ANALYTIC AX-KOCHEN-ERSOV THEORY WITH LIFTS OF THE RESIDUE FIELD AND VALUE GROUP

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Abstract. We develop an extension theory for *analytic* valuation rings in order to establish Ax-Kochen-Ersov type results for these structures. New is that we can add in salient cases lifts of the residue field and the value group and show that the induced structure on the lifted residue field is just its field structure, and on the lifted value group is just its ordered abelian group structure. This restores an analogy with the non-analytic AKE-setting that was missing in earlier treatments of analytic AKE-theory.

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1. Introduction

In the 1960s Ax and Kochen [2, 3, 4], and Ersov [16, 17, 18, 19] independently, developed a model theory for henselian valuation rings with significant applications to p-adic number theory. Since then there have been many generalizations and refinements, and AKE-theory remains a very active area of research. For example, in the 1980s Denef and van den Dries [11, 12] treated the ring of p-adic integers with *analytic* structure given by (restricted) power series. This led to the solution of a problem posed by Serre [27], and to a theory of p-adic subanalytic sets. Using "separated" power series this was upgraded to a theory of rigid subanalytic sets over henselian valuation rings equipped with a richer analytic structure, by L. Lipshitz, Z. Robinson, R. Cluckers, see [24, 25, 9, 7, 8].

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An interesting part of the original AKE-theory has so far not been extended to this analytic setting: in the equal characteristic 0 case one can add a predicate for a *coefficient field* (a lift of the residue field to the ambient field), and then the structure induced on this coefficient field can be shown to be just its pure field structure; likewise for a *monomial group*, that is, a lift of the value group.

In the analytic setting, there is only a partial result in this direction by Binyamini, Cluckers and Novikov [6, Proposition 2], and the usual approaches to analytic AKE-theory—based on direct reductions to ordinary AKE-theory by Weierstrass division "with parameters"—cannot be adapted to cover fully the induced structure aspect, as far as we know. Their partial result inspired us to try another approach.

We do indeed obtain the expected induced structure results in an analytic setting by developing a theory of analytic valuation rings in closer analogy with ordinary valuation theory. Weierstrass division is still key, as in [11, 12], but now in a different way. In an earlier version of the present paper, now in [5], this was done by elaborating, generalizing, and cleaning up substantial parts of [15]. The cleaning up was necessary because we noticed problems with [15, Lemma 3.1], and to remedy it we had to pass to finite extensions, with additional complications. We found subsequently that a somewhat different and more general approach to similar issues was already available in [9, 7]. So in our Section 9 we quote and slightly adapt instead relevant results from those sources, saving some 15 pages compared to this earlier version. Much of our analytic valuation theory is characteristic-free, but for the analytic AKE-results in Section 10 we require that the valued field be of equicharacteristic 0.

More on induced structure. Here we state in detail a typical case of our result on induced structure. First we say what it is in the classical (non-analytic) setting. Let C be a (coefficient) field. This yields the valuation ring $C[[t]]$ of formal power series in one variable t over C. We now expand the ring $C[[t]]$ to the structure $\big(C[[t]], C\big)$: a ring with a distinguished subset. Then a classical "induced structure" result is that if $\operatorname{char} C = 0$, any set $X \subseteq C^n$ which is definable in $\big(C[[t]], C\big)$ is even definable in the field C. (This can be proved along familiar lines, so we consider it as folklore knowledge, though we do not know an explicit reference. It seems this is still open for $\operatorname{char} C > 0$.) Here and below, n ranges over $\mathbb{N} = \{0, 1, 2, \dots\}$ and "definable" means "definable with parameters from the ambient structure".

We now equip $C[[t]]$ with analytic structure as follows: for each n we have the (Tate) ring $A\langle Y_1, \dots, Y_n\rangle$ of restricted power series in the distinct indeterminates $Y_1, \dots, Y_n$ over $A = C[[t]]$: it consists of the formal power series

$$f \;=\; f(Y_1, \dots, Y_n) \;=\; \sum_{\nu} a_\nu Y_1^{\nu_1} \cdots Y^{\nu_n}, \qquad \nu = (\nu_1, \dots, \nu_n) \text{ ranging over } \mathbb{N}^n,$$

with all $a_\nu \in A$ such that $a_\nu \to 0$, t-adically, as $|\nu| = \nu_1 + \cdots + \nu_n \to \infty$. Each such f gives rise to an n-ary operation on $C[[t]]$, namely

$$y \;=\; (y_1, \dots, y_n) \mapsto f(y_1, \dots, y_n) \;:\; C[[t]]^n \to C[[t]].$$

We expand the ring $C[[t]]$ to $C[[t]]_{\mathrm{an}}$ by taking each such f as a new n-ary function symbol that names the above n-ary operation on $C[[t]]$. Further expansion yields the structure $\big(C[[t]]_{\mathrm{an}}, C\big)$, and now our new induced structure result says that any set $X \subseteq C^n$ which is definable in $\big(C[[t]]_{\mathrm{an}}, C\big)$ is even definable in the field C. (For example, any subset of $\mathbb{C}$ definable in $\big(\mathbb{C}[[t]]_{\mathrm{an}}, \mathbb{C}\big)$ is finite or its complement in $\mathbb{C}$

is finite.) In fact, our induced structure result, Corollary 10.5, is stronger and more general in several ways, for example in also allowing $t^{\mathbb{N}}$ as a distinguished subset of $C[[t]]$. For various reasons it is more convenient to take the fraction field $C(\!(t)\!)$ of $C[[t]]$ as the ambient ring, equipped with its natural valuation to recover $C[[t]]$. For $C = \mathbb{C}$ we obtain [6, Proposition 2] as a special case, as explained in Section 10.

Contents of Sections 2–11. We begin with a brief section on henselianity. Next a section on ultranormed rings and restricted power series over them, including the Weierstrass theorems. At the beginning of Section 4 we define for any complete ultranormed ring A subject to mild conditions the notion of *A-analytic ring*: each n-variable restricted power series over A yields an n-ary operation on any A-analytic ring. Starting in Section 5 we specialize to the case that A is noetherian with an ideal $\smallo(A) \neq A$, such that $\bigcap_n \smallo(A)^n = \{0\}$ and A is $\smallo(A)$-adically complete.

In Section 6 we define A-analytic *valuation* rings and establish basic facts about them. In Section 7 we treat immediate extensions and prove an analytic version of Kaplansky's embedding theorem. In Section 8 (not needed for "induced structure" results, but included for its independent interest) we apply this to show that various extension procedures preserve truncation closedness. Section 9 uses [7, 9] to describe the function given by a univariate term in the language of analytic valued fields as analytic on the annuli of a suitable finite covering of the valuation ring. This allows us to complete the full array of extension results. Then we can prove in Section 10 an analytic AKE-type equivalence theorem, with an induced structure result for "coefficient field + monomial group" as a consequence. Together with our work on immediate extensions in Section 7, this yields *NIP transfer* in our analytic context. Section 11 proves a "separation of variables" result.

Notational and terminological conventions. Throughout d, m, n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$; *ring* means *commutative ring with* 1. From Section 6 onwards we consider valued fields. Let K be a *valued* field; it is specified by a valuation ring R of the field K. Let $v : K^\times \to \Gamma$ be a valuation on K with $R = \{a \in K : va \geqslant 0\}$. Here $\Gamma = v(K^\times)$ is the (ordered) value group, and we extend v to a function $v : K \to \Gamma_\infty = \Gamma \cup \{\infty\}$ by setting $v(0) := \infty$ and we extend the total ordering of Γ to a total ordering on Γ_∞ by $\Gamma < \infty$. It will be convenient to let $\preccurlyeq$, $\asymp$, $\prec$, $\succcurlyeq$, $\succ$, and $\sim$ denote the binary relations on K given for $x, y \in K$ by

$$\begin{gathered} x \preccurlyeq y \ :\Leftrightarrow\ vx \geqslant vy \ \Leftrightarrow\ x = yz \text{ for some } z \in R, \\ x \asymp y \ :\Leftrightarrow\ x \preccurlyeq y \text{ and } y \preccurlyeq x, \qquad x \prec y \ :\Leftrightarrow\ x \preccurlyeq y \text{ and } x \not\asymp y, \\ x \succcurlyeq y \ :\Leftrightarrow\ y \preccurlyeq x, \quad x \succ y \ :\Leftrightarrow\ y \prec x, \quad x \sim y \ \Leftrightarrow\ x - y \prec x. \end{gathered}$$

We let $\smallo(R)$ be the maximal ideal of R, and let $\operatorname{res} K := R/\smallo(R)$ be the residue field. For $a \in R$ we let $\operatorname{res} a$ be the residue class of a in $\operatorname{res} K$. If we need to indicate dependence on K we write R_K, v_K, Γ_K instead of R, v, Γ. The reason we use the letter R here instead of the more common $\mathcal{O}$ is that in Section 9 we follow [20] in denoting the algebra of analytic functions on a suitable set F by $\mathcal{O}(F)$; see the start of Section 9 for context and definitions of these notions.

Model theoretic arguments become important in Sections 9 and 10, although in earlier sections we already construe various mathematical structures as L-structures for various first-order languages L. We deal only with one-sorted structures, and "$\mathcal{M} \subseteq \mathcal{N}$" indicates that $\mathcal{M}$ is a substructure of $\mathcal{N}$, for L-structures $\mathcal{M}$ and $\mathcal{N}$. (One exception: we refer to a 3-sorted structure from [6] in Section 11.)

We cite many results of classical AKE-theory from the exposition [13]. We do so for convenience and do not suggest that the cited facts originate with [13].[1]

2. Henselianity

There are a few places where we need "henselianity" outside the usual pattern of a henselian local ring. Accordingly, this section proves basic facts about henselian pairs (which generalize henselian local rings). These facts are well-known, but our treatment is more elementary than we have seen in the literature.

Given a ring R we let $R^\times$ denote the multiplicative group of units of R. The *Jacobson radical* of a ring R is the intersection of the maximal ideals of R. For the Jacobson radical J of R, if $a \in R$ and $a + J \in (R/J)^\times$, then $a \in R^\times$. In this section X and Y are distinct indeterminates and I is an ideal of the ring R.

Lemma 2.1. *Let I be contained in the Jacobson radical of R and let $P(X) \in R[X]$ and $a \in R$ be such that $P'(a) \in R^\times$. Then $P(b) = 0$ for at most one $b \in a + I$.*

Proof. Let $b \in a + I$ and $P(b) = 0$. Then for $\epsilon \in I$ we have $r \in R$ such that

$$P(b+\epsilon) \ = \ P(b) + P'(b)\epsilon + r\epsilon^2 \ = \ P'(b)\epsilon + r\epsilon^2 \ = \ P'(b)\epsilon\big(1 + rP'(b)^{-1}\epsilon\big) \ = \ 0,$$

and $P'(b), 1 + rP'(b)^{-1}\epsilon \in R^\times$, so $\epsilon = 0$. □

The pair (R, I) is henselian means:

- I is contained in the Jacobson radical of R, equivalently, $1 + I \subseteq R^\times$;
- for all polynomials $P(X) \in R[X]$ and $a \in R$ with $P(a) \in I$ and $P'(a) \in R^\times$ there exists $b \in R$ such that $P(b) = 0$ and $a - b \in I$.

Thus given a maximal ideal $\mathfrak{m}$ of the ring R, the pair $(R, \mathfrak{m})$ is henselian iff R is a henselian local ring in the usual sense.

Lemma 2.2. *Assume $1 + I \subseteq R^\times$. Then the following conditions are equivalent:*

(i) *(R, I) is henselian;*
(ii) *each polynomial $1 + X + ea_2X^2 + \cdots + ea_nX^n$ with $n \geqslant 2$, $e \in I$, and $a_2, \dots, a_n \in R$ has a zero in R (obviously, such a zero lies in $-1 + I$);*
(iii) *each polynomial $Y^n + Y^{n-1} + ea_2Y^{n-2} + \cdots + ea_n$ with $n \geqslant 2$, $e \in I$, and $a_2, \dots, a_n \in R$ has a zero in $R^\times$;*
(iv) *given any polynomial $P(X) \in R[X]$ and $a \in R$, $e \in I$ such that $P(a) = eP'(a)^2$ there exists $b \in R$ such that $P(b) = 0$ and $b - a \in eP'(a)R$.*

Proof. (i)⇒(ii) is clear. For (ii)⇔ (iii): use that for $x \in R^\times$ and $y := x^{-1}$, x is a zero in (ii) iff y is a zero in (iii). Now assume (ii) and let P, a, e be as in the hypothesis of (iv). Let $x \in R$ and consider the expansion:

$$\begin{aligned} P(a+x) \ &= \ P(a) + P'(a)x + \sum_{i \geqslant 2} P_{(i)}(a)x^i \\ &= \ eP'(a)^2 + P'(a)x + \sum_{i \geqslant 2} P_{(i)}(a)x^i. \end{aligned}$$

[1]We take the opportunity to mention that the Chinese Remainder Theorem on [13, p. 86] is stated there in too great a generality. But it holds for ideals in a ring, which is how it gets applied.

Set $x = eP'(a)y$ where $y \in R$. Then

$$P(a+x) \;=\; eP'(a)^2\left(1 + y + \sum_{i\geqslant 2} ea_i y^i\right)$$

where the $a_i \in R$ do not depend on y. From (ii) we obtain $y \in R$ such that

$$1 + y + \sum_{i\geqslant 2} ea_i y^i \;=\; 0.$$

This yields an element $b = a + x = a + eP'(a)y$ as required. This shows (ii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (i) is clear. $\square$

Lemma 2.3. *Suppose every element of I is nilpotent. Then (R, I) is henselian.*

Proof. Consider a polynomial $P(X) = a + X + \sum_{i=2}^{n} ea_i X^i$ where $n \geqslant 2$ and

$$a, e, a_2, \dots, a_n \in R, \quad e^m = 0,\; m \geqslant 1.$$

By induction on m we show that $P(X)$ has a zero in R. The case $m = 1$ being trivial, let $m \geqslant 2$. Then

$$P(-a+eY) \;=\; a + (-a+eY) + \sum_{i=2}^{n} ea_i(-a+eY)^i \;=\; e\big(Y + \sum_{i=2}^{n} a_i(-a+eY)^i\big).$$

An easy computation gives $f, b, b_2, \dots, b_n \in R$ such that

$$Y + \sum_{i=2}^{n} a_i(-a+eY)^i \;=\; b + Y(1+ef) + \sum_{i=2}^{n} e^2 b_i Y^i.$$

Now use that $1 + ef \in R^\times$ and $(e^2)^{m-1} = 0$. $\square$

Lemma 2.4. *Let J be an ideal of R with $I \subseteq J$. Then the following are equivalent:*

(i) *(R, I) and $(R/I, J/I)$ are henselian;*
(ii) *(R, J) is henselian.*

Proof. The condition $1 + J \subseteq R^\times$ is easily seen to be equivalent to the conjunction of $1 + I \subseteq R^\times$ and $1 + (J/I) \subseteq (R/I)^\times$. This gives (ii)$\Rightarrow$(i). Now assume (i), and let $P(X) \in R[X]$ and $a \in R$ with $P(a) \in J$, $P'(a) \in R^\times$. Working modulo I this gives $b \in R$ such that $P(b) \in I$ and $a - b \in J$. Hence $P'(b) - P'(a) \in J$, and thus $P'(b) \in R^\times$, giving $c \in R$ with $P(c) = 0$ and $b - c \in I$. Hence $a - c \in J$. $\square$

Corollary 2.5. *Suppose (R, I) is henselian and J is an ideal of R contained in the nilradical $\sqrt{I}$ of I. Then (R, J) is henselian.*

Proof. Every element of $\sqrt{I}/I$ is nilpotent in R/I, so by Lemmas 2.3 and 2.4 the pair $(R, \sqrt{I})$ is henselian, and so is (R, J). $\square$

Recall also that a local ring R is said to be *henselian* if the pair $(R, \mathfrak{m})$ is henselian, where $\mathfrak{m}$ is the maximal ideal of R.

