

VALUATION SEMIGROUPS OF TWO DIMENSIONAL LOCAL RINGS

STEVEN DALE CUTKOSKY AND PHAM AN VINH

ABSTRACT. We consider the question of when a semigroup is the semigroup of a valuation dominating a two dimensional noetherian domain, giving some surprising examples. We give a necessary and sufficient condition for the pair of a semigroup S and a field extension L/\mathfrak{k} to be the semigroup and residue field of a valuation dominating a regular local ring R of dimension two with residue field \mathfrak{k} , generalizing the theorem of Spivakovsky for the case when there is no residue field extension.

1. INTRODUCTION

Suppose that (R, \mathfrak{m}_R) is a Noetherian local ring which is dominated by a valuation ν . The semigroup of ν in R is

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}.$$

$S^R(\nu)$ generates the value group of ν .

In this paper we give a classification of the semigroups and residue field extensions that may be obtained by a valuation dominating a regular local ring of dimension two. Our results are completely general, as we make no further assumptions on the ring or on the residue field extension of the valuation ring. This classification (given in Theorems 1.1 and 1.2) is very simple. The classification does not extend to more general rings. Theorem3*

We give an example showing that the semigroup of a valuation dominating a normal local ring of dimension two can be quite different from the semigroup of a regular local ring, even on an A_2 singularity (Example 9.2). In [17], [18] and [11], we give examples showing that the semigroups of valuations dominating regular local rings of dimension ≥ 3 can be very complicated. For instance, in Proposition 6.3 of [11], we show that there exists a regular local ring R of dimension 3 dominated by a rational rank 1 valuation ν which has the property that given $\varepsilon > 0$, there exists an i such that $\beta_{i+1} - \beta_i < \varepsilon$, where $\beta_0 < \beta_1 < \dots$ is the minimal set of generators of $S^R(\nu)$. In [17] and [18] we give examples showing that spectacularly strange behavior of the semigroup can occur for a higher rank valuation. The growth of valuation semigroups is however bounded by a polynomial whose coefficients are computed from the multiplicities of the centers of the composite valuations on R . This is proven in [18].

The possible value groups Γ of a valuation ν dominating a Noetherian local ring have been extensively studied and classified, including in the papers MacLane [35], MacLane and Schilling [36], Zariski and Samuel [48], and Kuhlmann [32]. Γ can be any ordered abelian group of finite rational rank (Theorem 1.1 [32]). The semigroup $S^R(\nu)$ is however not well understood, although it is known to encode important information about the topology and resolution of singularities of $\text{Spec}(R)$ [5], [6], [44], [45], [7], [16], [19], [21], [24], [37], [42], [27] to mention a few references, and the ideal theory of R [46], [47], [48] and its development in many subsequent papers.

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In Sections ^{RLR1}3 through ^{Poly}8 of this paper we analyze valuations dominating a regular local ring R of dimension two. Our analysis is constructive, being based on an algorithm which finds a generating sequence for the valuation. A generating sequence of ν in R is a set of elements of R whose initial forms are generators of the graded $\mathfrak{k} = R/\mathfrak{m}_R$ -algebra $\text{gr}_\nu(R)$ (Section ^{Pre1}2). The characteristic of the residue field of R does not appear at all in the proofs, although the proof may be simplified significantly if the assumption that R has equal characteristic is added; in this case we may reduce to the case where R is a polynomial ring over a field (Section ^{Poly}8). A construction of a generating sequence, and the subsequent classification of the semigroups, is classical in the case when the residue field of R is algebraically closed; this was proven by Spivakovsky in [41]. Besides the complete generality of our results, our proofs differ from those of Spivakovsky in that we only use elementary techniques, using nothing more sophisticated than the definition of linear independence in a vector space, and the definition of the minimal polynomial of an element in a field extension. In our proof we construct the residue field of the valuation ring as a tower of primitive extensions; the minimal polynomials of the primitive elements are used to construct the generating sequence for the valuation. It is not necessary for R to be excellent in our analysis; the only place in this paper where excellence manifests itself is in the possibility of ramification in the extension of a valuation to the completion of a non excellent regular local ring (Proposition ^{Prop1}3.4).

In a finite field extension, the quotient of the valuation group of an extension of a valuation by the value group is always a finite group (2nd corollary on page 52 of [48]). This raises the following question: Suppose that $R \rightarrow T$ is a finite extension of regular local rings, and ν is a valuation which dominates R . Is $S^T(\nu)$ a finitely generated module over the semigroup $S^R(\nu)$? We give a counterexample to this question in Example ^{Example3}9.4. This example is especially interesting in light of the results on relative finite generation in the papers ^{CHK}[22] of Ghezzi, Hà and Kashcheyeva, and ^{GK}[23] of Ghezzi and Kashcheyeva.

We now turn to a discussion of our results on regular local rings of dimension two. We obtain the following necessary and sufficient condition for a semigroup and field extension to be the semigroup and residue field extension of a valuation dominating a ^{complete} regular local ring of dimension two in the following theorem (proven in Section ^{Proof1}5):

Theorem3*

Theorem 1.1. *Suppose that R is a complete regular local ring of dimension two with residue field $R/\mathfrak{m}_R = \mathfrak{k}$. Let S be a subsemigroup of the positive elements of a totally ordered abelian group and L be a field extension of \mathfrak{k} . Then S is the semigroup of a valuation ν dominating R with residue field $V_\nu/\mathfrak{m}_\nu = L$ if and only if there exists a finite or countable index set I , of cardinality $\Lambda = |I| - 1 \geq 1$ and elements $\beta_i \in S$ for $i \in I$ and $\alpha_i \in L$ for $i \in I_+$, where $I_+ = \{i \in I \mid i > 0\}$, such that*

- 1) *The semigroup S is generated by $\{\beta_i\}_{i \in I}$ and the field L is generated over \mathfrak{k} by $\{\alpha_i\}_{i \in I_+}$.*
- 2) *Let*

$$\bar{n}_i = [G(\beta_0, \dots, \beta_i) : G(\beta_0, \dots, \beta_{i-1})]$$

and

$$d_i = [\mathfrak{k}(\alpha_1, \dots, \alpha_i) : \mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})].$$

Then there are inequalities

$$\beta_{i+1} > \bar{n}_i d_i \beta_i > \beta_i$$

with $\bar{n}_i < \infty$ and $d_i < \infty$ for $1 \leq i < \Lambda$ and if $\Lambda < \infty$, then either $\bar{n}_\Lambda = \infty$ and $d_\Lambda = 1$ or $\bar{n}_\Lambda < \infty$ and $d_\Lambda = \infty$.

Here $G(\beta_0, \dots, \beta_i)$ is the subgroup generated by β_0, \dots, β_i .

The case when R is not complete is more subtle, because of the possibility, when R is not complete, of the existence of a rank 1 discrete valuation which dominates R and such that the residue field extension V_ν/\mathfrak{m}_ν of $\mathfrak{k} = R/\mathfrak{m}_R$ is finite. For all other valuations ν which dominate R (so that ν is not rank 1 discrete with V_ν/\mathfrak{m}_ν finite over \mathfrak{k}) the analysis is the same as for the complete case, as there is then a unique extension of ν to a valuation dominating the completion of R which is an immediate extension; that is, there is no extension of the valuation semigroups or of the residue fields of the valuations. The differences between the complete and non complete cases are explained in more detail by [Theorem 3.1](#), [Corollary 3.2](#), [Example 3.3](#), [Proposition 3.4](#) and [Corollary 5.1](#) to [Theorem 1.1](#).

We give a necessary and sufficient condition for a semigroup to be the semigroup of a valuation dominating a regular local ring of dimension two in the following theorem, which is proven in [Section 6](#):

Theorem 1.2. *Suppose that R is a regular local ring of dimension two. Let S be a sub-semigroup of the positive elements of a totally ordered abelian group. Then S is the semi-group of a valuation ν dominating R if and only if there exists a finite or countable index set I , of cardinality $\Lambda = |I| - 1 \geq 1$ and elements $\beta_i \in S$ for $i \in I$ such that*

- 1) *The semigroup S is generated by $\{\beta_i\}_{i \in I}$.*
- 2) *Let*

$$\bar{n}_i = [G(\beta_0, \dots, \beta_i) : G(\beta_0, \dots, \beta_{i-1})].$$

There are inequalities

$$\beta_{i+1} > \bar{n}_i \beta_i$$

with $\bar{n}_i < \infty$ for $1 \leq i < \Lambda$. If $\Lambda < \infty$ then $\bar{n}_\Lambda \leq \infty$.

[Theorem 1.2](#) is proven by Spivakovsky when R has algebraically closed residue field in [\[41\]](#).

The proof in [Section 5](#) of [\[11\]](#), given for the case when the residue field of R is algebraically closed, now extends to arbitrary regular local rings of dimension two, using the conclusions of [Theorem 1.2](#), to prove the following:

Corollary 1.3. *Suppose that R is a regular local ring of dimension two and ν is a rank 1 valuation dominating R . Embed the value group of ν in \mathbb{R}_+ so that 1 is the smallest nonzero element of $S^R(\nu)$. Let $\varphi(n) = |S^R(\nu) \cap (0, n)|$ for $n \in \mathbb{Z}_+$. Then*

$$\lim_{n \rightarrow \infty} \frac{\varphi(n)}{n^2}$$

exists. The set of limits which are obtained by such valuations ν dominating R is the real half open interval $[0, \frac{1}{2})$.

As a consequence of [Theorem 1.1](#), we obtain the following example, which we prove in [Section 6](#), showing the subtlety of the criteria of [Theorem 1.1](#).

Example 1.4. *There exists a semigroup S which satisfies the sufficient conditions 1) and 2) of [Theorem 1.2](#), such that if (R, \mathfrak{m}_R) is a 2-dimensional regular local ring dominated by a valuation ν such that $S^R(\nu) = S$, then $R/\mathfrak{m}_R = V_\nu/\mathfrak{m}_\nu$; that is, there can be no residue field extension.*

The main technique we use in the proofs of the above theorems is the algorithm of [Theorem 4.2](#), which constructs a sequence of elements $\{P_i\}$ in R , starting with a given

regular system of parameters $P_0 = x, P_1 = y$ of R , which gives a generating sequence of ν in R . This fact is proven in Theorems [4.11](#) and [4.12](#). Corollary 3*

In Section [7](#), we develop the birational theory of the generating sequence $\{P_i\}$, generalizing to the case when R has arbitrary residue field the results of [\[41\]](#). RLR2

Suppose that R is a regular local ring of dimension two which is dominated by a valuation ν . Let $\mathfrak{k} = R/\mathfrak{m}_R$ and

eqX3*

$$(1) \quad R \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots$$

be the sequence of quadratic transforms along ν , so that $V_\nu = \cup T_i$, and $L = V_\nu/\mathfrak{m}_\nu = \cup T_i/\mathfrak{m}_{T_i}$. Suppose that x, y are regular parameters in R , and let $P_0 = x, P_1 = y$ and $\{P_i\}$ be the sequence of elements of R constructed in Theorem [4.2](#). Suppose there exists some smallest value i in the sequence [\(1\)](#) such that the divisor of xy in $\text{Spec}(T_i)$ has only one component. Let $R_1 = T_i$. By Theorem [7.1](#), a local equation of the exceptional divisor and a strict transform of P_2 in R_1 are a regular system of parameters in R_2 , and a local equation of the exceptional divisor and a strict transform of P_i in R_1 for $i \geq 2$ satisfy the conclusions of Theorem [4.2](#) on R_2 . Theorem 1*

We can repeat this construction, for this new sequence, to construct a sequence of quadratic transforms $R_1 \rightarrow R_2$ such that a local equation of the exceptional divisor and a strict transform of P_3 is a regular system of parameters in R_2 , and a local equation of the exceptional divisor and a strict transform of P_i for ≥ 3 satisfy the conclusions of Theorem [4.2](#) on R_2 . Theorem 1*

We thus have a sequence of iterated quadratic transforms

$$R \rightarrow R_1 \rightarrow R_2 \rightarrow \cdots$$

such that $V_\nu = \cup R_i$ and where a local equation of the exceptional divisor of $R_i \rightarrow R_{i+1}$ and the strict transform of P_{i+1} are a regular system of parameters in R_i for all i .

The notion of a generating sequence of a valuation already can be recognized in the famous algorithm of Newton to find the branches of a (characteristic zero) plane curve singularity. In more modern times, it has been developed by Maclane [\[35\]](#) (“key polynomials”), Zariski [\[46\]](#), Abhyankar [\[3\]](#), [\[4\]](#) (“approximate roots”), and Spivakovsky [\[41\]](#). Most recently, the construction and application of generating sequences of a valuation have appeared in many papers, including [\[13\]](#), [\[9\]](#), [\[15\]](#), [\[20\]](#), [\[21\]](#), [\[25\]](#), [\[22\]](#), [\[23\]](#), [\[34\]](#), [\[38\]](#), [\[43\]](#). The theory of generating sequences in regular local rings of dimension two is closely related to the configuration of exceptional curves appearing in the sequence of quadratic transforms along the center of the valuation. This subject has been explored in many papers, including [\[7\]](#) and [\[33\]](#). The extension of valuations to the completion of a local ring, which becomes extremely difficult in higher dimension and rank, is studied in [\[41\]](#), [\[28\]](#), [\[12\]](#), [\[10\]](#), [\[14\]](#), [\[8\]](#), [\[12\]](#) and [\[30\]](#). There is an extensive literature on the theory of complete ideals in local rings, beginning with Zariski’s articles [\[46\]](#) and [\[48\]](#). Z1

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Prel

2. PRELIMINARIES

Suppose that (R, \mathfrak{m}_R) is a Noetherian local domain and ν is valuation of the quotient field which dominates R . Let V_ν be the valuation ring of ν , and \mathfrak{m}_ν be its maximal ideal. Let Γ_ν be the value group of ν . Let $\mathfrak{k} = R/\mathfrak{m}_R$. The semigroup of ν on R is

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}.$$

For $\varphi \in \Gamma_\nu$, define valuation ideals

$$\mathcal{P}_\varphi(R) = \{f \in R \mid \nu(f) \geq \varphi\},$$

and

$$\mathcal{P}_\varphi^+(R) = \{f \in R \mid \nu(f) > \varphi\}.$$

We have that $\mathcal{P}_\varphi^+(R) = \mathcal{P}_\varphi(R)$ if and only if $\varphi \notin S^R(\nu)$. The associated graded ring of ν on R is

$$\text{gr}_\nu(R) = \bigoplus_{\varphi \in \Gamma_\nu} \mathcal{P}_\varphi(R)/\mathcal{P}_\varphi^+(R).$$

Suppose that $f \in R$ and $\nu(f) = \varphi$. Then the initial form of f in $\text{gr}_\nu(R)$ is

$$\text{in}_\nu(f) = f + \mathcal{P}_\varphi^+(R) \in [\text{gr}_\nu(R)]_\varphi = \mathcal{P}_\varphi(R)/\mathcal{P}_\varphi^+(R).$$

A set of elements $\{F_i\}_{i \in I}$ such that $\{\text{in}_\nu(F_i)\}$ generates $\text{gr}_\nu(R)$ as a \mathfrak{k} -algebra is called a generating sequence of ν in R .

We have that the vector space dimension

$$\dim_{R/\mathfrak{m}_R} \mathcal{P}_\varphi(R)/\mathcal{P}_\varphi^+(R) < \infty$$

and

$$\dim_{R/\mathfrak{m}_R} \mathcal{P}_\varphi(R)/\mathcal{P}_\varphi^+(R) \leq [V_\nu/\mathfrak{m}_\nu : R/\mathfrak{m}_R]$$

for all $\varphi \in \Gamma_\nu$.

$S^R(\nu)$ is countable and is well ordered of ordinal type $\leq \omega^2$ by Proposition 2, Appendix 3 [48]. Further, V_ν/\mathfrak{m}_ν is a countably generated field extension of $\mathfrak{k} = R/\mathfrak{m}_R$, since $\text{gr}_\nu(R)$ is a countably generated vector space over R/\mathfrak{m}_R , and if $0 \neq \alpha \in V_\nu/\mathfrak{m}_\nu$, then α is the residue of $\frac{f}{g}$ for some $f, g \in R$ with $\nu(f) = \nu(g)$.

We will make use of Abhyankar's Inequality ([1], Appendix 2 [48]):

$$\text{eq17} \quad (2) \quad \text{rat rank } \nu + \text{trdeg}_{R/\mathfrak{m}_R} V_\nu/\mathfrak{m}_\nu \leq \dim R$$

If equality holds then $\Gamma_\nu \cong \mathbb{Z}^m$ as an *unordered* group, where $m = \text{rat rank } \nu$, and V_ν/\mathfrak{m}_ν is a finitely generated field extension of R/\mathfrak{m}_R .

We have that

$$\text{rank } \nu \leq \text{rat rank } \nu \leq \dim R.$$

Let $n = \text{rank } \nu$. Then we have an order preserving embedding

$$\text{eqZ11} \quad (3) \quad \Gamma_\nu \subset \Gamma_\nu \mathbb{R} \cong (\mathbb{R}^n)_{\text{lex}}$$

(Proposition 2.10 [2]). We say that ν is discrete if Γ_ν is discrete in the Euclidean topology.

If I is an ideal in R , we may define $\nu(I) = \min\{\nu(f) \mid f \in I \setminus \{0\}\}$, since $S^R(\nu)$ is well ordered.

\mathbb{N} denotes the natural numbers $\{0, 1, 2, \dots\}$ and \mathbb{Z}_+ denotes the positive integers $\{1, 2, 3, \dots\}$.

Given elements z_1, \dots, z_n in a group G , let $G(z_1, \dots, z_n)$ be the subgroup generated by z_1, \dots, z_n . Let $S(z_1, \dots, z_n)$ be the semigroup generated by z_1, \dots, z_n .

