

A GENERALIZATION OF THE ABHYANKAR JUNG THEOREM TO ASSOCIATED GRADED RINGS OF VALUATIONS

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ABSTRACT. Suppose that $R \rightarrow S$ is an extension of local domains and ν^* is a valuation dominating S . We consider the natural extension of associated graded rings along the valuation $\text{gr}_{\nu^*}(R) \rightarrow \text{gr}_{\nu^*}(S)$. We give examples showing that in general, this extension does not share good properties of the extension $R \rightarrow S$, but after enough blow ups above the valuations, good properties of the extension $R \rightarrow S$ are reflected in the extension of associated graded rings. Stable properties of this extension (after blowing up) are much better in characteristic zero than in positive characteristic. Our main result is a generalization of the Abhyankar-Jung theorem which holds for extensions of associated graded rings along the valuation, after enough blowing up.

1. INTRODUCTION

Suppose that $K \rightarrow K^*$ is a finite field extension, ν^* is a valuation of K^* and $\nu = \nu^*|_K$. We will consider local rings R of K dominated by ν (we do not require local rings to be Noetherian) and the local ring S of K^* which is the localization of the integral closure of R in K^* at the center of ν^* (notation is given in Section 2). This extension $R \rightarrow S$ occurs in local uniformization with R regular as a first step in reducing the multiplicity of S .

Let $e = [\Gamma_{\nu^*} : \Gamma_{\nu}]$ be the reduced ramification index of ν^* over ν and $f = [V_{\nu^*}/m_{\nu^*} : V_{\nu}/m_{\nu}]$ be the residue degree of the valuation rings V_{ν^*} of ν^* and V_{ν} of ν and $\delta(\nu^*/\nu)$ be the defect of ν^* over ν (these concepts are reviewed in Section 2).

Bernard Teissier has defined the associated graded ring along a valuation of a ring R which is contained in a valuation ring in his work on resolution of singularities [24], [25]. There are antecedents in Teissier's appendix to Zariski's book on moduli of plane branches [27] and in Spivakovsky's paper [23]. For $\gamma \in \Gamma_{\nu}$, let $\mathcal{P}_{\gamma}(R) = \{f \in R \mid \nu(f) \geq \gamma\}$ and $\mathcal{P}_{\gamma}^+(R) = \{f \in R \mid \nu(f) > \gamma\}$. The associated graded ring of R along ν is defined as

$$\text{gr}_{\nu}(R) = \bigoplus_{\gamma \in \Gamma_{\nu}} \mathcal{P}_{\gamma}(R)/\mathcal{P}_{\gamma}^+(R).$$

This ring is almost always non Noetherian. The quotient field $\text{QF}(\text{gr}_{\nu}(R)) = \text{QF}(\text{gr}_{\nu}(V_{\nu}))$ (Lemma 3.1) and $[\text{QF}(\text{gr}_{\nu^*}(S)) : \text{QF}(\text{gr}_{\nu}(R))] = ef$ (Proposition 3.3).

In this paper, we show that after possibly replacing R with a sequence of blow ups along the valuation, the extension $\text{gr}_{\nu}(R) \rightarrow \text{gr}_{\nu^*}(S)$ shares many good properties with the extension $R \rightarrow S$. As is to be expected, stable properties under blowing up in characteristic zero are much better than in positive characteristic (compare Example 1.3 and Theorem 1.6).

We begin by giving four examples where R is a regular local ring and ν^* has rank 1, illustrating possible behavior of these extensions. Example 3.7 (in Section 3), which is valid over any field of characteristic $\neq 2$, is of a regular local ring R such that S is not regular and $\text{gr}_{\nu^*}(S) = \text{gr}_{\nu}(R)$. This behavior is caused by the fact that the rank of a valuation

dominating a local ring may increase when extending the valuation to the completion of the ring ([23], [17], [7], [14]). If there are immediate extensions of ν and ν^* to valuations dominating \hat{R} and \hat{S} , then we have that $\hat{R} = \hat{S}$ if $\text{gr}_{\nu^*}(S) = \text{gr}_{\nu}(R)$ (Proposition 3.6) so the pathology of Example 3.7 cannot hold in this case.

The next three examples show that the extension $\text{gr}_{\nu}(R) \rightarrow \text{gr}_{\nu^*}(S)$ does not in general reflect good properties of the extension $R \rightarrow S$, so blowing up along the valuation is necessary to obtain the results of this paper.

Example 1.1 is an example over any field k , which shows that it is possible for $\text{gr}_{\nu^*}(S)$ to not be integral over $\text{gr}_{\nu}(R)$.

Example 1.1. *Let k be a field, $K = k(u, y)$ and $K^* = k(x, y)$ be rational function fields over k with an inclusion $K \rightarrow K^*$ induced by the substitution $u = yx + x^2$. K^* is Galois over K with Galois group \mathbb{Z}_2 if $\text{char}(k) \neq 2$. Let $R = k[u, y]_{(u, y)}$ and $S = k[x, y]_{(x, y)}$. We have that S is the integral closure of R in K^* . Define a rank 1 valuation ν^* on K^* which dominates S by prescribing that $\nu^*(x) = 1$ and $\nu^*(y) = \frac{1}{\pi}$. We have semigroups*

$$S^S(\nu^*) = \{\nu^*(f) \mid f \in S \setminus \{0\}\} = \mathbb{Z}_{\geq 0} + \mathbb{Z}_{\geq 0} \frac{1}{\pi}$$

and

$$S^R(\nu) = \{\nu^*(f) \mid f \in R \setminus \{0\}\} = \mathbb{Z}_{\geq 0}(1 + \frac{1}{\pi}) + \mathbb{Z}_{\geq 0} \frac{1}{\pi}.$$

We have that $n\nu^*(x) \notin S^R(\nu)$ for all $n \in \mathbb{Z}_{>0}$, so $\text{gr}_{\nu^*}(S)$ is not integral over $\text{gr}_{\nu}(R)$.

Example 1.2 (from [11]), which is valid over any field, shows that it is possible for both R and S to be regular and the extension of associated graded rings to be integral but not finite.

Example 1.2. *Let k be a field, $K = k(u, v)$ and $K^* = k(x, y)$ be rational function fields over k with an inclusion $K \rightarrow K^*$ induced by the substitutions $u = x^2$ and $v = y^2$. K^* is Galois over K with Abelian Galois group if $\text{char}(k) \neq 2$. Let $R = k[u, v]_{(u, v)}$ and $S = k[x, y]_{(x, y)}$. We have that S is the integral closure of R in K^* . By Example 9.4 [11], there is a valuation ν^* of K^* with value group $\Gamma_{\nu^*} = \frac{1}{3\infty}\mathbb{Z}$ and $V_{\nu^*}/m_{\nu^*} = k$ such that $\text{gr}_{\nu^*}(S)$ is integral over $\text{gr}_{\nu}(R)$ but is not finite.*

Example 1.3 (from [10]) is an example of an immediate extension (so that $\text{gr}_{\nu^*}(S)$ and $\text{gr}_{\nu}(R)$ have the same quotient fields) in characteristic $p > 0$ such that $\text{gr}_{\nu^*}(S)$ is integral (purely inseparable) but not finite over $\text{gr}_{\nu}(R)$. In fact, $\text{gr}_{\nu^*}(S)^p \subset \text{gr}_{\nu}(R)$. In this example the property of being not finite and purely inseparable is stable under sequences of monoidal transforms along ν^* .

Example 1.3. *Let k be an algebraically closed field of characteristic $p > 0$, $K = k(u, v)$ and $K^* = k(x, y)$ be rational function fields over k with an inclusion $K \rightarrow K^*$ induced by the substitutions $u = \frac{x^p}{1-x^{p-1}}$ and $v = y^p - x^{p-1}y$. K^* is a tower of two Artin-Schreier extensions over K . Let $R = k[u, v]_{(u, v)}$ and $S = k[x, y]_{(x, y)}$. We have that S is the integral closure of R in K^* . In the main example of Section 7.11 [10], a valuation ν^* of K^* is constructed which dominates S . The value groups are $\Gamma_{\nu^*} = \frac{1}{p\infty}\mathbb{Z} = \Gamma_{\nu}$ and $V_{\nu^*}/m_{\nu^*} = k$ so that ν^*/ν is an immediate extension. Further, ν^* is the unique extension of $\nu = \nu^*|_K$ to K^* . Thus the defect $\delta(\nu^*/\nu) = 2$. Since the extension is immediate, we have that $\text{QF}(\text{gr}_{\nu^*}(S)) = \text{QF}(\text{gr}_{\nu}(R))$. In [10], the associated graded rings are calculated as*

$$\text{gr}_{\nu}(R) = k[U_0, U_1, \dots] / (U_1^{p^2} - U_0, \{U_j^{p^2} - U_0^{p^{2j-2}} U_{j-1}\}_{2 \leq j})$$

and

$$\mathrm{gr}_{\nu^*}(S) = k[X_0, X_1, \dots]/(X_1^{p^2} - X_0, \{X_j^{p^2} - X_0^{p^{2j-2}} X_{j-1}\}_{2 \leq j})$$

and the inclusion $\mathrm{gr}_{\nu}(R) \rightarrow \mathrm{gr}_{\nu^*}(S)$ of k -algebras is obtained by the substitutions $U_j = X_j^p$ for $0 \leq j$. In particular, $\mathrm{gr}_{\nu^*}(S)$ is integral (even purely inseparable) over $\mathrm{gr}_{\nu}(R)$ but is not finite. We have that $\mathrm{gr}_{\nu^*}(S)^p \subset \mathrm{gr}_{\nu}(R)$.

It is shown in Section 7.11 of [10] that this property (of the extension of associated graded rings being integral but not finite) is stable under sequences of monoidal transforms

$$\begin{array}{ccc} R' & \rightarrow & S' \subset V_{\nu^*} \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

above R and S which are dominated by ν^* .

The conclusions of the following theorem should be compared with Examples 1.1 - 1.3.

Theorem 1.4. *Suppose that K is an algebraic function field over an arbitrary field k and K^* is a finite extension of K . Suppose that ν^* is a rank one k -valuation of K^* . Let ν be the restriction of ν^* to K . Then there exist a finite set of elements $f_1, \dots, f_n \in V_{\nu}$ such that if R is a local ring of K which is dominated by ν which contains f_1, \dots, f_n and S is the localization at the center of ν^* on the integral closure of R in K^* , then $\mathrm{gr}_{\nu^*}(S)$ is integral over $\mathrm{gr}_{\nu}(R)$.*

Theorem 1.4 is proven in Section 3.

The conclusions of the following theorem should be compared with Example 1.3. Example 1.3 is a tower of two immediate Artin-Schreier extensions and ν^* is the unique extension of ν , so the following theorem ensures that $\mathrm{gr}_{\nu^*}(S)^{p^2} \subset \mathrm{gr}_{\nu}(R)$ in Example 1.3.

Theorem 1.5. *Suppose that K and K^* are fields, ν^* is a valuation of K^* with restriction ν to K . Let p be the characteristic of V_{ν^*}/m_{ν^*} if V_{ν^*}/m_{ν^*} has positive characteristic and let $p = 1$ if the characteristic of V_{ν^*}/m_{ν^*} is zero. Suppose that K^* is Galois over K , ν^* has rank 1, ν^* is the unique extension of ν to K^* , $[\Gamma_{\nu^*} : \Gamma_{\nu}]$ is a power of p and V_{ν^*}/m_{ν^*} is purely inseparable over V_{ν}/m_{ν} (so $[K^* : K] = p^n$ for some n). Suppose R is a normal local ring of K which is dominated by ν and S is the integral closure of R in K^* (which is necessarily a local ring). Then $\mathrm{gr}_{\nu^*}(S)^{p^n} \subset \mathrm{gr}_{\nu}(R)$.*

Theorem 1.5 is proven in Section 3.