3. Complete Ultranormed Rings and Restricted Power Series

We introduce here the restricted power series that will define operations on the valuation rings considered in later sections, where we develop an AKE-theory for these valuation rings with these extra operations. The coefficients of these restricted power series will be from a fixed coefficient ring A which is complete with respect to an ultranorm. We begin with defining ultranorms.

Ultranormed abelian groups. Let A be an additively written abelian group. An *ultranorm* on A is a function $a \mapsto |a| : A \to \mathbb{R}^{\geqslant}$ such that for all $a, b \in A$,

- $|a| = 0 \Leftrightarrow a = 0$;
- $|-a| = |a|$;
- $|a+b| \leqslant \max(|a|, |b|)$.

Let A be equipped with the ultranorm $|\cdot|$ on A. We make A a metric space with metric $(a, b) \mapsto |a - b|$. Then A is a topological group with respect to the topology on A induced by this metric. The ultranorm $|\cdot| : A \to \mathbb{R}$ and the group operations $- : A \to A$ and $+ : A \times A \to A$ are uniformly continuous.

In the rest of this subsection A is complete with respect to its ultranorm, that is, complete with respect to the metric above. We now discuss convergence of series with terms in A. Let $(a_i) = (a_i)_{i \in I}$ be a family in A (that is, all $a_i \in A$). We say (a_i) is *summable* if for every ϵ we have $|a_i| < \epsilon$ for all but finitely many $i \in I$. In that case the set of $i \in I$ with $a_i \neq 0$ is countable, and there is a unique $a \in A$ such that for every $\epsilon \in \mathbb{R}^{>}$ there is a finite $I(\epsilon) \subseteq I$ with $|a - \sum_{i \in J} a_i| < \epsilon$ for all finite $J \subseteq I$ with $I(\epsilon) \subseteq J$; this a is then denoted by $\sum_{i \in I} a_i$ (or $\sum_i a_i$ if I is understood from the context). Instead of saying that (a_i) is summable we also say that $\sum_i a_i$ exists, or that $\sum_i a_i$ converges. Of course, if I is finite, then $\sum_i a_i$ exists and is the usual sum. Here are simple rules, used throughout, for dealing with such (possibly infinite) sums, where $(a_i)_{i \in I}$ is a summable family in A:

- if $c \in \mathbb{R}^{>}$ and $|a_i| \leqslant c$ for all i, then $|\sum_i a_i| \leqslant c$;
- $(-a_i)$ is summable with $\sum_i -a_i = -\sum_i a_i$;
- if $(b_i)_{i \in I}$ is also a summable family in A, then so is $(a_i + b_i)$ with
$$\sum_i a_i + b_i \;=\; \sum_i a_i + \sum_i b_i;$$
- if $i \mapsto \lambda(i) : I \to \Lambda$ is a bijection and $(b_\lambda)_{\lambda \in \Lambda}$ is a family in A with $a_i = b_{\lambda(i)}$ for all $i \in I$, then $\sum_\lambda b_\lambda$ exists and equals $\sum_i a_i$;
- if the family $(a_j)_{j \in J}$ in A is also summable with $I \cap J = \emptyset$, then $(a_k)_{k \in I \cup J}$ is summable with $\sum_k a_k = \sum_i a_i + \sum_j a_j$;
- if $I = \dot{\bigcup}_{\lambda \in \Lambda} I_\lambda$ (disjoint union), then $\sum_{i \in I_\lambda} a_i$ exists for all $\lambda \in \Lambda$, and $\sum_\lambda \big(\sum_{i \in I_\lambda} a_i\big)$ exists and equals $\sum_{i \in I} a_i$.

Suppose E is a closed subgroup of A. Then
$$|a + E| \;:=\; \inf_{e \in E} |a + e| \qquad (a \in A)$$
yields an ultranorm on the quotient group A/E with respect to which A/E is complete; we call it the *quotient norm of* A/E. If the family (a_i) in A is summable, then so is the family $(a_i + E)$ in A/E with its quotient norm, and
$$\big(\sum_i a_i\big) + E \;=\; \sum_i (a_i + E).$$

Ultranormed rings. Let A be a ring. An *ultranorm on* A is a function

$$a \mapsto |a| \ : \ A \to \mathbb{R}^{\geqslant}$$

such that for all $a, b \in A$,

- $|a| = 0 \Leftrightarrow a = 0$, $|1| = |-1| = 1$;
- $|a+b| \leqslant \max(|a|, |b|)$;
- $|ab| \leqslant |a| \cdot |b|$.

Let A be equipped with the ultranorm $|\cdot|$ on A. Then $|-a| = |a|$ for all $a \in A$, so $|\cdot|$ is an ultranorm on the underlying additive group of A. The function $\cdot : A \times A \to A$ is continuous. If A is complete with respect to its ultranorm and $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are summable families in A, then $(a_i b_j)_{(i,j) \in I \times J}$ is summable, with $(\sum_i a_i)(\sum_j b_j) = \sum_{(i,j)} a_i b_j$.

From now on in this paper A is a ring with $1 \neq 0$, equipped with an ultranorm $|\cdot|$ such that $|a| \leqslant 1$ for all $a \in A$, and A is complete with respect to its ultranorm.

It follows that if $a \in A$ and $|a| < 1$, then $\sum_n a^n$ exists, with

$$(1-a) \sum_n a^n \ = \ 1.$$

We have the ideal $\mathcal{O}(A) := \{a \in A : \ |a| < 1\}$, and set $\overline{A} := A/\mathcal{O}(A)$, with the canonical ring morphism $a \mapsto \overline{a} = a + \mathcal{O}(A) : A \to \overline{A}$. We saw that $1 + \mathcal{O}(A)$ consists entirely of units of A. Thus $a \in A$ is a unit of A iff $\overline{a}$ is a unit of $\overline{A}$. In particular, $\mathcal{O}(A)$ is contained in the Jacobson radical of A. The completeness assumption now yields Hensel's Lemma as stated in [13, Section 2.2]: the pair $(A, \mathcal{O}(A))$ is henselian. It follows that $(A, \sqrt{\mathcal{O}(A)})$ is also henselian.

Passing to A/I. Suppose the proper ideal I of A is closed. Then the quotient norm of the quotient group A/I is an ultranorm on the ring A/I. Equipping A/I with the quotient norm, the canonical map $A \to A/I$ is norm decreasing, and $\mathcal{O}(A/I)$ is the image of $\mathcal{O}(A)$ under this canonical map.

If A is noetherian, then by [26, Theorem 8.14] every ideal of A is closed.

Restricted power series over an ultranormed ring. For distinct indeterminates $Y_1, \dots, Y_n$ we let $A\langle Y \rangle = A\langle Y_1, \dots, Y_n \rangle$ be the subalgebra of the A-algebra $A[[Y_1, \dots, Y_n]]$ consisting of the series $\sum_\nu a_\nu Y^\nu$ with $a_\nu \to 0$ as $|\nu| \to \infty$. Here and below, when using an expression like $\sum_\nu a_\nu Y^\nu$ for a series in $A\langle Y \rangle$ it is assumed that $a_\nu \to 0$ as $|\nu| \to \infty$. We extend $|\cdot|$ on A to an ultranorm on the ring $A\langle Y \rangle$ by

$$|\sum_\nu a_\nu Y^\nu| \ := \ \max_\nu |a_\nu|,$$

so with respect to this ultranorm, $A\langle Y \rangle$ is complete and $A[Y]$ is dense in it. Note that for $a_1, \dots, a_n \in \mathcal{O}(A)$ we have $|a_1 Y_1 + \cdots + a_n Y_n| < 1$, so $1 + a_1 Y_1 + \cdots + a_n Y_n$ is a unit of the ring $A\langle Y \rangle$.

For $f = \sum_\nu a_\nu Y^\nu \in A\langle Y \rangle$, the family $(a_\nu Y^\nu)$ in $A\langle Y \rangle$ is in fact summable with sum f. If $|ab| = |a| \cdot |b|$ for all $a, b \in A$, then $|fg| = |f| \cdot |g|$ for all $f, g \in A\langle Y \rangle$. For any $y = (y_1, \dots, y_n) \in A^n$ we have the evaluation map

$$f = \sum_\nu a_\nu Y^\nu \mapsto f(y) := \sum_\nu a_\nu y^\nu \ : \ A\langle Y \rangle \to A,$$

which is an A-algebra morphism with $|f(y)| \leqslant |f|$ for all $y \in A^n$. If $(f_i)_{i\in I}$ is a summable family in $A\langle Y\rangle$ and $y \in A^n$, then $\sum_i f_i(y)$ exists in A and equals $(\sum_i f_i)(y)$. The obvious inclusion of $A[[Y_1,\dots,Y_m]]$ in $A[[Y_1,\dots,Y_n]]$ for $m \leqslant n$ restricts to an inclusion of $A\langle Y_1,\dots,Y_m\rangle$ in $A\langle Y_1,\dots,Y_n\rangle$. For $f = f(Y) \in A\langle Y\rangle$ we have unique $f_j \in A\langle Y_1,\dots,Y_j\rangle$ for $j = 0,\dots,n$ such that

$$f(Y) \;=\; f_0 + Y_1 f_1 + \cdots + Y_n f_n.$$

Substitution. Besides $Y = (Y_1,\dots,Y_n)$, let $X = (X_1,\dots,X_m)$ also be a tuple of distinct indeterminates. Let $f = \sum_\mu a_\mu X^\mu \in A\langle X\rangle$ with $\mu = (\mu_1,\dots,\mu_m)$ ranging over $\mathbb{N}^m$, and $g_1,\dots,g_m \in A\langle Y\rangle$. Then $|a_\mu g_1^{\mu_1}\cdots g_m^{\mu_m}| \leqslant |a_\mu| \to 0$ as $|\mu| \to \infty$, so

$$f(g_1,\dots,g_m) \;:=\; \sum_\mu a_\mu g_1^{\mu_1}\cdots g_m^{\mu_m} \in A\langle Y\rangle,$$

and for fixed $g = (g_1,\dots,g_m) \in A\langle Y\rangle^m$ the map $f \mapsto f(g) : A\langle X\rangle \to A\langle Y\rangle$ is an A-algebra morphism with $|f(g)| \leqslant |f|$ and $f(g)(y) = f(g(y))$ for $y \in A^n$. Moreover, if (f_i) is a summable family in $A\langle X\rangle$ and $g \in A\langle Y\rangle^m$, then $\sum_i f_i(g)$ exists in $A\langle Y\rangle$ and equals $(\sum_i f_i)(g)$. It follows that the above kind of composition is associative in the following sense: let $Z = (Z_1,\dots,Z_p)$ be a third tuple of distinct indeterminates, $p \in \mathbb{N}$, and $h = (h_1,\dots,h_n) \in A\langle Z\rangle^n$. Then

$$\big(f(g)\big)(h) \;=\; f\big(g_1(h),\dots,g_m(h)\big) \text{ in } A\langle Z\rangle.$$

From now on $X_1, X_2, X_3, \dots, Y_1, Y_2, Y_3, \dots$ (two infinite sequences) are distinct indeterminates, and unless specified otherwise,

$$X := (X_1,\dots,X_m), \qquad Y := (Y_1,\dots,Y_n).$$

The natural $A[[X]]$-algebra isomorphism $A[[X]][[Y]] \to A[[X,Y]]$ restricts to the norm preserving $A\langle X\rangle$-algebra isomorphism $A\langle X\rangle\langle Y\rangle \to A\langle X,Y\rangle$ given by

$$\sum_\nu f_\nu Y^\nu \mapsto \sum_\nu f_\nu Y^\nu$$

where $f_\nu \in A\langle X\rangle$ for all ν and $f_\nu \to 0$ as $|\nu| \to \infty$, with righthand and lefthand side interpreted naturally in $A\langle X\rangle\langle Y\rangle$ and $A\langle X,Y\rangle$ respectively. We identify $A\langle X\rangle\langle Y\rangle$ and $A\langle X,Y\rangle$ via this isomorphism.

Polynomials as restricted power series. Let $p = p(T) \in A[T]$ be a monic polynomial of degree $d \geqslant 1$ over A, so $|p| = 1$ as an element of $A\langle T\rangle$.

Lemma 3.1. *For all $f \in A\langle T\rangle$ we have $|pf| = |f|$. Moreover, $pA\langle T\rangle$ is a proper ideal of $A\langle T\rangle$ and is closed in $A\langle T\rangle$.*

Proof. For $f = \sum_n a_n T^n \in A[T]^{\neq}$, take n maximal with $|a_n| = |f|$, and note that then the coefficient of T^{d+n} in pf is $a_n + b$ with $|b| < |a_n|$, so $|a_n + b| = |a_n| = |f|$. The rest follows easily. □

Lemma 3.2. *Let $f \in A\langle T\rangle$. Then there are unique $q \in A\langle T\rangle$ and $r \in A[T]$ with $\deg r < d$ such that $f = qp + r$; moreover, $|f| = \max(|q|,|r|)$ for these q, r.*

Proof. For each n we have $T^n = q_n p + r_n$ with $q_n, r_n \in A[T]$ and $\deg r_n < d$. Thus for $f = \sum_n a_n T^n \in A\langle T\rangle$ we have $f = qp + r$ with $q = \sum_n a_n q_n \in A\langle T\rangle$ and $r = \sum_n a_n r_n \in A[T]$ with $\deg r < d$, and $|f| = \max(|q|,|r|)$ for these q, r.

Uniqueness holds because for $g \in A\langle T\rangle$ with $gp \in A[T]$, $\deg gp < d$, we have $g = 0$ by the proof of Lemma 3.1. □

Corollary 3.3. *The composition $A[T] \to A\langle T\rangle \to A\langle T\rangle/pA\langle T\rangle$, with inclusion on the left and the canonical map on the right, is surjective and has kernel $pA[T]$, so induces an A-algebra isomorphism $A[T]/pA[T] \to A\langle T\rangle/pA\langle T\rangle$.*

Proof. Lemma 3.2 gives surjectivity. The uniqueness in that lemma and division with remainder in $A[T]$ (by p) yields kernel $pA[T]$. □

Division with Remainder. Let $n \geqslant 1$, set $Y' := (Y_1, \dots, Y_{n-1})$. The inclusion $A\langle Y'\rangle[Y_n] \subseteq A\langle Y'\rangle\langle Y_n\rangle = A\langle Y\rangle$ makes $A\langle Y'\rangle[Y_n]$ a subring of $A\langle Y\rangle$.

