H$ with $K \in \mathrm{V}[G_\alpha]$, there is $\beta \in H$ such that whenever $U \subset K$ is a downward closed subtree which is Souslin in $\mathrm{V}[G_\alpha]$, then U is not Soulsin in $\mathrm{V}[G_\beta]$.

In order to see this, fix $\alpha \in H$ with $K \in \mathrm{V}[G_\alpha]$. Using Jensen-Sclechta fact, if $U \subset K$ is a downward closed subtree which is Souslin in $\mathrm{V}[G_\alpha]$ then $U \in \mathrm{V}_\alpha[G_\alpha]$. So there are only $\aleph_1$ many such U's in $\mathrm{V}[G]$. Since H is cofinal in λ, there is $\beta \in H$ such that if $U \subset K$ is a downward closed subtree which is Souslin in $\mathrm{V}[G_\alpha]$ then U is not Souslin in $\mathrm{V}[G_\beta]$.

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Now define $f : H \longrightarrow H$ in $\mathrm{V}[G]$, by letting $f(\alpha)$ be the smallest $\beta \in H$ such that every Souslin $U \subset K$ which is in $\mathrm{V}[G_\alpha]$ is killed in $\mathrm{V}[G_\beta]$. Let $\mu \in H$ be an f-closed ordinal. Also note that $Q = Q_\mu * R_\mu$ for some Q_μ-name for a ccc poset R_μ. Since K has no Souslin subtree in $\mathrm{V}[G_\mu]$, R_μ can not add branches to K. Moreover, there is no ccc poset of size $\aleph_1$ in V which creates a Kurepa tree. Therefore, K has at most $\aleph_1$ many branches.

The large cardinal strength

Theorem (Todorcevic)

Assume M is an inner model of set theory which correctly computes ω_1. Then M contains a partition of ω_1 into infinitely many sets which are stationary in the universe of all sets V.

Theorem

If every Kurepa tree has an Aronszajn subtree then ω_2 is inaccessible in L.

Proof

Assume that ω_2, of V, is a successor cardinal in L. Then there is $X \subset \omega_1$ such that $\mathrm{L}[X]$ computes ω_1 and ω_2 correctly. Using Stevo's Theorem there are two disjoint subsets S, S' of $\lim(\omega_1)$ in $\mathrm{L}[X]$ such that $S \cup S' = \lim(\omega_1)$ and they are stationary in V. Let f be a function on $\lim(\omega_1)$ which is defined as follows.

- If $\alpha \in S$ then $f(\alpha)$ is the lease $\xi > \alpha$ such that $\mathrm{L}_\xi[X] \prec \mathrm{L}_{\omega_1}[X]$.
- If $\alpha \in S'$ then $f(\alpha)$ is the least $\xi > \alpha$ such that for some countable $N \prec L_{\omega_2}[X]$, N is isomorphic to $L_\xi[X]$.

Let T be an ω_1-tree in $\mathrm{L}[X]$ such that:

- $T = (\omega_1, <)$ and 0 is the smallest element of T,
- $T_{\alpha+1}$ is the first ω ordinals after T_α,
- every node $t \in T_\alpha$ has two extensions in $T_{\alpha+1}$,
- if $\alpha \in \lim(\omega_1)$ then a cofinal branch $b \subset T_{<\alpha}$ has a top element in T_α if and only if $b \in L_{f(\alpha)}[X]$.

Claim

The tree T is a Kurepa tree.

It suffices to show that T is Kurepa in $\mathrm{L}[X]$. Assume for a contradiction that $\langle b_\xi : \xi \in \omega_1 \rangle$ is an enumeration of all branches of T. Let $N \prec \mathrm{L}_{\omega_2}[X]$ be countable such that $T, \langle b_\xi : \xi \in \omega_1 \rangle, X$ are in N, $\alpha = N \cap \omega_1 \in S$ and $\varphi : N \longrightarrow L_\delta[X]$ is the transitive collapse isomorphism. Note that $\delta < f(\alpha)$. Then

$$\varphi(\langle b_\xi : \xi \in \omega_1 \rangle) = \langle b_\xi \restriction \alpha : \xi \in \alpha \rangle \in L_\delta[X].$$

But $L_\delta[X] \subset L_{f(\alpha)}[X]$. Therefore, using the sequence $\langle b_\xi \restriction \alpha : \xi \in \alpha \rangle$ one can find a cofinal branch b of $T_{<\alpha}$ which is in $L_{f(\alpha)}[X]$ and which is different from all $\langle b_\xi \restriction \alpha : \xi \in \alpha \rangle$. This contradicts the fact that $\langle b_\xi : \xi \in \omega_1 \rangle$ an enumeration of all branches of T.

T has no Aronszajn subtree in V.