Lemma2 **Lemma 2.1.** *Suppose that Γ is a totally ordered abelian group, I is a finite or countable index set of cardinality ≥ 2 and $\beta_i \in \Gamma$ are positive elements for $i \in I$. Let $\Lambda = |I| - 1$. Let*

$$\bar{n}_i = [G(\beta_0, \dots, \beta_i) : G(\beta_0, \dots, \beta_{i-1})] \in \mathbb{Z}_+ \cup \{\infty\}$$

for ≥ 1 . Assume that $\bar{n}_i \in \mathbb{Z}_+$ if $i < \Lambda$. Let s_i be the smallest positive integer t such that $t\beta_i \in S_{i-1}$ (or $s_i = \infty$ if $i = \Lambda$ and no such t exists).

Suppose that $1 \leq k < \Lambda$ and $\bar{n}_i \beta_i < \beta_{i+1}$ for $1 \leq i \leq k - 1$. Then

- 1) $s_i = \bar{n}_i$ for $1 \leq i \leq k$.
- 2) If $\gamma \in G(\beta_0, \dots, \beta_k)$ and $\gamma \geq \bar{n}_k \beta_k$ then $\gamma \in S(\beta_0, \dots, \beta_k)$.

Proof. We first prove 2). By repeated Euclidean division, we obtain an expansion $\gamma = a_0 \beta_0 + a_1 \beta_1 + \dots + a_k \beta_k$ with $a_0 \in \mathbb{Z}$ and $0 \leq a_i < \bar{n}_i$ for $1 \leq i \leq k$. Now we calculate, using the inequalities $\bar{n}_i \beta_i < \beta_{i+1}$,

$$a_1 \beta_1 + \dots + a_k \beta_k < \bar{n}_k \beta_k.$$

Thus $a_0 > 0$ and $\gamma \in S(\beta_0, \dots, \beta_k)$.

Now 1) follows from 2) and induction on k . □

A Laurent monomial in H_0, H_1, \dots, H_l is a product $H_0^{a_0} H_1^{a_1} \dots H_l^{a_l}$ with $a_0, a_1, \dots, a_l \in \mathbb{Z}$.

Suppose that R is a regular local ring with maximal ideal \mathfrak{m}_R . Suppose that $f \in R$. Then we define

$$\text{ord}(f) = \max\{n \in \mathbb{N} \mid f \in \mathfrak{m}_R^n\}.$$

3. REGULAR LOCAL RINGS OF DIMENSION TWO

RLR1

Suppose that (R, \mathfrak{m}_R) is a Noetherian local domain of dimension two. Up to order isomorphism, the value groups Γ_ν of a valuation ν which dominates R are by Abhyankar's inequality and Example 3, Section 15, Chapter VI [48]:

1. $\alpha\mathbb{Z} + \beta\mathbb{Z}$ with $\alpha, \beta \in \mathbb{R}$ rationally independent.
2. $(\mathbb{Z}^2)_{\text{lex}}$.
3. Any subgroup of \mathbb{Q} .

Suppose that N is a field, and V is a valuation ring of N . We say that the rank of V increases under completion if there exists an analytically normal local domain T with quotient field N such that V dominates T and there exists an extension of V to a valuation ring of the quotient field of \hat{T} which dominates \hat{T} and which has higher rank than the rank of V .

TheoremR4

Theorem 3.1. (Theorem 4.2, [14]^{CK}; [41]^S in the case when R/\mathfrak{m}_R is algebraically closed) Suppose that V dominates an excellent two dimensional local ring R . Then the rank of V increases under completion if and only if V/\mathfrak{m}_V is finite over R/\mathfrak{m}_R and V is discrete of rank 1.

CorollaryR2

Corollary 3.2. If R is complete and ν is a discrete rank one valuation which dominates R then $[V_\nu/\mathfrak{m}_\nu : R/\mathfrak{m}_R] = \infty$.

The following example shows an important distinction between the case when R is complete and when R is not.

ExampleR1

Example 3.3. Suppose that \mathfrak{k} is a field and $R = \mathfrak{k}[x, y]_{(x, y)}$ is a localization of a polynomial ring in two variables. Then there exists a rank one discrete valuation ν dominating R such that $V_\nu/\mathfrak{m}_\nu = \mathfrak{k}$.

Proof. Let $f(t) \in \mathfrak{k}[[t]]$ be a transcendental element over $\mathfrak{k}(t)$. Embed R into $\mathfrak{k}[[t]]$ by substituting t for x and $f(t)$ for y . The valuation ν on R obtained by restriction of the t -adic valuation to R has the desired properties. □

Suppose that ν is a valuation which dominates R . Let a be the smallest positive element in $S^R(\nu)$. Suppose that $\{f_i\}$ is a Cauchy sequence in R (for the \mathfrak{m}_R -adic topology). Then

either there exist $n_0 \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+$ and $\gamma \in S^R(\nu)$ such that $\gamma < ma$ and $\nu(f_i) = \gamma$ for $i \geq n_0$, or

eqZ12 (4) Given $m \in \mathbb{Z}_+$, there exists $n_0 \in \mathbb{Z}_+$ such that $\nu(f_i) > ma$ for $i > n_0$

Let $I_{\hat{R}}$ be the set of limits of Cauchy sequences $\{f_i\}$ satisfying ^{eqZ12}(4). Then $I_{\hat{R}}$ is a prime ideal in \hat{R} ([10], [13], [12], [41], [42]). The following proposition is well known.

Prop17 **Proposition 3.4.** *Suppose that R is a regular local ring of dimension two, and let ν be a valuation which dominates R . Then there exists an extension of ν to a valuation $\hat{\nu}$ which dominates the completion \hat{R} of R with respect to \mathfrak{m}_R , which has one of the following semigroups:*

1. $\text{rank } \nu = \text{rank } \hat{\nu} = 1$ and

eq14 (5) $S^R(\nu) = S^{\hat{R}}(\hat{\nu})$.

2. ν is discrete of rank 1, $\hat{\nu}$ is discrete of rank 2 and

eq15 (6) $S^{\hat{R}}(\hat{\nu})$ is generated by $S^R(\nu)$ and an element α such that $\alpha > \gamma$ for all $\gamma \in S^R(\nu)$.

3. ν and $\hat{\nu}$ are discrete of rank 2, there exists a height one prime I_R in R , and a discrete rank 1 valuation $\bar{\nu}$ which dominates the maximal ideal $\mathfrak{m}_R(R/I_R)$ of R/I_R such that

$S^R(\nu)$ is generated by $S^{R/I_R}(\bar{\nu})$ and an element α such that $\alpha > \gamma$ for all $\gamma \in S^{R/I_R}(\bar{\nu})$.

eq16 (7) $S^{\hat{R}}(\hat{\nu})$ is generated by $S^{R/I_R}(\bar{\nu})$ and an element β such that $\alpha - t\beta \in S^{R/I_R}(\bar{\nu})$, for some $t \in \mathbb{Z}_+$. If $R_{\mathfrak{m}}$ is excellent, then $t = 1$.

4. ν and $\hat{\nu}$ are discrete of rank 2, $I_{\hat{R}} = (0)$ and $S^R(\nu) = S^{\hat{R}}(\hat{\nu})$.

Proof. First suppose that ν has rank 1. Then $I_{\hat{R}} \cap R = (0)$, so we have an embedding $R \subset \hat{R}/I_{\hat{R}}$. We can then extend ν to a valuation $\bar{\nu}$ which dominates $\hat{R}/I_{\hat{R}}$ by defining for $f \notin I_{\hat{R}}$, $\bar{\nu}(f + I_{\hat{R}}) = \lim_{i \rightarrow \infty} \nu(f_i)$, where $\{f_i\}$ is a Cauchy sequence in R representing f . We have that $S^R(\nu) = S^{\hat{R}/I_{\hat{R}}}(\bar{\nu})$.

If $I_{\hat{R}} = (0)$ then we have constructed the desired extension $\hat{\nu} = \bar{\nu}$ of ν to \hat{R} . Suppose that $I_{\hat{R}} \neq (0)$. Then $\hat{R}/I_{\hat{R}}$ has dimension 1, so $\bar{\nu}$ is discrete of rank 1. We have that $I_{\hat{R}} = (v)$ is a height one prime ideal. We can extend $\bar{\nu}$ to a rank 2 valuation $\hat{\nu}$ which dominates \hat{R} by defining $\hat{\nu}(f) = (n, \bar{\nu}(g)) \in (\mathbb{Z} \oplus \Gamma_{\bar{\nu}})_{\text{lex}}$ if $f \in \hat{R}$ has a factorization $f = v^n g$ where $n \in \mathbb{N}$ and $v \nmid g$.

Now assume that ν has rank 2. Further assume that $I_{\hat{R}} \cap R \neq (0)$. Then ν has rank 2, and $I_R = I_{\hat{R}} \cap R$ is a height one prime ideal in R . Thus there exists an irreducible $g \in R$ such that $I_R = (g)$. We then have that $I_{\hat{R}}$ is a height one prime ideal in \hat{R} , so there exists an irreducible $v \in \hat{R}$ such that $I_{\hat{R}} = (v)$.

There exists a valuation $\bar{\nu}$ dominating R/I_R such that if $f \in R$ has a factorization $f = g^n h$ where $g \nmid h$, then

$$\nu(f) = n\nu(g) + \bar{\nu}(h).$$

Write $g = v^t \varphi$ where $t \in \mathbb{Z}_+$ and $v \nmid \varphi$. Thus $\varphi \notin I_{\hat{R}}$. If R is excellent, then g is reduced in \hat{R} (by Scholie IV 7.8.3 (vii) [26]), so $t = 1$. We have an inclusion $R/I_R \subset \hat{R}/I_{\hat{R}}$, and $\bar{\nu}$ extends to a valuation $\hat{\bar{\nu}}$ which dominates $\hat{R}/I_{\hat{R}}$. We then extend ν to a valuation $\hat{\nu}$ which dominates \hat{R} by setting

$$t\hat{\nu}(v) = \nu(g) - \hat{\bar{\nu}}(\varphi)$$

in $\Gamma_\nu \mathbb{R} \cong (\mathbb{R}^2)_{\text{lex}}$. Suppose that $0 \neq f \in \hat{R}$. Factor f as $f = v^n h$ where $n \in \mathbb{N}$ and $v \nmid h$. Then define

$$\hat{\nu}(f) = n\hat{\nu}(v) + \hat{\nu}(h).$$

We now show that $S^{R/I_R}(\bar{\nu}) = S^{\hat{R}/I_{\hat{R}}}(\hat{\nu})$. We have that $\hat{\nu}(\mathfrak{m}(\hat{R}/I_{\hat{R}})) = \bar{\nu}(\mathfrak{m}(R/I_R))$. Suppose that $0 \neq h \in \hat{R}/I_{\hat{R}}$, and that $\hat{\nu}(h) = \gamma$. There exists $n \in \mathbb{Z}_+$ such that $n\hat{\nu}(\mathfrak{m}(\hat{R}/I_{\hat{R}})) > \gamma$ and there exists $f \in R$ such that if \bar{f} is the image of f in R/I_R , then $\bar{f} - h \in \mathfrak{m}^n(\hat{R}/I_{\hat{R}})$. Thus $\nu(f) = \bar{\nu}(\bar{f}) = \hat{\nu}(h) = \gamma$.

Suppose that $\text{rank } \nu = 2$ and $I_{\hat{R}} \cap R = (0)$. We can extend ν to a valuation $\bar{\nu}$ dominating $R/I_{\hat{R}}$ by defining for $f \notin I_{\hat{R}}$, $\bar{\nu}(f + I_{\hat{R}}) = \lim_{i \rightarrow \infty} \nu(f_i)$ if $\{f_i\}$ is a Cauchy sequence in R converging to f . We must have that $I_{\hat{R}} = (0)$, since otherwise we would be able to extend $\bar{\nu}$ to a valuation $\tilde{\nu}$ dominating \hat{R} which is composite with the rank 2 extension $\bar{\nu}$ of ν to $\hat{R}/I_{\hat{R}}$; this extension would have rank ≥ 3 which is impossible by Abhyankar's inequality. Thus $I_{\hat{R}} = (0)$. □

Remark 3.5. Nagata gives an example in the Appendix to [39] of a regular local ring R of dimension two with an irreducible element $f \in R$ such that f is not reduced in \hat{R} .

4. THE ALGORITHM

In this section, we will suppose that R is a regular local ring of dimension two, with maximal ideal \mathfrak{m}_R and residue field $\mathfrak{k} = R/\mathfrak{m}_R$. For $f \in R$, let \bar{f} or $[f]$ denote the residue of f in \mathfrak{k} . Suppose that CS is a coefficient set of R . A coefficient set of R is a subset CS of R such that the mapping $CS \rightarrow \mathfrak{k}$ defined by $s \mapsto \bar{s}$ is a bijection. We further require that $0 \in CS$ and $1 \in CS$.

RemarkG4

Remark 4.1. Suppose that x, y are regular parameters in R , $a, b \in CS$ and $n \in \mathbb{Z}_+$. Let $c \in CS$ be defined by $\bar{a} + \bar{b} = \bar{c}$. Then there exist $e_{ij} \in CS$ such that

$$a + b = c + \sum_{i+j=1}^{n-1} e_{ij} x^i y^j + h$$

with $h \in \mathfrak{m}_R^n$. Let $d \in CS$ be defined by $\bar{a}\bar{b} = \bar{d}$. Then there exist $g_{ij} \in CS$ such that

$$ab = d + \sum_{i+j=1}^{n-1} g_{ij} x_i y_j + h'$$

with $h' \in \mathfrak{m}_R^n$.

Theorem1*

Theorem 4.2. Suppose that ν is a valuation of the quotient field of R dominating R . Let $L = V_\nu/\mathfrak{m}_\nu$ be the residue field of the valuation ring V_ν of ν . For $f \in V_\nu$, let $[f]$ denote the class of f in L . Suppose that x, y are regular parameters in R . Then there exist $\Omega \in \mathbb{Z}_+ \cup \{\infty\}$ and $P_i \in \mathfrak{m}_R$ for $i \in \mathbb{Z}_+$ with $i < \min\{\Omega + 1, \infty\}$ such that $P_0 = x$, $P_1 = y$ and for $1 \leq i < \Omega$, there is an expression

eq11*

$$(8) \quad P_{i+1} = P_i^{n_i} + \sum_{k=1}^{\lambda_i} c_k P_0^{\sigma_{i,0}(k)} P_1^{\sigma_{i,1}(k)} \dots P_i^{\sigma_{i,i}(k)}$$

with $n_i \geq 1$, $\lambda_i \geq 1$,

eq12*

$$(9) \quad 0 \neq c_k \in CS$$

for $1 \leq k \leq \lambda_i$, $\sigma_{i,s}(k) \in \mathbb{N}$ for all s, k , $0 \leq \sigma_{i,s}(k) < n_s$ for $s \geq 1$. Further,

$$n_i \nu(P_i) = \nu(P_0^{\sigma_{i,0}(k)} P_1^{\sigma_{i,1}(k)} \dots P_i^{\sigma_{i,i}(k)})$$

for all k .

For all $i \in \mathbb{Z}_+$ with $i < \Omega$, the following are true:

- 1) $\nu(P_{i+1}) > n_i \nu(P_i)$.
- 2) Suppose that $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \leq l \leq m$ and $0 \leq j_k(l) < n_k$ for $1 \leq k \leq r$ are such that $(j_0(l), j_1(l), \dots, j_r(l))$ are distinct for $1 \leq l \leq m$, and

$$\nu(P_0^{j_0(l)} P_1^{j_1(l)} \dots P_r^{j_r(l)}) = \nu(P_0^{j_0(1)} \dots P_r^{j_r(1)})$$

for $1 \leq l \leq m$. Then

$$1, \left[\frac{P_0^{j_0(2)} P_1^{j_1(2)} \dots P_r^{j_r(2)}}{P_0^{j_0(1)} P_1^{j_1(1)} \dots P_r^{j_r(1)}} \right], \dots, \left[\frac{P_0^{j_0(m)} P_1^{j_1(m)} \dots P_r^{j_r(m)}}{P_0^{j_0(1)} P_1^{j_1(1)} \dots P_r^{j_r(1)}} \right]$$

are linearly independent over \mathfrak{k} .

- 3) Let

$$\bar{n}_i = [G(\nu(P_0), \dots, \nu(P_i)) : G(\nu(P_0), \dots, \nu(P_{i-1}))].$$

Then \bar{n}_i divides $\sigma_{i,i}(k)$ for all k in (8). In particular, $n_i = \bar{n}_i d_i$ with $d_i \in \mathbb{Z}_+$

- 4) There exists $U_i = P_0^{w_0(i)} P_1^{w_1(i)} \dots P_{i-1}^{w_{i-1}(i)}$ for $i \geq 1$ with $w_0(i), \dots, w_{i-1}(i) \in \mathbb{N}$ and $0 \leq w_j(i) < n_j$ for $1 \leq j \leq i-1$ such that $\nu(P_i^{\bar{n}_i}) = \nu(U_i)$ and if

$$\alpha_i = \left[\frac{P_i^{\bar{n}_i}}{U_i} \right]$$

then

$$b_{i,t} = \left[\sum_{\sigma_{i,i}(k)=t\bar{n}_i} c_k \frac{P_0^{\sigma_{i,0}(k)} P_1^{\sigma_{i,1}(k)} \dots P_{i-1}^{\sigma_{i,i-1}(k)}}{U_i^{(d_i-t)}} \right] \in \mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$$

for $0 \leq t \leq d_i - 1$ and

$$f_i(u) = u^{d_i} + b_{i,d_i-1} u^{d_i-1} + \dots + b_{i,0}$$

is the minimal polynomial of α_i over $\mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$.