The classical Abhyankar Jung Theorem [18], [1], tells us that if R is regular of equicharacteristic zero and the discriminant of R in K^* is a simple normal crossing divisor, then S has only Abelian quotient singularities. A few references on this topic and related problems are [16], [21], [22] and [26]. Over an algebraically closed field of characteristic zero, the Abhyankar Jung Theorem can be summarized as follows. Suppose that $K \rightarrow K^*$ is a finite extension of algebraic function fields over an algebraically closed field k of characteristic 0. Let R be a regular algebraic local ring of K , with residue field k , S be a local ring of the integral closure of R in K^* . Suppose there exist regular parameters x_1, \dots, x_n in R such that the discriminant of R in K^* is a monomial in x_1, \dots, x_n . Then there exists an integer $d > 0$ and inclusions of k -algebras

$$\hat{R} = k[[x_1, \dots, x_n]] \subset \hat{S} \subset k[[x_1^{\frac{1}{d}}, \dots, x_n^{\frac{1}{d}}]]$$

such that \hat{S} is the invariant ring of a subgroup of \mathbb{Z}_d^n .

Our most difficult result is the following generalization of the Abhyankar Jung Theorem to associated graded rings of valuations. Theorem 1.6 is proven in Section 4.

Theorem 1.6. *Suppose that K is an algebraic function field over an algebraically closed field k of characteristic zero and K^* is a finite extension of K . Suppose that ν^* is a rank one k -valuation of K^* whose residue field is k . Let ν be the restriction of ν^* to K , and suppose that R' is an algebraic local ring of K which is dominated by ν . Then there exists a sequence of monoidal transforms $R' \rightarrow R$ along ν such that R is regular, and if S is the local ring of the center of ν^* on the integral closure of R in K^* , then*

- 1) $\text{gr}_{\nu^*}(S)$ is a free $\text{gr}_{\nu}(R)$ -module of finite rank $e = [\Gamma_{\nu^*}/\Gamma_{\nu}]$.
- 2) $\Gamma_{\nu^*}/\Gamma_{\nu}$ acts on $\text{gr}_{\nu^*}(S)$ with $\text{gr}_{\nu^*}(S)^{\Gamma_{\nu^*}/\Gamma_{\nu}} \cong \text{gr}_{\nu}(R)$.

The conclusions of 1) of Theorem 1.6 are proven in dimension 2 (and rational rank 1) by Ghezzi, Ha and Kashcheyeva [12] and established by Ghezzi and Kascheyeva [13] for two dimensional defectless extensions of positive characteristic algebraic function fields. The conclusions of 1) of the theorem are established for excellent local domains R' of dimension two under a defectless extension in [8]. Thus it is reasonable to ask if the conclusions of 1) of the theorem are true for defectless extensions (assuming resolution of singularities is true).

The assumption that ν has rank 1 is used in Theorem 1.6 to avoid some problems with extensions of valuations to the completion of an analytically irreducible local ring. The theorem could be true for arbitrary rank valuations.

Examples 1.1 and 1.2 satisfy the classical discriminant condition of the Abhyankar Jung Theorem, but do not satisfy the conclusions of the theorem. However, after some blowing up along the valuation,

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \subset V_{\nu^*} \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

$R_1 \rightarrow S_1$ must satisfy the conclusions of the theorem (if the characteristic of the ground field k is zero).

The characteristic $p > 0$ example Example 1.3 is much worse. Finiteness of the extension of graded rings never holds after blowing up.

The key point in the proof of Theorem 1.6 is to find $R \rightarrow S$ such that $\hat{S} = \bigoplus_{i=1}^e w_i \hat{R}$ where $\{\nu^*(w_i)\}$ is a complete set of representatives of the cosets of Γ_{ν} in Γ_{ν^*} .

In general, \hat{S} is not a free \hat{R} -module if R is regular and $\dim R > 2$, although the discriminant condition of the classical Abhyankar-Jung Theorem ensures this.

The proof of Theorem 1.6 uses the local monomialization theorem [7], an extension in [9] giving nice extensions of the valuations to the completions of the local rings and the classical Abhyankar Jung Theorem [1].

2. NOTATION AND PRELIMINARIES

2.1. Local algebra. All rings will be commutative with identity. A ring S is essentially of finite type over R if S is a local ring of a finitely generated R -algebra. We will denote the maximal ideal of a local ring R by m_R , and the quotient field of a domain R by $\text{QF}(R)$. (We do not require that a local ring be Noetherian). Suppose that $R \subset S$ is an inclusion of local rings. We will say that S dominates R if $m_S \cap R = m_R$. If the local ring R is a domain with $\text{QF}(R) = K$ then we will say that R is a local ring of K . If K is an algebraic function field over a field k (which we do not assume to be algebraically closed) and a local ring R of K is essentially of finite type over k , then we say that R is an algebraic local ring of k .

Suppose that $K \rightarrow K^*$ is a finite field extension, R is a local ring of K and S is a local ring of K^* . We will say that S lies over R if S is a localization of the integral closure T of R in K^* . If R is a local ring, \hat{R} will denote the completion of R by its maximal ideal m_R .

Suppose that R is a regular local ring. A monoidal transform $R \rightarrow R_1$ of R is a local ring of the form $R[\frac{P}{x}]_m$ where P is a regular prime ideal in R (R/P is a regular local ring), $0 \neq x \in P$ and m is a prime ideal of $R[\frac{P}{x}]$ such that $m \cap R = m_R$. R_1 is called a quadratic transform if $P = m_R$.

2.2. Valuation Theory. Suppose that ν is a valuation on a field K . We will denote by V_ν the valuation ring of ν :

$$V_\nu = \{f \in K \mid \nu(f) \geq 0\}.$$

We will denote the value group of ν by Γ_ν . Good treatments of valuation theory are Chapter VI of [28] and [4], which contain references to the original papers. If ν is a valuation ring of an algebraic function field over a field k , we insist that ν vanishes on $k \setminus \{0\}$, and say that ν is a k -valuation.

If ν is a valuation of a field K and R is a local ring of K we will say that ν dominates R if the valuation ring V_ν dominates R . Suppose that ν dominates R . A monoidal transform $R \rightarrow R_1$ is called a monoidal transform along ν if ν dominates R_1 . If A is a subring of V_ν , then the center of ν on A is the prime ideal $m_\nu \cap A$.

Suppose that K^*/K is a finite extension, ν^* is a valuation of K^* and ν is the restriction of ν^* to K . We define

$$e = e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu|$$

and

$$f = f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu].$$

ν^*/ν is an immediate extension if $e = f = 1$. The defect $\delta(\nu^*/\nu)$ is defined and its basic properties are developed in Section 11, Chapter VI [28], [19], [20] and Section 7.1 of [10]. If ν^* is the unique extension of ν to K^* and $p > 0$ is the residue characteristic of V_{ν^*} , then

$$[K^* : K] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

2.3. Galois theory of local rings. Suppose that K^*/K is a finite Galois extension, R is a normal local ring of K and S is a normal local ring of K^* which lies over R . We will denote the Galois group of K^*/K by $G(K^*/K)$. The splitting group $G^s(S/R)$, splitting field $K^s(S/R) = (K^*)^{G^s(S/R)}$ and inertia group $G^i(S/R)$, inertia field $K^i(S/R) = (K^*)^{G^i(S/R)}$ are defined and their basic properties developed in Section 7 of [4]. We have that

$$G^s(S/R) = \{\sigma \in G(K^*/K) \mid \sigma(s) = s\}$$

and

$$G^i(S/R) = \{\sigma \in G^s(S/R) \mid g(u) \equiv u \pmod{m_S} \text{ for all } u \in S\}.$$

2.4. Galois theory of valuations. The Galois theory of valuation rings is developed in Section 12 of Chapter VI of [28] and in Section 7 of [4]. Some of the basic results we need are surveyed in Section 7.1 [10]. If we take $S = V_{\nu^*}$ and $R = V_\nu$ where ν^* is a valuation of K^* and ν is the restriction of ν^* to K , then we obtain the splitting group $G^s(\nu^*/\nu)$, the splitting field $K^s(\nu^*/\nu)$ and the inertia group $G^i(\nu^*/\nu)$, inertia field $K^i(\nu^*/\nu) = (K^*)^{G^i(\nu^*/\nu)}$. In Section 12 of Chapter VI of [28], $G^s(\nu^*/\nu)$ is written as G_Z and called the decomposition group. $G^i(\nu^*/\nu)$ is written as G_T . The ramification group

G_V of ν^*/ν is defined in Section 12 of Chapter VI of [28] and is surveyed in Section 7.1 [10]. We will denote this group by $G^r(\nu^*/\nu)$. We have that

$$G_V = G^r(\nu^*/\nu) = \left\{ g \in G^i(\nu^*/\nu) \mid \nu^* \left(\frac{g(x)}{x} - 1 \right) > 0 \text{ for all } x \in K^* \setminus \{0\} \right\}.$$

2.5. Semigroups and associated graded rings of a local ring with respect to a valuation. Suppose that ν is a valuation of field K which dominates a local ring R of K . We will denote the semigroup of values of ν on S by

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}.$$

Suppose that $\gamma \in \Gamma_\nu$. We define ideals in R

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\}$$

and

$$\mathcal{P}_\gamma^+(R) = \{f \in R \mid \nu(f) > \gamma\}$$

and define (as in [24]) the associated graded ring of R with respect to ν by

$$\text{gr}_\nu(R) := \bigoplus_{\gamma \in \Gamma_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R).$$

3. RAMIFICATION OF ASSOCIATED GRADED RINGS OF VALUATIONS

Lemma 3.1. *Suppose that R is a (not necessarily Noetherian) local ring with quotient field K which is dominated by a valuation ν of K . Then*

$$\text{QF}(\text{gr}_\nu(R)) = \text{QF}(\text{gr}_\nu(V_\nu)).$$

Proof. For $\gamma \in \Gamma_\nu$, $\mathcal{P}_\gamma(V_\nu) \cap R = \mathcal{P}_\gamma(R)$ and $\mathcal{P}_\gamma^+(V_\nu) \cap R = \mathcal{P}_\gamma^+(R)$, so we have a natural graded inclusion $\text{gr}_\nu(R) \subset \text{gr}_\nu(V_\nu)$.

