Hyperfinite subequivalence relations of treed equivalence relations

Anush Tserunyan

University of Illinois at Urbana-Champaign

(Joint work with Robin Tucker-Drob)

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- This brings us to the study of countable Borel equivalence relations on a standard probability space (X, μ).

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$$\text{If } \Gamma \text{ is nonamenable, then } \exists\ E_{\mathbb{F}_2} \subseteq E_\Gamma,$$

where $E_{\mathbb{F}_2}$ arises from an a.e. free ergodic action of $\mathbb{F}_2$.

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- A graphing of E is a Borel graph G on X, whose connected components are exactly the E-classes a.e.

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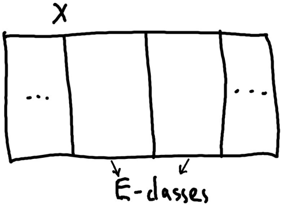

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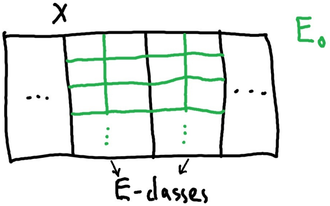

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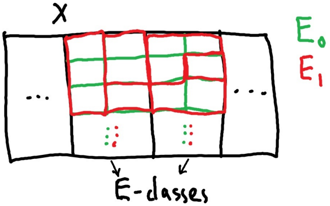

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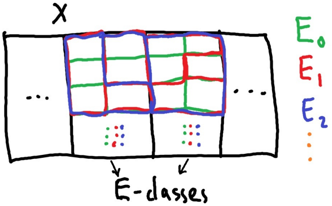

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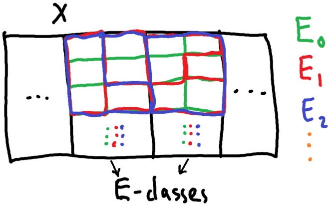

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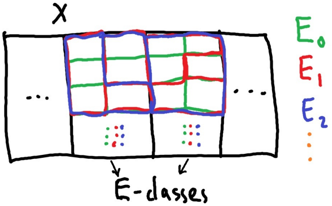

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Def. Admit an invariant mean, i.e. fin. additive probability measure.

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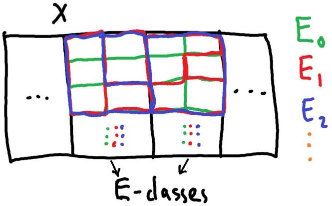

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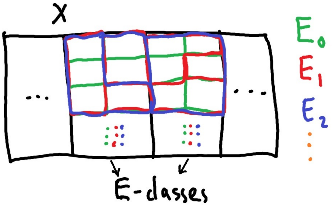

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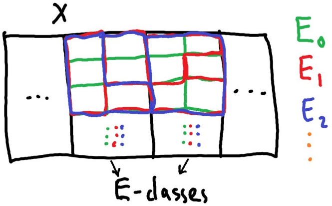
Def. Admit a measurable bundle of invariant means



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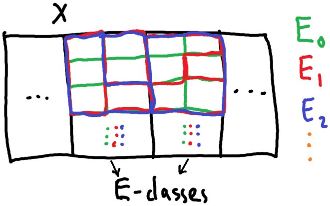

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$\Downarrow$ Schreier graph

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Examples. Virtually free groups, surface groups, limit groups.

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  Note: the Schreier graph of the action of ab is not a subgraph of T.

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Answer (Bowen): Yes in the pmp setting.

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Question: Do the analogues of these hold for the equivalence relations?

Answer (Bowen): Yes in the pmp setting.

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Question: Do these statements hold in the general (non-pmp) setting?

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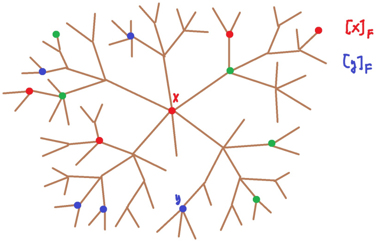
Question: But, more constructively/geometrically, what's so special about these particular ends?

- In the example of E being induced by an a.e. free $\mathbb{F}_2 \curvearrowright (X, \mu)$ and F by the action of the subgroup generated by ab, each F-class "spans" exactly two ends.