The algorithm terminates with $\Omega < \infty$ if and only if either

$$\boxed{\text{eqL15}} \quad (10) \quad \bar{n}_\Omega = [G(\nu(P_0), \dots, \nu(P_\Omega)) : G(\nu(P_0), \dots, \nu(P_{\Omega-1}))] = \infty$$

or

$$\boxed{\text{eqL10}} \quad (11) \quad \bar{n}_\Omega < \infty \text{ (so that } \alpha_\Omega \text{ is defined as in 4)} \text{ and } d_\Omega = [\mathfrak{k}(\alpha_1, \dots, \alpha_\Omega) : \mathfrak{k}(\alpha_1, \dots, \alpha_{\Omega-1})] = \infty.$$

If $\bar{n}_\Omega = \infty$, set $\alpha_\Omega = 1$.

Proof. Consider the following statements $A(i)$, $B(i)$, $C(i)$, $D(i)$ for $1 \leq i < \Omega$:

There exists $U_i = P_0^{w_0(i)} P_1^{w_1(i)} \dots P_{i-1}^{w_{i-1}(i)}$ for some $w_j(i) \in \mathbb{N}$ and $0 \leq w_j(i) < n_j$ for $1 \leq j \leq i-1$

such that $\bar{n}_i \nu(P_i) = \nu(U_i)$. Let $\alpha_i = [\frac{P_i^{\bar{n}_i}}{U_i}] \in L$ and

$A(i)$ $f_i(u) = u^{d_i} + b_{i,d_i-1} u^{d_i-1} + \dots + b_{i,0} \in \mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})[u]$
be the minimal polynomial of α_i .

Let d_i be the degree of $f_i(u)$, and $n_i = \bar{n}_i d_i$. Then there exist $a_{s,t} \in CS$

and $j_0(s,t), j_1(s,t), \dots, j_{i-1}(s,t) \in \mathbb{N}$ with $0 \leq j_k(s,t) < n_k$

for $k \geq 1$ and $0 \leq t < \bar{d}_i$ such that

$$\nu(P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)} P_i^{t \bar{n}_i}) = \bar{n}_i d_i \nu(P_i)$$

for all s, t and

eqM1

(12)

$$P_{i+1} := P_i^{\bar{n}_i d_i} + \sum_{t=0}^{d_i-1} \left(\sum_{s=1}^{\lambda_t} a_{s,t} P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)} \right) P_i^{t \bar{n}_i}$$

satisfies

$$b_{i,t} = \left[\sum_{s=1}^{\lambda_t} a_{s,t} \frac{P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)}}{U_i^{d_i-t}} \right]$$

for $0 \leq t \leq d_i - 1$. In particular,

eqM2

(13)

$$\nu(P_{i+1}) > n_i \nu(P_i).$$

$B(i)$ Suppose that M is a Laurent monomial in P_0, P_1, \dots, P_i and $\nu(M) = 0$. Then there exist $s_i \in \mathbb{Z}$ such that

$$M = \prod_{j=1}^i \left[\frac{P_j^{\bar{n}_j}}{U_j} \right]^{s_j},$$

so that

$$[M] \in \mathfrak{k}(\alpha_1, \dots, \alpha_i).$$

Suppose that $\lambda \in \mathfrak{k}(\alpha_1, \dots, \alpha_i)$ and N is a Laurent monomial in P_0, P_1, \dots, P_i such that $\gamma = \nu(N) \geq n_i \nu(P_i)$. Then there exists

$$C(i) \quad G = \sum_j c_j P_0^{\tau_0(j)} P_1^{\tau_1(j)} \dots P_i^{\tau_i(j)}$$

with $\tau_0(j), \dots, \tau_i(j) \in \mathbb{N}$, $0 \leq \tau_k(j) < n_k$ for $1 \leq k \leq i$ and $c_j \in CS$ such that

$$\nu(P_0^{\tau_0(j)} P_1^{\tau_1(j)} \dots P_i^{\tau_i(j)}) = \gamma \text{ for all } j$$

and

$$\left[\frac{G}{N} \right] = \lambda.$$

Suppose that $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \leq l \leq m$ and $0 \leq j_k(l) < n_k$ for $1 \leq k \leq i$ are such that the $(j_0(l), j_1(l), \dots, j_i(l))$ are distinct for $1 \leq l \leq m$, and $\nu(P_0^{j_0(l)} P_1^{j_1(l)} \dots P_i^{j_i(l)}) = \nu(P_0^{j_0(1)} \dots P_i^{j_i(1)})$

$D(i)$ for $1 \leq l \leq m$. Then

$$1, \left[\frac{P_0^{j_0(2)} P_1^{j_1(2)} \dots P_i^{j_i(2)}}{P_0^{j_0(1)} P_1^{j_1(1)} \dots P_i^{j_i(1)}} \right], \dots, \left[\frac{P_0^{j_0(m)} P_1^{j_1(m)} \dots P_i^{j_i(m)}}{P_0^{j_0(1)} P_1^{j_1(1)} \dots P_i^{j_i(1)}} \right]$$

are linearly independent over \mathfrak{k} .

We will leave the proofs of $A(1)$, $B(1)$, $C(1)$ and $D(1)$ to the reader, as they are an easier variation of the following inductive statement, which we will prove.

Assume that $i \geq 1$ and $A(i)$, $B(i)$, $C(i)$ and $D(i)$ are true. We will prove that $A(i+1)$, $B(i+1)$ and $C(i+1)$ and $D(i+1)$ are true. Let $\beta_j = \nu(P_j)$ for $0 \leq j \leq i+1$. By Lemma [Lemma 2.1](#), there exists $U_{i+1} = P_0^{w_0(i)} P_1^{w_1(i)} \dots P_i^{w_i(i)}$ for some $w_j(i) \in \mathbb{N}$ such that $0 \leq w_j(i) < n_j$ for $1 \leq j \leq i$ and $\nu(U_{i+1}) = \bar{n}_{i+1} \beta_{i+1}$ (where $\bar{n}_{i+1} = [G(\beta_0, \dots, \beta_{i+1}) : G(\beta_0, \dots, \beta_i)]$).

Let $f_{i+1}(u)$ be the minimal polynomial of

$$\alpha_{i+1} = \left[\frac{P_{i+1}^{\bar{n}_{i+1}}}{U_{i+1}} \right]$$

over $\mathfrak{k}(\alpha_1, \dots, \alpha_i)$. Let $d = d_{i+1} = \deg f_{i+1}$. Expand

$$f_{i+1}(u) = u^d + b_{d-1} u^{d-1} + \dots + b_0$$

with $b_j \in \mathfrak{k}(\alpha_1, \dots, \alpha_i)$. For $j \geq 1$,

$$\nu(U_{i+1}^j) = j \bar{n}_{i+1} \beta_{i+1} \geq \beta_{i+1} > n_i \beta_i.$$

In the inductive statement $C(i)$, take $N = U_{i+1}^{d-t}$ for $0 \leq t < d = d_{i+1}$, to obtain for $0 \leq t < d_{i+1}$,

$$\boxed{\text{eqF3}} \quad (14) \quad G_t = \sum_{s=1}^{\lambda_t} a_{s,t} P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_i^{j_i(s,t)}$$

with $a_{s,t} \in CS$, $j_k(s,t) \in \mathbb{N}$ and $0 \leq j_k(s,t) < n_k$ for $1 \leq k \leq i$ such that

$$\nu(G_t) = \nu(P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_i^{j_i(s,t)}) = (d-t) \bar{n}_{i+1} \beta_{i+1}$$

for all s, t and

$$\boxed{\text{eqF5}} \quad (15) \quad \left[\frac{G_t}{U_{i+1}^{d-t}} \right] = b_t.$$

Set

$$\boxed{\text{eqF1}} \quad (16) \quad \begin{aligned} P_{i+2} &= P_{i+1}^{\bar{n}_{i+1} d_{i+1}} + G_{d-1} P_{i+1}^{\bar{n}_{i+1} (d_{i+1}-1)} + \dots + G_0 \\ &= P_{i+1}^{\bar{n}_{i+1} d_{i+1}} + \sum_{t=0}^{d-1} \sum_{s=1}^{\lambda_t} a_{s,t} P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_i^{j_i(s,t)} P_{i+1}^{t \bar{n}_{i+1}}. \end{aligned}$$

We have established $A(i+1)$.

Suppose M is a Laurent polynomial in P_0, P_1, \dots, P_{i+1} and $\nu(M) = 0$. We have a factorization

$$M = P_0^{a_0} P_1^{a_1} \dots P_i^{a_i} P_{i+1}^{a_{i+1}}$$

with all $a_j \in \mathbb{Z}$. Thus $a_{i+1}\beta_{i+1} \in G(\beta_0, \dots, \beta_i)$, so that \bar{n}_{i+1} divides a_{i+1} . Let $s = \frac{a_{i+1}}{\bar{n}_{i+1}}$. Then

$$M = U_{i+1}^s (P_0^{a_0} P_1^{a_1} \dots P_i^{a_i}) \left(\frac{P_{i+1}^{\bar{n}_{i+1}}}{U_{i+1}} \right)^s.$$

Now $U_{i+1}^s P_0^{a_0} \dots P_i^{a_i}$ is a Laurent monomial in P_0, \dots, P_i of value zero, so the validity of $B(i+1)$ follows from the inductive assumption $B(i)$.

We now establish $C(i+1)$. Suppose $\lambda \in \mathfrak{k}(\alpha_1, \dots, \alpha_{i+1})$ and N is a Laurent monomial in P_0, P_1, \dots, P_{i+1} such that $\gamma = \nu(N) \geq n_{i+1}\nu(P_{i+1})$. We have

$$\gamma \geq n_{i+1}\beta_{i+1} = \bar{n}_{i+1}d_{i+1}\beta_{i+1} \geq \bar{n}_{i+1}\beta_{i+1}.$$

By Lemma [2.1](#) ^{Lemma2} there exist $r_0, r_1, \dots, r_i, k \in \mathbb{N}$ such that $0 \leq r_j < n_j$ for $1 \leq j \leq i$ and $0 \leq k < \bar{n}_{i+1}$ such that

$$\bar{N} = P_0^{r_0} P_1^{r_1} \dots P_i^{r_i} P_{i+1}^k$$

satisfies $\nu(\bar{N}) = \gamma$. Let $\tilde{N} = P_0^{r_0} P_1^{r_1} \dots P_i^{r_i}$, so that $\bar{N} = \tilde{N} P_{i+1}^k$. Let $\tau = \lfloor \frac{N}{\bar{N}} \rfloor$. We have that $0 \neq \tau \in \mathfrak{k}(\alpha_1, \dots, \alpha_{i+1})$ by $B(i+1)$.

Suppose $0 \leq j \leq d_{i+1} - 1$. Then

$$\begin{aligned} \nu \left(\frac{\tilde{N}}{U_{i+1}^j} \right) &= \nu(\tilde{N}) - j\nu(U_{i+1}) \\ \text{eqF2} \quad (17) \quad &\geq \gamma - (\bar{n}_{i+1} - 1)\beta_{i+1} - (d_{i+1} - 1)\bar{n}_{i+1}\beta_{i+1} \\ &\geq \bar{n}_{i+1}d_{i+1}\beta_{i+1} - \bar{n}_{i+1}\beta_{i+1} + \beta_{i+1} - d_{i+1}\bar{n}_{i+1}\beta_{i+1} + \bar{n}_{i+1}\beta_{i+1} \\ &\geq \beta_{i+1} > n_i\beta_i. \end{aligned}$$

Write

$$\tau\lambda = e_0 + e_1\alpha_{i+1} + \dots + e_{d_{i+1}-1}\alpha_{i+1}^{d_{i+1}-1}$$

with $e_j \in \mathfrak{k}(\alpha_1, \dots, \alpha_i)$. By the inductive statement $C(i)$ and [\(17\)](#) ^{eqF2}, there exist for $0 \leq j \leq d_{i+1} - 1$

$$H_j = \sum_k c_{k,j} P_0^{\delta_0(k,j)} P_1^{\delta_1(k,j)} \dots P_i^{\delta_i(k,j)}$$

with $\delta_0(k,j), \delta_1(k,j), \dots, \delta_i(k,j) \in \mathbb{N}$, $0 \leq \delta_l(k,j) < n_l$ for $1 \leq l$ and $c_{k,j} \in CS$ for all k, j such that

$$\nu(P_0^{\delta_0(k,j)} P_1^{\delta_1(k,j)} \dots P_i^{\delta_i(k,j)}) = \nu \left(\frac{\tilde{N}}{U_{i+1}^j} \right)$$

for all j, k and

$$\left[\frac{H_j}{\left(\frac{\tilde{N}}{U_{i+1}^j} \right)} \right] = e_j$$

for all j . Set

$$G = H_0 P_{i+1}^k + H_1 P_{i+1}^{\bar{n}_{i+1}+k} + \dots + H_{d_{i+1}-1} P_{i+1}^{\bar{n}_{i+1}(d_{i+1}-1)+k}.$$

We have

$$\bar{n}_{i+1}(d_{i+1} - 1) + k < \bar{n}_{i+1}(d_{i+1} - 1) + \bar{n}_{i+1} \leq \bar{n}_{i+1}d_{i+1} = n_{i+1},$$

and

$$\frac{G}{\bar{N}} = \frac{H_0}{\tilde{N}} + \left(\frac{H_1 U_{i+1}}{\tilde{N}} \right) \left(\frac{P_{i+1}^{\bar{n}_{i+1}}}{U_{i+1}} \right) + \cdots + \left(\frac{H_{d_{i+1}-1} U_{i+1}^{d_{i+1}-1}}{\tilde{N}} \right) \left(\frac{P_{i+1}^{\bar{n}_{i+1}}}{U_{i+1}} \right)^{d_{i+1}-1}.$$

We have

$$\left[\frac{G}{\bar{N}} \right] = e_0 + e_1 \alpha_{i+1} + \cdots + e_{d_{i+1}-1} \alpha_{i+1}^{d_{i+1}-1} = \tau \lambda.$$

Thus

$$\left[\frac{G}{\bar{N}} \right] = \left[\frac{G}{\bar{N}} \right] \left[\frac{\bar{N}}{\bar{N}} \right] = \tau \lambda \tau^{-1} = \lambda.$$

We have established $C(i+1)$.

Suppose that $D(i+1)$ is not true. We will obtain a contradiction. Under the assumption that $D(i+1)$ is not true, there exists $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \leq l \leq m$ with $0 \leq j_k(l) < n_k$ for $1 \leq k \leq i+1$ such that $(j_0(l), j_1(l), \dots, j_{i+1}(l))$ are distinct for $1 \leq l \leq m$, and

$$\nu(P_0^{j_0(l)} P_1^{j_1(l)} \cdots P_{i+1}^{j_{i+1}(l)}) = \nu(P_0^{j_0(1)} P_1^{j_0(1)} \cdots P_{i+1}^{j_{i+1}(1)})$$

for $1 \leq l \leq m$ and $\tilde{a}_l \in \mathfrak{k}$ for $1 \leq l \leq m$ not all zero such that

$$\tilde{a}_1 + \tilde{a}_2 \left[\frac{P_0^{j_0(2)} P_1^{j_1(2)} \cdots P_{i+1}^{j_{i+1}(2)}}{P_0^{j_0(1)} P_1^{j_1(1)} \cdots P_{i+1}^{j_{i+1}(1)}} \right] + \cdots + \tilde{a}_m \left[\frac{P_0^{j_0(m)} P_1^{j_1(m)} \cdots P_{i+1}^{j_{i+1}(m)}}{P_0^{j_0(1)} P_1^{j_1(1)} \cdots P_{i+1}^{j_{i+1}(1)}} \right] = 0.$$

$(j_{i+1}(l) - j_{i+1}(1))\beta_{i+1} \in G(\beta_0, \dots, \beta_i)$ for $1 \leq l \leq m$, so \bar{n}_{i+1} divides $(j_{i+1}(l) - j_{i+1}(1))$ for all l . Thus after possibly dividing all monomials $P_0^{j_0(l)} P_1^{j_1(l)} \cdots P_{i+1}^{j_{i+1}(l)}$ by a common power of P_{i+1} , we may assume that

$$\boxed{\text{eqH1}} \quad (18) \quad \bar{n}_{i+1} \text{ divides } j_{i+1}(l) \text{ for all } l.$$

After possibly reindexing the $P_0^{j_0(l)} P_1^{j_1(l)} \cdots P_{i+1}^{j_{i+1}(l)}$, we may assume that $j_{i+1}(1) = \bar{n}_{i+1} \varphi$ is the largest value of $j_{i+1}(l)$.

For $1 \leq l \leq m$, define $a_l \in CS$ by $\bar{a}_l = \tilde{a}_l$. Let

$$Q = \sum_{l=1}^m a_l P_0^{j_0(l)} P_1^{j_1(l)} \cdots P_{i+1}^{j_{i+1}(l)}.$$

Let

$$Q_s = \sum_{j_{i+1}(l) = s \bar{n}_{i+1}} a_l P_0^{j_0(l)} P_1^{j_1(l)} \cdots P_i^{j_i(l)}$$

for $0 \leq s \leq \varphi$. Then

$$\boxed{\text{eqH2}} \quad (19) \quad Q = \sum_{s=0}^{\varphi} Q_s P_{i+1}^{\bar{n}_{i+1} s}.$$

Let

$$c_s = \left[\frac{Q_s}{P_0^{j_0(1)} P_1^{j_1(1)} \cdots P_i^{j_i(1)} U_{i+1}^{(\varphi-s)}} \right] \in \mathfrak{k}(\alpha_1, \dots, \alpha_i)$$

by $B(i)$. We further have that $c_\varphi \neq 0$ by $D(i)$ since the monomials are all distinct.

