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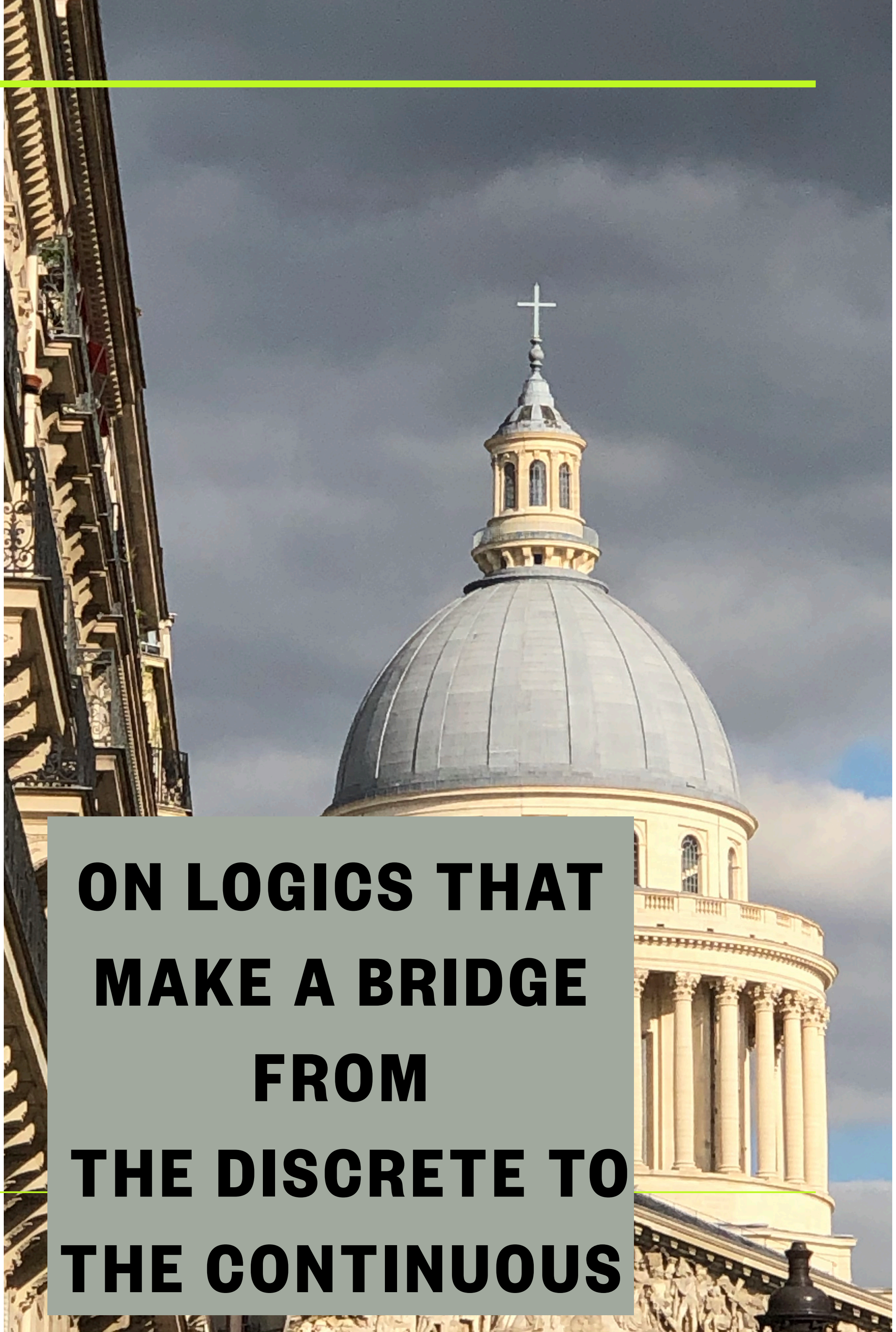