Lemma 3.4. *Let $f \in A\langle Y'\rangle[Y_n]$ be monic of degree $d \geq 1$ and $g \in A\langle Y\rangle$. Then there are unique $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with $\deg_{Y_n} r < d$ such that $g = qf + r$. Moreover, $|g| = \max(|q|, |r|)$ for these q, r.*

Proof. This is Lemma 3.2 applied to $A\langle Y'\rangle$ in the role of A. □

Consider $A\langle X, Y_1, \dots, Y_{j-1}\rangle[Y_j]$ likewise as a subring of $A\langle X, Y\rangle$ for $j = 1, \dots, n$. By a straightforward induction on n the previous lemma gives:

Lemma 3.5. *Let $f_j \in A\langle X, Y_1, \dots, Y_{j-1}\rangle[Y_j]$ be monic of degree d_j in Y_j for $j = 1, \dots, n$. Then*

$$A\langle X, Y\rangle \;=\; (f_1, \dots, f_n)A\langle X, Y\rangle + \bigoplus_{(j_1,\dots,j_n)} A\langle X\rangle Y_1^{j_1} \cdots Y_n^{j_n}$$

where $(j_1, \dots, j_n)$ ranges over the elements of $\mathbb{N}^n$ with $j_1 < d_1, \dots, j_n < d_n$.

Corollary 3.6. *Let $m = n$ and $f(X) \in A\langle X\rangle$. Then*

$$f(X) - f(Y) \in (X_1 - Y_1, \dots, X_n - Y_n)A\langle X, Y\rangle.$$

Proof. By Lemma 3.5 we have $f(X) - f(Y) = \sum_{j=1}^{n}(X_j - Y_j)q_j + r$ with all q_j in $A\langle X, Y\rangle$ and $r \in A\langle X\rangle$. Substituting X_j for Y_j gives $0 = r$. □

We extend $a \mapsto \overline{a} : A \to \overline{A}$ to the ring morphism

$$f = \sum_{\nu} a_\nu Y^\nu \;\mapsto \overline{f} := \sum_{\nu} \overline{a_\nu} Y^\nu \;:\; A\langle Y\rangle \to \overline{A}[Y],$$

whose kernel is $\mathcal{o}(A\langle Y\rangle) := \{f \in A\langle Y\rangle : \; |f| < 1\}$. Moreover,

$$\overline{f(g_1, \dots, g_m)} = \overline{f}(\overline{g_1}, \dots, \overline{g_m}), \qquad (f \in A\langle X\rangle,\; g_1, \dots, g_m \in A\langle Y\rangle).$$

For $d \in \mathbb{N}$, call $f \in A\langle Y\rangle$ *regular in Y_n of degree d* if $\overline{f} = f_0 + f_1 Y_n + \cdots + f_d Y_n^d$ with $f_0, \dots, f_d \in \overline{A}[Y']$ and f_d a unit in $\overline{A}[Y']$. We now extend Lemma 3.4:

Proposition 3.7 (Weierstrass Division)**.** *Suppose $f \in A\langle Y\rangle$ is regular in Y_n of degree d and $g \in A\langle Y\rangle$. Then there are $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with*

$$g \;=\; qf + r, \qquad \deg_{Y_n} r \;<\; d, \qquad |g| = \max(|q|, |r|).$$

Proof. Multiplying f by a unit of $A\langle Y'\rangle$ we arrange that $\overline{f} \in \overline{A}[Y]$ is monic in Y_n of degree d. Hence $f = f_0 + E$ where $f_0 \in A\langle Y'\rangle[Y_n]$ is monic of degree d in Y_n and $E \in A\langle Y\rangle$, $|E| < 1$. Now $g = q_0 f_0 + r_0$ with $q_0 \in A\langle Y\rangle$ and $r_0 \in A\langle Y'\rangle[Y_n]$,

$\deg_{Y_n} r_0 < d$ and $|g| = \max(|q_0|, |r_0|)$, so $g = q_0 f + r_0 + g_1$ with $g_1 = -Eq_0$, and thus $|g_1| \leqslant |E||g|$. With g_1 in the role of g and iterating:

$$\begin{aligned}
g &= q_0 f + r_0 + g_1, & g_1 &= -Eq_0, & |g_1| &\leqslant |E||g|,\\
g_1 &= q_1 f + r_1 + g_2, & g_2 &= -Eq_1, & |g_2| &\leqslant |E|^2|g|,\\
\ldots &= \ldots\\
\ldots &= \ldots\\
g_k &= q_k f + r_k + g_{k+1}, & g_{k+1} &= -Eq_k, & |g_{k+1}| &\leqslant |E|^{k+1}|g|,\\
\ldots &= \ldots
\end{aligned}$$

where $q_k \in A\langle Y\rangle$, $r_k \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r_k < d$ and $|g_k| = \max(|q_k|, |r_k|)$. It follows that $g_k, q_k, r_k \to 0$ as $k \to \infty$. Thus we can add the right and left-hand sides in the equalities above to obtain $g = qf + r$ where $q := \sum_k q_k \in A\langle Y\rangle$ and $r := \sum_k r_k \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r < d$, so $|g| = \max(|q|, |r|)$. $\square$

Corollary 3.8 (Weierstrass Preparation)**.** *Suppose $f \in A\langle Y\rangle$ is regular in Y_n of degree d. Then for some unit u of $A\langle Y\rangle$ we have: $uf \in A\langle Y'\rangle[Y_n]$, and uf is monic of degree d in Y_n.*

Proof. We have $Y_n^d = qf + r$ with $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$, $\deg_{Y_n} r < d$. Hence $Y_n^d - \overline{r} = \overline{q}\overline{f}$ in $\overline{A}[Y]$, so $\overline{q}$ is a unit of $\overline{A}[Y']$, hence q is a unit of $A\langle Y\rangle$, and thus $u := q$ has the desired property. $\square$

A somewhat twisted argument also gives uniqueness in the last two results:

Corollary 3.9. *Let $f \in A\langle Y\rangle$ be regular in Y_n of degree d. Then there is only one pair (q,r) with $q \in A\langle Y\rangle$ and $r \in A\langle Y'\rangle[Y_n]$ with $g = qf + r$ and $\deg_{Y_n} r < d$. There is also only one unit u of $A\langle Y\rangle$ such that $uf \in A\langle Y'\rangle[Y_n]$, and uf is monic of degree d in Y_n.*

Proof. By Corollary 3.8 (just the existence of u), the uniqueness of (q,r) follows from the uniqueness in Lemma 3.4. Next, the uniqueness of u follows from the proof of Corollary 3.8 and the uniqueness in Proposition 3.7. $\square$

Besides $n \geqslant 1$ we now also assume $d \geqslant 1$. Under an extra assumption on $\overline{A}$ (see Lemma 3.10) we can apply automorphisms to arrange regularity in Y_n. Set

$$T_d(Y) := \big(Y_1 + Y_n^{d^{n-1}}, \ldots, Y_{n-1} + Y_n^d, Y_n\big),$$

which gives a norm preserving automorphism $f(Y) \mapsto f\big(T_d(Y)\big)$ of the A-algebra $A\langle Y\rangle$ with inverse $g(Y) \mapsto g\big(T_d^{-1}(Y)\big)$, where

$$T_d^{-1}(Y) := \big(Y_1 - Y_n^{d^{n-1}}, \ldots, Y_{n-1} - Y_n^d, Y_n\big).$$

Lemma 3.10. *Assume $\overline{A}$ is a field. Let $f \in A\langle Y\rangle$ be such that $\overline{f} \neq 0$ in $\overline{A}[Y]$, and $d > \deg \overline{f}$. Then $f\big(T_d(Y)\big)$ is regular in Y_n of some degree.*

Proof. With $f = \sum_\nu a_\nu Y^\nu$, let $(\mu_1, \ldots, \mu_n)$ be lexicographically largest among the $\nu \in \mathbb{N}^n$ for which $\overline{a_\nu} \neq 0$. A straightforward computation shows that then for $\ell := \mu_1 d^{n-1} + \cdots + \mu_{n-1} d + \mu_n$ we have

$$\overline{f}\big(T_d(Y)\big) = \overline{a_\mu} Y_n^\ell + \text{terms in } \overline{A}[Y] \text{ of degree} < \ell \text{ in } Y_n.$$

Thus $f\big(T_d(Y)\big)$ is regular in Y_n of degree ℓ. $\square$

4. Rings with A-analytic Structure

Given a ring R and a set E we have the ring R^E of R-valued functions on E, where the ring operations are given pointwise. A *ring with A-analytic structure* is a ring R together with a ring morphism

$$\iota_n \ : \ A\langle Y_1, \dots, Y_n\rangle \to \ \text{ring of } R\text{-valued functions on } R^n$$

for every n, such that the following conditions are satisfied:

(A1) $\iota_n(Y_k)(y_1, \dots, y_n) = y_k$, for $k = 1, \dots, n$ and $y = (y_1, \dots, y_n) \in R^n$;
(A2) for $f \in A\langle Y_1, \dots, Y_n\rangle \subseteq A\langle Y_1, \dots, Y_n, Y_{n+1}\rangle$ and $(y_1, \dots, y_n, y_{n+1}) \in R^{n+1}$ we have $\iota_n(f)(y_1, \dots, y_n) = \iota_{n+1}(f)(y_1, \dots, y_n, y_{n+1})$;
(A3) for $n \geqslant 1$, $f \in A\langle Y_1, \dots, Y_n\rangle$, $g := f(Y_{n+1}, \dots, Y_{2n}) \in A\langle Y_1, \dots, Y_{2n}\rangle$, and $(y_1, \dots, y_{2n}) \in R^{2n}$ we have: $\iota_n(f)(y_{n+1}, \dots, y_{2n}) = \iota_{2n}(g)(y_1, \dots, y_{2n})$.

Let R be a ring with A-analytic structure as above. We set $h(y) := \iota_n(h)(y)$ for $h \in A\langle Y\rangle$ and $y \in R^n$, a notational convention that will be in force from now on. In other words, each $h \in A\langle Y\rangle$ defines a function $R^n \to R$ that we also denote by h. For $n = 0$ the above gives the ring morphism $\iota_0 : A \to R$ upon identifying a function $R^0 \to R$ with its only value, and so R is an A-algebra with structural morphism ι_0. Accordingly we denote for $a \in A$ the element $\iota_0(a)$ of R also by a when no confusion is likely. Simple example of a ring with A-analytic structure: A itself with $\iota_n(f)(y) := f(y)$ for $f \in A\langle Y\rangle$ and $y \in A^n$ and below we consider A to be equipped with this A-analytic structure.

Lemma 4.1. *Let $f, g_1, \dots, g_n \in A\langle Y\rangle$ and $y \in R^n$. Then*

$$f(g_1, \dots, g_n)(y) \ = \ f\big(g_1(y), \dots, g_n(y)\big).$$

Proof. The case $n = 0$ is trivial. Let $n \geqslant 1$ and set $B := A\langle Y_1, \dots, Y_{2n}\rangle$. In $A\langle Y\rangle$ we have $f(g_1, \dots, g_n)(Y) = f\big(g_1(Y), \dots, g_n(Y)\big)$, trivially. Also $A\langle Y\rangle \subseteq B$, $f(Y_{n+1}, \dots, Y_{2n}) \in B$, and by Corollary 3.6,

$$f(g_1, \dots, g_n)(Y) - f(Y_{n+1}, \dots, Y_{2n}) \in \big(g_1(Y) - Y_{n+1}, \dots, g_n(Y) - Y_{2n}\big)B.$$

Now apply ι_{2n} to this, and use (A1), (A2), (A3) to evaluate at the point

$$\big(y_1, \dots, y_n, g_1(y), \dots, g_n(y)\big) \in R^{2n}. \qquad \square$$

We abbreviate the expression *ring with A-analytic structure* to *A-analytic ring*, or just *A-ring*. A good feature of the above is that the A-rings naturally form an *equational class* (which is not the case for the narrower notion of rings with analytic A-structure defined in [12].) To back this up, we introduce the language $\mathcal{L}^A$ of A-rings: it is the language $\{0, 1, -, +, \cdot\}$ of rings augmented by an n-ary function symbol for each $f \in A\langle Y\rangle = A\langle Y_1, \dots, Y_n\rangle$, to be denoted also by f. We construe any A-ring R in the obvious way as an $\mathcal{L}^A$-structure, with f as above naming the function $y \mapsto f(y) : R^n \to R$, so the A-rings are exactly the models of an equational $\mathcal{L}^A$-theory, and for any $\mathcal{L}^A$-term $t(Z_1, \dots, Z_n)$ there is an $f \in A\langle Y\rangle$ such that $t(z) = f(z)$ for every A-ring R and $z \in R^n$.

The A-ring A is initial in this equational class:

Lemma 4.2. *Given any A-ring R there is a unique morphism $A \to R$ of A-rings, namely $\iota_0 : A \to R$.*

Proof. If $j : A \to R$ is an A-ring morphism, then clearly $j(a) = \iota_0(a)$ for $a \in A$. It remains to check that for $n \geqslant 1$, $f \in A\langle Y\rangle$, $Y = (Y_1, \dots, Y_n)$, and $a_1, \dots, a_n \in A$,

$$\iota_0\big(f(a_1, \dots, a_n)\big) \ = \ f\big(\iota_0(a_1), \dots, \iota_0(a_n)\big).$$

Take $a_1, \dots, a_n$ as elements of $A\langle Y\rangle$ and $f(a_1, \dots, a_n) \in A$ accordingly also as an element of $A\langle Y\rangle$. Fixing any $y \in R^n$ we obtain from (A2) that

$$\iota_0\big(f(a_1, \dots, a_n)\big) \ = \ \iota_n\big(f(a_1, \dots, a_n)\big)(y),$$

which by Lemma 4.1 equals $f\big(\iota_n(a_1)(y), \dots, \iota_n(a_n)(y)\big)$, and by (A2) again this equals $f\big(\iota_0(a_1), \dots, \iota_0(a_n)\big)$, as promised. □

Example. Let A_0 be a ring with $1 \neq 0$ and $A := A_0[[t]]$, the power series ring in one variable t over A_0, with the (complete) ultranorm given by $|f| = 2^{-n}$ for $f \in t^nA \setminus t^{n+1}A$. Let $\iota : A_0 \to \boldsymbol{k}$ be a ring morphism into a field $\boldsymbol{k}$, let Γ be an ordered abelian group with a distinguished element $1 > 0$. We identify $\mathbb{Z}$ with its image in Γ via $k \mapsto k \cdot 1$, which makes $\mathbb{Z}$ an ordered subgroup of Γ. (We do not assume here that 1 is the least positive element of Γ.) This yields the Hahn field $K = \boldsymbol{k}(\!(t^\Gamma)\!)$ with its valuation ring $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]] \supseteq \boldsymbol{k}[[t]]$. Now ι extends to the ring morphism $A \to \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$,

$$\sum_n c_n t^n \mapsto \sum_n \iota(c_n) t^n \in \boldsymbol{k}[[t]] \quad \text{(with all } c_n \in A_0).$$

We have a natural A-analytic structure (ι_n) on $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$, where ι_0 is the above ring morphism $A \to \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$, and more generally, for $f = \sum a_\nu Y^\nu$ in $A\langle Y\rangle$ and $y \in (\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]])^n$,

$$\iota_n(f)(y) \ := \ \sum_\nu \iota_0(a_\nu) y^\nu \in k[[t^{\Gamma^{\geqslant}}]].$$

Note that $\iota_0(A)$ is the subring $\iota(A_0)[[t]]$ of $\boldsymbol{k}[[t]]$.

Returning to the general setting, let R be an A-ring. Among its units are clearly the elements $1 + a_1y_1 + \cdots + a_ny_n$ for $a_1, \dots, a_n \in \smallfrown\!\!\!\!o(A)$ and $y_1, \dots, y_n \in R$.