Dividing Q by $P_0^{j_0(1)} P_1^{j_1(1)} \dots P_i^{j_i(1)} U_{i+1}^\varphi$, we have

$$0 = \sum_{s=0}^{\varphi} c_s \alpha_{i+1}^s.$$

Thus the minimal polynomial $f_{i+1}(u)$ of α_{i+1} divides $g(u) = \sum_{s=0}^{\varphi} c_s u^s$ in $\mathfrak{k}(\alpha_1, \dots, \alpha_i)[u]$. But then $\varphi \geq d_{i+1}$, so that $j_{i+1}(1) = \bar{n}_{i+1} \varphi \geq n_{i+1}$, a contradiction. \square

RemarkH1

Remark 4.3. Theorem ^{Theorem1*}4.2 can be stated without recourse to a coefficient set. To give this statement (which has the same proof) ^{eq12*}(9) must be replaced with “ c_k are units in R for $1 \leq k \leq \lambda_i$ ”. In the proof, the statement “ $a_{s,t} \in CS$ ” in $A(i)$ must be replaced with “ $a_{s,t}$ units in R or $a_{s,t} = 0$ ”. The statement “ $c_j \in CS$ ” in $C(i)$ must be replaced with “ c_j is a unit in R or $c_j = 0$ ”.

RemarkH10

Remark 4.4. For $i > 0$, there is an expression

$$P_{i+1} = y^{n_1 \dots n_i} + x \Theta_{i+1}$$

with $\Theta_{i+1} \in R$. This follows by considering the expression ^{eq11*}(8) and the various constraints on the values of the terms of the monomials in this expression.

RemarkH2

Remark 4.5. The algorithm of Theorem ^{Theorem1*}4.2 concludes with $\Omega < \infty$ if and only if $\nu(P_\Omega) \notin \mathbb{Q}\nu(x)$ (so that $\text{rank}(\nu) = 2$) or ν is discrete of rank 1 with $\text{trdeg}_{R/\mathfrak{m}_R} V_\nu/\mathfrak{m}_\nu = 1$ (so that ν is divisorial).

Proof. From Theorem ^{Theorem1*}4.2, we see that the algorithm terminates with $\Omega < \infty$ if and only if either

$$[G(\nu(P_0), \dots, \nu(P_\Omega)) : G(\nu(P_0), \dots, \nu(P_{\Omega-1}))] = \infty$$

or

$$[G(\nu(P_0), \dots, \nu(P_\Omega)) : G(\nu(P_0), \dots, \nu(P_{\Omega-1}))] < \infty \text{ and } [\mathfrak{k}(\alpha_1, \dots, \alpha_\Omega) : \mathfrak{k}(\alpha_1, \dots, \alpha_{\Omega-1})] = \infty.$$

\square

Remark2

Remark 4.6. Suppose that $\Omega = \infty$ and $n_i = 1$ for $i \gg 0$ in the conclusions of Theorem ^{Theorem1*}4.2. Then ν is discrete, and V_ν/\mathfrak{m}_ν is finite over \mathfrak{k} .

Proof. We first deduce a consequence of the assumption that $\Omega = \infty$ and $n_i = 1$ for $i \gg 0$. There exists $i_0 \in \mathbb{Z}_+$ such that $n_i = 1$ for all $i \geq i_0$. Thus for $i \geq i_0$, P_{i+1} is the sum of P_i and a \mathfrak{k} -linear combination of monomials M in x and the finitely many P_j with $j < i_0$, and with $\nu(M) = \nu(P_i)$. We see that the P_i form a Cauchy sequence in \hat{R} whose limit f in \hat{R} is nonzero (by Remark ^{RemarkH10}4.4), and such that $\lim_{i \rightarrow \infty} \nu(P_i) = \infty$.

Thus $I_{\hat{R}} \neq (0)$, ν is discrete and V_ν/\mathfrak{m}_ν is finite over \mathfrak{k} by the proof of Proposition ^{Prop17}3.4. \square

RemarkH5

Remark 4.7. Suppose that $V_\nu/\mathfrak{m}_\nu = R/\mathfrak{m}_R$ in the hypotheses of Theorem ^{Theorem1*}4.2 (so that there is no residue field extension). Then the P_i constructed by the algorithm are binomials for $i \geq 2$; ^{eq11*}(8) becomes

$$P_{i+1} = P_i^{\bar{n}_i} + cU_i = P_i^{\bar{n}_i} + cP_0^{w_0(i)} \dots P_{i-1}^{w_{i-1}(i)}$$

for some $0 \neq c \in CS$.

Example 4.8. *There exists a rank 2 valuation ν dominating $R = \mathbb{k}[x, y]_{(x, y)}$ such that the set*

$$\{\nu(P_0), \nu(P_1), \nu(P_2), \dots\}$$

does not generate the semigroup $S^R(\nu)$.

Proof. Suppose that \mathbb{k} is a field of characteristic zero. We define a rank 2 valuation $\hat{\nu}$ on $\mathbb{k}[[x, y]]$. Let $g(x, y) = y - x\sqrt{x+1}$. For $0 \neq f(x, y) \in \mathbb{k}[[x, y]]$, we have a factorization $f = g^n h$ where $n \in \mathbb{N}$ and $g \nmid h$. The rule

$$\hat{\nu}(f) = (n, \text{ord}(h(x, x\sqrt{1+x}))) \in (\mathbb{Z}^2)_{\text{lex}}$$

then defines a rank 2 valuation dominating $\mathbb{k}[[x, y]]$ with value group $(\mathbb{Z}^2)_{\text{lex}}$.

We have that $(g) \cap \mathbb{k}[x, y] = (y^2 - x^2 - x^3)$. Thus $\hat{\nu}$ restricts to a rank 2 valuation ν which dominates the maximal ideal $\mathfrak{n} = (x, y)$ of $\mathbb{k}[x, y]$. Expand

$$x\sqrt{1+x} = \sum_{j \geq 1} a_j x^j = x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \dots$$

as a series with all $a_j \in \mathbb{k}$ non zero. Applying the algorithm of Theorem ^{Theorem1*}4.2, we construct the infinite sequence of polynomials P_1, P_2, \dots where $P_0 = x$, $P_1 = y$ and $P_i = y - \sum_{j=1}^{i-1} a_j x^j$ for $i \geq 2$. We have that $\nu(P_i) = (0, i)$ for $i \geq 0$. However, $\nu(y^2 - x^2 - x^3) = (1, 1)$.

Thus the set $\{\nu(x), \nu(P_1), \nu(P_2), \dots\}$ does not generate the semigroup $S^R(\nu)$. □

LemmaG10

Lemma 4.9. *Suppose that ν is a valuation dominating R . Let*

$$P_0 = x, P_1 = y, P_2, \dots$$

be the sequence of elements of R constructed by Theorem ^{Theorem1}4.2. Set $\beta_i = \nu(P_i)$ for $i \geq 0$. Suppose that $P_0^{m_0} P_1^{m_1} \dots P_r^{m_r}$ is a monomial in P_0, \dots, P_r and $m_i \geq n_i$ for some $i \geq 1$. Let $\rho = \nu(P_0^{m_0} P_1^{m_1} \dots P_r^{m_r})$. Then with the notation of ^{eqM1}(12),*

eqW1

$$(20) \quad P_0^{m_0} \dots P_r^{m_r} = - \sum_{t=0}^{d_i-1} \sum_{s=1}^{\lambda_t} a_{s,t} P_0^{m_0+j_0(s,t)} \dots P_{i-1}^{m_{i-1}+j_{i-1}(s,t)} P_i^{m_i-n_i+t\bar{n}_i} P_{i+1}^{m_{i+1}} \dots P_r^{m_r} + P_0^{m_0} \dots P_i^{m_i-n_i} P_{i+1}^{m_{i+1}+1} \dots P_r^{m_r}.$$

All terms in the first sum of ^{eqW1}(20) have value ρ and $\nu(P_0^{m_0} \dots P_i^{m_i-n_i} P_{i+1}^{m_{i+1}+1} \dots P_r^{m_r}) > \rho$.

Suppose that W is a Laurent monomial in P_0, \dots, P_r such that $\nu(W) = \rho$. Then

eqW2

$$(21) \quad \left[\frac{P_0^{m_0} P_1^{m_1} \dots P_r^{m_r}}{W} \right] = - \sum_{t=0}^{d_i-1} \sum_{s=1}^{\lambda_t} \bar{a}_{s,t} \left[\frac{P_0^{m_0+j_0(s,t)} \dots P_{i-1}^{m_{i-1}+j_{i-1}(s,t)} P_i^{m_i-n_i+\bar{n}_i t} P_{i+1}^{m_{i+1}} \dots P_r^{m_r}}{W} \right]$$

and

eqW3

$$(22) \quad (m_0 + j_0(s, t)) + \dots + (m_{i-1} + j_{i-1}(s, t)) + (m_i - n_i + t\bar{n}_i) + m_{i+1} + \dots + m_r > m_0 + m_1 + \dots + m_r$$

for all terms in the first sum of ^{eqW1}(20).

Proof. We have

$$P_0^{m_0} \dots P_r^{m_r} = P_0^{m_0} \dots P_i^{n_i} P_i^{m_i-n_i} \dots P_r^{m_r}$$

where $m_i - n_i \geq 0$. Substituting (I2) for $P_i^{n_i}$, we obtain equation (20). We compute, from the first term of (20),

$$\begin{aligned}
& - \sum_{t=1}^{d_i-1} \sum_{s=1}^{\lambda_t} \bar{a}_{s,t} \left[\frac{P_0^{m_0+j_0(s,t)} \dots P_r^{m_r}}{W} \right] \\
&= - \left[\frac{P_0^{m_0} \dots P_i^{m_i-n_i} \dots P_r^{m_r} U_i^{d_i}}{W} \right] \left(\sum_{t=0}^{d_i-1} \sum_{s=1}^{\lambda_t} \bar{a}_{s,t} \left[\frac{P_0^{j_0(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)}}{U_i^{d_i-t}} \right] \left[\frac{P_i^{\bar{n}_i}}{U_i} \right]^t \right) \\
&= - \left[\frac{P_0^{m_0} \dots P_i^{m_i-n_i} \dots P_r^{m_r} U_i^{d_i}}{W} \right] \left(\sum_{t=0}^{d_i-1} b_{i,t} \alpha_i^t \right) \\
&= \left[\frac{P_0^{m_0} \dots P_i^{m_i-n_i} \dots P_r^{m_r} U_i^{d_i}}{W} \right] \alpha_i^{d_i} \\
&= \left[\frac{P_0^{m_0} \dots P_i^{m_i-n_i} \dots P_r^{m_r} U_i^{d_i}}{W} \right] \left[\frac{P_i^{\bar{n}_i}}{U_i} \right]^{d_i} \\
&= \left[\frac{P_0^{m_0} \dots P_i^{m_i} \dots P_r^{m_r}}{W} \right],
\end{aligned}$$

giving (21). For all s, t (with $0 \leq t \leq d_i - 1$),

$$\begin{aligned}
n_i \beta_i &= j_0(s, t) \beta_0 + j_1(s, t) \beta_1 + \dots + j_{i-1}(s, t) \beta_{i-1} + \bar{n}_i t \beta_i \\
&< (j_0(s, t) + j_1(s, t) + \dots + j_{i-1}(s, t) + \bar{n}_i t) \beta_i
\end{aligned}$$

so

$$n_i < j_0(s, t) + j_1(s, t) + \dots + j_{i-1}(s, t) + \bar{n}_i t.$$

(22) follows. □

TheoremG2

Theorem 4.10. *Suppose that ν is a valuation dominating R . Let*

$$P_0 = x, P_1 = y, P_2, \dots$$

be the sequence of elements of R constructed by Theorem 4.2. Set $\beta_i = \nu(P_i)$ for $i \geq 0$. Suppose that $f \in R$ and there exists $n \in \mathbb{Z}_+$ such that $\nu(f) < n\nu(\mathfrak{m}_R)$. Then there exists an expansion

$$f = \sum_I a_I P_0^{i_0} P_1^{i_1} \dots P_r^{i_r} + \sum_J \varphi_J P_0^{j_0} \dots P_r^{j_r} + h$$

where $r \in \mathbb{N}$, $a_I \in CS$, $I, J \in \mathbb{N}^{r+1}$, $\nu(P_0^{i_0} P_1^{i_1} \dots P_r^{i_r}) = \nu(f)$ for all I in the first sum, $0 \leq i_k < n_k$ for $1 \leq k \leq r$, $\nu(P_0^{j_0} \dots P_r^{j_r}) > \nu(f)$ for all terms in the second sum, $\varphi_J \in R$ and $h \in \mathfrak{m}_R^n$.

The first sum is uniquely determined by these conditions.

Proof. We first prove existence. We have an expansion

$$f = \sum a_{i_0, i_1} x^{i_0} y^{i_1} + h_0$$

with $a_{i_0, i_1} \in CS$ and $h_0 \in \mathfrak{m}_R^n$. More generally, suppose that we have an expansion

eqG5

$$(23) \quad f = \sum a_I P_0^{i_0} P_1^{i_1} \dots P_r^{i_r} + h$$

for some $r \in \mathbb{Z}_+$, $I = (i_0, \dots, i_r) \in \mathbb{N}^{r+1}$, $a_I \in CS$ and $h \in \mathfrak{m}_R^n$. Let

$$\rho = \min\{\nu(P_0^{i_0} P_1^{i_1} \dots P_r^{i_r}) \mid a_I \neq 0\}.$$

We can rewrite [\(23\)](#) ^{eqG5} as

$$\text{eqG6} \quad (24) \quad f = \sum_J a_J P_0^{j_0} P_1^{j_1} \cdots P_r^{j_r} + \sum_{J'} a_{J'} P_0^{j'_0} P_1^{j'_1} \cdots P_r^{j'_r} + h$$

where the terms in the first sum have minimal value $\nu(P_0^{j_0} P_1^{j_1} \cdots P_r^{j_r}) = \rho$ and the nonzero terms in the second sum have value $\nu(P_0^{j'_0} P_1^{j'_1} \cdots P_r^{j'_r}) > \rho$.

If we have that the first sum is nonzero and $0 \leq j_k < n_k$ for $1 \leq k \leq r$ for all terms in the first sum of [\(24\)](#) then $\rho = \nu(f)$ and we have achieved the conclusions of the theorem. So suppose that one of these conditions fails.

First suppose that $\sum_J a_J P_0^{j_0} \cdots P_r^{j_r} \neq 0$ and for some J , $j_i \geq n_i$ for some $i \geq 1$. Let

$$a = \min\{j_0 + \cdots + j_r \mid j_i \geq n_i \text{ for some } i \geq 1\}$$

and let b be the numbers of terms in $\sum_J a_J P_0^{j_0} \cdots P_r^{j_r}$ such that $j_i \geq n_i$ for some $i \geq 1$ and $j_0 + \cdots + j_r = a$. Let $\sigma = (a, b) \in (\mathbb{Z}^2)_{\text{lex}}$. Let $J_0 = (\bar{j}_0, \dots, \bar{j}_r)$ be such that $a_{J_0} \neq 0$ and $\bar{j}_0 + \cdots + \bar{j}_r = a$. Write

$$P_0^{\bar{j}_0} \cdots P_r^{\bar{j}_r} = P_0^{\bar{j}_0} \cdots P_i^{\bar{j}_i - n_i} P_i^{n_i} \cdots P_r^{\bar{j}_r}$$

and substitute [\(12\)](#) ^{eqM1} for $P_i^{n_i}$, to obtain an expression of the form [\(20\)](#) ^{eqW1} of Lemma [4.9](#) ^{LemmaG10}.

Substitute this expression [\(20\)](#) ^{eqW1} for $P_0^{\bar{j}_0} \cdots P_r^{\bar{j}_r}$ in [\(24\)](#) ^{eqG6} and apply Remark [4.1](#) ^{RemarkG4} to obtain an expression of the form [\(24\)](#) ^{eqG6} such that either the first sum is zero or the first sum is nonzero and all terms in the first sum satisfy $j_i < n_i$ for $1 \leq i$ so that $\nu(f) = \rho$ and we have achieved the conclusions of the theorem, or the first sum has a nonzero term which satisfies $j_i \geq n_i$ for some $i \geq 1$. By [\(22\)](#) ^{eqW3}, we have an increase in σ if this last case holds.

Since there are only finitely many monomials M in P_0, \dots, P_r which have the value ρ , after a finite number of iterations of this step we must either find an expression [\(24\)](#) ^{eqG6} where the first sum is zero, or attain an expression [\(24\)](#) ^{eqG6} satisfying the conclusions of the theorem.

If we obtain an expression [\(24\)](#) ^{eqG6} where the first sum is zero, then we have an expression [\(23\)](#) ^{eqG5} with an increase in ρ (and possibly an increase in r), and we repeat the last step, either attaining the conclusions of the theorem or obtaining another increase in ρ . Since there are only a finite number of monomials in the $\{P_i\}$ which have value $\leq \nu(f)$, we must achieve the conclusions of the theorem in a finite number of steps.

Uniqueness of the first sum follows from 2) of Theorem [4.2](#) ^{Theorem1*}.

□

Corollary1*

Theorem 4.11. *Suppose that ν is a rank 1 valuation which dominates R and $\nu(x) = \nu(\mathfrak{m}_R)$. Then*

- a) *The set $\{\text{in}_\nu(x)\} \cup \{\text{in}_\nu(P_i) \mid n_i > 1\}$ minimally generates $\text{gr}_\nu(R)$ as a \mathfrak{k} -algebra.*
- b) *The set*

$$\{\nu(x)\} \cup \{\nu(P_i) \mid \bar{n}_i > 1\}$$

minimally generates the semigroup $S^R(\nu)$.