Suppose $f \in \text{gr}_\nu(V_\nu)$ is homogeneous, so $f = \text{in}_\nu(a)$ for some $a \in V_\nu$. Write $a = \frac{b}{c}$ with $b, c \in R$. Since $ca = b$, we have that $\text{in}_\nu(c)\text{in}_\nu(a) = \text{in}_\nu(b)$, so

$$f = \frac{\text{in}_\nu(b)}{\text{in}_\nu(c)} \in \text{QF}(\text{gr}_\nu(R)).$$

Since every element of $\text{gr}_\nu(V_\nu)$ is a finite sum of homogeneous elements, we have that $\text{gr}_\nu(V_\nu) \subset \text{QF}(\text{gr}_\nu(R))$, and the conclusion of the lemma follows. \square

Lemma 3.2. *Suppose that $K \rightarrow K^*$ is a field extension, ν^* is a valuation of K^* and $\nu|_K$ is its restriction to K . Then the extension is immediate if and only if*

$$\text{gr}_\nu(V_\nu) = \text{gr}_{\nu^*}(V_{\nu^*}).$$

Proof. Suppose that the extension is immediate. We have that $\mathcal{P}_\gamma(V_{\nu^*}) \cap K = \mathcal{P}_\gamma(V_\nu)$ and $\mathcal{P}_\gamma^+(V_{\nu^*}) \cap K = \mathcal{P}_\gamma^+(V_\nu)$ so we have a natural graded inclusion $\text{gr}_\nu(V_\nu) \subset \text{gr}_{\nu^*}(V_{\nu^*})$. Since K^*/K is immediate, we have $\Gamma_{\nu^*} = \Gamma_\nu$ and

$$\mathcal{P}_0(V_{\nu^*})/\mathcal{P}_0^+(V_{\nu^*}) = V_{\nu^*}/m_{\nu^*} = V_\nu/m_\nu = \mathcal{P}_0(V_\nu)/\mathcal{P}_0^+(V_\nu).$$

Suppose $f \in \text{gr}_{\nu^*}(V_{\nu^*})$. We will show that $f \in \text{gr}_\nu(V_\nu)$. We may assume that f is homogeneous, so $f = \text{in}_{\nu^*}(a)$ for some $a \in V_{\nu^*}$. There exists $b \in V_\nu$ such that $\nu(b) = \nu^*(a)$. Thus

$$\text{in}_{\nu^*}\left(\frac{a}{b}\right) \in V_{\nu^*}/m_{\nu^*} = V_\nu/m_\nu$$

so there exists $c \in V_\nu$ such that

$$\text{in}_\nu(c) = \text{in}_{\nu^*}\left(\frac{a}{b}\right) = \text{in}_{\nu^*}(a) - \text{in}_\nu(b).$$

Thus

$$f = \text{in}_{\nu^*}(a) = \text{in}_\nu(b) + \text{in}_\nu(c) \in \text{gr}_\nu(V_\nu).$$

□

Proposition 3.3. *Suppose that $K \rightarrow K^*$ is a finite field extension, ν^* is a valuation of K^* and $\nu|_K$ is its restriction to K . Then*

$$[\text{QF}(\text{gr}_{\nu^*}(V_{\nu^*})) : \text{QF}(\text{gr}_\nu(V_\nu))] = ef.$$

Proof. Let $w_i \in V_{\nu^*}$ be such that the cosets of the $\nu^*(w_i)$ are a complete set of representatives of $\Gamma_{\nu^*}/\Gamma_\nu$, and let $c_j \in V_{\nu^*}$ be such that $\nu^*(c_j) = 0$ for all j , and the classes of the c_j in V_{ν^*}/m_{ν^*} are a basis of V_{ν^*}/m_{ν^*} over V_ν/m_ν . Let $\bar{w}_i = \text{in}_{\nu^*}(w_i)$ and $\bar{c}_j = \text{in}_{\nu^*}(c_j)$.

Suppose $f \in \text{gr}_{\nu^*}(V_{\nu^*})$ is homogeneous and nonzero. Then $f = \text{in}_{\nu^*}(a)$ for some $a \in V_{\nu^*}$. We have that $\nu^*(a) - \nu^*(w_i) \in \Gamma_\nu$ for some w_i . So

$$\nu^*(a) - \nu^*(w_i) = \nu(c) - \nu(d)$$

for some $c, d \in V_\nu$. We have that

$$\nu^*\left(\frac{ad}{w_i c}\right) = 0.$$

Thus there exists $h_j \in V_\nu$, with $\nu(h_j) = 0$, such that

$$\sum_j \bar{c}_j \text{in}_\nu(h_j) = \text{in}_{\nu^*}\left(\frac{ad}{w_i c}\right),$$

so

$$\text{in}_{\nu^*}(a) \text{in}_\nu(d) = \sum_j \text{in}_\nu(c) \bar{w}_i \bar{c}_j \text{in}_\nu(h_j)$$

and

$$f = \text{in}_{\nu^*}(a) \in \sum_j \text{QF}(\text{gr}_\nu(V_\nu)) \bar{c}_j \bar{w}_i,$$

and thus

$$\sum_{i,j} \text{QF}(\text{gr}_\nu(V_\nu)) \bar{c}_j \bar{w}_i = \text{QF}(\text{gr}_{\nu^*}(V_{\nu^*})).$$

Suppose there exist $h_{ij} \in \text{QF}(\text{gr}_\nu(V_\nu))$ not all zero such that $\sum_{i,j} h_{ij} \bar{c}_j \bar{w}_i = 0$. Multiplying by an appropriate nonzero element of $\text{gr}_\nu(V_\nu)$, we may assume that $h_{ij} \in \text{gr}_\nu(V_\nu)$ for all i, j . Writing $h_{ij} = \sum_\gamma h_{ij,\gamma}$ where $h_{ij,\gamma}$ is homogeneous of degree γ in $\text{gr}_\nu(V_\nu)$, we see that $\sum_j h_{ij} \bar{c}_j = 0$ for all i . Since $\bar{c}_j \in \mathcal{P}_0(V_{\nu^*})/\mathcal{P}_0^+(V_{\nu^*})$, we have that $\sum_j h_{ij,\gamma} \bar{c}_j = 0$ for all $\gamma \in \Gamma_\nu$. Suppose some $h_{ij,\gamma}$ is not zero, say $h_{ij_0,\gamma}$. Then

$$(1) \quad \sum_j \left(\frac{h_{ij,\gamma}}{h_{ij_0,\gamma}} \right) \bar{c}_j = 0.$$

There exists $a_j \in V_\nu$ such that $\nu(a_j) = \gamma$ and $\text{in}_\nu(a_j) = h_{ij_0,\gamma}$. We have that $\nu\left(\frac{a_j}{a_{j_0}}\right) = 0$, and

$$\text{in}_\nu\left(\frac{a_j}{a_{j_0}}\right) = \frac{h_{ij,\gamma}}{h_{ij_0,\gamma}}.$$

Thus (1) is a relation of linear dependence of the \bar{c}_j over V_ν/m_ν , which is impossible. We have that $\{\bar{c}_j\bar{w}_i\}$ are linearly independent over $\text{QF}(\text{gr}_\nu(V_\nu))$, and are thus a basis of $\text{QF}(\text{gr}_{\nu^*}(V_{\nu^*}))$ over $\text{QF}(\text{gr}_\nu(V_\nu))$. \square

Theorem 3.4. *Suppose that $K \rightarrow K^*$ is a finite field extension, ν^* is a valuation of K^* , and $\nu = \nu^*|_K$. Suppose that $V_\nu/m_\nu = V_{\nu^*}/m_{\nu^*}$ is algebraically closed of characteristic zero. Then $\text{QF}(\text{gr}_{\nu^*}(V_{\nu^*}))$ is a finite Galois extension of $\text{QF}(\text{gr}_\nu(V_\nu))$ with Galois group $\Gamma_{\nu^*}/\Gamma_\nu$.*

Proof. We will first prove the theorem with the assumptions that K^* is Galois over K and the splitting group $G^s(\nu^*/\nu)$ of ν^* over ν is equal to the Galois group $G(K^*/K)$ of K^* over K . With our assumption that $V_\nu/m_\nu = V_{\nu^*}/m_{\nu^*}$, we have that $G^s(\nu^*/\nu)$ is equal to the inertia group $G^i(\nu^*/\nu)$ of ν^*/ν (Theorem 21, page 69 [28]).

Choose $w_i \in V_{\nu^*}$ which are a complete set of representatives of $\Gamma_{\nu^*}/\Gamma_\nu$. Let $K_1 := \text{QF}(\text{gr}_\nu(V_\nu))$ and $K_2 := \text{QF}(\text{gr}_{\nu^*}(V_{\nu^*}))$. By the Corollary to Theorem 25, page 78 [28], the w_i are a basis of K^* as a K -vector space and by the proof of Proposition 3.3, letting $\bar{w}_i = \text{in}_{\nu^*}(w_i)$, the \bar{w}_i are a basis of K_2 as a K_1 -vector space.

Suppose that $\sigma \in G(K^*/K)$. Since $G^s(\nu^*/\nu) = G(K^*/K)$, we have that ν^* is the unique extension of ν to K^* , so the valuations $\nu^*\sigma$ and ν^* are equal (formula (3) on page 68 [28]). Thus $\nu^*(\sigma(a)) = \nu^*(a)$ for all $a \in K^*$ and so

$$\sigma(w_i) = c_i w_i$$

with $c_i \in V_{\nu^*}$ such that $\nu^*(c_i) = 0$.

Since the w_i are a basis of K^* over K and they are a complete set of representatives of $\Gamma_{\nu^*}/\Gamma_\nu$, there exist uniquely determined index $\lambda(i, j)$, $g_{ij} \in K$ and $h_{ij} \in \sum_{k \neq \lambda(i, j)} K w_k$ such that

$$w_i w_j = g_{ij} w_{\lambda(i, j)} + h_{ij}$$

with

$$\nu^*(g_{ij} w_{\lambda(i, j)}) = \nu^*(w_i w_j)$$

and $\nu^*(h_{ij}) > \nu^*(w_i w_j)$. We compute

$$(2) \quad \sigma(w_i w_j) = \sigma(g_{ij} w_{\lambda(i, j)} + h_{ij}) = g_{ij} \sigma(w_{\lambda(i, j)}) + \sigma(h_{ij}) = g_{ij} c_{\lambda(i, j)} w_{\lambda(i, j)} + \sigma(h_{ij})$$

and

$$(3) \quad \sigma(w_i w_j) = \sigma(w_i) \sigma(w_j) = c_i c_j w_i w_j = c_i c_j (g_{ij} w_{\lambda(i, j)} + h_{ij}).$$

Comparing (2) and (3), and since

$$\nu^*(\sigma(h_{ij})) = \nu^*(h_{ij}) > \nu^*(g_{ij} w_{\lambda(i, j)}),$$

we obtain

$$c_i c_j \equiv c_{\lambda(i, j)} \pmod{m_{\nu^*}}.$$

Define

$$\bar{\sigma}(\bar{w}_i) = \bar{c}_i \bar{w}_i.$$

where $\bar{c}_i = \text{in}_{\nu^*}(c_i) \in V_{\nu^*}/m_{\nu^*}$. $\bar{\sigma}$ extends naturally to a K_1 -vector space isomorphism of K_2 . We will show that $\bar{\sigma}$ preserves the algebra structure on K_2 , so that $\bar{\sigma}$ is actually a K_1 -algebra isomorphism of K_2 . To check this, we observe that there exist $a_{ij}, b_{ij} \in V_\nu$ such that $g_{ij} = \frac{a_{ij}}{b_{ij}}$ and so

$$\bar{w}_i \bar{w}_j = \frac{\text{in}_\nu(a_{ij})}{\text{in}_\nu(b_{ij})} \bar{w}_{\lambda(i, j)}.$$

Thus

$$\bar{\sigma}(\bar{w}_i \bar{w}_j) = \frac{\text{in}_\nu(a_{ij})}{\text{in}_\nu(b_{ij})} \sigma(\bar{w}_{\lambda(i,j)}) = \frac{\text{in}_\nu(a_{ij})}{\text{in}_\nu(b_{ij})} \bar{c}_{\lambda(i,j)} \bar{w}_{\lambda(i,j)} = \bar{c}_i \bar{w}_i \bar{c}_j \bar{w}_j = \bar{\sigma}(\bar{w}_i) \bar{\sigma}(\bar{w}_j).$$

Now from the facts that $G(K^*/K) = G^s(\nu^*/\nu) = G^i(\nu^*/\nu)$, and $G^r(\nu^*/\nu) = \{\text{id}\}$ (by Theorem 24, page 77 [28]), and that if $\bar{c}_i = 1$ for all i then $\bar{\sigma} \in G^r(\nu^*/\nu)$ by equation (17), page 75 [28], we have an injection of groups

$$G(K^*/K) \rightarrow \text{Aut}(K_2/K_1).$$

From the natural isomorphism

$$G(K^*/K) = G^i(\nu^*/\nu) = G^s(\nu^*/\nu) \cong \Gamma_{\nu^*}/\Gamma_\nu$$

of the Corollary of page 77 [28], and the fact that $[K_2 : K_1] = ef = e = |\Gamma_{\nu^*}/\Gamma_\nu|$ by Proposition 3.3, we have that K_2 is Galois over K_1 with Galois group $\Gamma_{\nu^*}/\Gamma_\nu$.