ON LOGICS THAT MAKE A BRIDGE FROM THE DISCRETE TO THE CONTINUOUS

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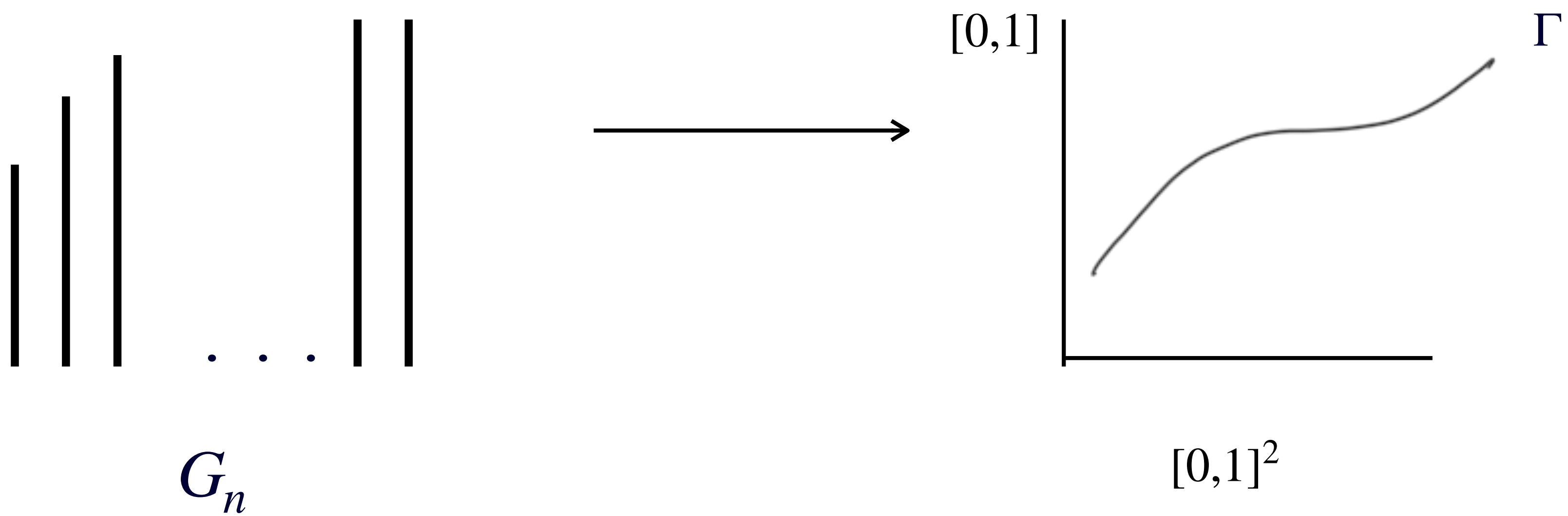
1. Introduction

One of the main discoveries in discrete mathematics recently has been that of a

GRAPHON

by Lovász and his group around 2006.

A *graphon* is an uncountable limit of a sequence of finite graphs.



In fact, a graphon is a a measurable function $\Gamma : [0,1] \times [0,1] \to [0,1]$ *which* represents the sequence $\langle G_n : n < \omega \rangle$ *in the sense that certain graph invariants are transferred.*

For example, the graph homomorphism density *is transferred:*

$$\lim_{n\to\infty} t(F,G_n) = \int_{[0,1]^{v(F)}} \Pi_{i,j\in E(F)}\Gamma(x_i,x_j)\Pi_{i\in v(F)}dx_i \; ,$$

for every finite graph F, where $t(F,G) = \dfrac{\hom(F,G)}{|G|^{|F|}}$

There is a notion of metric convergence for the sequence $\langle G_n :\ n < \omega\rangle$ *associated to this,* cut metric.