Any ideal I of R yields a *congruence relation* for the A-analytic structure of R. This means: for any $f \in A\langle Y\rangle$ and any $x, y \in R^n$ with $x \equiv y \mod I$ (that is, $x_1 - y_1, \dots, x_n - y_n \in I$), we have $f(x) \equiv f(y) \mod I$, an immediate consequence of Corollary 3.6. Thus R/I is an A-ring, given by

$$f(y_1 + I, \dots, y_n + I) \ := \ f(y_1, \dots, y_n) + I \qquad (f \in A\langle Y\rangle,\ (y_1, \dots, y_n) \in R^n).$$

This construal of R/I as an A-ring is part of our goal of developing some algebra for A-rings analogous to ordinary facts about rings. But we need some extra notational flexibility in dealing with indeterminates, as we already tacitly used in this argument about R/I: we do not want to be tied down to the particular sequence of indeterminates $Y_1, Y_2, \dots$ used in the definition of *A-analytic structure.* Namely, for any tuple $Z = (Z_1, \dots, Z_n)$ of distinct indeterminates, not necessarily among the $X_1, X_2, \dots, Y_1, Y_2, \dots$, any $f = f(Z) = \sum_\nu a_\nu Z^\nu \in A\langle Z\rangle$ and $z \in R^n$ we set $f(z) := \big(\iota_n f(Y)\big)(z)$, where $f(Y) := \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$.

This is in harmony with other notational conventions: Let $V, Z_1, \dots, Z_n$ be distinct variables. Identifying $A\langle Z\rangle = A\langle Z_1, \dots, Z_n\rangle$ as usual with a subring of $A\langle V, Z\rangle$, this harmony means that for $f \in A\langle Z\rangle$ and $(v, z_1, \dots, z_n)$ in R^{n+1} we have $f(z_1, \dots, z_n) = f(v, z_1, \dots, z_n)$, where the last f refers to the image of the series $f \in A\langle Z\rangle$ in $A\langle V, Z\rangle$. Thus we can add dummy variables on the left. We can also add them at other places: identifying $f \in A\langle Z\rangle$ with its image in $A\langle Z_1,\dots,Z_i,V,Z_{i+1},\dots,Z_n\rangle$ as usual, where $1 \leqslant i \leqslant n$, we have likewise

$$f(z_1,\dots,z_n) \ = \ f(z_1,\dots,z_i,v,z_{i+1},\dots,z_n)$$

for $(z_1,\dots,z_i,v,z_{i+1},\dots,z_n) \in R^{n+1}$. We shall tacitly use these facts.

Henselianity Again. Let R be an A-ring. Note that $\mathcal{O}(A)R$, that is, the ideal of R generated by $\iota_0\big(\mathcal{O}(A)\big)$, is contained in the Jacobson radical of R, because for $a_1,\dots,a_n \in \mathcal{O}(A)$ the series $1+a_1Y_1+\cdots+a_nY_n$ is a unit of $A\langle Y\rangle$. In later sections we consider the case that R is a valuation ring whose maximal ideal is $\sqrt{\mathcal{O}(A)R}$, and then the following is relevant:

Lemma 4.3. *The pair $(R, \mathcal{O}(A)R)$ is henselian, hence so is $(R, \sqrt{\mathcal{O}(A)R})$.*

Proof. Let $n = 1$, so $Y = Y_1$. We show that any polynomial

$$f(Y) \ = \ 1 + Y + z_2Y^2 + \cdots + z_NY^N \in R[Y], \qquad (N \in \mathbb{N}^{\geqslant 2})$$

with $z_2,\dots,z_N \in \mathcal{O}(A)R$ has a zero in R. Take m and $x \in R^m$ such that

$$z_2 \ = \ g_2(x),\dots,z_N \ = \ g_N(x), \quad g_2,\dots,g_N \in \mathcal{O}(A)A[X] \subseteq \mathcal{O}(A)A\langle X\rangle.$$

Then $F(X,Y) := 1 + Y + g_2(X)Y^2 + \cdots + g_N(X)Y^N \in A[X,Y] = A\langle X,Y\rangle$ is regular in Y of degree 1, so $F(X,Y) = E\cdot(Y-c)$ for a unit E of $A\langle X,Y\rangle$ and $c \in A\langle X\rangle$. Thus $f(Y)$ has a zero $c(x)$ in R. □

Extensions of A-rings. Let R be an A-ring. When referring to an A-ring R^* as *extending* R this means of course that R is a subring of R^*, but also includes the requirement that the A-analytic structure of R^* extends that of R.

A set $S \subseteq R$ is said to be A-closed (in R) if for all m, $f \in A\langle X\rangle$ and $x_1,\dots,x_m$ in S we have $f(x_1,\dots,x_m) \in S$. Then S is a subring of R and the A-analytic structure of R restricts to an A-analytic structure on S. We view such S as an A-ring so as to make the A-ring R extend S. For $S \subseteq R$, the *A-closure of S in R* is the smallest (with respect to inclusion) A-closed subset of R that contains S.

Lemma 4.4. *Let R^* be an A-ring extending R, and $y = (y_1,\dots,y_n) \in (R^*)^n$. Let $R\langle y\rangle$ be the A-closure of $R \cup \{y_1,\dots,y_n\}$ in R^*. Then*

$$R\langle y\rangle \ = \ \bigcup_m \{g(x,y) : \ x \in R^m, \ g \in A\langle X,Y\rangle\}.$$

Here is a consequence of Lemma 3.5:

Lemma 4.5. *Suppose R^* is an A-ring that extends R. Let $f \in A\langle X, Y_1,\dots,Y_n\rangle$ and $x \in R^m$, and assume $y_1,\dots,y_n \in R^*$ are integral over R. Then*

$$f(x,y_1,\dots,y_n) \ \in \ R[y_1,\dots,y_n].$$

Proof. By increasing m and accordingly extending x with extra coordinates we arrange that for $j = 1,\dots,n$ we have a polynomial $f_j(X,Y_j) \in A[X,Y_j]$, monic in Y_j, with $f_j(x,y_j) = 0$. Now apply Lemma 3.5. □

Lemma 4.6. *Let R^* be a ring extension of R with $z \in R^*$ integral over R. Then at most one A-analytic structure on $R[z]$ makes $R[z]$ an A-ring extending R.*

Proof. We can assume $R^* = R[z]$. Take a monic polynomial $\phi \in R[Z]$, say of degree $d \geqslant 1$, with $\phi(z) = 0$. Let R^* be equipped with an A-analytic structure extending that of R, and let $g \in A\langle Y_1, \dots, Y_n\rangle$, $n \geqslant 1$, and let $y_1, \dots, y_n \in R^*$; we have to show that then the element $g(y_1, \dots, y_n) \in R^*$ does not depend on the given A-analytic structure on R^*. We have $\phi(Z) = x_{00} + x_{01}Z + \cdots + x_{0,d-1}Z^{d-1} + Z^d$ with $x_{00}, \dots, x_{0,d-1} \in R$ and $y_j = x_{j0} + x_{j1}z + \cdots + x_{j,d-1}z^{d-1}$, $x_{j0}, \dots, x_{j,d-1} \in R$, for $j = 1, \dots, n$. We now set $m := (1+n)d$ and

$$\begin{aligned} x &:= (x_{00}, \dots, x_{0,d-1}, x_{10}, \dots, x_{1,d-1}, \dots, x_{n0}, \dots, x_{n,d-1}) \in R^m, \\ X &= (X_1, \dots, X_m) := \big(X_{00}, \dots, X_{0,d-1}, \dots, X_{n0}, \dots, X_{n,d-1}\big), \end{aligned}$$

so $\phi(Z) = F(x, Z)$, $F(X, Z) := X_{00} + X_{01}Z + \cdots + X_{0,d-1}Z^{d-1} + Z^d \in A[X, Z]$. Let $G(X, Z) \in A\langle X, Z\rangle$ be the following substitution instance of g:

$$g\big(X_{10} + X_{11}Z + \cdots + X_{1,d-1}Z^{d-1}, \dots, X_{n0} + X_{n1}Z + \cdots + X_{n,d-1}Z^{d-1}\big).$$

Lemma 3.4 gives $G(X, Z) = Q(X, Z)F(X, Z) + R_0 + R_1Z + \cdots + R_{d-1}Z^{d-1}$ with $R_0, \dots, R_{d-1} \in A\langle X\rangle$, and so $g(y) = G(x, z) = R_0(x) + R_1(x)z + \cdots + R_{d-1}(x)z^{d-1}$, which uses only the A-analytic structure on R, not that on R^*. □

Proposition 4.7. *Let R^* be a ring extension of R and integral over R. Then some A-analytic structure on R^* makes R^* an A-ring extending R.*

Proof. In view of Lemmas 4.5 and 4.6 this reduces to the case $R^* = R[z]$ where $z \in R^*$ is integral over R. Let $\phi(Z) \in R[Z]$ be as in the proof of Lemma 4.6, in particular monic of degree $d \geqslant 1$ in Z. If the ring extension $R[Z]/\phi(Z)R[Z]$ of R can be given an A-analytic structure extending that of R, then this is also the case for its image $R[z]$ under the R-algebra morphism $R[Z] \to R[z]$ sending Z to z. Thus replacing $R[z]$ by $R[Z]/\phi(Z)R[Z]$ if necessary we arrange that $R[z]$ is free as an R-module with basis $1, z, \dots, z^{d-1}$. We now adopt other notation from the proof above, where $n \geqslant 1$ and where we introduced a tuple

$$X = (X_1, \dots, X_m) = (X_{00}, \dots, X_{n,d-1})$$

of $m = (n+1)d$ distinct variables, the polynomial $F(X, Z) \in A[X, Z]$, and for any $g \in A\langle Y\rangle = A\langle Y_1, \dots, Y_n\rangle$ the series $G = G(X, Z) \in A\langle X, Z\rangle$, and the series $R_0, \dots, R_{d-1} \in A\langle X\rangle$. To indicate their dependence on g we set

$$\begin{aligned} G_g &:= G, \quad R_{g,0} := R_0, \quad \dots, \quad R_{g,d-1} := R_{d-1}, \\ R_g &:= R_{g,0} + R_{g,1}Z + \cdots + R_{g,d-1}Z^{d-1} \in A\langle X\rangle[Z]. \end{aligned}$$

We claim that setting $g(y) := R_g(x, z)$ for any $n \geqslant 1$ and $g \in A\langle Y\rangle$ yields an A-analytic structure on $R[z]$ extending that on R. We just verify two items that are part of this claim: let $f, g, h, g_1, \dots, g_n \in A\langle Y\rangle$ and $y \in R^n$; then

(1) $gh(y) = g(y)h(y)$;
(2) $f(g_1, \dots, g_n)(y) = f\big(g_1(y), \dots, g_n(y)\big)$.

As to (1), we have $G_{gh} = G_g G_h$, so $R_{gh} \equiv R_g R_h \mod F$ in $A\langle X, Z\rangle$. We also have $R \in A\langle X\rangle[Z]$ of degree $< d$ in Z such that $R_g R_h \equiv R \mod F$ in $A\langle X\rangle[Z]$. Hence $R_{gh} = R$, and thus $gh(y) = R(x, z) = R_g(x, z)R_h(x, z) = g(y)h(y)$. As to (2), by Corollary 3.6 we have in $A\langle X, Z\rangle$,

$$G_{f(g_1, \dots, g_n)} = f\big(G_{g_1}, \dots, G_{g_n}\big) \equiv f(R_{g_1}, \dots, R_{g_n}) \mod F.$$

Note that $f(R_{g_1},\dots,R_{g_n})$ is obtained by substituting $R_{g_j,i}$ for X_{ji} in G_f, for $j=1,\dots,n$ and $i=0,\dots,d-1$ (and Z for Z), that is,

$$f(R_{g_1},\dots,R_{g_n}) \;=\; G_f\big(R_{g_1,0},\dots,R_{g_1,d-1},\dots,R_{g_n,0},\dots,R_{g_n,d-1},Z\big).$$

Making the same substitution in the congruence $G_f \equiv R_f \mod F$, using that the variables X_{ji} with $j=1,\dots,n$ and $i=0,\dots,d-1$ do not occur in F, we obtain

$$G_f\big(R_{g_1,0},\dots,R_{g_1,d-1},\dots,R_{g_n,0},\dots,R_{g_n,d-1},Z\big)$$

is congruent in $A\langle X,Z\rangle$ modulo F to

$$R_f(X_{00},\dots,X_{0,d-1},R_{g_1,0},\dots,R_{g_1,d-1},\dots,R_{g_n,0},\dots,R_{g_n,d-1},Z\big),$$

which is in $A\langle X\rangle[Z]$ of degree $< d$ in Z, and thus equals $R_{f(g_1,\dots,g_n)}$. Since

$$f\big(g_1(y),\dots,g_n(y)\big) \;=\; R_f\big(x_{00},\dots,x_{0,d-1},R_{g_1,0}(x),\dots,R_{g_n,d-1}(x),z\big),$$

this yields $f\big(g_1(y),\dots,g_n(y)\big) = f(g_1,\dots,g_n)(y)$, as required. □

Corollary 4.8. *If R^* is a ring extension of R and integral over R, then there is a unique A-analytic structure on R^* that makes R^* an A-ring extending R.*

Corollary 4.9. *Let R_1 and R_2 be A-rings extending R and let $\phi : R_1 \to R_2$ be an R-algebra morphism such that $\phi(R_1)$ is integral over R. Then ϕ is a morphism of A-rings (that is, a homomorphism in the sense of $\mathcal{L}^A$-structures).*

Proof. The kernel of ϕ is a congruence relation on R as A-ring, so $\phi(R_1)$ has an A-analytic structure making $\phi : R_1 \to \phi(R_1)$ a morphism of A-rings. Since $\phi(R_1)$ is A-closed as a subset of R_2 it follows from Corollary 4.8 that this A-analytic structure on $\phi(R_1)$ coincides with the one that makes the inclusion $\phi(R_1) \to R_2$ a morphism of A-rings. Thus ϕ is a morphism of A-rings. □

Corollary 4.10. *Suppose the A-ring R^* extends R, and $z_i \in R^*$ for $i \in I$ is integral over R. Then $R[z_i : i \in I]$ is A-closed in R^*.*

Proof. Let $f(Y) \in A\langle Y\rangle$ and suppose $y_1,\dots,y_n \in R^*$ are integral over R; it suffices to show that then $f(y) \in R[y]$ where $y = (y_1,\dots,y_n)$. Take $x \in R^m$ and monic $f_j \in A\langle X\rangle[Y_j]$ such that $f_j(x,y_j) = 0$ for $j = 0,\dots,n$, and apply Lemma 3.5. □

Defining $R\langle Y\rangle$. Let R be an A-ring. To define a ring $R\langle Y\rangle$ analogous to the polynomial ring $R[Y]$, observe that polynomials over R arise from polynomials over $\mathbb{Z}$ by specializing: for $f(X,Y) \in \mathbb{Z}[X,Y]$ and $x \in R^m$ we have $f(x,Y) \in R[Y]$. We take this as a hint and with A instead of $\mathbb{Z}$, we define for $f(X,Y) \in A\langle X,Y\rangle$ and $x \in R^m$ the power series $f(x,Y) \in R[[Y]]$ as follows: $f(X,Y) = \sum_\nu f_\nu(X)Y^\nu$ with all $f_\nu \in A\langle X\rangle$, and then

$$f(x,Y) \;:=\; \sum_\nu f_\nu(x)Y^\nu.$$

Thus for fixed $x \in R^m$ the map $f(X,Y) \mapsto f(x,Y) : A\langle X,Y\rangle \to R[[Y]]$ is an A-algebra morphism. We define

$$R\langle Y\rangle \;:=\; \bigcup_m \{f(x,Y) :\; f = f(X,Y) \in A\langle X,Y\rangle,\; x \in R^m\} \;\subseteq\; R[[Y]].$$