We now establish the theorem in the general case of $K \rightarrow K^*$. Let K' be a Galois closure of K . Let ν' be an extension of ν^* to K' . Let

$$K^s = K^{G^s(\nu'/\nu)}$$

where $G^s(\nu'/\nu) \leq G(K'/K)$ is the splitting group of ν' over ν , and let

$$(K^s)^* = K^{G^s(\nu'/\nu^*)}$$

where $G^s(\nu'/\nu^*) \leq G(K'/K^*)$ is the splitting group of ν' over ν^* . The Galois group

$$G(K'/K^s) \cong G^s(\nu'/\nu) \cong \Gamma_{\nu'}/\Gamma_\nu$$

is Abelian ($G^i(\nu'/\nu) = G^s(\nu'/\nu)$ since V_ν/m_ν is algebraically closed) so $(K^s)^*$ is Galois over K^s , with Galois group $\Gamma_{\nu^*}/\Gamma_\nu$.

Let $\bar{\nu}^* = \nu'|_{(K^s)^*}$ and $\bar{\nu} = \nu'|_{K^s}$. We have that $G^s(\nu^*/\bar{\nu}) = G((K^s)^*/K^s)$ by Proposition 1.46 [4], since $V_{\nu'}$ is the only local ring of K' lying over $V_{\bar{\nu}}$. Thus the analysis of the first case holds for $(K^s)^*/K^s$, and we have that $\text{QF}(\text{gr}_{\bar{\nu}^*}(V_{\bar{\nu}^*}))$ is Galois over $\text{QF}(\text{gr}_{\bar{\nu}}(V_{\bar{\nu}}))$ with Galois group $\Gamma_{\nu^*}/\Gamma_\nu$. By Lemma 3.2 or Proposition 3.3, $\text{QF}(\text{gr}_\nu(V_\nu)) = \text{QF}(\text{gr}_{\bar{\nu}}(V_{\bar{\nu}}))$ and $\text{QF}(\text{gr}_{\nu^*}(V_{\nu^*})) = \text{QF}(\text{gr}_{\bar{\nu}^*}(V_{\bar{\nu}^*}))$ so the theorem holds. \square

Suppose that R is a (Noetherian) local ring which is dominated by a rank 1 valuation ν . For $f \in \hat{R}$, we write $\nu(f) = \infty$ if there exists a Cauchy sequence $\{f_n\}$ in R which converges to f , and such that $\lim_{n \rightarrow \infty} \nu(f_n) = \infty$. We define (Definition 5.2 [9]) a prime ideal

$$P(\hat{R})_\infty = \{f \in \hat{R} \mid \nu(f) = \infty\}$$

in \hat{R} . We then have a canonical immediate extension $\hat{\nu}$ of ν to $\text{QF}(\hat{R}/P(\hat{R})_\infty)$ which dominates $\hat{R}/P(\hat{R})_\infty$.

The following Lemma appears in [14].

Lemma 3.5. *Suppose that ν is a rank 1 valuation of a field K and R is a (Noetherian) local ring which is dominated by ν . Let $\hat{\nu}$ be the canonical extension of ν to $\text{QF}(\hat{R}/P(\hat{R})_\infty)$ which dominates $\hat{R}/P(\hat{R})_\infty$. Then the inclusion $R \rightarrow \hat{R}/P(\hat{R})_\infty$ induces an isomorphism*

$$\text{gr}_\nu(R) \cong \text{gr}_{\hat{\nu}}(\hat{R}/P(\hat{R})_\infty).$$

Proof. Suppose $h \in \hat{R} \setminus P(\hat{R})_\infty$. There exists a Cauchy sequence $\{f_n\}$ in R such that $\lim_{n \rightarrow \infty} f_n = h$. Let m be a positive integer such that $m\nu(m_R) > \hat{\nu}(h)$ (where $\nu(m_R) = \min\{\nu(g) \mid g \in m_R\}$). There exists n_0 such that $f_n - h \in m_R^m \hat{R}$ for $n \geq n_0$. Then $\text{in}_\nu(f_n) = \text{in}_\nu(h)$ for $n \geq n_0$. \square

Proposition 3.6. *Suppose that $K \rightarrow K^*$ is a finite field extension, ν^* is a rank 1 valuation of K^* and $\nu = \nu^*|_K$. Suppose that R and S are respective local rings of K and K^* such that ν^* dominates S and S lies over R . Further suppose that $\mathcal{P}(\hat{S})_\infty = (0)$ (so that $\mathcal{P}(\hat{R})_\infty = (0)$) and $\text{gr}_{\nu^*}(S) = \text{gr}_\nu(R)$. Then $\hat{S} = \hat{R}$.*

Proof. For $n \in \mathbb{Z}_{\geq 0}$, let

$$I_n = \{f \in R \mid \nu(f) \geq n\} \text{ and } J_n = \{g \in S \mid \nu^*(g) \geq n\}.$$

Since $\mathcal{P}(\hat{R})_\infty = (0)$ and $\mathcal{P}(\hat{S})_\infty = (0)$, $\cap_{n \geq 1} (J_n \hat{S}) = (0)$ and $\cap_{n \geq 1} (I_n \hat{R}) = (0)$. By Chevalley's theorem (Theorem 13, page 270 [28]), the topology on \hat{S} induced by $\{J_n \hat{S}\}$ is equivalent to the $m_S \hat{S}$ -adic topology and the topology on \hat{R} induced by $\{I_n \hat{R}\}$ is equivalent to the $m_R \hat{R}$ -adic topology. Since $\text{gr}_{\nu^*}(S) = \text{gr}_\nu(R)$, $\hat{S}/J_n \hat{S} \cong S/J_n \cong R/I_n \cong \hat{R}/I_n \hat{R}$ for all n , so $\hat{S} \cong \hat{R}$. \square

The assumption that $\mathcal{P}(\hat{S})_\infty = (0)$ is necessary in Proposition 3.6, as is shown by the following example.

Example 3.7. *There exists a finite extension $K \rightarrow K^*$ of algebraic function fields over any ground field k of characteristic $\neq 2$, a discrete rank 1 k -valuation ν^* of K^* with restriction ν to K , and an algebraic regular local ring R of K which is dominated by ν such that the local ring S obtained by localizing the integral closure of R in K^* at the center of ν^* is not a regular local ring but the natural inclusion $\text{gr}_\nu(R) \rightarrow \text{gr}_{\nu^*}(S)$ is an isomorphism.*

Proof. Let $A = k[x, y]$ and $p(x) = x + a_2x^2 + a_3x^3 + \dots$ be a transcendental series in $k[[x]]$. Let $K = \text{QF}(A) = k(x, y)$. Define a valuation ν on K which dominates $R = A_{(x, y)}$ by $\nu(f) = n$ if $f \in A$ and $\text{ord } f(x, p(x)) = n$ (computed in $k[[x]] \cong \hat{R}/(y - p(x))$). We have that $\Gamma_\nu \cong \mathbb{Z}$ and $V_\nu/m_\nu \cong k$. (The valuation ν is the restriction of the order valuation of $k[[x]]$ to R by the natural inclusion $R \rightarrow \hat{R}/(y - p(x)) \cong k[[x]]$. We have that $P(\hat{R})_\infty = (y - p(x))$.)

Let $B = k[x, y, z]/(z^2 - xy)$. B is the integral closure of A in the quotient field K^* of B . Since $\text{char } k \neq 2$, in $k[[x, z]]$, we have a factorization

$$z^2 - xp(x) = z^2 - x^2 - a_2x^3 - \dots = (z - \varphi(x))(z - \psi(x))$$

where $\varphi(x) = x + b_2x^2 + \dots$ and $\psi(x) = -x + c_2x^2 + \dots$ for some $b_i, c_i \in k$. Define a valuation ν^* on K^* by $\nu^*(f) = n$ if $f \in B$ and $\text{ord}_x f(x, p(x), \varphi(x)) = n$. We have that ν^* is the restriction of the order valuation of $k[[x]]$ to $S = B_{(x, y, z)}$ by the natural inclusion $S \rightarrow \hat{S}/(y - p(x), z - \varphi(x)) \cong k[[x]]$. We have that $\Gamma_{\nu^*} \cong \mathbb{Z}$, $V_{\nu^*}/m_{\nu^*} = k$ and $P(\hat{S})_\infty = (y - p(x), z - \varphi(x))$. We have that $\nu = \nu^*|_K$, and the natural inclusion of R into S induces an isomorphism $\hat{R}/P(\hat{R})_\infty \cong \hat{S}/P(\hat{S})_\infty \cong k[[x]]$. Thus the natural inclusion $\text{gr}_\nu(R) \cong \text{gr}_{\nu^*}(S)$ is an isomorphism by Lemma 3.5. In fact, we have that both graded rings are isomorphic to $\text{gr}_{xk[[x]]} k[[x]] \cong k[x]$, a \mathbb{Z} -graded ring with $\deg(x) = 1$. \square

The following lemma isolated an argument in the proof of Theorem 4.9 [4].

Lemma 3.8. *Suppose that $K \rightarrow K^*$ is a finite Galois extension, ν^* is a valuation of K^* and $\nu = \nu^*|_K$. Then there exists a finite set of elements $a_1, \dots, a_m \in V_\nu$ such that if R is a local ring of K which contains a_1, \dots, a_m and S is the local ring of the integral closure of R in K^* obtained by localizing at the center of ν^* , then*

$$G^s(S/R) = G^s(\nu^*/\nu).$$

Proof. Let $\nu_1^* = \nu^*, \nu_2^*, \dots, \nu_n^*$ be the distinct extensions of ν to K^* , and let $U = \bigcap_{i=1}^n V_{\nu_i^*}$ be the integral closure of V_ν in K^* . Let $H_i = m_{\nu_i^*} \cap U$ for $1 \leq i \leq n$ be the maximal ideals of U . There exists $u \in U$ such that $u \in H_1$ and $u \notin H_i$ for $2 \leq i \leq n$ (Lemma 1.3 [4]). Let $u^m + a_1 u^{m-1} + \dots + a_m = 0$ with $a_i \in V_\nu$ be an equation of integral dependence of u over V_ν .