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Volume 60

Large Networks and Graph Limits

László Lovász

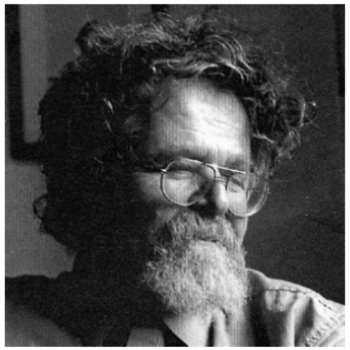
Many other notions of combinatorial limits have been introduced since, many applications found and many awards gained. We mention one, as an impressive example :

ERC Synergy Grant 2018, 1st in Mathematics

Project: Dynamics and Structure of Networks (DYNASNET)
ERC funding: 9.315 million for 6 years
Researchers and Host institutions:

Albert-László Barabási
Central European University, Budapest

Laszlo Lovasz
Hungarian Academy of Science



Jaroslav Nesetril
Charles University in Prague

An example of a generalisation of graphons is the idea of modeling:

The original idea

$$\lim_{n\to\infty} t(F, G_n) = \int_{[0,1]^{v(F)}} \Pi_{i,j\in E(F)} \Gamma(x_i, x_j) \Pi_{i\in v(F)} dx_i \ ,$$

for every finite graph F, where $t(F,G) = \dfrac{\hom(F,G)}{|G|^{|F|}}$

works well for sequences of dense graphs but is rather information-free for sparse graphs, as we get 0 *in the limit.*

To capture sparse graphs, a new theory was needed, developed by Benjamini-Schramm and further by Nešetřil and Ossona de Mendez. A unifying theory was given by the latter authors through the notion of

FIRST ORDER CONVERGENCE

which leads to the limit notion called modeling. *In the case of a sequence of dense graphs, a modeling reduces to a graphon.*

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A Unified Approach to Structural Limits and Limits of Graphs with Bounded Tree-Depth

Jaroslav Nešetřil
Patrice Ossona de Mendez

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2. Connections to model theory

2.1 FO convergence

Let τ be a finite relational language and $\langle A_n : n < \omega \rangle$ a sequence of finite τ-structures.

If $\varphi(x_1, \ldots, x_k)$ is a FO formula we define the Stone pairings $\langle \varphi, A_n \rangle$, which for any n is the probability that a random k-element subset of A_n satisfies φ :

$$\langle \varphi, A_n \rangle = \frac{|\{\bar{a} \in A_n^k : A_n \vDash \varphi[\bar{a}]\}|}{|A_n^k|}$$

Then $\langle A_n : n < \omega \rangle$ is a FO-convergent if $\lim_{n \to \infty} \langle \varphi, A_n \rangle$ exists for all φ.

In various situations there is a standard Borel space (so uncountable) A which is a τ-structure and which satisfies $\langle \varphi, A \rangle = \lim_{n \to \infty} \langle \varphi_n, A \rangle$ for all φ. This is the modeling. The notion encapsulates graphons.

2.2. Ultrapowers and Loeb measures

While developing the notion of a hypergraphon, Elek and Szegedy (2012) considered an ultraproduct $\Pi_{n\in\omega}(H_n, \mu_n)/\mathcal{U}$, where μ_n is the counting measure on H_n, $\mathcal{U}$ is a non-principal ultrafilter on ω and the hypergraphon is obtained through a certain separable quotient.

In fact, this construction is a special case of the classical Loeb's measure on ultraproducts (1975) and a countably generated substructure.

In particular, any graphon can be obtained in this way.

Ultraproducts ⟶ Graphons

This idea has been extended to measure preserving actions by Conley, Kechris and Tucker-Drob in ***Ultraproducts of measure preserving actions and graph combinatorics (2012)***

2.3 Pseudofinite objects ≡ ultrapowers of finite objects

Model theory of such objects is well understood (see the work of Hrushovski and others)and has been used to obtain deep combinatorial results (for example the work of Chernikov). We shall review one, and then mention a result of ours with Tomašić that connected that with graphons.