- c) *$V_\nu/\mathfrak{m}_\nu = \mathfrak{k}(\alpha_i \mid d_i > 1)$ where α_i is defined by 4) (and possibly [\(II\)](#) ^{eqL10}) of Theorem [4.2](#) ^{Theorem1*}.*

Proof. Theorem [4.10](#) ^{TheoremG2} implies that the set $\{\text{in}_\nu(x)\} \cup \{\text{in}_\nu(P_i) \mid n_i > 1\}$ generates $\text{gr}_\nu(R)$ as a \mathfrak{k} -algebra. We will show that the set generates $\text{gr}_\nu(R)$ minimally. Suppose that it doesn't. Then there exists an $i \in \mathbb{N}$ such that $n_i > 1$ if $i > 0$ and a sum

$$\text{eqZ30} \quad (25) \quad H = \sum_J c_J P_0^{j_0} \cdots P_r^{j_r}$$

for some $r \in \mathbb{N}$ with $c_J \in CS$ such that the monomials $P_0^{j_0} \cdots P_r^{j_r}$ have value $\nu(P_0^{j_0} \cdots P_r^{j_r}) = \nu(P_i)$ with $j_i = 0$ and $j_k = 0$ if $n_k = 1$ for $1 \leq k \leq r$ for all J , and

$$\nu\left(\sum_J c_J P_0^{j_0} \cdots P_r^{j_r} - P_i\right) > \nu(P_i).$$

We thus have by 1) of Theorem [4.2](#) and since $\nu(P_0) = \nu(\mathfrak{m}_R)$, that $r \leq i - 1$. Thus $i \geq 1$. By Theorem [4.10](#) applied to H , we have an expression

$$\text{eqZ31} \quad (26) \quad P_i = \sum_K d_K P_0^{k_0} \cdots P_s^{k_s} + f$$

where $s \in \mathbb{N}$, $d_K \in CS$, $0 \leq k_l < n_l$ for $1 \leq l$, some $d_K \neq 0$, $f \in R$ is such that $\nu(f) > \nu(P_i)$, and

$$\nu(P_0^{k_0} \cdots P_s^{k_s}) = \nu(H) = \nu(P_i)$$

for all monomials in the first sum of [\(26\)](#). Since the minimal value terms of the expression of H in [\(25\)](#) only involve P_0, \dots, P_{i-1} and all these monomials have the same value $\rho = \nu(H)$, the algorithm of Theorem [4.10](#) ends with $s \leq i - 1$ in [\(26\)](#). But then we obtain from [\(26\)](#) a contradiction to 2) of Theorem [4.2](#).

Now a) and 3) of Theorem [4.2](#) imply statement b).

Suppose that $\lambda \in L = V_\nu/\mathfrak{m}_\nu$. Then $\lambda = \left[\frac{f}{f'}\right]$ for some $f, f' \in R$ with $\nu(f) = \nu(f')$. By Theorem [4.10](#), there exist $r \in \mathbb{Z}_+$ and expressions

$$f = \sum_{i=1}^m a_i P_0^{\sigma_0(i)} P_1^{\sigma_1(i)} \cdots P_r^{\sigma_r(i)} + h,$$

$$f' = \sum_{j=1}^n b_j P_0^{\tau_0(j)} P_1^{\tau_1(j)} \cdots P_r^{\tau_r(j)} + h'$$

with $a_i, b_j \in CS$, $0 \leq \sigma_k(i) < n_k$ for $1 \leq k$ and $0 \leq \tau_k(j) < n_k$ for $1 \leq k$, the $P_0^{\sigma_0(i)} P_1^{\sigma_1(i)} \cdots P_r^{\sigma_r(i)}$, $P_0^{\tau_0(j)} P_1^{\tau_1(j)} \cdots P_r^{\tau_r(j)}$ all have the common value

$$\rho := \nu(f) = \nu(f'),$$

$h, h' \in R$ and $\nu(h) > \rho$, $\nu(h') > \rho$.

$$\lambda = \left(\sum_i \bar{a}_i [P_0^{\sigma_0(i)-\sigma_0(1)} \cdots P_r^{\sigma_r(i)-\sigma_r(1)}]\right) \left(\sum_j \bar{b}_j [P_0^{\tau_0(j)-\tau_0(1)} \cdots P_r^{\tau_r(j)-\tau_r(1)}]\right)^{-1} \in \mathfrak{k}(\alpha_1, \dots, \alpha_r)$$

by $B(r)$ of the proof of Theorem [4.2](#). □

If V_ν/\mathfrak{m}_ν is transcendental over \mathfrak{k} then $\Gamma_\nu \cong \mathbb{Z}$ by Abhyankar's inequality. Zariski called such a valuation a "prime divisor of the second kind". By c) of Theorem [4.11](#), $V_\nu/\mathfrak{m}_\nu = \mathfrak{k}(\alpha_i \mid d_i > 1)$. There thus exists an index i such that $\mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$ is algebraic over \mathfrak{k} and α_i is transcendental over $\mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$. Thus $\Omega = i$ in the algorithm of Theorem [4.2](#), since α_i does not have a minimal polynomial over $\mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$. □

Corollary3*

Theorem 4.12. *Suppose that ν is a rank 2 valuation which dominates R and $\nu(x) = \nu(\mathfrak{m}_R)$. Let I_ν be the height one prime ideal in V_ν . Then one of the following three cases hold:*

1. $I_\nu \cap R = \mathfrak{m}_R$. Then

a) the finite set

$$\{\text{in}_\nu(x)\} \cup \{\text{in}_\nu(P_i) \mid n_i > 1\}$$

minimally generates $\text{gr}_\nu(R)$ as an \mathfrak{k} -algebra and

b) the finite set

$$\{\nu(x)\} \cup \{\nu(P_i) \mid \bar{n}_i > 1\}$$

minimally generates the semigroup $S^R(\nu)$.

c) $V_\nu/\mathfrak{m}_\nu = \mathfrak{k}(\alpha_i \mid d_i > 1)$.

2. $I_\nu \cap R = (P_\Omega)$ is a height one prime ideal in R and

a) the finite set

$$\{\text{in}_\nu(x)\} \cup \{\text{in}_\nu(P_i) \mid n_i > 1\}$$

minimally generates $\text{gr}_\nu(R)$ as a \mathfrak{k} -algebra, and

b) The finite set

$$\{\nu(x)\} \cup \{\nu(P_i) \mid \bar{n}_i > 1\}$$

minimally generates the semigroup $S^R(\nu)$.

c) $V_\nu/\mathfrak{m}_\nu = \mathfrak{k}(\alpha_i \mid d_i > 1)$.

3. $I_\nu \cap R = (g)$ is a height one prime ideal in R and

a) the finite set

$$\{\text{in}_\nu(x)\} \cup \{\text{in}_\nu(P_i) \mid n_i > 1\} \cup \{\text{in}_\nu(g)\}$$

minimally generates $\text{gr}_\nu(R)$ as a \mathfrak{k} -algebra, and

b) The finite set

$$\{\nu(x)\} \cup \{\nu(P_i) \mid \bar{n}_i > 1\} \cup \{\nu(g)\}$$

minimally generates the semigroup $S^R(\nu)$.

c) $V_\nu/\mathfrak{m}_\nu = \mathfrak{k}(\alpha_i \mid d_i > 1)$.

Proof. Since ν has rank 2, the set $\{P_i \mid n_i > 1\}$ is a finite set since otherwise either Γ_ν is not a finitely generated group or V_ν/\mathfrak{m}_ν is not a finitely generated field extension of \mathfrak{k} , by 3) and 4) of Theorem [Theorem1*](#), which is a contradiction to Abhyankar's inequality.

The case when $I_\nu \cap R = \mathfrak{m}_R$ now follows from Theorem [Theorem2](#) [4.10](#) and 2), 3) of Theorem [Theorem1*](#) [4.2](#); the proof of c) is the same as the proof of c) of Theorem [Corollary1*](#) [4.11](#).

Suppose that $I_\nu \cap R = (g)$ is a height one prime ideal in R . Suppose that $f \in R$. Then there exists $n \in \mathbb{N}$ and $u \in R$ such that $f = g^n u$ with $u \notin (g)$. Thus

$$\text{eqZ15} \quad (27) \quad \nu(f) = n\nu(g) + \nu(u).$$

Assume that $\Omega < \infty$. Then $\nu(P_\Omega) \notin \mathbb{Q}\nu(\mathfrak{m}_R)$ by Remark [RemarkH2](#) [4.5](#). Then $P_\Omega = gf$ for some $f \in R$. We will show that f is a unit in R . Suppose not. Then $\nu(g) < \nu(P_\Omega)$. Let $t = \text{ord}(g)$. There exists $c \in \mathbb{Z}_+$ such that if $j_0, j_1, \dots, j_{\Omega-1} \in \mathbb{N}$ are such that $\nu(P_0^{j_0} P_1^{j_1} \dots P_{\Omega-1}^{j_{\Omega-1}}) \geq c\nu(\mathfrak{m}_R)$ then $\text{ord}(P_0^{j_0} P_1^{j_1} \dots P_{\Omega-1}^{j_{\Omega-1}}) > t$. We may assume that c is larger than t . Write

$$g = \sum_{i,j=1}^c a_{ij} x^i y^j + \Lambda$$

with $\Lambda \in \mathfrak{m}_R^c$ and $a_{ij} \in CS$. g has an expression of the form

$$\text{eq*} \quad (28) \quad g = \sum_J a_J P_0^{j_0} \dots P_\Omega^{j_\Omega} + \sum_{J'} a_{J'} P_0^{j'_0} \dots P_\Omega^{j'_{\Omega'}} + h$$

with $a_J, a_{J'} \in CS$ and $h \in \mathfrak{m}_R^c$, and the terms in the first sum all have a common value ρ , which is smaller than the values of the terms in the second sum.

Now we draw some conclusions which must hold for an expression of the form $(28)^{\text{eq*}}$. We must have that

$$\text{eq1} \quad (29) \quad \rho < c\nu(\mathfrak{m}_R),$$

since otherwise, by our choice of c and our assumption that $\text{ord}(f) > 0$, so that $\text{ord}(P_\Omega) > \text{ord}(g) = t$, we would have that the right hand side of $(28)^{\text{eq*}}$ has order larger than t , which is impossible. In particular, we have

$$\text{eq2} \quad (30) \quad j_\Omega = 0$$

in all terms in the first sum.

We also must have that

$$\text{eq**} \quad (31) \quad j_i \geq n_i \text{ for some } i \text{ with } 1 \leq i < \Omega \text{ for all terms in the first sum.}$$

This follows since otherwise we would have $\nu(g) = \rho < c\nu(\mathfrak{m}_R)$, which is impossible.

We apply the algorithm of Theorem 4.9 to $(28)^{\text{eq*}}$, and apply a substitution of the form (20) to a monomial in the first sum. As shown in the proof of Theorem 4.9, we must obtain an expression $(28)^{\text{eq*}}$ with an increase in ρ after a finite number of iterations, since $(31)^{\text{eq**}}$ must continue to hold. Since there are only finitely many values in the semigroup $S^R(\nu)$ between 0 and $c\nu(\mathfrak{m}_R)$, after finitely many iterations of the algorithm we obtain an expression $(28)^{\text{eq*}}$ with $\rho \geq c\nu(\mathfrak{m}_R)$, which is a contradiction to $(29)^{\text{eq1}}$. This contradiction shows that P_Ω is a unit times g , so we may replace g with P_Ω , and we are in Case 2 of the conclusions of the corollary.

If $\Omega = \infty$ then $\nu(P_i) \in \mathbb{Q}\nu(\mathfrak{m}_R)$ for all i (by Remark 4.5^{RemarkH2}) and we are in Case 3 of the conclusions of the corollary.

The conclusions of a) and b) of Cases 2 and 3 of the corollary now follow from applying Theorem 4.10 and 2), 3) of Theorem 4.2 to u in $(27)^{\text{eq15}}$.

Suppose that $\lambda \in V_\nu/\mathfrak{m}_\nu$. Then $\lambda = \left[\frac{f}{f'} \right]$ for some $f, f' \in R$ with $\nu(f) = \nu(f')$. We may assume (after possibly dividing out a common factor) that $g \nmid f$ and $g \nmid f'$. Then the proof of c) of cases 2 and 3 proceeds as in the proof of c) of Theorem 4.11^{Corollary1*}. □

5. VALUATION SEMIGROUPS AND RESIDUE FIELD EXTENSION ON A TWO DIMENSIONAL REGULAR LOCAL RING

Proof1

In this section, we prove Theorem 1.1^{Theorem3*} which is stated in the introduction. Theorem 1.1^{Theorem3*} gives necessary and sufficient conditions for a semigroup and field extension to be the valuation semigroup and residue field of a valuation dominating a regular local ring of dimension two.

Suppose that ν is a valuation dominating R . Let $S = S^r(\nu)$ and $L = V_\nu/\mathfrak{m}_\nu$. Let x, y be regular parameters in R such that $\nu(x) = \nu(\mathfrak{m}_R)$. Set $P_0 = x$ and $P_1 = y$. Let $\{P_i\}$ be the sequence of elements of R defined by the algorithm of Theorem 4.2^{Theorem1*}. We have by Remark 4.6^{Remark2} and its proof, that if

$$\Omega = \infty \text{ and } n_i = 1 \text{ for } i \gg 0,$$

then $I_{\hat{R}} \neq (0)$ (where $I_{\hat{R}}$ is the prime ideal in \hat{R} of Cauchy sequences in R satisfying $(4)^{\text{eqZ12}}$). Thus ν has rank 2 since R is complete, and ν must satisfy Case 3 of Theorem 4.12^{Corollary3*}.

Set $\sigma(0) = 0$ and inductively define

$$\sigma(i) = \min\{j \mid j > \sigma(i-1) \text{ and } n_j > 1\}.$$

This defines an index set I of finite or infinite cardinality $\Lambda = |I| - 1 \geq 1$. Suppose that either ν has rank 1 or ν has rank 2 and one of the first two cases of Theorem 4.12 hold for the P_i . Let

$$\beta_i = \nu(P_{\sigma(i)}) \in S^R(\nu)$$

for $i \in I$ and

$$\gamma_i = \left[\frac{P_{\sigma(i)}^{\bar{n}_{\sigma(i)}}}{U_{\sigma(i)}} \right] \in V_\nu/\mathfrak{m}_\nu$$

if $i > 0$ and $\sigma(i) < \Omega$ or $\sigma(i) = \Omega$ and $\bar{n}_\Omega < \infty$. Set $\gamma_\Lambda = 1$ if $\sigma(\Lambda) = \Omega$ and $\bar{n}_\Omega = \infty$.

By Theorem 4.2 and Theorem 4.11 or 4.12, $\{\beta_i\}$ and $\{\gamma_i\}$ satisfy the conditions 1) and 2) of Theorem 1.1.

Suppose that ν has rank 2 and the third case of Theorem 4.12 holds for the P_i . Then $\Lambda < \infty$. Let $I_\nu \cap R = (g)$ (where I_ν is the height one prime ideal of V_ν). Let $\bar{\Lambda} = \Lambda + 1$. Define $\beta_i = \nu(P_{\sigma(i)})$ for $i < \bar{\Lambda}$ and $\beta_{\bar{\Lambda}} = \nu(g)$. Define

$$\gamma_i = \left[\frac{P_{\sigma(i)}^{\bar{n}_{\sigma(i)}}}{U_{\sigma(i)}} \right] \in V_\nu/\mathfrak{m}_\nu$$

for $0 < i < \bar{\Lambda}$ and define $\gamma_{\bar{\Lambda}} = 1$. By Theorem 4.2 and Case 3 of Theorem 4.12, $\{\beta_i\}$ and $\{\gamma_i\}$ satisfy conditions 1) and 2) of Theorem 1.1.

Now suppose that S and L and the given sets $\{\beta_i\}$ and $\{\alpha_i\}$ satisfy conditions 1) and 2) of the theorem. We will construct a valuation ν which dominates R with $S^R(\nu) = S$ and $V_\nu/\mathfrak{m}_\nu = L$.

Let

$$f_i(u) = u^{d_i} + b_{i,d_i-1}u^{d_i-1} + \cdots + b_{i,0}$$

be the minimal polynomial of α_i over $\mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})$, and let $n_i = \bar{n}_i d_i$.

We will inductively define $P_i \in R$, a function ν on Laurent monomials in P_0, \dots, P_i such that

$$\nu(P_0^{a_0} P_1^{a_1} \cdots P_i^{a_i}) = a_0 \beta_0 + a_1 \beta_1 + \cdots + a_i \beta_i$$

for $a_0, \dots, a_i \in \mathbb{Z}$ and monomials U_i in P_0, \dots, P_{i-1} , such that

$$\nu(U_i) = \bar{n}_i \beta_i,$$

a function res on the Laurent monomials $P_0^{a_0} P_1^{a_1} \cdots P_i^{a_i}$ which satisfy $\nu(P_0^{a_0} P_1^{a_1} \cdots P_i^{a_i}) = 0$, such that

$$\text{eqL40} \quad (32) \quad \text{res} \left(\frac{P_j^{\bar{n}_j}}{U_j} \right) = \alpha_j$$

for $1 \leq j \leq i$.

Let x, y be regular parameters in R . Define $P_0 = x$, $P_1 = y$, $\beta_0 = \nu(P_0)$, and $\beta_1 = \nu(P_1)$. We inductively construct the P_i by the procedure of the algorithm of Theorem 4.2. We must modify the inductive statement $A(i)$ of the proof of Theorem 4.2 as follows:

There exists $U_i = P_0^{w_0(i)} P_1^{w_1(i)} \dots P_{i-1}^{w_{i-1}(i)}$ for some $w_j(i) \in \mathbb{N}$ and $0 \leq w_j(i) < n_j$ for $1 \leq j \leq i-1$
 $\bar{A}(i)$ such that $\bar{n}_i \nu(P_i) = \nu(U_i)$. There exist $a_{s,t} \in CS$ and $j_0(s,t), j_1(s,t), \dots, j_{i-1}(s,t) \in \mathbb{N}$ with $0 \leq j_k(s,t) < n_k$ for $k \geq 1$ and $0 \leq t < \bar{d}_i$ such that
 $\nu(P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)} P_i^{t\bar{n}_i}) = \bar{n}_i d_i \nu(P_i)$
for all s, t and

$$\boxed{\text{eqM1*}} \quad (33) \quad P_{i+1} := P_i^{\bar{n}_i d_i} + \sum_{t=0}^{d_i-1} \left(\left(\sum_{s=1}^{\lambda_t} a_{s,t} P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)} \right) P_i^{t\bar{n}_i} \right)$$

satisfies

$$b_{i,t} = \sum_{s=1}^{\lambda_t} \bar{a}_{s,t} \text{res} \left(\frac{P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)}}{U_i^{d_i-t}} \right)$$

for $0 \leq t \leq d_i - 1$.