Suppose that R is a local ring of K which is dominated by ν and contains a_1, \dots, a_m . Let T be the integral closure of R in K^* , and let $P_1 = T \cap m_{\nu^*} = T \cap H_1$, so that $S = T_{P_1}$. Then $u \in S \cap H_1 = P_1$ and $u \notin S \cap H_i$ for $2 \leq i \leq n$. Suppose $\sigma \in G^s(S/R)$. Then $\sigma(P_1) = P_1$ so $\sigma(H_1) = H_1$. Thus $G^s(S/R) \subset G^s(\nu^*/\nu)$. By Proposition 1.50 [4], we must have that $G^s(\nu^*/\nu) \subset G^s(S/R)$. Thus $G^s(\nu^*/\nu) = G^s(S/R)$. \square

Theorem 3.9. *Suppose that K is an algebraic function field over an arbitrary field k and K^* is a finite extension of K . Suppose that ν^* is a rank one k -valuation of K^* . Let ν be the restriction of ν^* to K . Then there exist a finite set of elements $f_1, \dots, f_n \in V_\nu$ such that if R is a normal algebraic local ring of K which is dominated by ν which contains f_1, \dots, f_n and S is the localization at the center of ν^* on the integral closure of R in K^* , then $\text{gr}_{\nu^*}(S)$ is integral over $\text{gr}_\nu(R)$.*

Proof. First suppose that K^* is Galois over K and ν^* is the unique extension of ν to K^* . Then S is the integral closure of R in K^* . Suppose that $z \in S$. Index $G = G(K^*/K)$ as

$$G(K^*/K) = \{\sigma_1, \dots, \sigma_r\}.$$

Let

$$f(x) = \prod_{i=1}^r (x - \sigma_i(z)) \in R[x].$$

We expand

$$f(x) = x^n + S_1 x^{n-1} + \dots + S_n$$

where $n = |G|$ and S_i is the i -th elementary symmetric function in $\{\sigma_1(z), \sigma_2(z), \dots, \sigma_r(z)\}$. Since ν^* is the unique extension of ν to K^* , $\nu^*(\sigma(z)) = \nu^*(z)$ for all $\sigma \in G$, so $\nu^*(S_i) \geq i\nu^*(z)$ for all i and $\nu^*(S_i z^{n-i}) \geq \nu^*(z^n)$. We thus have a relation

$$\text{in}_{\nu^*}(z)^n + \sum_{\nu(S_i) = i\nu^*(z)} \text{in}_\nu(S_i) \text{in}_{\nu^*}(z)^{n-i} = 0$$

in $\mathcal{P}_{n\nu^*(z)}(S)/\mathcal{P}_{n\nu^*(z)}^+(S)$. Thus $\text{in}_{\nu^*}(z)$ is integral over $\text{gr}_\nu(R)$.

Now suppose that K^*/K is separable, but with no other restrictions. Let K' be a Galois closure of K^* over K . Let ν' be an extension of ν to K' and let T be the localization of the integral closure of R in K' at the center of ν' . Let $K^s = (K^*)^{G^s(\nu'/\nu)}$ be the splitting field of ν' over ν . By Lemma 3.8, there exist $f_1, \dots, f_n \in V_\nu$ such that if $f_1, \dots, f_n \in R$, then the splitting group $G^s(T/R) = G^s(\nu'/\nu)$. Let U be the localization at the center of ν' of the integral closure of R in K^s . $\text{gr}_{\nu'}(T)$ is integral over $\text{gr}_{\nu'}(U)$ by the first part of this proof.

Since $\text{gr}_{\nu^*}(S)$ is contained in $\text{gr}_{\nu'}(T)$, we have reduced to establishing that $\text{gr}_{\nu'}(U)$ is integral over $\text{gr}_\nu(R)$. By Theorem 1.47 [4], we have that $R/m_R = U/m_U$ and $m_R U = m_U$,

so by (10.14) and (10.1) [5], $\hat{R} = \hat{S}$. Let $\hat{\nu}$ be an extension of ν to $\text{QF}(\hat{R})$ which dominates \hat{R} . Then Γ_ν is an isolated subgroup of $\Gamma_{\hat{\nu}}$. We have

$$\text{gr}_\nu(R) \cong \bigoplus_{\gamma \in \Gamma_\nu} \mathcal{P}_\gamma(\hat{R})/\mathcal{P}_\gamma^+(\hat{R}) \cong \bigoplus_{\gamma \in \Gamma_{\hat{\nu}}} \mathcal{P}_\gamma(\hat{U})/\mathcal{P}_\gamma^+(\hat{U}) \cong \text{gr}_{\hat{\nu}}(U).$$

The remaining case is of a general finite extension K^* over K . We have a factorization $K \rightarrow \bar{K} \rightarrow K^*$ where \bar{K} is separable over K and K^* is purely inseparable over \bar{K} . Let $\bar{\nu}$ be the restriction of ν^* to \bar{K} , and let A be the localization of the integral closure of R in \bar{K} at the center of $\bar{\nu}$. By the first two parts of the proof, we have reduced to showing that $\text{gr}_{\nu^*}(S)$ is integral over $\text{gr}_{\bar{\nu}}(A)$. S is the integral closure of A in K^* since S is the only local ring of K^* lying over A . Suppose that $z \in S$. Then there exists an exponent z^{p^n} (where p is the characteristic of k) such that $z^{p^n} \in K$. Thus $z^{p^n} \in S \cap K = R$. so we have that $\text{in}_{\nu^*}(z^{p^n}) \in \text{gr}_{\bar{\nu}}(A)$. \square

Theorem 3.10. *Suppose that K and K^* are fields, ν^* is a valuation of K^* with restriction ν to K , K^* is Galois over K , ν^* has rank 1, ν^* is the unique extension of ν to K^* , $[\Gamma_{\nu^*} : \Gamma_\nu]$ is a power of p where p is the residue characteristic of V_{ν^*} and V_{ν^*}/m_{ν^*} is purely inseparable over V_ν/m_ν (so $[K^* : K] = p^n$ for some n). Suppose R is a normal local ring of K which is dominated by ν and S is the integral closure of R in K^* (which is necessarily a local ring). Then $\text{gr}_{\nu^*}(S)^{p^n} \subset \text{gr}_\nu(R)$.*

Proof. With our assumptions, we have that $G(K^*/K) = G^r(\nu^*/\nu)$ (page 68 [28], Theorem 21, page 69 [28] and Theorem 25, page 76 [28]) and $|G^r(\nu^*/\nu)| = p^n$ for some n (Theorem 24, page 77 [28]). Thus $r = [K^* : K] = p^n$ for some positive integer n . By page 68 [28],

$$(4) \quad \nu^* \sigma(x) = \nu^*(x) \text{ for all } \sigma \in G(K^*/K) \text{ and } x \in (K^*)^\times$$

and by page 75 [28],

$$(5) \quad \nu^*(\sigma(x) - x) > \nu^*(x) \text{ for all } \sigma \in G(K^*/K) \text{ and } x \in (K^*)^\times.$$

Index $G(K^*/K)$ as $G(K^*/K) = \{\sigma, \dots, \sigma_r\}$. For $z \in V_{\nu^*}$, let $S_i(z)$ be the i -th symmetric functions in $\sigma_1(z), \dots, \sigma_r(z)$. By (5),

$$S_i(z) = \binom{p^n}{i} z^i + h_i$$

with $\nu^*(h_i) > i\nu^*(z)$. so

$$(6) \quad \nu^*(S_i(z)) > i\nu^*(z) \text{ for } 0 < i < p^n.$$

Now suppose $h \in \text{gr}_{\nu^*}(S)$. Since $\text{gr}_{\nu^*}(S)$ has characteristic $p > 0$, we may assume that h is homogeneous to establish that $h^{p^n} \in \text{gr}_\nu(R)$. Then $h = \text{in}_{\nu^*}(z)$ for some $z \in S$. Let

$$f(x) = \prod_{i=1}^r (x - \sigma_i(z)) = \sum_{i=0}^r S_{r-i}(z) x^i = x^r + a_1 x^{r-1} + \dots + a_r$$

with $a_i = S_i(z) \in K \cap S = R$. We have

$$z^r + a_1 z^{r-1} + \dots + a_r = 0.$$

By (6), $\text{in}_{\nu^*}(z)^r + \text{in}_\nu(a_r) = 0$ in $\mathcal{P}_{r\nu^*(z)}(S)/\mathcal{P}_{r\nu^*(z)}^+(S)$. Thus $h^r \in \text{gr}_\nu(R)$. \square

4. AN ABHYANKAR JUNG THEOREM FOR ASSOCIATED GRADED RINGS OF VALUATIONS

Theorem 4.1. *Suppose that K is an algebraic function field over an algebraically closed field k of characteristic zero and K^* is a finite extension of K . Suppose that ν^* is a rank one k -valuation of K^* whose residue field is k . Let ν be the restriction of ν^* to K , and suppose that R^* is an algebraic local ring of K^* which is dominated by ν^* . Then there exists a sequence of monoidal transforms $R^* \rightarrow R_0$ along ν^* such that R_0 is regular, and if T is the local ring of the center of ν^* on the integral closure of R_0 in K^* , then*

- 1) $\text{gr}_{\nu^*}(S)$ is a free $\text{gr}_{\nu}(R_0)$ -module of finite rank $e = [\Gamma_{\nu^*} : \Gamma_{\nu}]$.
- 2) $\Gamma_{\nu^*}/\Gamma_{\nu}$ acts on $\text{gr}_{\nu^*}(T)$ with $\text{gr}_{\nu^*}(T)^{\Gamma_{\nu^*}/\Gamma_{\nu}} \cong \text{gr}_{\nu}(R_0)$.

With the assumptions of Theorem 4.1, R_0 being a regular local ring with the branch divisor of R_0 in K^* being a SNC divisor is not enough to obtain the conclusions of Theorem 4.1. Such an example is given of a rational rank 1 valuation in two dimensional, characteristic zero, algebraic function fields, in Example 9.4 [11]. In this example,

$$R_0 = k[x^2, y^2]_{(x^2, y^2)} \rightarrow T = k[x, y]_{(x, y)}$$

and $\text{gr}_{\nu^*}(T)$ is not a finitely generated $\text{gr}_{\nu}(R_0)$ -module. However, by Theorem 4.1, we will find a finitely generated extension after some blowing up above R_0 .

In Theorem 7.38 [10], an example is given of rational rank 1 valuations in two dimensional algebraic function fields, over an algebraically closed field k of positive characteristic, where the branch locus of R_0 in K^* is a SNC divisor, R_0 and T are regular local rings, the extension is immediate ($e = f = 1$) and $\text{gr}_{\nu^*}(T)$ is not a finitely generated $\text{gr}_{\nu}(R_0)$ -module. In fact, lack of finite generation continues to hold (in this example) after any amount of blowing up above R_0 .

We summarize some results on ramification. Suppose that R is a normal algebraic local ring with quotient field K and K^* is a finite separable extension of K .

Let $B(K^*/R) = \sqrt{D(K^*/R)}$ where $D(K^*/R)$ is the discriminant ideal of $R \rightarrow K^*$ (page 31, [4]). For \mathfrak{p} a prime ideal of R , $R_{\mathfrak{p}}$ is unramified in K^* if and only if $B(K^*/R) \not\subset \mathfrak{p}$ (Theorems 1.44 and 1.44 A [4]).

Suppose that S is the localization of the integral closure of R in K^* at a maximal ideal. If R has residue characteristic zero, then the ramification index of S over R is

$$r(S : R) = \frac{[\text{QF}(\hat{S}) : \text{QF}(\hat{R})]}{[S/m_S : R/m_R]}$$

(Definition 1 [3] or Definition 1 [1]). Further, $r(S : R) = 1$ if and only if $R \rightarrow S$ is unramified (Lemma 4 [3]).