Fact: The space of graphons is compact in the cut metric.

This is proved using Szemeredi Regularity Lemma

Tao (2012) proved Tao's algebraic regularity Lemma,

as shown on the next slide.

Lemma 1 (Algebraic regularity lemma) Let F be a finite field, let V, W be definable non-empty sets of complexity at most M, and let $E \subset V \times W$ also be definable with complexity at most M. Assume that the characteristic of F is sufficiently large depending on M. Then we may partition $V = V_1 \cup \ldots \cup V_m$ and $W = W_1 \cup \ldots \cup W_n$ with $m, n = O_M(1)$, with the following properties:

- (Definability) Each of the $V_1, \ldots, V_m, W_1, \ldots, W_n$ are definable of complexity $O_M(1)$.
- (Size) We have $|V_i| \gg_M |V|$ and $|W_j| \gg_M |W|$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
- (Regularity) We have

$$|E \cap (A \times B)| = d_{ij}|A||B| + O_M(|F|^{-1/4}|V||W|) \qquad (2)$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $A \subset V_i$, and $B \subset W_j$, where d_{ij} is a rational number in $[0, 1]$ with numerator and denominator $O_M(1)$.

Starchenko and Pillay (unpublished preprint) and independently Hrushovski (letter to Tao), gave a proof using the theory of pseudofinite fields which removes the requirement of large characteristics.

Džamonja-Tomašić (submitted) gave a proof using graphons, which inspired us to prove the following general theorem, regarding

0-1 graphons.

Theorem. In the space of graphons, the set of accumulation points of the family of realisations of a definable bipartite graph over the structures ranging in an asymptotic class is a finite set of stepfunctions.

(suggested in the private correspondance of Hrushovski to Tao)

What is an asymptotic class ?

It is a certain hereditary class of finite structures.

Definition 3.5. Let $\mathcal{C}$ be a class of finite structures (considered a category with substructure embeddings). We say that $\mathcal{C}$ is an *asymptotic class* (in the sense of [8] and [3]), if, for every definable set $\mathbf{X}$ over $\mathbf{S}$, there exist

(1) a definable function $\boldsymbol{\mu}_{\mathbf{X}} : \mathbf{S} \to \mathbb{Q}$,
(2) a definable function $\mathbf{d}_{\mathbf{X}} : \mathbf{S} \to \mathbb{N}$,

so that, for every $\epsilon > 0$ there exists a constant $N > 0$ such that for every $F \in \mathcal{C}$ with $|F| > N$ and every $s \in \mathbf{S}(F)$,

$$\big||\mathbf{X}_s(F)| - \boldsymbol{\mu}_{\mathbf{X}}(s)|F|^{\mathbf{d}_{\mathbf{X}}(s)}\big| \leq \epsilon |F|^{\mathbf{d}_{\mathbf{X}}(s)}.$$

Macpherson

and Steinhorn

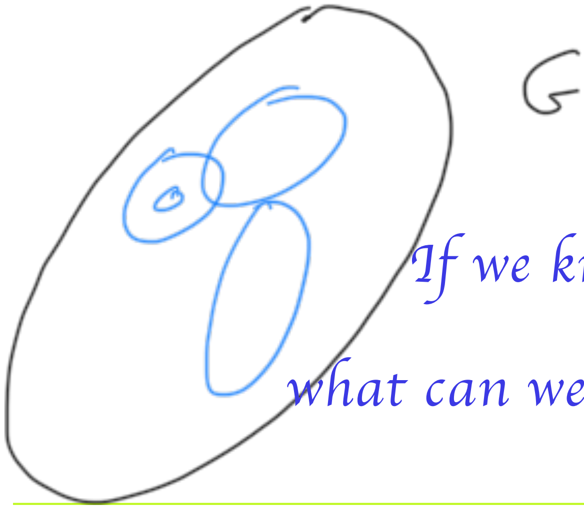
“The graphons generated by graphs coming from a certain hereditary class are simple”

2.4 Connections with ages and classification theory

Question: Suppose that $\mathcal{C}$ is a hereditary class of graphs. Which conditions on $\mathcal{C}$ guarantee that the graphons generated by graphs in $\mathcal{C}$ are "simple" ? For example, have values 0 and 1.