We inductively verify $\bar{A}(i)$ for $1 \leq i < \Lambda$ and the statements $B(i)$, $C(i)$ and $D(i)$ (with the residues $[M]$ replaced with $\text{res}(M)$). We observe from $B(i)$ that the function res is determined by (32) . The inequality in 2) of the assumptions of the theorem is necessary to allow us to apply Lemma 2.1.

We now show that if $\Lambda = \infty$, then given $\sigma \in \mathbb{Z}_+$, there exists $\tau \in \mathbb{Z}_+$ such that

$$\boxed{\text{eqZ20}} \quad (34) \quad \text{ord}(P_i) > \sigma \text{ if } i > \tau.$$

We establish (34) by induction on σ . Suppose that $\text{ord}(P_i) > \sigma$ if $i > \tau$. There exists λ such that $\beta_0 < \beta_i$ if $i \geq \lambda$. Let $\tau' = \max\{\sigma + \tau + 1, \tau + 1, \lambda\}$. We will show that $\text{ord}(P_i) > \sigma + 1$ if $i > \tau'$. From (33) , we must show that if $i > \tau'$ and $(a_0, \dots, a_{i-1}) \in \mathbb{N}^i$ are such that

$$a_0 \beta_0 + a_1 \beta_1 + \dots + a_{i-1} \beta_{i-1} = n_{i-1} \beta_{i-1}$$

then

$$\boxed{\text{eqZ21}} \quad (35) \quad a_0 \text{ord}(P_0) + a_1 \text{ord}(P_1) + \dots + a_{i-1} \text{ord}(P_{i-1}) > \sigma + 1.$$

If $a_{\tau+1} + \dots + a_{i-1} \geq 2$ then (35) follows from induction. If $a_{\tau+1} + \dots + a_{i-1} = 1$ then some $a_j \neq 0$ with $0 \leq j \leq \tau$ since $n_{i-1} > 1$, so (35) follows from induction. If $a_j = 0$ for $j \geq \tau + 1$ then

$$n_{i-1} \beta_{i-1} = a_0 \beta_0 + \dots + a_\tau \beta_\tau < (a_0 + \dots + a_\tau) \beta_\tau.$$

Thus

$$(a_0 + \dots + a_\tau) > \frac{n_{i-1} \beta_{i-1}}{\beta_\tau} \geq 2^{i-\tau} > \sigma + 1.$$

Thus (35) holds in this case.

We first suppose that for all P_i , there exists $m_i \in \mathbb{Z}_+$ such that $m_i \nu(P_i) > \min\{\beta_0, \beta_1\}$.

We now establish the following:

Suppose that $f \in R$. Then there exists an expansion

$$\boxed{\text{fund}} \quad (36) \quad f = \sum_I a_I P_0^{i_0} P_1^{i_1} \dots P_r^{i_r} + \sum_J \varphi_J P_0^{j_0} \dots P_r^{j_r}$$

for some $r \in \mathbb{N}$ where $\nu(P_0^{i_0} P_1^{i_1} \cdots P_r^{i_r})$ have a common value ρ for all terms in the first sum, all $a_I \in CS$, $I, J \in \mathbb{N}^{r+1}$ and some $a_I \neq 0$, $0 \leq i_k < n_k$ for $1 \leq k \leq r$ $\nu(P_0^{j_0} \cdots P_r^{j_r}) > \rho$ for all terms in the second sum, and $\varphi_J \in R$ for all terms in the second sum. The first sum $\sum_I a_I P_0^{i_0} P_1^{i_1} \cdots P_r^{i_r}$ is uniquely determined by these conditions.

The proof of (36) follows from the proofs of Lemma 4.9 and Theorem 4.10, observing that all properties of a valuation which ν is required to satisfy in these proofs hold for the function ν on Laurent monomials in the P_i which we have defined above, and replacing $[M]$ in Lemma 4.9 with the function $\text{res}(M)$ for Laurent monomials M with $\nu(M) = 0$.

The n in the statement of Theorem 4.10 is chosen so that if M is a monomial in the P_i with $\text{ord}(M) = \text{ord}(f)$, then $\nu(M) < n \min\{\beta_0, \beta_1\}$ (such an n exists trivially if $\Lambda < \infty$ and by (34) if $\Lambda = \infty$).

We can thus extend ν to R by defining

$$\nu(f) = \rho \text{ if } f \text{ has an expansion (36).}$$

Now we will show that ν is a valuation. Suppose that $f, g \in R$. We have expansions

$$\text{eqL20} \quad (37) \quad f = \sum_I a_I P_0^{i_0} P_1^{i_1} \cdots P_r^{i_r} + \sum_J \varphi_J P_0^{j_0} \cdots P_r^{j_r}$$

and

$$\text{eqL21} \quad (38) \quad g = \sum_K b_K P_0^{k_0} P_1^{k_1} \cdots P_r^{k_r} + \sum_L \varphi_L P_0^{l_0} \cdots P_r^{l_r}$$

of the form (36). Let $\rho = \nu(f)$ and $\rho' = \nu(g)$. The statement that $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ follows from Remark 4.1 and the algorithm of Theorem 4.10.

Let V be a monomial in P_0, \dots, P_r such that $\nu(V) = \nu(P_0^{i_0} \cdots P_r^{i_r})$ for all I in the first sum of f in (37) and let W be a monomial in P_0, \dots, P_r such that $\nu(W) = \nu(P_0^{k_0} \cdots P_r^{k_r})$ for all K in the first sum of g in (38). We have that

$$\sum \bar{a}_I \text{res} \left(\frac{P_0^{i_0} \cdots P_r^{i_r}}{V} \right) \neq 0 \text{ in } L$$

and

$$\sum \bar{b}_K \text{res} \left(\frac{P_0^{k_0} \cdots P_r^{k_r}}{W} \right) \neq 0 \text{ in } L$$

by $D(r)$.

We have (applying Remark 4.1) an expansion

$$\text{eqL22} \quad (39) \quad fg = \sum_M d_M P_0^{m_0} P_1^{m_1} \cdots P_r^{m_r} + \sum_Q \psi_Q P_0^{q_0} \cdots P_r^{q_r}$$

with $d_M \in S$ for all M , $\psi_Q \in R$ for all Q , $\nu(P_0^{m_0} P_1^{m_1} \cdots P_r^{m_r}) = \rho + \rho'$ for all terms in the first sum, and some $d_M \neq 0$ and $\nu(P_0^{q_0} \cdots P_r^{q_r}) > \rho + \rho'$ for all terms in the second sum, which satisfies all conditions of (36) except that we only have that $m_0, m_1, \dots, m_r \in \mathbb{N}$. We have

$$\sum_M \bar{d}_M \text{res} \left(\frac{P_0^{m_0} \cdots P_r^{m_r}}{VW} \right) = \left(\sum_I \bar{a}_I \text{res} \left(\frac{P_0^{i_0} \cdots P_r^{i_r}}{V} \right) \right) \left(\sum_K \bar{b}_K \text{res} \left(\frac{P_0^{k_0} \cdots P_r^{k_r}}{W} \right) \right) \neq 0.$$

By (21) of Lemma 4.9 (with $[M]$ replaced with $\text{res}(M)$ for a Laurent monomial N with $\nu(M) = 0$) we see that the algorithm of Theorem 4.10 which puts the expansion (39) into

the form $\frac{\text{fund}}{(36)}$ converges to an expression $\frac{\text{fund}}{(36)}$ where the terms in the first sum all have $\nu(P_0^{i_0} \cdots P_r^{i_r}) = \rho + \rho'$ with

$$\sum_I \bar{a}_I \operatorname{res} \left(\frac{P_0^{i_0} \cdots P_r^{i_r}}{VW} \right) = \sum_M \bar{d}_M \operatorname{res} \left(\frac{P_0^{m_0} \cdots P_r^{m_r}}{VW} \right) \neq 0.$$

Thus $\nu(fg) = \nu(f) + \nu(g)$. We have established that ν is a valuation.

By Theorem $\frac{\text{Corollary1*}}{4.11}$ or Case 1 of Theorem $\frac{\text{Corollary3*}}{4.12}$, we have that $S = S^R(\nu)$ and $L = V_\nu/\mathfrak{m}_\nu$.

Finally, we suppose that Λ is finite and $\bar{n}_\Lambda = \infty$. Given $g \in R$, write

$$\text{eqL31} \quad (40) \quad g = P_\Lambda^t f$$

where $P_\Lambda \nmid f$. Choose $n \in \mathbb{Z}_+$ so that if M is a monomial in $P_0, \dots, P_{\Lambda-1}$ with $\operatorname{ord}(M) = \operatorname{ord}(f)$ then $\nu(M) < n \min\{\beta_0, \beta_1\}$.

The argument giving the expansion $\frac{\text{fund}}{(36)}$ now provides an expansion

$$\text{eqL30} \quad (41) \quad f = \sum_I a_I P_0^{i_0} \cdots P_\Lambda^{i_\Lambda} + \sum_J \varphi_J P_0^{j_0} \cdots P_\Lambda^{j_\Lambda} + h_1$$

where $\nu(P_0^{i_0} \cdots P_\Lambda^{i_\Lambda})$ has a common value ρ for all monomials in the first sum, $a_I \in CS$ for all I , $\nu(P_0^{j_0} \cdots P_\Lambda^{j_\Lambda}) > \rho$ for all monomials in the second sum, $\varphi_J \in R$ for all J and $h_1 \in \mathfrak{m}_R^n$.

If $i_\Lambda = 0$ for all monomials in the first sum, then we obtain an expansion of f of the form $\frac{\text{fund}}{(36)}$. Suppose that $i_\Lambda \neq 0$ for some monomial in the first sum. Then $i_\Lambda \neq 0$ for all terms in the first sum, $j_\Lambda \neq 0$ for all terms in the second sum, and we have an expression $f = P_\Lambda t_1 + h_1$ for some $t_1 \in R$. Repeating this argument for increasingly large values of n , we either obtain an n giving an expression $\frac{\text{fund}}{(36)}$ for f , or we obtain the statement that

$$f \in \cap_{n=1}^\infty ((P_\Lambda) + \mathfrak{m}_R^n) = (P_\Lambda),$$

which is impossible. Thus we can extend ν to R by defining $\nu(g) = t\beta_\Lambda + \rho$ if $g = P_\Lambda^t f$ where $P_\Lambda \nmid f$ and f has an expansion $\frac{\text{fund}}{(36)}$.

It follows that ν is a valuation, by an extension of the proof of the previous case. By Case 2 of Theorem $\frac{\text{Corollary3*}}{4.12}$, we have that $S = S^R(\nu)$ and $L = V_\nu/\mathfrak{m}_\nu$.

CorollaryN1

Corollary 5.1. *Suppose that R is a regular local ring of dimension two and ν is a valuation dominating R . Then the semigroup $S^R(\nu)$ has a generating set $\{\beta_i\}_{i \in I}$ and V_ν/\mathfrak{m}_ν is generated over $\mathfrak{k} = R/\mathfrak{m}_R$ by a set $\{\alpha_i\}_{i \in I_+}$ such that 1) and 2) of Theorem $\frac{\text{Theorem3*}}{1.1}$ hold, but the additional case that $\bar{n}_\Lambda < \infty$ and $d_\Lambda < \infty$ if $\Lambda < \infty$ may hold if R is not complete.*

Proof. The only case we have not considered in Theorem $\frac{\text{Theorem3*}}{1.1}$ is the analysis in the case when $\Omega = \infty$, $n_i = 1$ for $i \gg 0$, $I_{\hat{R}} \neq 0$ and $I_{\hat{R}} \cap R = (0)$ (so that R is not complete). In this case ν is discrete of rank 1, $\Lambda < \infty$, $\bar{n}_\Lambda < \infty$ and $d_\Lambda < \infty$ by Remark $\frac{\text{Remark2}}{4.6}$, giving the additional possibility stated in the Corollary. \square

Proof2

6. VALUATION SEMIGROUPS ON A REGULAR LOCAL RING OF DIMENSION TWO

In this section we prove Theorem $\frac{\text{Corollary4*}}{1.2}$ which is stated in the introduction. Theorem $\frac{\text{Corollary4*}}{1.2}$ gives necessary and sufficient conditions for a semigroup to be the valuation semigroup of a valuation dominating a regular local ring of dimension two.

If $S = S^R(\nu)$ for some valuation ν dominating R , then 1) and 2) of Theorem $\frac{\text{Corollary4*}}{1.2}$ hold by Corollary $\frac{\text{CorollaryN1}}{5.1}$. Observe that the construction in the proof of Theorem $\frac{\text{Theorem3*}}{1.1}$ of a valuation

ν with a prescribed semigroup S and residue field L satisfying the conditions 1) and 2) of Theorem [1.1](#) is valid for any regular local ring R of dimension 2 (with residue field \mathfrak{k}). Taking $L = \mathfrak{k}$ (or $L = \mathfrak{k}(t)$ where t is an indeterminate), we may thus construct a valuation ν dominating R with semigroup $S^R(\nu) = S$ whenever S satisfies the conditions 1) and 2) of Theorem [1.2](#).

Definition 6.1. *Suppose that S is a semigroup such that the group G generated by S is isomorphic to \mathbb{Z} . S is symmetric if there exists $m \in G$ such that $s \in S$ if and only if $m - s \notin S$ for all $s \in G$.*

We deduce from Theorem [1.2](#) a [Corollary4*](#) generalization of a result of Noh [\[40\]](#).

Symmetric

Corollary 6.2. *Suppose that R is a regular local ring of dimension two and ν is a valuation dominating R such that ν is discrete of rank 1. Then $S^R(\nu)$ is symmetric.*

Proof. By Theorem [1.2](#), and since ν is discrete of rank 1, there exists a finite set

$$\beta_0 < \beta_1 < \cdots < \beta_\Lambda$$

such that $S^\nu(R) = S(\beta_0, \beta_1, \dots, \beta_\Lambda)$ and $\beta_{i+1} > \bar{n}_i \beta_i$ for $1 \leq i < \Lambda$, where $\bar{n}_i = [G(\beta_0, \dots, \beta_i) : G(\beta_0, \dots, \beta_{i-1})]$. We identify the value group Γ_ν with \mathbb{Z} . Then we calculate that

$$\text{lcm}(\text{gcd}(\beta_0, \dots, \beta_{i-1}), \beta_i) = \bar{n}_i \beta_i$$

for $1 \leq i \leq \Lambda$. We have that $\bar{n}_i \beta_i \geq \beta_i > \bar{n}_{i-1} \beta_{i-1}$ for $2 \leq i \leq \Lambda$. By Lemma [2.1](#), we have that $\bar{n}_i \beta_i \in S(\beta_0, \dots, \beta_{i-1})$ for $2 \leq i \leq \Lambda$. Since β_0 and β_1 are both positive, we have that $\bar{n}_1 \beta_1 \in S(\beta_0)$. Thus the criteria of Proposition 2.1 [\[29\]](#) is satisfied, so that $S^R(\nu)$ is symmetric. \square

Example 6.3. *There exists a semigroup S which satisfies the sufficient conditions 1) and 2) of Theorem [1.2](#), such that if (R, \mathfrak{m}_R) is a 2-dimensional regular local ring dominated by a valuation ν such that $S^R(\nu) = S$, then $R/\mathfrak{m}_R = V_\nu/\mathfrak{m}_\nu$; that is, there can be no residue field extension.*

Proof. Define $\beta_i \in \mathbb{Q}$ by

$$(42) \quad \beta_0 = 1, \beta_1 = \frac{3}{2}, \text{ and } \beta_i = 2\beta_{i-1} + \frac{1}{2^i} \text{ for } i \geq 2.$$

Let $S = S(\beta_0, \beta_1, \dots)$ be the semigroup generated by β_0, β_1, \dots . Observe that $\bar{n}_i = 2, \forall i \geq 1$, $\beta_0 < \beta_1 < \cdots$ is the minimal sequence of generators of S and S satisfies conditions 1) and 2) of Theorem [1.2](#). The group $\Gamma = G(\beta_0, \beta_1, \dots)$ generated by S is $\Gamma = \frac{1}{2^\infty} \mathbb{Z} = \cup_{i=0}^\infty \frac{1}{2^i} \mathbb{Z}$.

Now suppose that (R, \mathfrak{m}_R) is a regular local ring of dimension 2, with residue field \mathfrak{k} and ν is a valuation of the quotient field of R which dominates R such that $S^R(\nu) = S$. Since $\Gamma_\nu = \frac{1}{2^\infty}$ is not discrete, we have by Proposition [3.4](#) that ν extends uniquely to a valuation $\hat{\nu}$ of the quotient field of \hat{R} which dominates \hat{R} and $S^{\hat{\nu}}(\hat{R}) = S$.

We will now show that $V_\nu/\mathfrak{m}_\nu = V_{\hat{\nu}}/\mathfrak{m}_{\hat{\nu}}$. Suppose that $f \in \hat{R}$. Since $\hat{\nu}$ has rank 1, there exists a positive integer n such that $\hat{\nu}(f) < n\nu(\mathfrak{m})$. There exists $f' \in R$ such that $f'' = f - f' \in \mathfrak{m}_R^n \hat{R}$. Thus $\nu(f) = \nu(f')$. Suppose that $h \in V_{\hat{\nu}}/\mathfrak{m}_{\hat{\nu}}$. Then $h = \left[\frac{f}{g} \right]$ where $f, g \in \hat{R}$ and $\nu(f) = \nu(g)$. Write $f = f' + f''$ and $g = g' + g''$ where $f', g' \in R$ and $f'', g'' \in \hat{R}$ satisfy $\nu(f'') > \nu(f)$ and $\nu(g'') > \nu(g)$. Then $\left[\frac{f}{g} \right] = \left[\frac{f'}{g'} \right] \in V_\nu/\mathfrak{m}_\nu$.