An easy consequence is that inside the ambient ring $R[[Y]]$ we have for $i \leqslant n$:

$$R\langle Y_1,\dots,Y_i\rangle \;=\; R\langle Y\rangle \cap R[[Y_1,\dots,Y_i]].$$

Lemma 4.11. *Given any $g_1,\dots,g_k \in R\langle Y\rangle$, $k\in\mathbb{N}$, there exists m, $x\in R^m$, and $f_1,\dots,f_k\in A\langle X,Y\rangle$, such that $g_1=f_1(x,Y),\dots,g_k=f_k(x,Y)$.*

Proof. Let $m_1,\dots,m_k\in\mathbb{N}$ and $x^1\in R^{m_1},\dots,x^k\in R^{m_k}$ be such that

$$\begin{aligned} g_1 &= f^1(x^1,Y),\dots,g_k = f^k(x^k,Y), \quad f^1\in A\langle X^1,Y\rangle,\dots,f^k\in A\langle X^k,Y\rangle,\\ x^1 &= (x_{11},\dots,x_{1m_1}),\ \dots,\ x^k = (x_{k1},\dots,x_{km_k}),\\ X^1 &:= (X_{11},\dots,X_{1m_1}),\ \dots,\ X^k := (X_{k1},\dots,X_{km_k}). \end{aligned}$$

We can also arrange for $m:=m_1+\cdots+m_k$ that

$$X = (X_1,\dots,X_m) = (X^1,\dots,X^k).$$

For $f_1(X,Y):=f^1(X^1,Y)\in A\langle X,Y\rangle,\dots,f_k(X,Y):=f^k(X^k,Y)\in A\langle X,Y\rangle$ we then have $f_1(x,Y)=g_1,\dots,f_k(x,Y)=g_k$ for $x=(x^1,\dots,x^k)\in R^m$. □

Corollary 4.12. *$R\langle Y\rangle$ is a subring of $R[[Y]]$ with $R[Y]\subseteq R\langle Y\rangle$. If R is a domain, then so is $R\langle Y\rangle$; if R has no nilpotents other than 0, then neither does $R\langle Y\rangle$. For an A-ring R^* extending R the inclusion $R[[Y]]\to R^*[[Y]]$ maps $R\langle Y\rangle$ into $R^*\langle Y\rangle$, so $R\langle Y\rangle$ is a subring of $R^*\langle Y\rangle$.*

Proof. The claim about domains holds because it holds with $R[[Y]]$ in place of $R\langle Y\rangle$. Suppose R has no nilpotents. With $\mathfrak{p}$ ranging over the prime ideals of R this yields an injective "diagonal" ring morphism $R[[Y]]\to\prod_{\mathfrak{p}}(R/\mathfrak{p})[[Y]]$ into a ring with no nilpotents other than 0, so $R[[Y]]$ has no such nilpotents either. □

By the remark following the definition of $R\langle Y\rangle$ we have for $i\leqslant n$ the subring $R\langle Y_1,\dots,Y_i\rangle$ of $R\langle Y\rangle$. The ring $A\langle Y\rangle$ as defined in Section 3 is the same as the ring $A\langle Y\rangle$ as defined just now for $R=A$ viewed as an A-ring.

Corollary 4.13. *Suppose the A-ring R^* extends R and is integral over R. Then $R^*\langle Y\rangle$ is generated as a ring over its subring $R\langle Y\rangle$ by R^*.*

Proof. Using Corollary 4.10 it suffices to consider the case $R^*=R[z]$ where $z\in R^*$ is integral over R and to show $R^*\langle Y\rangle = R\langle Y\rangle[z]$. Let $f(X,Y)\in A\langle X,Y\rangle$ and $x\in(R^*)^m$. Towards proving $f(x,Y)\in R\langle Y\rangle[z]$, let $\phi(z)=0$ where

$$\phi(Z) = Z^d+u_{0,d-1}Z^{d-1}+\cdots+u_{0,0},\quad d\geqslant 1,\quad u_{0,0},\dots,u_{0,d-1}\in R.$$

Then for $i=1,\dots,m$ we have $x_i=u_{i0}+u_{i1}z+\cdots+u_{i,d-1}z^{d-1}$ with all $u_{ij}\in R$. Let $U=(U_{ij})_{0\leqslant i\leqslant m, j<d}$ be a tuple of distinct variables different also from the Y_j and Z and set $u:=(u_{ij})$. Then $f(x,Y)=g(u,z,Y)$ where

$$g(U,Z,Y) = f\big(\sum_{j<d}U_{1j}Z^i,\dots,\sum_{j<d}U_{mj}Z^i,Y\big) \in A\langle U,Z,Y\rangle.$$

In the ring $A\langle U,Z,Y\rangle$, $g(U,Z,Y)$ is congruent modulo $Z^d+U_{0,d-1}Z^{d-1}+\cdots+U_{0,0}$ to $g_0(U,Y)+g_1(U,Y)Z+\cdots+g_{d-1}(U,Y)Z^{d-1}$ for suitable $g_0,\dots,g_{d-1}\in A\langle U,Y\rangle$, by Lemma 3.4, and for such g_j we have

$$g(u,z,Y) = g_0(u,Y)+g_1(u,Y)z+\cdots+g_{d-1}(u,Y)z^{d-1}\in R\langle Y\rangle[z]. \qquad \square$$

Let $x\in R^m$. Then we equip R with the $A\langle X\rangle$-analytic structure $(\iota_{n,x})$ given for $f(X,Y)\in A\langle X\rangle\langle Y\rangle = A\langle X,Y\rangle$ by

$$\iota_{n,x}f : R^n\to R,\qquad y\mapsto f(x,y).$$

We refer to R with this $A\langle X\rangle$-analytic structure as *the* (A,x)*-ring* R. Construing R as an (A,x)-ring gives the same subring $R\langle Y\rangle$ of $R[[Y]]$ as when considering R as an A-ring.

$R\langle Z\rangle$ as an A-ring. Let $Z_1,\dots,Z_N$ with $N\in\mathbb{N}$ be distinct variables different from $X_1,X_2,\dots,$ and set $Z:=(Z_1,\dots,Z_N)$. We define $R\langle Z\rangle = R\langle Z_1,\dots,Z_N\rangle$ in the same way as $R\langle Y_1,\dots,Y_N\rangle$, with $Z_1,\dots,Z_N$ in the role of $Y_1,\dots,Y_N$. We make $R\langle Z\rangle$ an A-ring extending R as follows. Let $f\in A\langle Y\rangle$ and $u_1(x,Z),\dots,u_n(x,Z)$ in $R\langle Z\rangle$, where $u_1(X,Z),\dots,u_n(X,Z)\in A\langle X,Z\rangle$ and $x\in R^m$. Set

$$g(X,Z) \ := \ f\big(u_1(X,Z),\dots,u_n(X,Z)\big)\in A\langle X,Z\rangle.$$

Our aim is to define $f(u_1(x,Z),\dots,u_n(x,Z)) := g(x,Z)\in R\langle Z\rangle$. In order for this to make sense as a definition we first show:

Lemma 4.14. *Suppose $v_j(X,Z)\in A\langle X,Z\rangle$ and $u_j(x,Z)=v_j(x,Z)$ for $j=1,\dots,n$. Set $h(X,Z):=f\big(v_1(X,Z),\dots,v_n(X,Z)\big)\in A\langle X,Z\rangle$. Then*

$$g(x,Z) \ = \ h(x,Z).$$

Proof. By Corollary 3.6 we have for distinct variables $U_1,\dots,U_n,V_1,\dots,V_n$,

$$f(U_1,\dots,U_n)-f(V_1,\dots,V_n)\in(U_1-V_1,\dots,U_n-V_n)A\langle U,V\rangle.$$

Substituting $u_j(X,Z)$ and $v_j(X,Z)$ for U_j and V_j gives

$$g(X,Z)-h(X,Z)\in\big(u_1(X,Z)-v_1(X,Z),\dots,u_n(X,Z)-v_n(X,Z)\big)A\langle X,Z\rangle$$

from which we obtain the desired result by substituting x for X. $\square$

Now the next lemma shows the above does define $f\big(u_1(x,Z),\dots,u_n(x,Z)\big)$:

Lemma 4.15. *Let $m_1,m_2\in\mathbb{N}$, $m:=m_1+m_2$, and*

$$\begin{aligned} X^1 \ &:= \ (X_1,\dots,X_{m_1}), \quad X^2 \ := \ (X_{m_1+1},\dots,X_m),\\ x^1 \ &= \ (x_1,\dots,x_{m_1})\in R^{m_1}, \quad x^2 \ = \ (x_{m_1+1},\dots,x_m)\in R^{m_2}. \end{aligned}$$

Suppose that the series

$$u^1(X^1,Z),\dots,u^n(X^1,Z)\in A\langle X^1,Z\rangle, \quad v^1(X^2,Z),\dots,v^n(X^2,Z)\in A\langle X^2,Z\rangle$$

are such that $u^1(x^1,Z)=v^1(x^2,Z),\dots,u^n(x^1,Z)=v^n(x^2,Z)$. Then for

$$\begin{aligned} g^1(X^1,Z) \ &:= \ f\big(u^1(X^1,Z),\dots,u^n(X^1,Z)\big)\in A\langle X^1,Z\rangle,\\ g^2(X^2,Z) \ &:= \ f\big(v^1(X^2,Z),\dots,v^n(X^2,Z)\big)\in A\langle X^2,Z\rangle \end{aligned}$$

we have $g^1(x^1,Z)=g^2(x^2,Z)$ in $R\langle Z\rangle$.

Proof. Set $X:=(X^1,X^2)=(X_1,\dots,X_m)$, and for $j=1,\dots,n$,

$$\begin{aligned} u_j(X,Z) \ &:= \ u^j(X^1,Z)\in A\langle X,Z\rangle, \quad v_j(X,Z) \ := \ v^j(X^2,Z)\in A\langle X,Z\rangle,\\ g(X,Z) \ &:= \ f\big(u_1(X,Z),\dots,u_n(X,Z)\big) \ = \ g^1(X^1,Z)\in A\langle X,Z\rangle,\\ h(X,Z) \ &:= \ f\big(v_1(X,Z),\dots,v_n(X,Z)\big) \ = \ g^2(X^2,Z)\in A\langle X,Z\rangle. \end{aligned}$$

Then for $x=(x_1,\dots,x_m)$ we have $u_j(x,Z)=v_j(x,Z)$ for $j=1,\dots,n$, so $g(x,Z)=h(x,Z)$ by Lemma 4.14, and thus $g^1(x^1,Z)=g^2(x^2,Z)$. $\square$

We have now defined for $f \in A\langle Y\rangle$ a corresponding operation

$$(u_1,\dots,u_n) \mapsto f(u_1,\dots,u_n) \ : \ R\langle Z\rangle^n \to R\langle Z\rangle.$$

This makes $R\langle Z\rangle$ an A-ring extending R. For $f \in A\langle X,Z\rangle$ and $x \in R^m$ we can interpret $f(x,Z)$ on the one hand as the element of $R[[Z]]$ defined in the beginning of this subsection (with Y instead of Z), but also as the element of $R\langle Z\rangle$ obtained by evaluating f at the point $(x,Z) \in R\langle Z\rangle^{m+N}$ according to the A-analytic structure we gave $R\langle Z\rangle$; one checks easily that these two interpretations give the same element of $R\langle Z\rangle$, so there is no conflict of notation. This also shows that $R\langle Z\rangle$ is generated as an A-ring by its subset $R \cup \{Z_1,\dots,Z_N\}$.

For $R = A$ as an A-ring the above yields the A-ring $A\langle Z\rangle$ extending A. Let $f = f(Y) = \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$. One checks easily that for $(g_1,\dots,g_n) \in A\langle Z\rangle^n$ the convergent sum $f(g_1,\dots,g_n) = \sum_\nu a_\nu g_1^{\nu_1}\cdots g_n^{\nu_n} \in A\langle Z\rangle$ equals $f(g_1,\dots,g_n)$ as defined above for $R = A$, so this causes no conflict of notation. It is routine to check that for any A-ring R and $z \in R^N$ the evaluation map $g \mapsto g(z) : A\langle Z\rangle \to R$ is a morphism of A-rings. For $N = 0$ this is just $\iota_0 : A \to R$.

Corollary 4.16. *Let $J := \sqrt{\mathcal{o}(A)R}$. Then $\big(R\langle Z\rangle, JR\langle Z\rangle\big)$ is henselian.*

Proof. Applying Lemma 4.3 to the A-ring $R\langle Z\rangle$, the pair $\big(R\langle Z\rangle, \sqrt{\mathcal{o}(A)R\langle Z\rangle}\big)$ is henselian. Now use that $JR\langle Z\rangle \subseteq \sqrt{\mathcal{o}(A)R\langle Z\rangle}$. □

Our next goal is to define for $z \in R^N$ an evaluation map $g \mapsto g(z) : R\langle Z\rangle \to R$. We do this in the next section under a further noetherian assumption on A.

5. The case of noetherian A

Let A be a noetherian ring with an ideal $\mathcal{o}(A) \neq A$ such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ (with e ranging here and below over $\mathbb{N}$) and A is $\mathcal{o}(A)$-adically complete. Taking $0 < \delta < 1$ and defining $|a| := \delta^n$ if $a \in \mathcal{o}(A)^n \setminus \mathcal{o}(A)^{n+1}$ for $a \in A^{\neq}$ and $|0| := 0$ gives an ultranorm on A with respect to which A is complete, with $\mathcal{o}(A) = \{a \in A : |a| < 1\}$. Then the $\mathcal{o}(A)$-adic topology is the norm-topology. Take $t_1,\dots,t_r \in A$, $r \in \mathbb{N}$, such that $\mathcal{o}(A) = (t_1,\dots,t_r)$. Below, $n \geqslant 1$ and $Y = (Y_1,\dots,Y_n)$ as before, and λ,μ,ν range over $\mathbb{N}^n$.