We now summarize some results on toric rings from [6]. Suppose that M is a finitely generated submonoid (subsemigroup) of \mathbb{Z}^n for some $n \geq 0$. Let

$$\tilde{M}_{\mathbb{Z}^n} = \{v \in \mathbb{Z}^n \mid mv \in M \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

We have that $\tilde{M}_{\mathbb{Z}^n} = (\mathbb{R}_{\geq 0}M) \cap \mathbb{Z}^n$ (Proposition 2.2 [6]).

Proposition 4.2. (Proposition 2.43 [6]) *Suppose $v_1, \dots, v_n \in \mathbb{Z}_{\geq 0}^n$ are linearly independent. Let*

$$\text{par}(v_1, \dots, v_n) = \{q_1 v_1 + \dots + q_n v_n \mid 0 \leq q_i < 1 \text{ for } i = 1, \dots, n\}.$$

Let M be the submonoid of \mathbb{Z}^n generated by v_1, \dots, v_n . Then

- a) $\Lambda = \mathbb{Z}^n \cap \text{par}(v_1, \dots, v_n)$ is a system of generators of the M -module $\tilde{M}_{\mathbb{Z}^n}$ (that is, $\tilde{M}_{\mathbb{Z}^n} = \cup_{x \in \Lambda} x + M$).

- b) $(a + M) \cap (b + M) = \emptyset$ for $a, b \in \Lambda$ with $a \neq b$.
c) $|\Lambda| = [\mathbb{Q}U \cap \mathbb{Z}^n : U]$ where U is the sublattice of \mathbb{Z}^n generated by M .

Lemma 4.3. (Lemma 4.40 [6]) Suppose that k is a ring, M is a finitely generated submonoid of \mathbb{Z}^n and $k[z_1, \dots, z_n]$ is a polynomial ring over k . Then $k[z^v \mid v \in \tilde{M}_{\mathbb{Z}^n}]$ is the integral closure of $k[v \mid v \in M]$ in $k[z_1, \dots, z_n]$.

The prime ideals $P(\hat{R})_\infty$ are defined before Lemma 3.5.

Theorem 4.4. Let k be a field of characteristic zero, K an algebraic function field over k , K^* a finite algebraic extension of K , ν^* a k -valuation of K^* , $\nu = \nu^*|_K$ such that $\text{rank } \nu = 1$ and $\text{rat rank } \nu = s$. Let $n = \text{trdeg}_k K - \text{trdeg}_k V_\nu/m_\nu$, $e = e(\nu^*/\nu)$, $f = f(\nu^*/\nu)$. Let g_1, \dots, g_f be a basis of V_{ν^*}/m_{ν^*} over V_ν/m_ν .

Suppose that S^* is an algebraic local ring of K^* which is dominated by ν^* and R^* is an algebraic local ring of K which is dominated by ν and S^* . Then there exists a commutative diagram

$$\begin{array}{ccc} R_0 & \rightarrow & S \subset V_{\nu^*} \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

where $S^* \rightarrow S$ and $R^* \rightarrow R_0$ are sequences of monoidal transforms along ν^* such that R_0 has regular parameters (x_1, \dots, x_n) and S has regular parameters (y_1, \dots, y_n) such that there are units $\delta_1, \dots, \delta_s \in S$ and a $s \times s$ matrix $A = (a_{ij})$ of natural numbers (elements of $\mathbb{Z}_{\geq 0}$) such that $\text{Det}(A) \neq 0$,

$$(7) \quad \begin{array}{rcl} x_1 & = & \delta_1 y_1^{a_{11}} \cdots y_s^{a_{1s}} \\ & \vdots & \\ x_s & = & \delta_s y_1^{a_{s1}} \cdots y_s^{a_{ss}} \\ x_{s+1} & = & y_{s+1} \\ & \vdots & \\ x_n & = & y_n \end{array}$$

and $\{\nu(x_1), \dots, \nu(x_s)\}$, $\{\nu^*(y_1), \dots, \nu^*(y_s)\}$ are rational bases of $\Gamma_\nu \otimes \mathbb{Q}$ and $\Gamma_{\nu^*} \otimes \mathbb{Q}$ respectively. Furthermore, there exists λ with $\lambda < n - s$ such that

$$(8) \quad P(\hat{R}_0)_\infty = (h_1, \dots, h_{n-\lambda})$$

with

$$(9) \quad h_i \equiv x_{s+i} \pmod{m_{\hat{R}_0}^2}$$

for $1 \leq i \leq n - \lambda$ and

$$(10) \quad P(\hat{S})_\infty = P(\hat{R}_0)_\infty \hat{S}$$

are regular primes ($\hat{R}_0/P(\hat{R}_0)_\infty$ and $\hat{S}/P(\hat{S})_\infty$ are regular local rings).

There exists a (unique) normal algebraic local ring R of K which is dominated by ν such that S is the localization of the integral closure of R in K^* at the center of ν^* . We have that

$$(11) \quad [S/m_S : R/m_R] = f, |\text{Det}(A)| = e, [\text{QF}(\hat{S}) : \text{QF}(\hat{R})] = ef$$

and

$$(12) \quad \{g_1, \dots, g_f\} \text{ is a basis of } S/m_S \text{ over } R/m_R = R_0/m_{R_0}.$$

We further have that

$$(13) \quad \mathbb{Z}^s/A^t\mathbb{Z}^s \cong \Gamma_{\nu^*}/\Gamma_\nu$$

by the map $(b_1, \dots, b_s) \mapsto b_1\nu^*(y_1) + \dots + b_s\nu^*(y_s)$.

Let T be the localization of the integral closure of R_0 in K^* at the center of ν^* . We have that

$$(14) \quad [T/m_T : R_0/m_{R_0}] = f$$

and $\{g_1, \dots, g_f\}$ is a basis of T/m_T over R_0/m_{R_0} . Further,

$$(15) \quad P(\hat{T})_\infty = P(\hat{R}_0)_\infty \hat{T},$$

and

$$(16) \quad [\text{QF}(\hat{T}) : \text{QF}(\hat{R}_0)] = ef.$$

The branch ideal $B(K^*/R_0)$ of $R_0 \rightarrow K^*$ contains $\prod_{i=1}^s x_i$.

Proof. Suppose that

$$(17) \quad \begin{array}{ccccc} R_0 & \rightarrow & S & \subset & V_{\nu^*} \\ \uparrow & & \uparrow & & \\ R^* & \rightarrow & S^* & & \end{array}$$

is a commutative diagram of local homomorphisms, where $S^* \rightarrow S$ and $R^* \rightarrow R$ are sequences of monoidal transforms along ν^* , R_0 is regular with regular parameters x_1, \dots, x_n , S is regular with regular parameters y_1, \dots, y_n such that

$$x_i = \delta_i \prod_{j=1}^n y_j^{a_{ij}}$$

for $1 \leq j \leq n$ where $\delta_i \in S$ are units and $\text{Det}(a_{ij}) \neq 0$. By Theorem 4.2 [10] and its proof, there exists a unique algebraic normal local ring R of K which is the localization of the blow up of an ideal in R_0 such that S is a local ring of the integral closure of R in K^* .

Let A be the ring R' of Theorem 6.1 [10] and B be the ring R_0 of Theorem 4.10 [10]. The rings A and B are algebraic local rings of K . They are constructed so that if R_0 in the commutative diagram (17) dominates A then by Theorem 6.1 [10],

$$[S/m_S : R/m_R] = f, \quad |\text{Det}(a_{ij})| = e, \quad [\text{QF}(\hat{S}) : \text{QF}(\hat{R})] = ef$$

and $\{g_1, \dots, g_f\}$ is a basis of S/m_S over $R/m_R = R_0/m_{R_0}$.

If R_0 in the commutative diagram (17) dominates B then, by Theorem 4.10 [10] and its proof, $\mathbb{Z}^n/A^t\mathbb{Z}^n \cong \Gamma_{\nu^*}/\Gamma_\nu$ by the map $(b_1, \dots, b_n) \mapsto b_1\nu^*(y_1) + \dots + b_n\nu^*(y_n)$.

After performing sequences of monoidal transforms of R^* and S^* along ν^* we may assume that R^* dominates A and B .

By Theorem 5.1 [7] there exists a commutative diagram

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

where R_1 and S_1 are regular local rings which are dominated by ν^* such that S_1 dominates R_1 , the vertical arrows are products of monoidal transforms, V_{ν^*}/m_{ν^*} is algebraic over S_1/m_{S_1} and R_1/m_{R_1} , there is a regular system of parameters $x_1(1), \dots, x_n(1)$ in R_1 and a regular system of parameters $y_1(1), \dots, y_n(1)$ in S_1 , an $s \times s$ matrix $A(1) = (a_{ij}(1))$ of

nonnegative integers with $\text{Det}(A(1)) \neq 0$ and units $\delta_i(1) \in S_1$ for $1 \leq i \leq s$ such that $x_i(1) = \delta_i(1) \prod_{j=1}^s y_j(1)^{a_{ij}(1)}$ for $1 \leq i \leq s$ and $x_i(1) = y_i(1)$ for $s+1 \leq i \leq n$.

Let T_1 be the integral closure of R_1 in K^* . As R_1 is regular, $B(K^*/R_1)$ is a principal ideal by the Purity of the Branch Locus. Let f be such that $B(K^*/R_1) = (f)$. By Theorems 4.7, 4.8 and 4.10 of [7], there exists a commutative diagram

$$\begin{array}{ccc} R_2 & \rightarrow & S_2 \\ \uparrow & & \uparrow \\ R_1 & \rightarrow & S_1 \end{array}$$

and there exists a unit $\varepsilon(2) \in R_2$ and $d_i(2) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq s$ such that

$$f = \varepsilon(2) \prod_{i=1}^s x_i(2)^{d_i(2)}.$$

Further, the exceptional divisor of $R_1 \rightarrow R_2$ is supported on $Z(\prod_{i=1}^s x_i(2))$.

Now applying Theorem 6.5 [9], we next construct a diagram

$$\begin{array}{ccc} R_0 & \rightarrow & S \\ \uparrow & & \uparrow \\ R_2 & \rightarrow & S_2 \end{array}$$

where R_0 and S are regular local rings which are dominated by ν^* such that S dominates R_0 , the vertical arrows are products of monoidal transforms, there is a regular system of parameters x_1, \dots, x_n in R_0 and a regular system of parameters y_1, \dots, y_n in S , an $s \times s$ matrix $A = (a_{ij}(2))$ of nonnegative integers with $\text{Det}(A) \neq 0$ and units $\delta_i \in S$ for $1 \leq i \leq s$ such that $x_i = \delta_i \prod_{j=1}^s y_j^{a_{ij}}$ for $1 \leq i \leq s$ and $x_i = y_i$ for $s+1 \leq i \leq n$. Further, there exists λ with $\lambda < n - s$ such that

$$P(\hat{R}_0)_\infty = (h_1, \dots, h_{n-\lambda})$$

with

$$h_i \equiv x_{s+i} \pmod{m_{\hat{R}_0}^2}$$

for $1 \leq i \leq n - \lambda$ and

$$P(\hat{S})_\infty = P(\hat{R}_0)_\infty \hat{S}$$

are regular primes. The monoidal transforms used in Theorem 6.5 [9] are such that

$$f = \varepsilon \prod_{i=1}^s x_i^{d_i}$$

where ε is a unit in R_0 and the exceptional locus of $R_1 \rightarrow R_0$ is supported in $Z(\prod_{i=1}^s x_i)$. Thus the branch ideal $B(K^*/R_0)$ of $R_0 \rightarrow K^*$ contains $\prod_{i=1}^s x_i$.