We also have $\mathfrak{k} = R/\mathfrak{m}_R = \hat{R}/\mathfrak{m}_{\hat{R}}$. By Theorem [1.1](#), [Theorem3*](#) there exists $\alpha_i \in V_{\hat{\nu}/M_{\hat{\nu}}}$ for $i \geq 1$ such that $V_{\hat{\nu}/M_{\hat{\nu}}} = \mathfrak{k}(\alpha_1, \alpha_2, \dots)$ and if $d_i = [\mathfrak{k}(\alpha_1, \dots, \alpha_i) : \mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})]$ then

$$\text{eqV2} \quad (43) \quad \beta_{i+1} \geq \bar{n}_i d_i \beta_i, \forall i \geq 1,$$

so that

$$\text{eqV3} \quad (44) \quad [V_{\hat{\nu}}/\mathfrak{m}_{\hat{\nu}} : \mathfrak{k}] = \prod_{i=1}^{\infty} [\mathfrak{k}(\alpha_1, \dots, \alpha_i) : \mathfrak{k}(\alpha_1, \dots, \alpha_{i-1})] = \prod_{i=1}^{\infty} d_i.$$

On the other hand, since $\beta_i \geq \beta_1 = \frac{3}{2}, \forall i \geq 1$, we have

$$\text{eqV4} \quad (45) \quad \beta_{i+1} = 2\beta_i + \frac{1}{2^{i+1}} \leq 4\beta_i + \frac{1}{2^{i+1}} - 3 < 4\beta_i.$$

From [\(43\)](#), [\(44\)](#) and [\(45\)](#) we have $d_i = 1, \forall i \geq 1$ so that $[V_{\hat{\nu}}/\mathfrak{m}_{\hat{\nu}} : \mathfrak{k}] = 1$. □

7. BIRATIONAL EXTENSIONS

[RLR2](#)

Suppose that R is a regular local ring of dimension two which is dominated by a valuation ν . Let $\mathfrak{k} = R/\mathfrak{m}_R$. The quadratic transform R_1 of R along ν is defined as follows. Let u, v be a system of regular parameters in R , where we may assume that $\nu(u) \leq \nu(v)$. Then $R[\frac{v}{u}] \subset V_{\nu}$. Let

$$R_1 = R \left[\frac{v}{u} \right]_{R[\frac{v}{u}] \cap \mathfrak{m}_{\nu}}.$$

R_1 is a two dimensional regular local ring which is dominated by ν . Let

$$\text{eqX3} \quad (46) \quad R \rightarrow T_1 \rightarrow T_2 \cdots$$

be the sequence of quadratic transforms along ν , so that $V_{\nu} = \cup T_i$ ([Ab1](#)), and $L = V_{\nu}/\mathfrak{m}_{\nu} = \cup T_i/\mathfrak{m}_{T_i}$. Suppose that x, y are regular parameters in R .

[birat](#)

Theorem 7.1. [Theorem1*](#) *Let $P_0 = x, P_1 = y$ and $\{P_i\}$ be the sequence of elements of R constructed in Theorem [4.2](#). Suppose that $\Omega \geq 2$. Then there exists some smallest value i in the sequence [\(46\)](#) such that the divisor of xy in $\text{Spec}(T_i)$ has only one component. Let $R_1 = T_i$. Then $R_1/\mathfrak{m}_{R_1} \cong \mathfrak{k}(\alpha_1)$, and there exists $x_1 \in R_1$ and $w \in \mathbb{Z}_+$ such that $x_1 = 0$ is a local equation of the exceptional divisor of $\text{Spec}(R_1) \rightarrow \text{Spec}(R)$, and $Q_0 = x_1, Q_1 = \frac{P_2}{x_1^{w\bar{n}_1}}$ are regular parameters in R_1 . We have that*

$$Q_i = \frac{P_{i+1}}{Q_0^{w\bar{n}_1 \cdots \bar{n}_i}}$$

[Remark1](#) *for $1 \leq i < \max\{\Omega, \infty\}$ satisfy the conclusions of Theorem [4.2](#) (as interpreted by Remark [4.3](#)) for the ring R_1 .*

Proof. We use the notation of Theorem [4.2](#) and its proof for R and the $\{P_i\}$. Recall that $U_1 = U^{w_0(1)}$. Let $w = w_0(1)$. Since \bar{n}_1 and w are relatively prime, there exist $a, b \in \mathbb{N}$ such that

$$\varepsilon := \bar{n}_1 b - wa = \pm 1.$$

Define elements of the quotient field of R by

$$x_1 = (x^b y^{-a})^{\varepsilon}, y_1 = (x^{-w} y^{\bar{n}_1})^{\varepsilon}.$$

We have that

$$\text{eqZ1} \quad (47) \quad x = x_1^{\bar{n}_1} y_1^a, y = x_1^w y_1^b.$$

Since $\bar{n}_1\nu(y) = w\nu(x)$, it follows that

$$\bar{n}_1\nu(x_1) = \nu(x), \bar{\nu}(y_1) = 0.$$

We further have that

$$\text{eqZ3} \quad (48) \quad \alpha_1 = [y_1]^\varepsilon \in L.$$

Let $A = R[x_1, y_1] \subset V_\nu$ and $\mathfrak{m}_A = \mathfrak{m}_\nu \cap A$. $R \rightarrow A_{\mathfrak{m}_A}$ factors as a product of quadratic transforms such that xy has two distinct irreducible factors in all intermediate rings. Thus $A = R_1$. Recall that

$$f_1(u) = u^{d_1} + b_{1,d_1-1}u^{d_1-1} + \cdots + b_{1,0}$$

is the minimal polynomial of $\alpha_1 = \left[\frac{y^{\bar{n}_1}}{x^w}\right]$ over \mathfrak{k} , and from (II2) of $A(1)$,

$$\text{eqZ2} \quad (49) \quad P_2 = y^{\bar{n}_1 d_1} + a_{1,d_1-1}x^w y^{\bar{n}_1(d_1-1)} + \cdots + a_{1,0}x^{d_1 w}.$$

Substituting (47) into (49), we find that

$$P_2 = x_1^{wn_1} \left(y_1^{b\bar{n}_1 d_1} + a_{1,d_1-1} y_1^{aw+b\bar{n}_1(d_1-1)} + \cdots + a_{1,0} y_1^{ad_1 w} \right).$$

Thus

$$Q_1 = \frac{P_2}{x_1^{wn_1}} \in R_1.$$

We calculate

$$\text{eqZ4} \quad (50) \quad \nu(Q_1) = \nu(P_2) - wn_1\nu(x_1) = \nu(P_2) - n_1\nu(P_1) > 0$$

Thus $x_1, Q_1 \in \mathfrak{m}_{R_1}$.

Suppose that $\varepsilon = 1$. Then since

$$Q_1 = y_1^{awd_1} \left(y_1^{d_1} + a_{1,d_1-1} y_1^{d_1-1} + \cdots + a_{1,0} \right)$$

and y_1 is a unit in R_1 , we have that

$$R_1/(x_1, Q_1) \cong \mathfrak{k}[y_1]/(f(y_1)) \cong \mathfrak{k}(\alpha_1).$$

Suppose that $\varepsilon = -1$. Let

$$h(u) = y_1^{d_1} + \frac{b_{1,1}}{b_{1,0}} y_1^{d_1-1} + \cdots + \frac{1}{b_{1,0}},$$

which is the minimal polynomial of α_1^{-1} over \mathfrak{k} . Since

$$Q_1 = y_1^{b\bar{n}_1 d_1} \left(1 + a_{1,d_1-1} y_1 + \cdots + a_{1,0} y_1^{d_1} \right)$$

and y_1 is a unit in R_1 , we have that

$$R_1/(x_1, Q_1) \cong \mathfrak{k}[y_1]/(h(y_1)) \cong \mathfrak{k}(\alpha_1^{-1}) = \mathfrak{k}(\alpha_1).$$

Now define $\beta_i = \nu(P_i)$ and $\hat{\beta}_i = \nu(Q_i)$ for $i \geq 0$. We have

$$\hat{\beta}_i = \nu(P_{i+1}) - wn_1 \cdots n_i \hat{\beta}_0$$

for $i \geq 1$.

Since $\gcd(w, \bar{n}_1) = 1$, we have that $G(\hat{\beta}_0) = G(\beta_0, \beta_1)$. Thus

$$\bar{n}_{i+1} = [G(\hat{\beta}_0, \dots, \hat{\beta}_i) : G(\hat{\beta}_0, \dots, \hat{\beta}_{i-1})]$$

for $i \geq 1$.

We will leave the proof that the analogue of $A(1)$ of Theorem ^{Theorem1*}4.2 holds for Q_1 in R_1 for the reader, as is an easier variation of the following inductive statement, which we will prove.

Assume that $2 \leq i < \Omega - 1$ and the analogue of $A(j)$ of Theorem ^{Theorem1*}4.2 holds for Q_j in R_1 for $j < i$. We will prove that the analogue of $A(i)$ of Theorem ^{Theorem1*}4.2 holds for Q_i in R_1 .

In particular, we assume that

$$\text{eqJ1} \quad (51) \quad \hat{\beta}_{j+1} > n_{j+1}\hat{\beta}_j$$

for $1 \leq j \leq i - 1$.

Define

$$\text{eqJ6} \quad (52) \quad \begin{aligned} V_i &= U_{i+1}Q_0^{-wn_1n_2 \cdots n_i \bar{n}_{i+1}} y_1^{-(aw_0(i+1)+bw_1(i+1))} \\ &= Q_0^{\hat{w}_0(i+1)} Q_1^{w_2(i+1)} \cdots Q_{i-1}^{w_i(i+1)} \end{aligned}$$

where

$$\hat{w}_0(i+1) = \bar{n}_1 w_0(i+1) + w w_1(i+1) + w n_1 w_2(i+1) + \cdots + w n_1 n_2 \cdots n_{i-1} w_i(i+1) - w n_1 n_2 \cdots n_i \bar{n}_{i+1}.$$

We have that

$$\nu(Q_i^{\bar{n}_{i+1}}) = \bar{n}_{i+1} \hat{\beta}_i = \bar{n}_{i+1} \nu(P_{i+1}) - w n_1 n_2 \cdots n_i \bar{n}_{i+1} \hat{\beta}_0 = \nu(V_i).$$

Thus

$$\bar{n}_{i+1} \hat{\beta}_i = \hat{w}_0(i+1) \hat{\beta}_0 + \hat{w}_2(i+1) \hat{\beta}_1 + \hat{w}_3(i+1) \hat{\beta}_2 + \cdots + w_i(i+1) \hat{\beta}_{i-1}.$$

Recall that $0 \leq w_j(i+1) < n_j$ for $1 \leq j \leq i$ and apply ^{eqJ1}(51) to obtain

$$\text{eqJ2} \quad (53) \quad \begin{aligned} \hat{w}_0(i+1) \hat{\beta}_0 &= \bar{n}_{i+1} \hat{\beta}_i - w_i(i+1) \hat{\beta}_{i-1} - \cdots - w_3(i+1) \hat{\beta}_2 - w_2(i+1) \hat{\beta}_1 \\ &\geq \hat{\beta}_i - (n_i - 1) \hat{\beta}_{i-1} - \cdots - (n_3 - 1) \hat{\beta}_2 - (n_2 - 1) \hat{\beta}_1 \\ &> \hat{\beta}_i - (n_i - 1) \hat{\beta}_{i-1} - \cdots - (n_4 - 1) \hat{\beta}_3 - n_3 \hat{\beta}_2 \\ &\quad \vdots \\ &\geq \hat{\beta}_i - n_i \hat{\beta}_{i-1} > 0. \end{aligned}$$

Thus $V_i \in R_1$. We have

$$\text{eqJ3} \quad (54) \quad \frac{Q_i^{\bar{n}_{i+1}}}{V_i} = \left(\frac{P_{i+1}^{\bar{n}_{i+1}}}{U_{i+1}} \right) y_1^{aw_0(i+1)+bw_1(i+1)}.$$

Let

$$\text{eqJ4} \quad (55) \quad \hat{\alpha}_i = \left[\frac{Q_i^{\bar{n}_{i+1}}}{V_i} \right] = \alpha_{i+1} \alpha_1^{\varepsilon(aw_0(i+1)+bw_1(i+1))} \in L$$

From the minimal polynomial $f_{i+1}(u)$ of α_{i+1} , we see that

$$g_i(u) = u^{d_{i+1}+b_{i+1,d_{i+1}-1}} \alpha_1^{\varepsilon(aw_0(i+1)+bw_1(i+1))d_{i+1}} u^{d_{i+1}-1} + \cdots + b_{i+1,0} \alpha_1^{\varepsilon(aw_0(i+1)+bw_1(i+1))d_{i+1}}$$

is the minimal polynomial ^{eqJ1} of $\hat{\alpha}_i$ over $\mathfrak{k}(\alpha_1)(\hat{\alpha}_1, \dots, \hat{\alpha}_{i-1})$.

Now from equation (I2) of $A(i+1)$ determining P_{i+1} , we obtain

$$\text{eqJ5} \quad (56) \quad \begin{aligned} Q_{i+1} &= \frac{P_{i+2}}{Q_0^{wn_1n_2 \cdots n_{i+1}}} \\ &= Q_i^{\bar{n}_{i+1}d_{i+1}} + \sum_{t=0}^{d_{i+1}-1} \left(\sum_{s=1}^{\lambda_t} a_{s,t} y_1^{aj_0(s,t)+bj_1(s,t)} Q_0^{\hat{j}_0(s,t)} Q_1^{j_2(s,t)} \cdots Q_{i-1}^{j_i(s,t)} \right) Q_i^{t\bar{n}_{i+1}} \end{aligned}$$

where

$$\hat{j}_0(s,t) = \bar{n}_1 j_0(s,t) + w j_1(s,t) + w n_1 j_2(s,t) + \cdots + w n_1 n_2 \cdots n_{i-1} j_i(s,t) - (d_{i+1}-t) w n_1 n_2 \cdots n_i \bar{n}_{i+1}.$$

Recall that $0 \leq j_k(s, t) < n_k$ for $1 \leq k \leq i$. We further have that

$$\nu(Q_0^{\hat{j}_0(s,t)} Q_1^{j_2(s,t)} \dots Q_{i-1}^{j_i(s,t)}) = (d_{i+1} - t) \bar{n}_{i+1} \hat{\beta}_i \geq \hat{\beta}_i.$$

By a similar argument to [\(53\)](#), we obtain that $\hat{j}_0(s, t) > 0$ for all s, t .

By the definition of Q_{i+1} , [\(52\)](#) and [\(56\)](#), we have

eqJ10

$$(57) \quad y_1^{(aw_0(i+1)+bw_1(i+1))d_{i+1}} \frac{P_{i+2}}{U_{i+1}^{d_{i+1}}} = \frac{Q_{i+1}}{V_i^{d_{i+1}}} \\ = \left(\frac{Q_{i+1}}{V_i} \right)^{d_{i+1}} + \sum_{t=0}^{d_{i+1}-1} \left(\sum_{s=1}^{\lambda_t} y_1^{a_{j_0(s,t)}+b_{j_1(s,t)}} \frac{Q_0^{\hat{j}_0(s,t)} Q_1^{j_2(s,t)} \dots Q_{i-1}^{j_i(s,t)}}{V_i^{d_{i+1}-t}} \right) \left(\frac{Q_{i+1}}{V_i} \right)^t$$

We have

$$\left[\sum_{s=1}^{\lambda_t} a_{s,t} y_1^{a_{j_0(s,t)}+b_{j_1(s,t)}} \frac{Q_0^{\hat{j}_0(s,t)} Q_1^{j_2(s,t)} \dots Q_{i-1}^{j_i(s,t)}}{V_i^{d_{i+1}-t}} \right] \\ = \left[\sum_{s=1}^{\lambda_t} a_{s,t} y_1^{(aw_0(i+1)+bw_1(i+1))(d_{i+1}-t)} \frac{P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_i^{j_i(s,t)}}{U_{i+1}^{d_{i+1}-t}} \right] \\ = b_{i+1,t} \alpha_1^{\varepsilon(aw_0(i+1)+bw_1(i+1))(d_{i+1}-t)}$$

for $0 \leq t \leq d_{i+1} - 1$ and

$$\left[\frac{Q_{i+1}}{V_i^{d_{i+1}}} \right] = g_i(\hat{\alpha}_i) = 0.$$

Thus

$$\hat{\beta}_{i+1} = \nu(Q_{i+1}) > d_{i+1} \nu(V_i) = d_{i+1} (\nu(U_{i+1}) - wn_1 n_2 \dots n_i \bar{n}_{i+1} \hat{\beta}_0) \\ = n_{i+1} (\nu(P_{i+1}) - wn_1 n_2 \dots n_i \hat{\beta}_0) = n_{i+1} \hat{\beta}_i.$$

We have thus established that $A(i)$ holds for Q_i in R_1 . By induction on i , we have that $A(i)$ of [Theorem 4.2](#) holds for Q_i in R_1 for $1 \leq i < \Omega - 1$.