Lemma 5.1. *Let $f = \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$. Then there is $d \in \mathbb{N}^{\geqslant 1}$ such that for all ν with $|\nu| \geqslant d$ we have $a_\nu = \sum_{|\mu|<d} a_\mu b_{\mu\nu}$ where the $b_{\mu\nu} \in \mathcal{o}(A)$ can be chosen such that $b_{\mu\nu} \to 0$ as $|\nu| \to \infty$ for each fixed μ with $|\mu| < d$.*

Proof. Since $a_\nu \to 0$ as $|\nu| \to \infty$, we have $a_\nu \in \mathcal{o}(A)^{e(\nu)}$ with $e(\nu) \in \mathbb{N}$, $e(\nu) \to \infty$ as $|\nu| \to \infty$. So $a_\nu = P_\nu(t_1,\dots,t_r)$ with $P_\nu \in A[T_1,\dots,T_r]$ homogeneous of degree $e(\nu)$. Take $d_0 \in \mathbb{N}$ such that the ideal of $A[T_1,\dots,T_r]$ generated by the P_ν is already generated by the P_μ with $|\mu| < d_0$. Next take $d \geqslant d_0$ in $\mathbb{N}^{\geqslant 1}$ so large that $e(\nu) > e(\mu)$ for all μ,ν with $|\mu| < d_0$ and $|\nu| \geqslant d$. Let $|\nu| \geqslant d$. Then $P_\nu = \sum_{|\mu|<d_0} P_\mu Q_{\mu\nu}$ with each $Q_{\mu\nu} \in A[T_1,\dots,T_r]$ homogeneous of degree $e(\nu) - e(\mu)$. Hence

$$a_\nu \ = \sum_{|\mu|<d_0} a_\mu b_{\mu\nu}, \qquad b_{\mu\nu} \ := \ Q_{\mu\nu}(t_1,\dots,t_r) \in \mathcal{o}(A),$$

which yields the desired result. □

Let f, d, and the $b_{\mu\nu}$ be as in the lemma. For μ with $|\mu| < d$ we set

$$f_\mu \ := \ Y^\mu + \sum_{|\nu|\geqslant d} b_{\mu\nu} Y^\nu \in A\langle Y\rangle, \quad \text{so} \quad f \ = \ \sum_{|\mu|<d} a_\mu f_\mu.$$

Therefore $\mathcal{o}(A\langle Y\rangle) = (t_1, \dots, t_r)A\langle Y\rangle$ and the ultranorm on $A\langle Y\rangle$ induced by the above ultranorm on A has the property that for all $f \in A\langle Y\rangle$ and n,

$$|f| \leqslant \delta^n \iff f \in \mathcal{o}(A\langle Y\rangle)^n,$$

so the norm-topology of $A\langle Y\rangle$ is the same as its $\mathcal{o}(A\langle Y\rangle)$-adic topology. Moreover,

$$A[Y] \cap \mathcal{o}(A\langle Y\rangle)^n \ = \mathcal{o}(A)^n A[Y], \text{ for all } n,$$

so $A\langle Y\rangle$ with the $\mathcal{o}(A\langle Y\rangle)$-adic topology is a completion of its noetherian subring $A[Y]$ with the $\mathcal{o}(A)A[Y]$-adic topology. Then $A\langle Y\rangle$ is noetherian by [26, Theorem 8.12], so $A\langle Y\rangle$ inherits the conditions we imposed on A at the beginning of this section.

Passing to A/I. Let I be a proper ideal of A. Since A is noetherian, I is closed in A by [26, Theorem 8.14]. We equip A/I with its quotient norm, and observe that the ring morphism

$$A\langle Y\rangle \to (A/I)\langle Y\rangle, \qquad f := \sum_\nu a_\nu Y^\nu \mapsto f/I := \sum_\nu (a_\nu + I)Y^\nu$$

is surjective and that its kernel contains $IA\langle Y\rangle$. Moreover:

Lemma 5.2. *The ring morphism $A\langle Y\rangle \to (A/I)\langle Y\rangle$ has the following properties:*

(i) *its kernel is $IA\langle Y\rangle$, a closed proper ideal of $A\langle Y\rangle$;*

(ii) *the induced ring isomorphism*

$$A\langle Y\rangle/IA\langle Y\rangle \to (A/I)\langle Y\rangle$$

is norm preserving, with the quotient norm on $A\langle Y\rangle/IA\langle Y\rangle$;

(iii) *for $f, g_1, \dots, g_n \in A\langle Y\rangle$ we have $(f/I)(g_1/I, \dots, g_n/I) = f(g_1, \dots, g_n)/I$.*

Proof. Suppose $f(Y) = \sum_\nu a_\nu Y^\nu \in A\langle Y\rangle$ is in the kernel. Then all $a_\nu \in I$. For some $d \geqslant 1$ we have an equality $f(Y) = \sum_{|\mu|<d} a_\mu f_\mu(Y)$ with μ ranging over $\mathbb{N}^n$ and all $f_\mu(Y) \in A\langle Y\rangle$, hence $f(Y) \in IA\langle Y\rangle$. Verifying (ii) is routine using the definitions of the norms involved. Item (iii) is an easy consequence of (ii). □

The effect on A-rings. Our noetherian assumption on A has consequences for A-rings. *In the rest of this section R is an A-ring, and $Z = (Z_1, \dots, Z_N)$ as before.* With (ι_n) the A-analytic structure of R, here is a corollary of Lemma 5.2(i):

Corollary 5.3. *Suppose the proper ideal I of A is contained in the kernel of ι_0. Then we have an (A/I)-analytic structure $(\iota_n/I)_n$ on R given by*

$$(\iota_n/I)(f/I) \ := \ \iota_n(f) \quad \text{for } f \in A\langle Y\rangle.$$

Lemma 5.4. *Let $f = \sum_\nu a_\nu(X)Y^\nu \in A\langle X\rangle\langle Y\rangle = A\langle X, Y\rangle$. Suppose $x \in R^m$ and $f(x, Y) = 0$, that is, $a_\nu(x) = 0$ for all ν. Then $f(x, y) = 0$ for all $y \in R^n$.*

Proof. With $A\langle X\rangle$ in the role of A, the above gives a finite sum decomposition $f \ = \ \sum_{|\mu|<d} a_\mu f_\mu$ with the $f_\mu \in A\langle X, Y\rangle$, which yields the desired conclusion. □

We can now prove the following key universal property of the A-ring $R\langle Z\rangle$:

Theorem 5.5. *Let $\phi : R \to R^*$ be an A-ring morphism and $z = (z_1, \ldots, z_N) \in (R^*)^N$. Then ϕ extends uniquely to an A-ring morphism $R\langle Z\rangle \to R^*$ sending $Z_1, \ldots, Z_N$ to $z_1, \ldots, z_N$, respectively.*

Proof. Let $g(Z) \in R\langle Z\rangle$. Take $f(X, Z) \in A\langle X, Z\rangle$ and $x \in R^m$ such that $g(Z) = f(x, Z)$, and set $g(z) := f(\phi(x), z) \in R^*$. By Lemma 5.4 (with Z instead of Y) and the usual arguments with dummy variables, this element of R^* depends only on $g(Z)$ and z, not on the choice of m, f, x. Moreover, the map $g(Z) \mapsto g(z) : R\langle Z\rangle \to R^*$ is a ring morphism that extends ϕ and sends Z_j to z_j for $j = 1, \ldots, N$. One also verifies easily that for $F \in A\langle Y\rangle$ and $g_1, \ldots, g_n \in R\langle Z\rangle$ we have

$$F(g_1, \ldots, g_n)(z) \;=\; F\big(g_1(z), \ldots, g_n(z)\big),$$

so this map $R\langle Z\rangle \to R^*$ is an A-ring morphism. □

We retain the notation $g(z)$ introduced in the proof above. In Theorem 5.5, $g \in R\langle Z_1, \ldots, Z_i\rangle$ with $i \leqslant N$ gives $g(z_1, \ldots, z_i) = g(z_1, \ldots, z_N)$ where on the right we take g as an element of $R\langle Z\rangle$. For $R = R^*$ and ϕ the identity on R this theorem gives the evaluation map $g \mapsto g(z) : R\langle Z\rangle \to R$ promised earlier as a morphism of A-rings. It is also a morphism of R-algebras.

Lemma 5.6. *Let $z \in R^N$. Then the kernel of the morphism $g \mapsto g(z) : R\langle Z\rangle \to R$ of R-algebras is the ideal $(Z_1 - z_1, \ldots, Z_N - z_N)R\langle Z\rangle$ of $R\langle Z\rangle$.*

Proof. For $N \geqslant 1$, Lemma 3.4 gives $R\langle Z\rangle = (Z_N - z_N)R\langle Z\rangle + R\langle Z_1, \ldots, Z_{N-1}\rangle$. Proceeding inductively we obtain $R\langle Z\rangle = (Z_1 - z_1, \ldots, Z_N - z_N)R\langle Z\rangle + R$, which gives the desired result. □

Note also that the map

$$R\langle Z\rangle \to \text{ring of } R\text{-valued functions on } R^N$$

assigning to each $g \in R\langle Z\rangle$ the function $z \mapsto g(z)$ is an R-algebra morphism.

Another special case of Theorem 5.5: let R^* be an A-ring extending R, let ϕ be the resulting inclusion $R \to R^*\langle Z\rangle$, and $z_j := Z_j \in R^*\langle Z\rangle$ for $j = 1, \ldots, N$. Then the corresponding A-ring morphism $R\langle Z\rangle \to R^*\langle Z\rangle$ is a restriction of the inclusion $R[[Z]] \to R^*[[Z]]$ and sends $f(x, Z) \in R\langle Z\rangle$ for $f \in A\langle X, Z\rangle$ and $x \in R^m$ to $f(x, Z) \in R^*\langle Z\rangle$. We identify $R\langle Z\rangle$ with an A-subring of $R^*\langle Z\rangle$ via this morphism. Thus in the situation of Lemma 4.4 we have

$$R\langle y\rangle \;=\; \{g(y) :\; g \in R\langle Y\rangle\}.$$

Let I be an ideal of R. Then the canonical map $R \to R/I$ extends to the morphism $R\langle Z\rangle \to (R/I)\langle Z\rangle$ of A-rings sending Z_j to Z_j for $j = 1, \ldots, N$, and we have:

Lemma 5.7. *The kernel of the above morphism $R\langle Z\rangle \to (R/I)\langle Z\rangle$ is $IR\langle Z\rangle$.*

Proof. With β ranging over $\mathbb{N}^N$, this morphism is a restriction of the ring morphism

$$\sum_\beta c_\beta Z^\beta \mapsto \sum_\alpha (c_\beta + I)Z^\beta \;:\; R[[Z]] \to (R/I)[[Z]] \qquad (\text{ all } c_\beta \in R),$$

so $IR\langle Z\rangle$ is contained in the kernel. Suppose $f(x, Z)$ is in the kernel where $x \in R^m$ and $f(X, Z) = \sum_\beta a_\beta(X)Z^\beta \in A\langle X, Z\rangle$. Then all $a_\beta(x) \in I$, and since for some $d \geqslant 1$ we have an equality $f(X, Z) = \sum_{|\alpha|<d} a_\alpha(X)f_\alpha(X, Z)$ with α ranging over $\mathbb{N}^N$ and all $f_\alpha(X, Z) \in A\langle X, Z\rangle$, substitution of x for X gives $f(x, Z) \in IR\langle Z\rangle$. □

Corollary 5.8. *Let* $J := \sqrt{\mathcal{o}(A)R}$. *Then* $JR\langle Z\rangle = \sqrt{\mathcal{o}(A)R\langle Z\rangle}$.

Proof. We have $\mathcal{o}(A)R\langle Z\rangle \subseteq JR\langle Z\rangle \subseteq \sqrt{\mathcal{o}(A)R\langle Z\rangle}$. It remains to note that $JR\langle Z\rangle$ is a radical ideal of $R\langle Z\rangle$, by Lemma 5.7 and a part of Corollary 4.12. □

Let $x \in R^m$, construe R as an (A, x)-ring, so R is equipped with a certain $A\langle X\rangle$-analytic structure, and let $\phi : R \to R^*$ be an A-ring morphism. Then $\phi : R \to R^*$ is also an $A\langle X\rangle$-ring morphism where we construe R^* as an $(A, \phi(x))$-ring, with $\phi(x) := \big(\phi(x_1), \dots, \phi(x_m)\big)$. For $z \in (R^*)^N$ the unique extension of ϕ to an A-ring morphism $R\langle Z\rangle \to R^*$ sending $Z_1, \dots, Z_N$ to $z_1, \dots, z_N$ is also an $A\langle X\rangle$-ring morphism. In other words, for $g \in R\langle Z\rangle$ and $z \in (R^*)^N$ the two ways of interpreting $g(z)$ give the same element of R^*, and so this raises no conflict of notation.

Substituting elements of $R\langle Z\rangle$ in elements of $R\langle Y\rangle$. Here is another case of Theorem 5.5: let $g_1, \dots, g_n \in R\langle Z\rangle$ and $\phi : R \to R\langle Z\rangle$ the inclusion map. Then ϕ extends uniquely to the A-ring morphism

$$f \mapsto f(g_1, \dots, g_n) \ : \ R\langle Y\rangle \to R\langle Z\rangle$$

that sends $Y_1, \dots, Y_n$ to $g_1, \dots, g_n$. For $z \in R^N$ we have

$$f(g_1, \dots, g_n)(z) \ = \ f\big(g_1(z), \dots, g_n(z)\big).$$

This follows for example from the uniqueness in Theorem 5.5.

Let now $n, d \geqslant 1$. For $N = n$ and $Y = Z$ this yields the automorphism

$$f(Y) \mapsto f\big(T_d(Y)\big)$$

of the A-ring $R\langle Y\rangle$ and the R-algebra $R\langle Y\rangle$, with inverse $g(Y) \mapsto g\big(T_d^{-1}(Y)\big)$.

Let $f \in A\langle X, Z\rangle$, $g_1(X, Z), \dots, g_N(X, Z) \in A\langle X, Z\rangle$, and set

$$h(X, Z) \ := \ f\big(X, g_1(X, Z), \dots, g_N(X, Z)\big) \in A\langle X, Z\rangle.$$

Then for $x \in R^m$ we can interpret $f\big(x, g_1(x, Z), \dots, g_N(x, Z)\big)$ on the one hand as $h(x, Z) \in R\langle Z\rangle$, and on the other hand as the element of $R\langle Z\rangle$ obtained by evaluating f at the point $\big(x, g_1(x, Z), \dots, g_N(x, Z)\big) \in R\langle Z\rangle^{m+N}$ according to the A-analytic structure we gave $R\langle Z\rangle$. By the uniqueness in Theorem 5.5 these two elements of $R\langle Z\rangle$ are equal, so this raises no conflict of notation.

Introducing $K\langle Y\rangle$. Let the A-ring R be a domain with fraction field K. Set

$$K\langle Y\rangle \ := \ \{c^{-1}g(Y) : \ c \in R^{\neq}, \ g(Y) \in R\langle Y\rangle \subseteq K[[Y]]\}.$$

Thus $K\langle Y\rangle$ is a subring of $K[[Y]]$ and contains $R\langle Y\rangle$ as a subring. For $n, d \geqslant 1$ the automorphism $g(Y) \mapsto g\big(T_d(Y)\big)$ of the A-ring $R\langle Y\rangle$ extends (uniquely) to an automorphism of the K-algebra $K\langle Y\rangle$, also to be indicated by $g \mapsto g\big(T_d(Y)\big)$.

Lemma 5.9. $K\langle Y\rangle \cap R[[Y]] = R\langle Y\rangle$, *inside the ambient ring* $K[[Y]]$.

Proof. The inclusion $\supseteq$ is clear. For the reverse inclusion, let $g \in K\langle Y\rangle \cap R[[Y]]$. Now $g = c^{-1}\sum_\nu a_\nu(x)Y^\nu$ with $c \in R^{\neq}$ and $\sum_\nu a_\nu(X)Y^\nu \in A\langle X, Y\rangle$, $x \in R^m$. By Lemma 5.1 applied to $A\langle X\rangle$ instead of A we have $d \in \mathbb{N}^{\geqslant 1}$ such that for all ν with $|\nu| \geqslant d$ we have $a_\nu(X) = \sum_{|\mu|<d} a_\mu(X)b_{\mu\nu}(X)$ where the $b_{\mu\nu} \in \mathcal{o}(A\langle X\rangle)$ are chosen such that $b_{\mu\nu} \to 0$ as $|\nu| \to \infty$ for each fixed μ with $|\mu| < d$. Put

$u_\mu := c^{-1}a_\mu(x) \in R$ for $|\mu| < d$. Then with a tuple $U = (U_\mu)_{|\mu|<d}$ of new variables and setting

$$F(X,U,Y) \ := \ \sum_{|\mu|<d} U_\mu Y^\nu + \sum_{|\nu|\geqslant d} \Big(\sum_{|\mu|<d} b_{\mu\nu}(X)U_\mu \Big) Y^\nu \in A\langle X,U,Y\rangle$$

we have $g(Y) = F(x,u,Y) \in R\langle Y\rangle$. □

For $y \in R^n$ and $f(Y) = c^{-1}g(Y) \in K\langle Y\rangle$ with $c \in R^{\neq}, g(Y) \in R\langle Y\rangle$, the element $c^{-1}g(y) \in K$ depends only on f, y, not on c, g, and so we can define $f(y) := c^{-1}g(y)$. The map $f \mapsto f(y) : K\langle Y\rangle \to K$ is a K-algebra morphism, extending the evaluation maps $K[Y] \to K$ and $R[Y] \to R$ sending $Y_1,\dots,Y_n$ to $y_1,\dots,y_n$, respectively. By Lemma 5.6 its kernel is the maximal ideal $(Y_1 - y_1,\dots,Y_n - y_n)K\langle Y\rangle$ of $K\langle Y\rangle$.

