Type Omission and Subcompact cardinals

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5. *κ is inaccessible for every λ, and every $P_\kappa\lambda$-tree has a branch.*

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3. *If M is a model of set theory of size λ, ${}^{<\kappa}M \subseteq M$, then there is a transitive model N and an elementary embedding $j\colon M \to N$, with $\operatorname{crit} j = \kappa$, $j[M] \subseteq s \in N$, $|s|^N < j(\kappa)$.*

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For example, take $\kappa = \lambda$.

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We want to have a *normal* analogue to each of the other characterizations of strong compactness.

Type Omission

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Theorem (Henkin-Orey)

Let T be a consistent theory and let $p(x)$ be a complete type (over a countable language). If there is no φ such that $T \vdash \exists x \varphi(x)$ and for all $\psi(x) \in p(x)$, $T \vdash \forall x(\varphi(x) \to \psi(x))$ then there is a model M of T that omits p.

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What is the $\mathcal{L}_{\kappa,\kappa}$-analogue?

Compactness of type omission

Let T be an $\mathcal{L}_{\kappa,\kappa}$-theory and let $p(x)$ be an $\mathcal{L}_{\kappa,\kappa}$-type with a single variable x. We say that T can omit p if there is a model of T that omits p.

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Theorem (Benda, 1976)

κ is supercompact if and only if for every $\mathcal{L}_{\kappa,\kappa}$-theory T and $\mathcal{L}_{\kappa,\kappa}$-type such that for club many $T' \cup p' \in P_\kappa(T \cup p)$, T' can omit p', then T can omit p.

We call this property κ-compactness for type omission.

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1. *κ-compactness for type omissions over $\mathcal{L}_{\kappa,\kappa}$ with a language of size λ.*
2. *For every transitive model M of size λ, ${}^{<\kappa}M \subseteq M$, there is an elementary embedding $j\colon M \to N$, N transitive, $\operatorname{crit} j = \kappa$, $j[M] \in N$.*

Supercompactness by omitting first order types and transitivity

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In particular, the supercomapct analogue of ω_1-compactness is simply supercompactness.

The strong tree property

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Definition

Let κ be a regular cardinal, $\lambda \geq \kappa$. A $P_\kappa\lambda$-tree $\mathcal{T}$ is a function, with domain $P_\kappa\lambda$ and $\mathcal{T}(x) \subseteq \mathcal{P}(x)$, $|\mathcal{T}(x)| < \kappa$.
Moreover, for every x, $|\mathcal{T}(x)| \neq \emptyset$ and if $x \subseteq y$ and $z \in \mathcal{T}(y)$ then $z \cap x \in \mathcal{T}(x)$.
A cofinal branch in $\mathcal{T}$ is a set $b \subseteq \lambda$, such that $b \cap x \in \mathcal{T}(x)$ for all x.

Ineffable Tree Property

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But this is not the right *normalized* version of the strong tree property, since when taking $\lambda = \kappa$, we get weakly compact on one hand and ineffable cardinal in the other.

The normalized strong tree property

Let $\mathcal{T}$ be a $P_\kappa\lambda$ tree. We say that L is a *ladder system* on $\mathcal{T}$ if

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- $\mathrm{dom}\ L \subseteq P_\kappa\lambda$ and contains a club,
- $L(x) \subseteq \mathcal{T}(x)$ non-empty, and
- for every $y \in L(x)$ such that $\mathrm{cf}(|x \cap \kappa|) > \omega$ there is a club $E_{x,y} \subseteq P_{|x\cap\kappa|}x$, such that for all $z \in E_{x,y}$, z belongs to the domain of L and $y \cap z \in L(z)$.

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Definition

Let $\kappa \leq \lambda$ be regular cardinals. We say that κ has the $P_\kappa\lambda$-tree property with ladder systems catching if every $P_\kappa\lambda$-tree $\mathcal{T}$ and a ladder system L, there is a cofinal branch b such that $\{x \in P_\kappa\lambda \mid b \cap x \in L(x)\}$ is cofinal.

Π^1_1-subcompactness for tree property

Theorem (H. and Magidor)

Let $\kappa \leq \lambda = \lambda^{<\lambda}$ be regular cardinals. The following are equivalent:

- *κ is λ-Π^1_1-subcompact.*
- *κ has the $P_\kappa\lambda$-tree property with ladder systems catching.*

The Subcompactness Hierarchy

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Strong compactness	Supercompactness
Fine measure on $P_\kappa\lambda$	Normal measure on $P_\kappa\lambda$
	Ineffable tree property for $P_\kappa\lambda$
$\mathcal{L}_{\kappa,\kappa}$-compactness for size λ	Π^1_1-λ-subcompactness
	λ-subcomapctness