An example of a hereditary class is the age= all finite substructures Age(G) of a countably infinite first order structure G.

For example, a structure obtained through a Fraïssé construction.



If we know a model-theoretic classification of G,

what can we say about the graphons generated by Age(G)?

Very interesting theorems have been proven by Lovász-Szegedy (2012), which, translated in the language of model theory, imply things like:

Theorem (Lovász-Szegedy 2012) Suppose that G is a NIP graph. Then every graphon obtained from Age(G) is 0-1 valued almost everywhere.

There is no mention of NIP in their paper, they rather speak of Vapnik-Červonenkis dimension. But some translation using theorems from model theory gives the above.

Fact. *Stable graphs are NIP.*

A very extensive study of model theory in metric structures, including measure algebras with the mesure of the set difference metric was given by Ben Yacov, Beresnikov, Henson nd Usvyatsov in "Model theory for metric structures" 2008

3. Countable versus uncountable limits

The discussion about the connection between the properties of a hereditary class versus the shape of the of graphon space that it generates, illustrates that there is a connection between the countable limits and the uncountable ones.

Countable limits we have seen so far: a simple union or a Fraïssé limit

Uncountable limits we have seen so far:

ultraproducts, graphons, modelings

The connection exists but is not simple.

Idea: change the countable limit to better reflect the properties of the uncountable limit, notably through changing the logic.

What is a logic?

Most people would answer that it is some syntactical way of generating formulas, usually through a recursive definition, such as FO logic, connected to a semantical notion of interpreting these formulas, again recursively, by a definition like Tarski (Vaught)'s definition of truth.

However, in abstract model theory, a subject indeed started by Tarski and Vaught in the 1950s, there is much more variety as to what a logic might be and the semantic and syntax are not necessarily connected. We were much inspired by the work of Karol Carp from 1959 to 1974, on chain logic. (Chain logic has nothing to do with this context, it was invented for singular cardinals).

In a Džamonja-Väänänen paper on connections between chain logics and Shelah's logic L^1_κ (to appear in Israel Journal of Mathematics), we used the following way of framing abstract logics and a way to compare them using Chu transforms. The concepts in the abstract were studied by Garcia-Matos and Väänänen (2005).

Definition 1.1 *A* logic *is a triple of the form* $\mathfrak{L} = (L, \models_{\mathfrak{L}}, S)$ *where* $\models_{\mathfrak{L}} \subseteq S \times L$ *and* S *comes with a notion of isomorphism, usually understood from the context. We think of* L *as the set or class of sentences of* $\mathfrak{L}$, S *as a set or class of models of* $\mathfrak{L}$ *and of* $\models_{\mathfrak{L}}$ *as the satisfaction relation. The classes* L *and* S *can be proper classes.*

Following Shelah:

Definition 1.2 *A logic* $(L, \models_{\mathfrak{L}}, S)$ *is* nice *iff it satisfies the following requirements:*