Structure (in the pmp setting)

- Let $F \subseteq E$ and T be as before, and suppose F is hyperfinite.
- For each $x \in X$, let $T_{[x]_F}$ be the subtree of T spanned by the convex hull of $[x]_F$:



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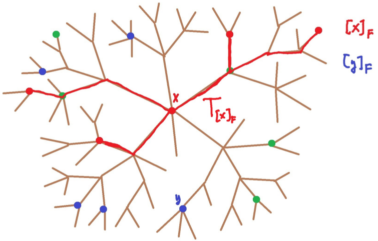

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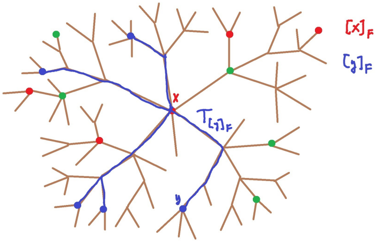

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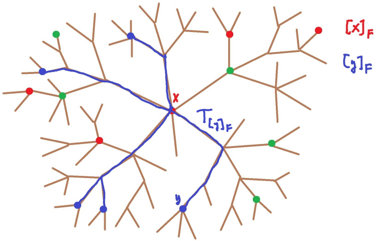

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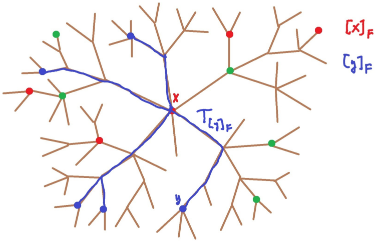

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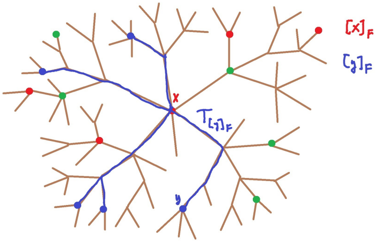

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- Thus if C_0 spans two ends, so does C_1.
- If C_0 spans one end, C_1 cannot span another end in addition to this, because then C_0 would see it too and would be able to select, contradicting C_0 maximally selecting only one end. $\square$

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Question: Do these statements hold in the general (non-pmp) setting?

Corollaries (in the pmp setting)

Observation (Ts.–Tucker-Drob)

Let $F \subseteq E$ be hyperfinite. If E is pmp, then a.e. F-class spans exactly the ends that it maximally selects.

This immediately implies:

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Problem: The observation above fails in the non-pmp setting!

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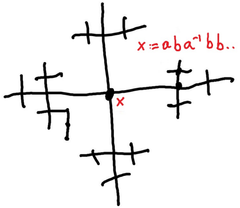

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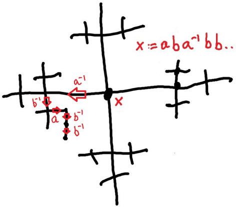

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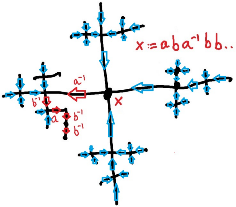

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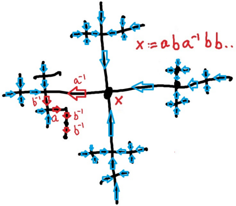

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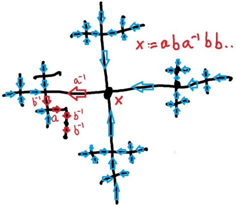

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- We can take $F := E$, so each F-class spans continuum-many ends, yet selects one!

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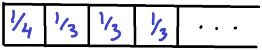

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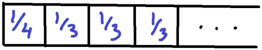

- The Radon–Nikodym cocycle $\frac{d\mu(y)}{d\mu(x)}$ grows in the direction of the shift: $\frac{d\mu(\theta^n(x))}{d\mu(x)} = 3^n \to \infty$.

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3. In fact, if $\liminf > 0$ or $\limsup < \infty$, then ρ is a coboundary.

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That's why ξ_x^+ was selected!

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Then Σ generates a free group, whose action on V is free.

Thanks!