The conclusions (11), (12) and (13) hold since R_0 dominates A and B . It remains to verify (14), (15) and (16).

The argument of the proof of Theorem 6.1 [10] shows that $[T/m_T : R_0/m_{R_0}] = f$ and $\{g_1, \dots, g_f\}$ is a basis of T/m_T over R_0/m_{R_0} (since R_0 dominates A).

It remains to show that $[\text{QF}(\hat{T}) : \text{QF}(\hat{R}_0)] = ef$. The ring R' from Theorem 6.1 [10] (our A) was constructed in that proof so that it has the following property (as R_0 dominates A). There is a Galois closure K' of K^*/K with extension ν' of ν^* to K' such that if T' is the localization of the integral closure of T in K' at the center of ν' , then we have equality of splitting groups

$$(18) \quad G^s(T'/T) = G^s(\nu'/\nu^*) \text{ and } G^s(T'/R_0) = G^s(\nu'/\nu).$$

By the Corollary to Theorem 25, page 78 [28], we have that

$$[K' : K^s(\nu'/\nu)] = e(\nu'/\nu)f(\nu'/\nu) \text{ and } [K' : K^s(\nu'/\nu^*)] = e(\nu'/\nu^*)f(\nu'/\nu^*).$$

Let $T_1 = K^s(\nu'/\nu) \cap T'$ and $T_2 = K^s(\nu'/\nu^*) \cap T'$. The extensions $R_0 \rightarrow T_1$ and $T \rightarrow T_2$ are unramified with $T_1/m_{T_1} = R_0/m_{R_0}$ and $T_2/m_{T_2} = T/m_T$ by equation (18) and Theorem 1.47 [4]. Thus $\hat{T}_1 = \hat{R}_0$ and $\hat{T}_2 = \hat{T}$ by (10.14) and (10.1) [5]. We have that

$$[K' : K^s(\nu'/\nu)] = [\text{QF}(\hat{T}_1) : \text{QF}(\hat{R}_0)] \text{ and } [K' : K^s(\nu'/\nu^*)] = [\text{QF}(\hat{T}_2) : \text{QF}(\hat{T})]$$

by II of Proposition 1 (page 498) [3], since there is a unique local ring in K' lying above T_1 and a unique local ring in K' lying above T_2 . Finally, we have that

$$[\text{QF}(\hat{T}) : \text{QF}(\hat{R}_0)] = \frac{[\text{QF}(\hat{T}') : \text{QF}(\hat{R}_0)]}{[\text{QF}(\hat{T}') : \text{QF}(\hat{T})]} = e(\nu^*/\nu)f(\nu^*/\nu)$$

since e and f are multiplicative. \square

We will now give the proof of Theorem 4.1. Taking S^* to be the localization of the integral closure of R^* in K^* at the center of ν^* , we may assume that the assumptions and conclusions of Theorem 4.4 (with the additional assumptions that k is algebraically closed and $V_{\nu^*}/m_{\nu^*} = k$, so that $f = 1$).

We have a canonical immediate extension of ν^* to the the quotient field of $\hat{S}/P(\hat{S})_\infty$ which dominates $\hat{S}/P(\hat{S})_\infty$, and identifying this extension with ν^* , we can define an extension $\hat{\nu}^*$ of ν to the quotient field of \hat{S} which dominates \hat{S} with value group $\mathbb{Z}^{n-\lambda} \times \Gamma_{\nu^*}$ in the lex order. The value $\hat{\nu}^*(f)$ for $0 \neq f \in \hat{S}$ is defined as follows. Let m_1 be such that $h_1^{m_1}$ is the largest power of h_1 which divides f in \hat{S} . Let f_1 be the residue of $\frac{f}{h_1^{m_1}}$ in the regular local ring $\hat{S}/h_1\hat{S}$. Let m_2 be the largest power of (the residue of) h_2 in $\hat{S}/h_1\hat{S}$ such that $h_2^{m_2}$ divides f_1 in $\hat{S}/h_1\hat{S}$. Continue this way constructing $f_1, \dots, f_{n-\lambda}$ and $m_1, \dots, m_{n-\lambda}$, where $f_{n-\lambda}$ is the residue of $f_{n-\lambda+1}$ in $\hat{S}/P(\hat{S})_\infty$. Define

$$(19) \quad \hat{\nu}^*(f) = (m_1, \dots, m_{n-\lambda}, \nu^*(f_{n-\lambda})) \in \mathbb{Z}^{n-\lambda} \times \Gamma_{\nu^*} \text{ in the lex order.}$$

There exists $z_1, \dots, z_s \in \hat{S}$ and units $\varepsilon_i \in \hat{S}$ such that $z_i = \varepsilon_i y_i$ and $x_i = z_1^{a_{i1}} \dots z_s^{a_{is}}$ for $1 \leq i \leq s$. Define $z_i = y_i$ for $s+1 \leq i \leq n$.

The following lemma can also be established by the argument before (25).

Lemma 4.5. *There exist Laurent monomials N_1, \dots, N_ℓ in x_1, \dots, x_n such that $\hat{R} = k[[N_1, \dots, N_\ell]]$.*

Proof. As shown in Theorem 4.2 [10], R is the localization of the integral closure of $R_0[f_1, \dots, f_s]$ in K at the center of ν , where $f_j = \prod_{i=1}^s x_i^{b_{ij}}$, with (b_{ij}) the adjoint matrix of $A = (a_{ij})$. By Lemma 4.3, we can extend f_1, \dots, f_s to a set of Laurent monomials f_1, \dots, f_ℓ ($\ell \geq s$) such that $k[x_1, \dots, x_n, f_1, \dots, f_\ell]$ is the integral closure of $k[x_1, \dots, x_n, f_1, \dots, f_s]$ in $k(x_1, \dots, x_n)$. Let

$$C = k[x_1, \dots, x_n, f_1, \dots, f_\ell]_{(x_1, \dots, x_n, f_1, \dots, f_\ell)} \text{ and } D = R_0[x_1, \dots, x_n, f_1, \dots, f_\ell]_{(x_1, \dots, x_n, f_1, \dots, f_\ell)}.$$

Since $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = \hat{R}_0$, $\hat{C} = \hat{D}$. Since C and D are excellent, by (v) of IV.7.8.3 [15] C normal implies $\hat{C} = \hat{D}$ is normal and so D is normal. Thus $D = R$ by Zariski's main theorem (10.8) [5]. \square

The next part of the proof (through the paragraph before (20)) is as in the proofs of Theorems 2 and 3 of [1].

Let K' be a Galois closure of K^*/K . Then $B(K'/R_0) = B(K^*/R_0)$ (by Proposition 1 [1]), which contains $x_1 \cdots x_s$, by the last statement of Theorem 4.4.

Let ν' be an extension of ν^* to K' and let R' be the center of ν' on the integral closure of R_0 in K' . Let K_s be the splitting field of R' over R_0 and K_s^* be the splitting field of R' over T . Let \overline{K}' be the Galois closure of K_s^* over K_s in K' . Let R_s be the localization of the integral closure of R_0 in K_s at the center of ν' and let T_s be the localization of the integral closure of the integral closure of T in K'_s at the center of ν' . Now $x_1 \cdots x_s \in B(K'/R_s)$ and so $x_1 \cdots x_s \in B(\overline{K}'/R_s)$ (since $B(K'/R_s) \subset B(\overline{K}'/R_s)$).

We have that $\hat{R}_0 \cong \hat{R}_s$ and $\hat{T} \cong \hat{T}_s$ (Theorem 1.47 [4] and (10.14), (10.1) [5]). For $1 \leq j \leq s$, set $n_j = r(A_j, (R_s)_{(x_j)})$, where A_j is any local ring of the integral closure of R_s in the Galois extension \overline{K}'/K_s which dominates $(R_s)_{(x_j)}$.

Let $K_1 = K_s(x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}})$ and $K_2 = \overline{K}'(x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}})$. Let ν'' be an extension of ν' to K_2 . Let R_1 be the localization of the integral closure of R_s in K_1 at the center of ν'' . By Lemma 6 [1], R_1 is unramified in K_2 in codimension 1. Now R_1 is a regular local ring with $m_{\hat{R}_1} = (x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}}, x_{s+1}, \dots, x_n)$ (by Proposition 1 [3] and Lemma 5 [1]). Thus $R_1 \rightarrow K_2$ is unramified by the purity of the branch locus (Theorem 1 [1]). We have that $\hat{R}_1 = k[[x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}}, x_{s+1}, \dots, x_n]]$. Let R_2 be the localization of the integral closure of R_1 in K_2 at the center of ν'' . Then $\hat{R}_1 \cong \hat{R}_2$ since R_1 is unramified in K_2 . Let

$$F_0 = \text{QF}(\hat{R}_0) = k((x_1, \dots, x_n)) \text{ and } E_0 = \text{QF}(\hat{T}).$$

Let

$$L_0 = \text{QF}(\hat{R}_2) = k((x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}}, x_{s+1}, \dots, x_n)).$$

L_0 is a Galois extension of F_0 , with abelian Galois group $G(L_0/F_0) \cong \bigoplus_{i=1}^s \mathbb{Z}_{n_i}$, acting diagonally on $x_i^{\frac{1}{n_i}}$ by multiplication by n_i -th roots of unity.

Thus E_0 is Galois over F_0 and E_0 has a basis over F_0 of the form

$$(20) \quad \{M_j = \prod_{i=1}^s x_i^{\frac{b_{ij}}{n_j}}\} \text{ for } 1 \leq j \leq e$$

with all $b_{ij} \in \mathbb{Z}_{\geq 0}$. We have that $[E_0 : F_0] = e$ by (16). Recall that R is defined before equation (11) in Theorem 4.4. Let

$$F_1 = \text{QF}(\hat{R}) \text{ and } E_1 = \text{QF}(\hat{S}) = k((z_1, \dots, z_n)).$$

We have that $E_1 = F_1(K^*)$ and $E_0 = F_0(K^*)$ (by Proposition 1 [3]). Thus M_1, \dots, M_e generate E_1 over F_1 . Since $e = [E_1 : F_1]$ (by (11)), $\{M_1, \dots, M_e\}$ is a basis of E_1 over F_1 . We have that z_i is a Laurent monomial in $x_1^{\frac{1}{e}}, \dots, x_s^{\frac{1}{e}}$ for $1 \leq i \leq s$, so by Lemma 4.5, for each $1 \leq i \leq s$, we have an expression

$$(21) \quad z_i = N_i M_{\sigma(i)}$$

where N_i is a Laurent monomial in x_1, \dots, x_n and $\sigma(i) \in \{1, \dots, e\}$. In particular, $z_1, \dots, z_n \in E_0$.

Let $a_i = (a_{i1}, \dots, a_{is}, 0, \dots, 0)$ for $1 \leq i \leq s$ (where $A = (a_{ij})$) and let e_j be the row vector of length n with a 1 in the j -th place and zeros everywhere else. Let M be the

submonoid of \mathbb{Z}^n generated by $a_1, \dots, a_s, a_{s+1} = e_{s+1}, \dots, a_n = e_n$. $k[z^\nu \mid \nu \in \tilde{M}_{\mathbb{Z}^n}]$ is the integral closure of $k[z^\nu \mid \nu \in M] = k[x_1, \dots, x_n]$ in $k[z_1, \dots, z_n]$ by Lemma 4.3.