Note that there are stationary many countable $N \prec H_{\omega_2}$ such that $N \cap \mathrm{L}[X] \prec L_{\omega_2}[X]$ and $\alpha = N \cap \omega_1 \in S'$. Fix such an N.

Definition (translated from Ishiu-Moore for trees)

Assume T is an ω_1-tree, κ is a large enough regular cardinal, $t \in T \cup \mathcal{B}(T)$, and $N \prec H_\kappa$ is countable such that $T \in N$. We say that N captures t if there is a chain $c \subset T$ in N which contains all elements of $T_{<N\cap\omega_1}$ bellow t, or equivalently $t \restriction (\delta_N) \subset c$.

Back to the proof

In order to see that N captures all elements of T_α, note that the transitive collapse of $N \cap L[X]$ is equal to $L_\delta[X]$ for some $\delta \geq f(\alpha)$. Let π be the transitive collapse map for $N \cap L[X]$. Assume $t \in T_\alpha$. Then there is a branch $b \subset T_{<\alpha}$ which is in $L_\delta[X]$ such that t is the top element of b. Since α is the first uncountable ordinal in $L_\delta[X]$, ${\omega_1}^{L_\delta[X]} = \alpha$. Since π is an isomorphism there is $b_1 \in N$ such that b_1 is an uncountable branch of T and $\pi(b_1) = b$. But then $b_1 \cap T_{<\alpha} = b$. This means that N captures t via b_1, as desired.

What can we say without using large cardinals?

Definition (Ishiu-Moore + some modification)

Assume $T = (\omega_1, <)$ is an ω_1-tree , $x \in T \cup \mathcal{B}(T)$ and $N \prec H_\theta$ is countable with $T \in N$. We say that x is weakly external to N if there is a stationary $\Sigma \subset [H_{2^{\omega_1+}}]^\omega$ in N such that

$$\forall M \in N \cap \Sigma,\ M \textit{ does not capture } x.$$

Proposition (Ishiu-Moore + some modification)

Let $T = (\omega_1, <)$ be an ω_1-tree, $\kappa = 2^{\omega_1+}$ and $\Sigma \subset [H_\kappa]^\omega$ be stationary. Assume for all large enough regular cardinal θ there are $x \in T$ and countable $N \prec H_\theta$ such that x is weakly external to N witnesses by Σ. In other words, for all $M \in \Sigma \cap N$, M does not capture x. Then T has an Aronszajn subtree.

Some of the complete suborders of Q

Definition

Q_c is the poset consisting of all conditions p in Q with the additional condition that for all $\alpha \in dom(p)$, $cf(\alpha) \leq \omega$. In general, if $A \subset \omega_2$, Q_A denotes the set of all conditions q in Q such that $dom(q) \subset A$.

Lemma

The poset Q_c is a complete suborder of Q. Moreover, if $X \subset \omega_2$ is a set of ordinals of cofinality ω_1, then $Q_{\omega_2 \setminus X}$ is a complete suborder of Q.

Fact

Assume $cf(\mu) = \omega$, $\mu \in \omega_2$, for some $\beta > \mu$, μ is a limit point of C_β and the set of all limit points of C_μ is cofinal in μ. Then Q_μ is not a complete suborder of Q.

Climbing Soulin trees to see ρ

Lemma

Let $2^{\omega_1 +} < \kappa_0 < \kappa < \theta$ be regular cardinals such that $2^{\kappa_0 +} < \kappa$, and $2^{\kappa +} < \theta$. Let S be the set of all $X \in [\omega_2]^\omega$ such that $C_{\alpha_X} \subset X$ and $\lim(C_{\alpha_X})$ is cofinal in X. Assume $\mathcal{A}$ is the set of all countable $N \prec H_\theta$ with the property that if $N \cap \omega_2 \in S$ then there is a club of countable elementary submodels $E \subset [H_{\kappa_0}]^\omega$ in N such that for all $M \in E \cap N$,

$$\rho(\alpha_M, \alpha_N) \leq M \cap \omega_1.$$

Then $\mathcal{A}$ contains a club.

ρ introduces Aronszajn subtrees everywhere in the P-generic tree T

Definition

P is the poset as in Definition of Q, but instead of condition 4, the elements $p \in P$ have the property that for all $\alpha < \beta$ in $\mathrm{dom}(p)$, $\max(p(\alpha) \cap p(\beta)) < \rho(\alpha, \beta)$. Moreover, $b_\xi = \bigcup\{p(\xi) : p \in G\}$, whenever G is a generic filter for P.

Definition

Assume G is generic for P. Then $b_\xi = \bigcup\{p(\xi) : p \in G\}$.

Lemma

Assume T is the generic tree for P. Then $\{b_\xi : \xi \in \omega_2\}$ is the set of all branches of T.

Theorem

It is consistent that there is a Kurepa tree T such that every Kurepa subset of T has an Aronszajn subtree.

Thank You