Suppose the A-ring S extends R, and is a domain with fraction field L taken as a field extension of K. Then $L[[Y]]$ has subrings $R\langle Y\rangle$, $K\langle Y\rangle$, $S\langle Y\rangle$, $L\langle Y\rangle$ with $K\langle Y\rangle \subseteq L\langle Y\rangle$. In this situation we have:

Lemma 5.10. *Assume S is integral over R and $b_1,\dots,b_m$ is a basis of the K-linear space L. Then $L\langle Y\rangle$ is a free $K\langle Y\rangle$-module with basis $b_1,\dots,b_m$.*

Proof. Let $g \in L\langle Y\rangle$ and take $c \in L^\times$ such that $cg \in S\langle Y\rangle$. Corollary 4.13 tells us that $S\langle Y\rangle$ is generated as a ring over its subring $R\langle Y\rangle$ by S, so $cg = \sum_{j\in J} a_j g_j$ with finite J, and $a_j \in S$, $g_j \in R\langle Y\rangle$ for all $j \in J$, so $g = \sum_j c^{-1}a_j g_j$. Each $c^{-1}a_j$ is a K-linear combination of $b_1,\dots,b_m$, so $g = b_1 f_1 + \cdots + b_m f_m$ with $f_1,\dots,f_m \in K\langle Y\rangle$. Moreover, if $f_1,\dots,f_m \in K\langle Y\rangle$ are not all zero, then $b_1 f_1 + \cdots + b_m f_m \neq 0$, by considering a monomial Y^ν for which one of the f_i has a nonzero coefficient. □

6. Valuation rings with A-analytic structure

We begin with generalities that do not require A to be noetherian. A *valuation A-ring* is an A-ring whose underlying ring is a valuation ring. An *A-field* is a valued field whose valuation ring is equipped with an A-analytic structure making it a valuation A-ring. The language $\mathcal{L}^A_{\preccurlyeq,D}$ is the language $\mathcal{L}^A$ of A-rings augmented with a binary relation symbol $\preccurlyeq$ (to encode the valuation) and a binary function symbol D (for restricted division). We construe an A-field K as an $\mathcal{L}^A_{\preccurlyeq,D}$-structure as follows, where R is the valuation A-ring of K:

- any $f \in A\langle Y\rangle$ is interpreted as the n-ary operation on K giving $f(y)$ its A-ring value in R for $y \in R^n$, and $f(y) := 0$ for $y \notin R^n$;
- $y_1 \preccurlyeq y_2 \ \Leftrightarrow y_1 \in y_2 R$;
- $D(y_1,y_2) = y_1/y_2$ if $y_1 \preccurlyeq y_2 \neq 0$, and $D(y_1,y_2) := 0$ otherwise.

This makes R an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of K. We define an *A-extension of* K to be a valued field extension of K whose valuation ring is equipped with an A-analytic structure that makes it an extension of the A-ring R. Thus any A-extension L of K is an A-field and naturally an $\mathcal{L}^A_{\preccurlyeq,D}$-structure so that K is a substructure of L.

Lemma 6.1. *Let K be an A-field with valuation A-ring R. Suppose E is an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of K. Then:*

(i) $R_E := \{a \in E : \ a \preccurlyeq 1\}$ *is an A-subring of R and is a valuation ring dominated by R;*
(ii) $E \subseteq \operatorname{Frac}(R_E) \subseteq K$, *and E is a valuation ring.*

Proof. It is clear that R_E is an A-subring of R. For $a, b \in R_E$, either $a \preccurlyeq b$, and then $a = D(a,b)b$, $D(a,b) \in R_E$, so $a \in R_E b$, or $b \preccurlyeq a$, and then likewise $b \in R_E a$. Thus R_E is a valuation ring. It is also clear that R dominates R_E. For $a \in E$, if $a \preccurlyeq 1$, then $a \in R_E$, so $a \in \mathrm{Frac}(R_E)$, and if $a \succ 1$, then $D(1,a) = a^{-1} \in R_E$, and so again $a \in \mathrm{Frac}(R_E)$. Since R_E is a valuation ring of $\mathrm{Frac}(R_E)$, so is E. $\square$

The A-extension generated over K by an element z. The language $\mathcal{L}^{A,K}_{\preccurlyeq,D}$ is $\mathcal{L}^{A}_{\preccurlyeq,D}$ augmented by names (constant symbols), one for each element of K, and we construe K and any A-extension of it accordingly as an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-structure. Let L be an A-extension of K. Let Z be an indeterminate and $z \in L$. Then any $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$ yields an element $\tau(z) \in L$. The set $\{\tau(z) : \tau(Z)$ is an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term$\}$ underlies a substructure of the $\mathcal{L}^{A}_{\preccurlyeq,D}$-structure L, namely the smallest substructure of the $\mathcal{L}^{A}_{\preccurlyeq,D}$-structure L that contains $K \cup \{z\}$; we do not claim this set is the underlying set of a subfield of L. Instead we call attention to the A-closed subring R_z of R_L with underlying set

$$\{\tau(z) : \ \tau(Z) \text{ is an } \mathcal{L}^{A,K}_{\preccurlyeq,D}\text{-term and } \tau(z) \preccurlyeq 1\}.$$

Note: $R \subseteq R_z$; if $z \preccurlyeq 1$, then $z \in R_z$; if $z \succ 1$, then $z^{-1} \in R_z$.

Lemma 6.2. *R_z is a valuation ring dominated by R_L.*

Proof. For L in the role of K this is a special case of Lemma 6.1(i). $\square$

Let K_z be the fraction field of R_z inside L, equipped with the valuation A-ring R_z. This makes K_z into an A-extension of K, and an $\mathcal{L}^{A}_{\preccurlyeq,D}$-substructure of L. Thus $\tau(z) \in K_z$ for every $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$.

Corollary 6.3. *K_z is the smallest substructure of the $\mathcal{L}^{A}_{\preccurlyeq,D}$-structure L that contains $K \cup \{z\}$ and whose underlying ring is a field. As a consequence, if $z \preccurlyeq 1$, then R_z is the smallest A-closed subring of R_L that contains $R \cup \{z\}$ and whose underlying ring is a valuation ring dominated by R_L.*

In the rest of this section, A is noetherian with an ideal $\mathcal{o}(A) \neq A$, such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ and A is $\mathcal{o}(A)$-adically complete. Also, R is always a valuation A-ring, and we let $\mathcal{o}(R)$ denote the maximal ideal of R. Thus $\mathcal{o}(A)R = tR$ for some $t \in \mathcal{o}(A)$. We let $\boldsymbol{k} = R/\mathcal{o}(R)$ denote the residue field of R, and K its fraction field. We construe the A-field K as an $\mathcal{L}^{A}_{\preccurlyeq,D}$-structure as described earlier.

Viability. We define R to be *viable* if $\mathcal{o}(A)R = \mathcal{o}(R) \neq \{0\}$ (so R is not a field). (This notion of viability is simpler and stricter than in [5], where extra flexibility was needed to be able to pass to A-extensions of K of finite degree. In the present set-up we we don't need to do that.) In order to make our Weierstrass preparation and division theorems useful for the model theory of R as a valuation A-ring we assume in the rest of this section:

R is viable.

Thus we have $t \in \mathcal{o}(A)$ with $\iota_0(t) \neq 0$ and $\mathcal{o}(R) = tR$, and then $vt := v\big(\iota_0(t)\big)$ is the smallest positive element of Γ. Below we fix such t and identify $\mathbb{Z}$ with its image in Γ via $k \mapsto k \cdot vt$, so $vt = 1$ and $\mathbb{Z}$ is a convex subgroup of Γ. It is clear that viability is inherited by A-subfields:

Lemma 6.4. *Suppose K_0 is a valued subfield of K and its valuation ring $R_0 = R \cap K_0$ is an A-subring of R. Then the valuation A-ring R_0 is viable.*

Note that R is henselian, by Lemma 4.3, so for any field extension F of K which is algebraic over K there is a unique valuation ring of F lying over R, and this valuation ring is the integral closure of R in F. Thus by Corollary 4.8:

Corollary 6.5. *If L is a valued field extension of K and is algebraic over K, then L has a unique expansion to an A-extension of K.*

In this corollary L might be an algebraic closure of K, in which case its valuation ring is the integral closure of R in L, and unlike the maximal ideal of R, the maximal ideal of this integral closure is not principal.

Corollary 6.6. *If z is algebraic over K, then $K(z)$ is the underlying field of K_z.*

Proof. Suppose z is algebraic over K. Then the valued subfield $K(z)$ of L expands uniquely to an A-extension of K by Lemma 6.5. This A-extension is then an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of L by Corollary 4.8. Now use Corollary 6.3. □

By the viability assumption on R the model theoretic results at the end of this paper do not apply to algebraically closed valued fields whose valuation ring is equipped with an A-analytic structure. To avoid this viability assumption one could replace the restricted power series rings over A with rings of separated power series over A where some variables range as before over the valuation ring and the other (formal) variables only over its maximal ideal. This is the direction taken by Lipshitz [24]; see also Lipshitz and Robinson [25]. Our treatment can probably be extended in this direction as well, but this will not be done here.

An A-extension of K is said to be *viable* if its valuation A-ring is viable.

Weierstrass preparation and division with parameters. Let

$$f \;=\; \sum_{\nu} a_\nu(X)Y^\nu \in A\langle X, Y\rangle, \qquad n \geqslant 1.$$

We now study how Weierstrass preparation applies to $f(x, Y)$ for $x \in R^m$, and how this depends on x. Lemma 5.1 with $A\langle X\rangle$ in the role of A gives $d \geqslant 1$ and $b_{\mu\nu} \in \mathcal{o}(A\langle X\rangle)$ for $|\mu| < d$ and $|\nu| \geqslant d$. As before we set for $|\mu| < d$,

$$f_\mu \;:=\; Y^\mu + \sum_{|\nu|\geqslant d} b_{\mu\nu}Y^\nu \in A\langle X, Y\rangle, \qquad f \;=\; \sum_{|\mu|<d} a_\mu f_\mu.$$

We order $\mathbb{N}^n$ lexicographically and for μ with $|\mu| < d$ we set

$$I(\mu) \;:=\; \{\lambda :\ |\lambda| < d,\ \lambda < \mu\}, \quad J(\mu) \;:=\; \{\lambda :\ |\lambda| < d,\ \lambda > \mu\}, \quad \text{so}$$

$$(*) \qquad f \;=\; \sum_{\lambda\in I(\mu)} a_\lambda f_\lambda + a_\mu f_\mu + \sum_{\lambda\in J(\mu)} a_\lambda f_\lambda.$$

Now fix μ with $|\mu| < d$ and introduce tuples

$$U_\mu := \big(U_{\lambda\mu} : \lambda \in I(\mu)\big), \qquad V_\mu := \big(V_{\lambda\mu} : \lambda \in J(\mu)\big)$$

of indeterminates, different from each other and from the X_i and Y_j. Set

$$\tilde{F}_\mu \;:=\; \sum_{\lambda\in I(\mu)} U_{\lambda\mu} f_\lambda + f_\mu + \sum_{\lambda\in J(\mu)} tV_{\lambda\mu} f_\lambda \in A\langle U_\mu, V_\mu, X, Y\rangle,$$

$$F_\mu \;:=\; \tilde{F}_\mu\big(U_\mu, V_\mu, X, T_d(Y)\big) \in A\langle U_\mu, V_\mu, X, Y\rangle.$$

Note that for $n = 1$ we have $T_d(Y) = Y$, so $F_\mu = \tilde{F}_\mu$.

Lemma 6.7. *F_μ is regular of degree $\ell := \mu_1 d^{n-1} + \cdots + \mu_n$ in Y_n, and so*

$$F_\mu \;=\; E \cdot (Y_n^\ell + G_1 Y_n^{\ell-1} + \cdots + G_\ell)$$

for a unit E of $A\langle U_\mu, V_\mu, X, Y\rangle$ and suitable $G_1, \ldots, G_\ell \in A\langle U_\mu, V_\mu, X, Y'\rangle$.

Here is a consequence of Lemma 6.7 for $n = 1$ (so $Y = Y_1$):

Corollary 6.8. *Let $n = 1$ and $g(Y) = \sum_{j=0}^{\infty} c_j Y^j \in K\langle Y\rangle$, $g \neq 0$. Then:*

(i) *there is $\mu \in \mathbb{N}$ with $c_i \preccurlyeq c_\mu \succ c_j$ whenever $i \leqslant \mu < j$;*
(ii) *for the unique μ in* (i) *we have $g(Y) = c \cdot r(Y) \cdot (Y^\mu + g_1 Y^{\mu-1} + \cdots + g_\mu)$ with $c = c_\mu \in K^\times$, $r(Y) \in R\langle Y\rangle^\times$, and $g_1, \ldots, g_\mu \in R$.*

Proof. We multiply g by an element of $K^\times$ to arrange $g \in R\langle Y\rangle$. Then $g(Y) = f(x, Y)$ with $x \in R^m$ and $f = f(X, Y) = \sum_j a_j(X) Y^j$ in $A\langle X, Y\rangle$, so $c_j = a_j(x)$ for all j. Lemma 5.1 with $A\langle X\rangle$ in the role of A gives $d \geqslant 1$ and $b_{ij} \in \mathcal{O}(A\langle X\rangle)$ for $i < d \leqslant j$ such that $a_j = \sum_{i<d} a_i b_{ij}$ for all $j \geqslant d$. Set

$$\gamma \;:=\; \min_{i<d} v(c_i), \qquad \mu \;:=\; \max\{i < d :\ v(c_i) = \gamma\}.$$

Then (i) holds for this μ: for $\mu < j$, distinguish the cases $j < d$ and $j \geqslant d$.

For (ii) we use the identities above for $n = 1$ and our f. The identity $(*)$ yields $f = \sum_{i<\mu} a_i f_i + a_\mu f_\mu + \sum_{\mu<i<d} a_i f_i$. Substituting x for X and factoring out $c := c_\mu = a_\mu(x)$ (possible because $c \neq 0$) gives

$$c^{-1} g(Y) \;=\; \sum_{i<\mu} (c_i/c) f_i(x, Y) + f_\mu(x, Y) + \sum_{\mu<i<d} (c_i/c) f_i(x, Y),$$

so for $u := \big(c_i/c : i < \mu\big) \in R^\mu$ and $v := \big(c_i/tc :\ \mu < i < d\big) \in R^{d-1-\mu}$ we have $c^{-1} g(Y) = F_\mu(u, v, x, Y)$. Now applying Lemma 6.7 for $n = 1$ shows that (ii) holds with $r(Y) = E(u, v, x, Y)$ and $g_i = G_i(u, v, x)$ for $i = 1, \ldots, \mu$. $\square$

Note that the proof above uses in a crucial way that $\mathcal{O}(R) = tR$.