The ring \hat{T} is integrally closed since T is. Let $A = k[z^\nu \mid \nu \in \tilde{M}_{\mathbb{Z}^n}] \subset k(z_1, \dots, z_n) \subset E_0$. All elements of A are integral over $k[x_1, \dots, x_n]$ and hence over $\hat{R}_0 = k[[x_1, \dots, x_n]]$. Thus $A \subset \hat{T}$.

Let m_A be the maximal ideal of A generated by the monomials of A of positive degree (in z_1, \dots, z_n). \hat{T} is the invariant ring of the action of the Galois group $G(L_0/E_0)$ on $k[[x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}}, x_{s+1}, \dots, x_n]]$, so the maximal ideal of \hat{T} is generated by monomials of positive degree in $x_1^{\frac{1}{n_1}}, \dots, x_s^{\frac{1}{n_s}}, x_{s+1}, \dots, x_n$. Thus $m_A \subset M_{\hat{T}}$ by (21). In particular, the completion \hat{A} at the maximal ideal m_A is contained in \hat{T} . \hat{A} is integrally closed since A is.

With the notation of Proposition 4.2, let $\Lambda = \mathbb{Z}^n \cap \text{par}(a_1, \dots, a_n)$. We have that

$$|\Lambda| = [\mathbb{Z}^s : AZ^s] = |\text{Det}(A)| = e$$

by c) of Proposition 4.2 and equation (11). Index the elements of $\{z^\nu \mid \nu \in \Lambda\}$ as $w_1 = 1, w_2, \dots, w_e$. We have that $A = \bigoplus_{i=1}^e k[x_1, \dots, x_n]w_i$ as a $k[x_1, \dots, x_n]$ -module by a) and b) of Proposition 4.2. Completing with respect to the maximal ideal (x_1, \dots, x_n) of $k[x_1, \dots, x_n]$, we have that

$$A \otimes_{k[x_1, \dots, x_n]} \hat{R}_0 \cong \bigoplus_{i=1}^e w_i \hat{R}_0.$$

We have that $\sqrt{(x_1, \dots, x_n)A} = m_A$, so that completion of A with respect to m_A is $\hat{A} \cong A \otimes_{k[x_1, \dots, x_n]} \hat{R}_0$ (Theorem 16, page 277 [28]). Now w_1, \dots, w_e is a basis of $\text{QF}(\hat{A})$ over $\text{QF}(\hat{R}_0) = F_0$, so $[\text{QF}(\hat{A}) : F_0] = e$. As $\text{QF}(\hat{A}) \subset E_0$ and $[E_0 : F_0] = e$, we have that $\text{QF}(\hat{A}) = E_0$. As \hat{A} is integrally closed, we have that $\hat{A} = \hat{T}$. Thus

$$\hat{T} = \bigoplus_{i=1}^e w_i \hat{R}_0.$$

Recall the extension $\hat{\nu}^*$ of ν^* to the quotient field E_1 of \hat{S} defined by (19). Notice that $\Gamma_{\hat{\nu}^*} = \mathbb{Z}^{n-\lambda} \times \Gamma_{\nu^*}$ contains Γ_{ν^*} as an isolated subgroup (page 40 [28]).

Lemma 4.6. *We have that $\hat{\nu}^*(w_i) \in \Gamma_{\nu^*}$ for $1 \leq i \leq e$ and $\hat{\nu}^*(w_i)$ for $1 \leq i \leq e$ are a complete set of representatives of the cosets of Γ_{ν} in Γ_{ν^*} .*

Proof. Each w_i is a monomial in z_1, \dots, z_s by construction, so $\hat{\nu}^*(w_i) \in \Gamma_{\nu^*}$ for $1 \leq i \leq e$ (since $\hat{\nu}^*(z_i) = \hat{\nu}^*(y_i)$ for all i). Write

$$(22) \quad w_i = z_1^{c_{i1}} \dots z_s^{c_{is}}$$

for $1 \leq i \leq e$. Suppose $\hat{\nu}^*(w_i) - \hat{\nu}^*(w_j) \in \Gamma_{\nu}$ for some $i \neq j$. Then

$$\hat{\nu}^*\left(\frac{w_i}{w_j}\right) \in \Gamma_{\nu}$$

implies

$$\frac{w_i}{w_j} = x_1^{f_1} \dots x_s^{f_s}$$

for some $f_1, \dots, f_s \in \mathbb{Z}$ by (13). Thus

$$\left(\prod_{f_i > 0} x_i^{f_i} \right) w_j = \left(\prod_{f_i < 0} x_i^{-f_i} \right) w_i.$$

But this is impossible since the w_i are a free basis of \hat{T} over \hat{R}_0 . Thus the classes of $\hat{\nu}^*(w_i)$ in $\Gamma_{\nu^*}/\Gamma_{\nu}$ are all distinct.

The lemma now follows since $e = |\Gamma_{\nu^*}/\Gamma_{\nu}|$. \square

Lemma 4.7. *There are natural isomorphisms of graded rings*

$$\mathrm{gr}_{\nu}(R_0) \cong \bigoplus_{\gamma \in \Gamma_{\nu}} \mathcal{P}_{\gamma}(\hat{R}_0)/\mathcal{P}_{\gamma}^+(\hat{R}_0)$$

and

$$\mathrm{gr}_{\nu^*}(T) \cong \bigoplus_{\gamma \in \Gamma_{\nu^*}} \mathcal{P}_{\gamma}(\hat{T})/\mathcal{P}_{\gamma}^+(\hat{T}).$$

Proof. Suppose that $\gamma \in \Gamma_{\nu}$. Then $\mathcal{P}_{\gamma}(\hat{R}_0) \cap R_0 = \mathcal{P}_{\gamma}(R_0)$ and $\mathcal{P}_{\gamma}^+(\hat{R}_0) \cap R_0 = \mathcal{P}_{\gamma}^+(R_0)$. Thus the natural map

$$(23) \quad \mathcal{P}_{\gamma}(R_0)/\mathcal{P}_{\gamma}^+(R_0) \rightarrow \mathcal{P}_{\gamma}(\hat{R}_0)/\mathcal{P}_{\gamma}^+(\hat{R}_0)$$

is well defined and injective. Suppose $F \in \mathcal{P}_{\gamma}(\hat{R}_0)$. Then there exists $m \in \mathbb{Z}_{>0}$ such that $\hat{\nu}^*(F) = \gamma < m\nu(m_{R_0})$ (since $\gamma \in \Gamma_{\nu}$) and there exist $F' \in R_0$, $h \in m_{\hat{R}_0}^m$ such that $F' = F + h$. Then $\hat{\nu}^*(F') = \hat{\nu}^*(F) = \gamma$ and

$$F' \equiv F \pmod{\mathcal{P}_{\gamma}^+(\hat{R}_0)}.$$

Thus (23) is an isomorphism for all $\gamma \in \Gamma_{\nu}$. The same argument applied to T establishes the lemma. \square

Now we return to the proof of Theorem 4.1. Suppose $f \in \hat{T}$ and $\hat{\nu}^*(f) \in \Gamma_{\nu^*}$. Write

$$f = \sum_{i=1}^e f_i w_i$$

with $f_i \in \hat{R}_0$. By Lemma 4.6, we have that

$$(24) \quad \hat{\nu}^*(f) = \min_i \{\hat{\nu}^*(f_i) + \hat{\nu}^*(w_i)\}$$

and there is a unique value of i giving this value. In particular, the classes

$$[w_i] := \mathrm{in}_{\nu^*}(w_i) \in \mathcal{P}_{\hat{\nu}^*(w_i)}(\hat{T})/\mathcal{P}_{\hat{\nu}^*(w_i)}^+(\hat{T})$$

for $1 \leq i \leq e$ generate

$$G_2 := \bigoplus_{\gamma \in \Gamma_{\nu^*}} \mathcal{P}_{\gamma}(\hat{T})/\mathcal{P}_{\gamma}^+(\hat{T})$$

as a

$$G_1 := \bigoplus_{\gamma \in \Gamma_{\nu}} \mathcal{P}_{\gamma}(\hat{R}_0)/\mathcal{P}_{\gamma}^+(\hat{R}_0)$$

module. $[w_1], \dots, [w_e]$ are a free basis of G_2 over G_1 since $\hat{\nu}^*(w_i) - \hat{\nu}^*(w_j) \notin \Gamma_{\nu}$ if $i \neq j$.

It follows from Theorem 4.10 and Lemma 4.4 [10] that E_1/F_1 is Galois with Galois group $G(E_1/F_1) \cong \mathbb{Z}^n/A\mathbb{Z}^n$. We summarize the construction given there of a natural isomorphism. Let ω be a primitive e -th root of unity in k . An element $\sigma \in G(E_1/F_1)$ is determined by its action on z_i for $1 \leq i \leq s$. Since $z_i^e \in \hat{R}$ for $1 \leq i \leq s$ we have that $\sigma(z_i) = \omega^{c_i} z_i$ for some $c_i \in \mathbb{Z}$. From the relation $x_i = z_1^{a_{i1}} \cdots z_s^{a_{is}}$ for $1 \leq i \leq s$, we have that

$$G(E_1/F_1) = \{c \in \mathbb{Z}^s \mid Ac \in e\mathbb{Z}^s\}/e\mathbb{Z}^s$$

and the natural map

$$(25) \quad \Psi : \mathbb{Z}^s / A\mathbb{Z}^s \rightarrow G(E_1/F_1)$$

defined by $\Psi(c) = (\text{adj}A)c$ is an isomorphism. We have shown that E_0 is Galois over F_0 (shown before (20)) and $[E_0 : F_0] = e$ by (16). Suppose $\sigma \in G(E_1/F_1)$. Since E_0 is Galois over F_0 , we have that $\sigma|_{E_0} \in G(E_0/F_0)$. Suppose $\sigma|_{E_0} = \text{id}$. $E_0 = F_0(K^*)$ then implies $\sigma|_{K^*} = \text{id}$, and $E_1 = F_1(K^*)$ implies $\sigma = \text{id}$. Now $|G(E_1/F_1)| = |G(E_0/F_0)| = e$ implies the restriction map $G(E_1/F_1) \rightarrow G(E_0/F_0)$ is an isomorphism. Comparing with the isomorphism (25) and (13) we have a natural group isomorphism

$$\Gamma_{\nu^*}/\Gamma_{\nu} \cong \mathbb{Z}^s / A^t \mathbb{Z}^s \cong \mathbb{Z}^s / A\mathbb{Z}^s \cong G(E_0/F_0).$$

This gives us a natural ‘‘diagonal’’ action of $\Gamma_{\nu}/\Gamma_{\nu^*}$ on $\hat{T} = \bigoplus_{i=1}^e \hat{R}_0 w_i$ and $\text{gr}_{\nu^*}(T) = \bigoplus_{i=1}^e \text{gr}_{\nu}(R_0)[w_i]$ such that

$$(\hat{T})^{\Gamma_{\nu}/\Gamma_{\nu^*}} \cong \hat{R}_0$$

and

$$\text{gr}_{\nu^*}(T)^{\Gamma_{\nu^*}/\Gamma_{\nu}} \cong \text{gr}_{\nu}(R_0),$$

completing the proof of Theorem 4.1.

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