We now will show that $D(r)$ of [Theorem 4.2](#) holds for the Q_i in R_1 for all r . We begin by establishing the following statement:

Suppose that $\lambda \geq n_1 w$ is as integer. Then there exist $\delta_0, \delta_1 \in \mathbb{N}$ with $0 \leq \delta_1 < \bar{n}_1$ such that

eqZ5

$$(58) \quad x^{\delta_0+iw} y^{\delta_1+(d_1-1-i)\bar{n}_1} = x_1^\lambda y_1^{z-i\varepsilon}$$

for $0 \leq i \leq d_1 - 1$ where $z = a\delta_0 + b(\delta_1 + (d_1 - 1)\bar{n}_1)$.

We first prove [\(58\)](#). We have that

$$(\lambda\varepsilon b - rw)\bar{n}_1 + (r\bar{n}_1 - \lambda\varepsilon a)w = \lambda$$

for all $r \in \mathbb{Z}$. Choose r so that $\delta_1 = r\bar{n}_1 - \lambda\varepsilon a$ satisfies $0 \leq \delta_1 < \bar{n}_1$. Set

$$\delta_0 = (\lambda\varepsilon b - rw) - (d_1 - 1)w.$$

Then

$$(\lambda\varepsilon b - rw)\bar{n}_1 = \lambda - \delta_1 w \geq n_1 w - (\bar{n}_1 - 1)w = (n_1 - \bar{n}_1 + 1)w$$

so

$$\delta_0 \geq (n_1 - \bar{n}_1 - d_1 + 2)w = ((\bar{n}_1 - 1)(d_1 - 1) + 1)w \geq w.$$

Substituting [\(47\)](#) in $x^{\delta_0+iw} y^{\delta_1+(d_1-1-i)\bar{n}_1}$, we obtain the formula [\(58\)](#).

We now will prove that statement $D(r)$ of Theorem [Theorem1*](#) [4.2](#) holds for the Q_i in R_1 for all r . Suppose that we have monomials $Q_0^{j_0(l)} Q_1^{j_1(l)} \dots Q_r^{j_r(l)}$ for $1 \leq l \leq m$ such that

$$\nu(Q_0^{j_0(l)} Q_1^{j_1(l)} \dots Q_r^{j_r(l)}) = \nu(Q_0^{j_0(1)} Q_1^{j_1(1)} \dots Q_r^{j_r(1)})$$

for $1 \leq l \leq m$, and that we have a dependence relation in $L = V_\nu/\mathfrak{m}_\nu$.

$$0 = e_1 + e_2 \left[\frac{Q_0^{j_0(2)} Q_1^{j_1(2)} \dots Q_r^{j_r(2)}}{Q_0^{j_0(1)} Q_1^{j_1(1)} \dots Q_r^{j_r(1)}} \right] + \dots + e_m \left[\frac{Q_0^{j_0(m)} Q_1^{j_1(m)} \dots Q_r^{j_r(m)}}{Q_0^{j_0(1)} Q_1^{j_1(1)} \dots Q_r^{j_r(1)}} \right]$$

with $e_i \in \mathfrak{k}(\alpha_1)$ (and some $e_i \neq 0$). Multiplying the $Q_0^{j_0(l)} Q_1^{j_1(l)} \dots Q_r^{j_r(l)}$ for $1 \leq l \leq m$ by a common term Q_0^t with t a sufficiently large positive integer, we may assume that

$$\hat{j}_0(l) = j_0(l) - j_1(l)wn_1 - j_2(l)wn_1n_2 - \dots - j_r(l)wn_1n_2 \dots n_r \geq n_1w$$

for $1 \leq l \leq m$. We have that

$$Q_0^{j_0(l)} Q_1^{j_1(l)} \dots Q_r^{j_r(l)} = Q_0^{\hat{j}_0(l)} P_2^{j_1(l)} \dots P_{r+1}^{j_r(l)}.$$

Since $\hat{j}_0(l) \geq wn_1$, [\(58\)](#) ^{eqZ5} implies that for each l with $1 \leq l \leq w$, there exist $\delta_0(l), \delta_1(l)$ with $\delta_0(l), \delta_1(l) \in \mathbb{N}$ and $0 \leq \delta_1(l) < \bar{n}_1$ such that

$$P_0^{\delta_0(l)+iw} P_1^{\delta_1(l)+(d_1-1-i)\bar{n}_1} = y_1^{z(l)-i\varepsilon} Q_0^{\hat{j}_0(l)}$$

for $0 \leq i \leq d_1 - 1$. The ordered set

$$\{\alpha_1^{\varepsilon(z(l)-z(1))}, \alpha_1^{\varepsilon(z(l)-z(1))-1}, \dots, \alpha_1^{\varepsilon(z(l)-z(1))-(d_1-1)}\}$$

is a \mathfrak{k} -basis of $\mathfrak{k}(\alpha_1)$ for all l (since multiplication by $\alpha_1^{\varepsilon(z(l)-z(1))+(d_1-1)}$ is a \mathfrak{k} -vector space isomorphism of $\mathfrak{k}(\alpha_1)$, and thus takes a basis to a basis). Thus there exists $e_{l,i} \in \mathfrak{k}$ such that

$$e_l = \sum_{i=0}^{d_1-1} e_{l,i} \alpha_1^{\varepsilon(z(l)-z(1))-i}.$$

Since some $e_{l,i} \neq 0$, we have a dependence relation

$$0 = \sum_{l=1}^m \sum_{i=0}^{d_1-1} e_{l,i} \left[\frac{P_0^{\delta_0(l)+iw} P_1^{\delta_1(l)+(d_1-1-i)\bar{n}_1} P_2^{j_1(l)} \dots P_{r+1}^{j_r(l)}}{P_0^{\delta_0(1)} P_1^{\delta_1(1)+(d_1-1)\bar{n}_1} P_2^{j_1(1)} \dots P_{r+1}^{j_r(1)}} \right],$$

a contradiction to [D\(r+1\)](#) of Theorem [Theorem1*](#) [4.2](#) for the P_i in R . Thus we have established [D\(r\)](#) of Theorem [Theorem1*](#) [4.2](#) for the Q_i in R_1 . \square

8. POLYNOMIAL RINGS IN TWO VARIABLES

Poly

The algorithm of Theorem [Theorem1*](#) [4.2](#) is applicable when $R = \mathfrak{k}[x, y]$ is a polynomial ring over a field and ν is a valuation which dominates the maximal ideal (x, y) of R . In this case many of the calculations in this paper become much simpler, as we now indicate (of course we take the coefficient set CF to be the field \mathfrak{k}). In the case when R is equicharacteristic, we can establish from the polynomial case the results of this paper using Cohen's structure theorem and Proposition [Prop17](#) [3.4](#) to reduce to the case of a polynomial ring in two variables.

If $f \in R = \mathfrak{k}[x, y]$ is a nonzero polynomial, then we have an expansion $f = a_0(x) + a_1(x)y + \dots + a_r(x)y^r$ where $a_i(x) \in \mathfrak{k}[x]$ for all i and $a_r(x) \neq 0$. We define $\text{ord}_y(f) = r$, and say that f is monic in y if $a_r(x) \in \mathfrak{k}$. We first establish the following formula.

eqZ60

$$(59) \quad P_i \text{ is monic in } y \text{ with } \deg_y P_i = n_1 n_2 \dots n_{i-1} \text{ for } i \geq 2.$$

We establish [\(59\)](#) by induction. In the expansion [\(12\)](#) of P_{i+1} , we have for $0 \leq t \leq d_i - 1$ and whenever $a_{s,t} \neq 0$, that $0 \leq j_k(s,t) < n_k$ for $1 \leq k \leq i - 1$. Thus

$$\begin{aligned} \deg_y(P_0^{j_0(s,t)} P_1^{j_1(s,t)} \dots P_{i-1}^{j_{i-1}(s,t)} P_i^{t\bar{n}_i}) \\ = j_1(s,t) + j_2(s,t)n_1 + j_3(s,t)n_1n_2 + \dots + j_{i-1}(s,t)n_1n_2 \dots n_{i-2} + t\bar{n}_in_1n_2 \dots n_{i-1} \\ < n_1n_2 \dots n_i. \end{aligned}$$

Thus $\deg_y P_{i+1} = \deg_y P_i^{n_i} = n_1n_2 \dots n_i$. We further see that P_{i+1} is monic in y .

Set $\sigma(0) = 0$ and for $i \geq 1$ let

$$\sigma(i) = \min\{j \mid j > \sigma(i-1) \text{ and } n_j > 1\}.$$

Let $Q_i = P_{\sigma(i)}$. We calculate (as long as we are not in the case $\Omega = \infty$ and $n_i = 1$ for $i \gg 0$) that for $d \in \mathbb{Z}_+$, there exists a unique $r \in \mathbb{Z}_+$ and $j_1, \dots, j_r \in \mathbb{Z}_+$ such that $0 \leq j_k < n_k$ for $1 \leq k \leq r$ and $\deg_y Q_1^{j_1} \dots Q_r^{j_r} = d$. Let M_d be this monomial. Since the monomials M_d are monic in y , we see (continuing to assume that we are not in the case $\Omega = \infty$ and $n_i = 1$ for $i \gg 0$) that if $f \in R = \mathbb{k}[x, y]$ is nonzero with $\deg_y(f) = d$, then there is a unique expression

$$\text{eqZ61} \quad (60) \quad f = \sum_{i=0}^d A_i(x)M_i$$

where $A_i(x) \in \mathbb{k}[x]$, and

$$\text{eqZ62} \quad (61) \quad \nu(f) = \min_i \{\text{ord}(A_i)\nu(Q_0) + \nu(M_i)\}.$$

In the case when $\Omega = \infty$ and $n_i = 1$ for $i \gg 0$ we have a similar statement, but we may need to introduce a new polynomial g of “infinite value” as in Case 3 of Theorem [4.12](#). [Corollary3*](#)

9. THE A_2 SINGULARITY

[norm](#)
[Lemma3](#)

Lemma 9.1. *Let \mathbb{k} be an algebraically closed field, and let $A = \mathbb{k}[x^2, xy, y^2]$, a subring of the polynomial ring $B = \mathbb{k}[x, y]$. Let $\mathfrak{m} = (x^2, xy, y^2)A$ and $\mathfrak{n} = (x, y)B$. Suppose that ν is a rational nondiscrete valuation dominating $B_{\mathfrak{n}}$, such that ν has a generating sequence*

$$P_0 = x, P_1 = y, P_2, \dots$$

in $\mathbb{k}[x, y]$ of the form of the conclusions of Theorem [4.2](#), [Theorem1*](#) such that each P_i is a \mathbb{k} -linear combinations of monomials in x and y of odd degree, and

$$\beta_0 = \nu(x), \beta_1 = \nu(y), \beta_2 = \nu(P_2), \dots$$

is the increasing sequence of minimal generators of $S^{B_{\mathfrak{n}}}(\nu)$, with $\beta_{i+1} > \bar{n}_i\beta_i$ for $i \geq 1$, where

$$\bar{n}_i = [G(\beta_0, \dots, \beta_i) : G(\beta_0, \dots, \beta_{i-1})].$$

Then

$$S^{A_{\mathfrak{m}}}(\nu) = \left\{ \begin{array}{l} a_0\beta_0 + a_1\beta_1 + \dots + a_i\beta_i \mid i \in \mathbb{N}, a_0, \dots, a_i \in \mathbb{N} \\ \text{and } a_0 + a_1 + \dots + a_i \equiv 0 \pmod{2} \end{array} \right\}.$$

Proof. For $f \in \mathbb{k}[x, y]$, let $t = \deg_y(f)$. By [\(60\)](#), [eqZ61](#), f has a unique expansion

$$f = \sum_{i=0}^t \left(\sum_k b_{k,i} x^k \right) P_1^{j_1(i)} \dots P_r^{j_r(i)}$$

where $b_{k,i} \in \mathbb{k}$, $0 \leq j_k(i) < \bar{n}_k$ for $1 \leq k$ and

$$\deg_y P_1^{j_1(i)} \dots P_r^{j_r(i)} = i$$

for all i . Looking first at the $t = \deg_y(f)$ term, and then at lower order terms, we see that $f \in \mathfrak{k}[x^2, xy, y^2]$ if and only if $k + j_1(i) + \cdots + j_r(i) \equiv 0 \pmod{2}$ whenever $b_{k,i} \neq 0$. \square

Example 1

Example 9.2. Suppose that \mathfrak{k} is a field and R is the localization of $\mathfrak{k}[u, v, w]/uv - w^2$ at the maximal ideal (u, v, w) . Then there exists a rational nondiscrete valuation ν dominating R such that if

$$\gamma_0 < \gamma_1 < \cdots$$

is the increasing sequence of minimal generators of the semigroup $S^R(\nu)$, then given $n \in \mathbb{Z}_+$, there exists $i > n$ such that $\gamma_{i+1} = \gamma_i + \frac{\gamma_0}{3}$ and γ_{i+1} is in the group generated by $\gamma_0, \dots, \gamma_i$.

Proof. Let $A = \mathfrak{k}[x, y]$ be a polynomial ring with maximal ideal $\mathfrak{n} = (x, y)\mathfrak{k}[x, y]$. We will use the criterion of Theorem 1.2 to construct a rational nondiscrete valuation ν dominating $T = A_{\mathfrak{n}}$, with a generating sequence

$$P_0 = x, P_1 = y, P_2, \dots$$

such that

$$\beta_0 = \nu(x), \beta_1 = \nu(y), \beta_2 = \nu(P_2), \dots$$

is the increasing set of minimal generators of the semigroup $S^T(\nu)$. We will construct the P_i so that each P_i is a \mathfrak{k} -linear combination of monomials in x and y of odd degree.

We define the first part of a generating sequence by setting

$$P_0 = x, P_1 = y, P_2 = y^3 - x^5,$$

with $\beta_0 = \nu(x) = 1$, $\beta_1 = \nu(y) = \frac{5}{3}$. Set $b_1 = 0$.

We now inductively define

$$P_{i+1} = P_i^3 - x^{a_i} P_{i-1}$$

with a_i an even positive integer, and $\beta_i = \nu(P_i) = b_i + \frac{5}{3^i}$ with $b_i \in \mathbb{Z}_+$, for $i \geq 2$, by requiring that 3 divides $a_i + b_{i-1}$ and

$$b_i = \frac{a_i + b_{i-1}}{3} > 3b_{i-1} + 5$$

for $i \geq 2$. a_i, b_i satisfying these relations can be constructed inductively from b_{i-1} .

Now let $B = \mathfrak{k}[x^2, xy, y^2]$, $\mathfrak{m} = (x^2, xy, y^2)B$, so that $R \cong B_{\mathfrak{m}}$. With this identification, the semigroup $S^R(\nu)$ is generated by $\{\beta_i + \beta_j \mid i, j \in \mathbb{N}\}$. From $3\beta_i < \beta_{i+1}$ for $i \geq 1$ and $\beta_i < \beta_j$ if $j > i$, we conclude that if $i \leq j$, $k \leq l$ and $j < l$, then

eq10

$$(62) \quad \beta_i + \beta_j < \beta_k + \beta_l.$$

Let

$$\gamma_0 = 2 < \gamma_1 < \cdots$$

be the sequence of minimal generators of the semigroup $S^R(\nu)$. By [\(62\)](#), for $n \in \mathbb{Z}_+$, there exists an index l such that $\gamma_l = \beta_0 + \beta_n$. We have $l \geq n$. The semigroup $S(\gamma_0, \gamma_1, \dots, \gamma_l)$ is generated by

$$\{\beta_i + \beta_j \mid i \leq j \text{ and } j \leq n - 1\}$$

and $\beta_0 + \beta_n$.

Suppose $\beta_1 + \beta_n \in S(\gamma_0, \gamma_1, \dots, \gamma_l)$. Since $S(\gamma_0, \dots, \gamma_{l-1}) \subset \frac{1}{3^{n-1}}\mathbb{Z}$, we have an expression $\beta_1 + \beta_n = r\gamma_l + \tau$ with r a positive integer, and $\tau \in S(\gamma_0, \dots, \gamma_{l-1})$. Now

$$\gamma_l = \beta_0 + \beta_n = 1 + b_n + \frac{5}{3^n}$$

and

$$\beta_1 + \beta_n = \frac{5}{3} + b_n + \frac{5}{3^n}$$

implies $\tau \leq \frac{5}{3} - 1 = \frac{2}{3}$, which is impossible, since $\gamma_0 = \beta_0 + \beta_0 = 2$. Thus $\beta_1 + \beta_n \notin S(\gamma_0, \gamma_1, \dots, \gamma_l)$ and $\beta_1 + \beta_n = \gamma_{l+1}$ is the next largest minimal generator of $S^R(\nu)$.

We have that $\gamma_{l+1} = \beta_1 + \beta_n = (\beta_0 + \beta_1) + (\beta_0 + \beta_n) - 2\beta_0$ is in the group generated by $\gamma_0, \dots, \gamma_l$. \square

Example2

Example 9.3. Let notation be as in [Example 9.2](#) and its proof. Then $R \rightarrow T$ is finite, but $S^T(\nu)$ is not a finitely generated $S^R(\nu)$ module.

Proof. Suppose $S^T(\nu)$ is a finitely generated $S^R(\nu)$ module. Then there exists $n > 0$ such that $S^T(\nu)$ is generated by β_0, \dots, β_n and $\{\beta_i + \beta_j \mid i, j \in \mathbb{N}\}$. For $l > n$, β_l cannot be in this semigroup. \square

Example3

Example 9.4. Let $A = \mathfrak{k}[u, v]_{(u, v)}$. Then $A \rightarrow T$ is a finite extension of regular local rings, but $S^T(\nu)$ is not a finitely generated $S^A(\nu)$ module.

Proof. Since A is a subring of R , $S^A(\nu)$ is a subsemigroup of $S^R(\nu)$. Since $S^T(\nu)$ is not a finitely generated $S^R(\nu)$ -module, by [Example 9.3](#), $S^T(\nu)$ cannot be a finitely generated $S^A(\nu)$ -module. \square

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STEVEN DALE CUTKOSKY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: cutkoskys@missouri.edu

PHAM AN VINH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: vapnnc@mizzou.edu