- *for any* n*-ary relation symbol* P *and constant symbols* $c_0, \ldots c_{n-1}$ *in* τ, $P[c_0, \ldots c_{n-1}]$ *is a sentence in* L,
- L *is closed under negation, conjunction and disjunction,*
- *for any* $\varphi \in L$ *and* $M \in S$, $M \not\models_{\mathfrak{L}} \varphi$ *if and only if* $M \models_{\mathfrak{L}} \neg\varphi$,
- $M \models_{\mathfrak{L}} \varphi_1 \wedge \varphi_2$ *iff* $M \models_{\mathfrak{L}} \varphi_1$ *and* $M \models_{\mathfrak{L}} \varphi_2$, *and similarly for disjunction,*
- *for any* $M \in S$, $a \in M$ *and a sentence* $\psi[a] \in S$ *such that* $M \models_{\mathfrak{L}} \psi[a]$, *we have that* $M \models_{\mathfrak{L}} (\exists x)\psi(x)$, *and conversely, if* $M \models_{\mathfrak{L}} (\exists x)\psi(x)$ *then there is* $a \in M$ *such that* $M \models_{\mathfrak{L}} \psi[a]$,
- *if* M_0 *and* M_1 *are isomorphic models of* τ, *by some isomorphism* f, *and if both* M_0, M_1 *are in* S, *then for every* $\varphi \in L$ *we have* $M_0 \models_{\mathfrak{L}} \varphi[a_0, \ldots a_{n_1}]$ *iff* $M_1 \models_{\mathfrak{L}} \varphi[f(a_0), \ldots f(a_{n_1})]$.

I have been interested to use these ideas to introduce new logics on countable models which will be used to relate them to uncountable models obtained as combinatorial limits. The following is my work in progress on this subject.

3.1 The ultrafilter logic, a simple example

Let $\mathcal{U}$ be a non-principal ultrafilter on ω and let τ be a finite relational language.

Let L be the set of all FO sentences in τ.

S consists of all countably infinite τ-structures M accompanied with an increasing decomposition $\langle M_n : n < \omega \rangle$ of M.

For $\varphi \in L$ and $M \in S$ we define $M \models_{\mathcal{U}} \varphi$ iff $\{n < \omega : M_n \models \varphi\} \in \mathcal{U}$.

Then one can check that $(L, \models_{\mathcal{U}}, S)$ forms a nice logic. A simple consequence of Łos's transfer theorem for FO logic is the following

Observation. $M \models_{\mathcal{U}} \varphi$ iff $\Pi_{n<\omega} M_n / \mathcal{U} \models \varphi$.

Therefore we obtain a way to interpret the ultrafilter through a countable model.

3.2 The modeling logic

Let τ be a finite relational language and let L be as in the previous example, the set of all FO sentences over τ. Let S be as in the previous example, the set of all countable infinite τ-structures M along with an increasing decomposition $\langle M_n : n < \omega \rangle$.

Now we define the modeling satisfaction relation by saying

$$M \vDash_{\mathscr{M}} \varphi \textit{ iff } \lim_{n \to \infty} \langle \varphi, M_n \rangle = 1.$$

Lemma : If there is a modeling A of $\langle M_n : n < \omega \rangle$, then $M \vDash_{\mathscr{M}} \varphi$ iff $A \vDash \varphi$. In this case, the modeling logic $(L, \vDash_{\mathscr{M}}, S)$ is a nice logic.

So now we have a countable 'mirror' of the uncountable modeling.

3.3 Comparing logics

We shall take a line from computer scientists, who study Chu transforms.

In set theory a similar concept is called Gallois-Tukey transform.

Garcia-Matos and Väänänen used Chu transforms to compare logics.

Definition. Let $\mathscr{L} = (L, \models, S)$ and $\mathscr{L}' = (L', \models', S')$ be two logics. We say that $(L, \models, S) \leq (L', \models', S')$ iff there is a pair of functions (f, g) such that $f: L \to L'$, $g: S' \to S$ onto, and the adjointness condition holds, which means $M' \models' f(\varphi) \iff g(M') \models \varphi$

Heuristic truth supported by various theorems. If $\mathscr{L} \leq \mathscr{L}'$, then the "nice properties" of $\mathscr{L}'$ are inherited by $\mathscr{L}$.

Lemma. The modeling logic is $\leq$ the ultrafilter logic.

Transfer principles à la Łos ...

3.4 The algorithmic aspects

We can define various others logics which measure how fast the sequence of finite models converges and use methods of finite model theory.

Material for another talk ...

Fast Track to Forcing

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