Corollary 6.9. *Let R^* be an A-ring extending R, and suppose $y \in R^*$ is not integral over R. Then $R\langle y\rangle$ has the following properties, with $n = 1$ in* (i)*:*

(i) *the morphism $g(Y) \mapsto g(y) : R\langle Y\rangle \to R\langle y\rangle$ of A-rings is an isomorphism;*
(ii) *$R\langle y\rangle$ is a domain but not a valuation ring;*
(iii) *inside the ambient field $\operatorname{Frac}(R\langle y\rangle)$ we have $R\langle y\rangle \not\subseteq K(y)$.*

Proof. For (ii), use that $Y \notin tR\langle Y\rangle$ and $t \notin YR\langle Y\rangle$. For (iii), if $\operatorname{char} \boldsymbol{k} \neq 2$, then the polynomial $Z^2 - (1 + ty)$ has a zero in $R\langle y\rangle$ by Corollary 4.16, but has no zero in $K(y)$. If $\operatorname{char} \boldsymbol{k} = 2$, use instead the polynomial $Z^3 - (1 + ty)$. $\square$

We return to our $f(X, Y) \in A\langle X, Y\rangle$ with $n \geqslant 1$. To find out how Weierstrass preparation for $f(x, Y)$ depends on $x \in R^m$, we now introduce the quantifier-free $\mathcal{L}^A_{\preccurlyeq}$-formulas $Z(X)$ and $S_\mu(X)$ (for $|\mu| < d$) in the variables X:

$$Z(X) \;:=\; \bigwedge_{|\mu|<d} a_\mu(X) = 0,$$

$$S_\mu(X) \;:=\; a_\mu(X) \neq 0 \wedge \Big(\bigwedge_{\lambda \in I(\mu)} a_\lambda(X) \preccurlyeq a_\mu(X) \Big) \wedge \Big(\bigwedge_{\mu \in J(\mu)} a_\lambda(X) \prec a_\mu(X) \ \Big).$$

Lemma 6.10. *For the $\mathcal{L}^A_{\preccurlyeq}$-structure R we have the following:*

(i) *for all $x \in R^m$, $Z(x)$ holds or $S_\mu(x)$ holds for some μ with $|\mu| < d$;*

(ii) *suppose $x \in R^m$, $|\mu| < d$, and $S_\mu(x)$ holds; so $u_{\lambda\mu} := a_\lambda(x)/a_\mu(x) \in R$ for $\lambda \in I(\mu)$ and $v_{\lambda\mu} := a_\lambda(x)/ta_\mu(x) \in R$ for $\lambda \in J(\mu)$. Then with*

$$u_\mu \ := \ \big(u_{\lambda\mu} : \lambda \in I(\mu)\big), \qquad v_\mu \ := \ \big(v_{\lambda\mu} : \lambda \in J(\mu)\big),$$

and $E, G_1, \dots, G_\ell$ as in Lemma 6.7 we have

$$f\big(x, T_d(Y)\big) \ = \ a_\mu(x) F_\mu(u_\mu, v_\mu, x, Y) \text{ in } R\langle Y\rangle$$

and $F_\mu(u_\mu, v_\mu, x, Y)$ equals, in $R\langle Y\rangle$, the product

$$E(u_\mu, v_\mu, x, Y) \cdot \big(Y_n^\ell + G_1(u_\mu, v_\mu, x, Y')Y_n^{\ell-1} + \cdots + G_\ell(u_\mu, v_\mu, x, Y')\big).$$

We can now prove a converse of Lemma 5.4:

Lemma 6.11. *Suppose $x \in R^m$ and $f(x, y) = 0$ for all $y \in R^n$. Then $f(x, Y) = 0$.*

Proof. If $Z(x)$ holds, then $a_\nu(x) = 0$ for all ν, that is, $f(x, Y) = 0$. Next assume $|\mu| < d$ and $S_\mu(x) \neq 0$. Then by (ii) of Lemma 6.10 we have a monic polynomial in $R[Y_n]$ vanishing identically on R. This is impossible as R is infinite. □

Corollary 6.12. *If $g \in R\langle Y\rangle$ and $g(y) = 0$ for all $y \in R^n$, then $g = 0$.*

By the last corollary, the map

$$K\langle Y\rangle \to \text{ring of } K\text{-valued functions on } R^n$$

that assigns to each $g \in K\langle Y\rangle$ the function $y \mapsto g(y)$ on R^n is an injective morphism of K-algebras.

Consequences for $K\langle Y\rangle$ of Weierstrass division. For an algebraic closure K_{alg} of K, the integral closure R_{alg} of R in K_{alg} is the unique valuation ring of K_{alg} dominating R, and has a unique A-analytic structure extending that of R.

More generally, we fix below an algebraically closed valued field extension K^{a} of K (not necessarily an algebraic closure of K), whose valuation ring R^{a} is equipped with an A-analytic structure extending that of R. This gives rise to $K\langle Y\rangle \subseteq K^{\text{a}}\langle Y\rangle$ and for $y \in (R^{\text{a}})^n$ we have the evaluation map $g \mapsto g(y) : K^{\text{a}}\langle Y\rangle \to K^{\text{a}}$, which for $y \in R^n$ extends the previous evaluation map $K\langle Y\rangle \to K$.

Lemma 6.13. *If E is a unit of $R\langle Y\rangle$, then $E(y) \asymp 1$ for all $y \in (R^{\text{a}})^n$.*

This is clear. The next two lemmas follow easily from $(*)$ and Lemma 6.10.

Lemma 6.14. *Let $g(Y) = \sum_\nu c_\nu Y^\nu \in R\langle Y\rangle$, $g \neq 0$. Then:*

(i) *there is a $d \geqslant 1$ and an index $\mu \in \mathbb{N}^n$ with $|\mu| < d$ such that*

$$c_\nu \preccurlyeq c_\mu \text{ whenever } |\nu| < d, \quad c_\nu \prec c_\mu \text{ whenever } |\nu| \geqslant d;$$

(ii) *if $c_\nu \prec 1$ for all ν, then $g(y) \prec 1$ for all $y \in (R^{\text{a}})^n$.*

Lemma 6.15. *Let $g(Y) \in K\langle Y\rangle^{\neq}$, $n \geqslant 1$. Then for some $d \in \mathbb{N}^{\geqslant 1}$ and $\ell \in \mathbb{N}$,*

(i) $g\big(T_d(Y)\big) \ = \ c \cdot E(Y) \cdot \big(Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \cdots + c_l(Y')\big)$
where $c \in K^\times$, $E(Y) \in R\langle Y\rangle$ is a unit, and $c_1(Y'), \dots, c_\ell(Y') \in R\langle Y'\rangle$.

(ii) $R\langle Y\rangle \ = \ \big(Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \cdots + c_l(Y')\big)R\langle Y\rangle + \sum_{i<\ell} R\langle Y'\rangle Y_n^i$ *and*

$$K\langle Y\rangle \ = \ g\big(T_d(Y)\big)K\langle Y\rangle + \sum_{i<\ell} K\langle Y'\rangle Y_n^i.$$

Proof. For (ii), use a reduction to $A\langle X, Y\rangle$ and appeal to Lemma 3.4. □

Weierstrass division leads in the usual way to noetherianity of $K\langle Y\rangle$ and more:

Theorem 6.16. *The integral domain $K\langle Y\rangle$ has the following properties:*

(i) *$K\langle Y\rangle$ is noetherian,*

and for every proper ideal I of $K\langle Y\rangle$:

(ii) *there is an injective K-algebra morphism $K\langle Y_1,\dots,Y_m\rangle \to K\langle Y\rangle/I$ with $m \leqslant n$, making $K\langle Y\rangle/I$ into a finitely generated $K\langle Y_1,\dots,Y_m\rangle$-module;*

(iii) *there is $y \in (R^{\mathrm{a}})^n$ such that $f(y) = 0$ for all $f \in I$.*

Proof. By induction on n. The case $n = 0$ being obvious, let $n \geqslant 1$. Recall that for $d \in \mathbb{N}^{\geqslant 1}$ we have the automorphism $g(Y) \mapsto g\big(T_d(Y)\big)$ of the K-algebra $K\langle Y\rangle$. Let I be an ideal of $K\langle Y\rangle$, $I \neq \{0\}$. Take a nonzero $g \in I$. To show I is finitely generated we apply an automorphism as above and use Lemma 6.15 to arrange $g = Y_n^\ell + c_1(Y')Y_n^{\ell-1} + \dots + c_\ell(Y')$ with $\ell \in \mathbb{N}$, $c_1,\dots,c_\ell \in R\langle Y'\rangle$, and

$$R\langle Y\rangle \;=\; gR\langle Y\rangle + \sum_{i<\ell} R\langle Y'\rangle Y_n^i, \qquad K\langle Y\rangle \;=\; gK\langle Y\rangle + \sum_{i<\ell} K\langle Y'\rangle Y_n^i.$$

For $\ell = 0$ this means $g = 1$, and we are done, so assume $\ell \geqslant 1$. Then the inclusion $K\langle Y'\rangle \to K\langle Y\rangle$ followed by the canonical map $K\langle Y\rangle \to K\langle Y\rangle/(g)$ makes $K\langle Y\rangle/(g)$ a $K\langle Y'\rangle$-module that is generated by the images of the Y_n^i with $i < \ell$. Assuming inductively that $K\langle Y'\rangle$ is noetherian, it follows that $K\langle Y\rangle/(g)$ is noetherian as a $K\langle Y'\rangle$-module, and thus as a ring. Hence the image of I in $K\langle Y\rangle/(g)$ is finitely generated, say by the images of $g_1,\dots,g_k \in I$, $k \in \mathbb{N}$. Then I is generated by $g, g_1,\dots,g_k$. This proves noetherianity of $K\langle Y\rangle$. Let now I also be proper, that is, $1 \notin I$, and set $I' := I \cap K\langle Y'\rangle$. The natural K-algebra embedding $K\langle Y'\rangle/I' \to K\langle Y\rangle/I$ makes $K\langle Y\rangle/I$ a finitely generated $K\langle Y'\rangle/I'$-module by the above. Assuming inductively that (ii) holds for $n-1$, $K\langle Y'\rangle$, I' instead of $n, K\langle Y\rangle, I$ yields (ii). For (iii) we can arrange that I is a maximal ideal of $K\langle Y\rangle$. Then in (ii) we have $m = 0$, so $K\langle Y\rangle/I$ is finite-dimensional as a vector space over K, hence algebraic over K as a field extension of K. This gives a K-algebra morphism $\phi : K\langle Y\rangle \to K^{\mathrm{a}}$ with kernel I and $\phi(K\langle Y\rangle)$ algebraic over K. We set $y := (y_1,\dots,y_n) = \big(\phi(Y_1),\dots,\phi(Y_n)\big) \in (K^{\mathrm{a}})^n$. We claim that $\phi(R\langle Y\rangle) \subseteq R^{\mathrm{a}}$ (and thus $\phi(R\langle Y\rangle)$ is integral over R).

Using $\phi(g) = 0$ gives $\phi(R\langle Y\rangle) = \sum_{i<\ell} \phi(R\langle Y'\rangle)y_n^i$. Since I' is a maximal ideal of $K\langle Y'\rangle$ we can assume inductively that $\phi(R\langle Y'\rangle) \subseteq R^{\mathrm{a}}$, so $\phi(R\langle Y\rangle) \subseteq \sum_{i<\ell} R^{\mathrm{a}} y_n^i$. Now $\phi(g) = 0$ means

$$y_n^\ell + \phi\big(c_1(Y')\big)y_n^{\ell-1} + \dots + \phi\big(c_\ell(Y')\big) \;=\; 0,$$

with $\phi\big(c_j(Y')\big) \in R^{\mathrm{a}}$ for $j = 1,\dots,\ell$. Hence $y_n \in R^{\mathrm{a}}$, which proves the claim. Therefore $y \in (R^{\mathrm{a}})^n$, and by Corollary 4.9 the restriction of ϕ to a map $R\langle Y\rangle \to R^{\mathrm{a}}$ is a morphism of A-rings. Thus for $f(Y) \in R\langle Y\rangle$ we have $\phi\big(f(Y)\big) = f(y)$, in particular, $f(y) = 0$ for all $f \in I$. □

7. Immediate A-Extensions

The study of immediate extensions of valued fields plays a key role in proving AKE-results via model theory and valuation theory. We try to follow this pattern. By Lemma 4.3 and Corollary 6.5, the case of algebraic immediate extensions is under control (at least in the equicharacteristic 0 case), so we are left with proving that a pseudocauchy sequence of transcendental type "generates" an immediate extension. The problem is that the valuation ring of such an extension should now be an A-ring, and thus closed under many more operations than in the non-analytic setting. In this section we show how to overcome this problem. This section uses only the material of Section 6 that precedes Lemma 6.10.

Below we assume some familiarity with [13, Section 4]; when using a result from those lecture notes we shall indicate the specific reference.

We continue with the previously set assumptions on A and R: *A is noetherian with an ideal $\mathcal{o}(A) \neq A$ such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ and A is $\mathcal{o}(A)$-adically complete; R is a viable valuation A-ring.* We fix $t \in \mathcal{o}(A)$ with $\mathcal{o}(R) = tR$, and adopt the notations and terminology concerning R and its fraction field K from Section 6, with the valuation $v : K^\times \to \Gamma$ on K such that $R = \{a \in K : va \geqslant 0\}$, so vt is the least positive element of Γ. For any valued field extension L of K we let $\Gamma_L \supseteq \Gamma$ be the value group of L and denote the valuation of L also by v, so that $v : L^\times \to \Gamma_L$ extends $v : K^\times \to \Gamma$.

By [13, Lemma 4.3] and the remark following its proof, any pc-sequence in K has a pseudolimit in some elementary $\mathcal{L}^A_\preccurlyeq$-extension of K; any such extension is an A-extension of K whose valuation A-ring inherits the conditions we imposed on R.

Immediate A-extensions generated by a pseudocauchy sequence. In this subsection L is an A-extension of K. Thus the valuation A-ring S of L extends the A-ring R and dominates R. We also view any subfield F of L as a *valued* subfield of L, and thus as a valued field extension of K if $K \subseteq F$.

With *pc* abbreviating *pseudocauchy*, let (a_ρ) be a pc-sequence in K of transcendental type over K, with all $a_\rho \in R$, and with pseudolimit $a \in L$. Then $a \in S$, a is transcendental over K, and the valued subfield $K(a)$ of L is an immediate extension of K, by [13, Theorem 4.9]. But the valuation ring of $K(a)$ does not contain $R\langle a\rangle$ by Corollary 6.9, and so is not A-closed in S.

Is there a valued subfield $K_a \supseteq K(a)$ of L that is an immediate extension of K and whose valuation ring R_a is A-closed in S? Such R_a must contain $R\langle a\rangle$, but has to be strictly larger, since $R\langle a\rangle$ is not a valuation ring, by Corollary 6.9.

To answer the question above affirmatively we proceed as follows. Take an index ρ_0 such that for $\rho > \rho_0$,

$$a \;=\; a_\rho + t_\rho u_\rho, \quad t_\rho \in K^\times,\ t_\rho \prec 1,\ u_\rho \in K(a),\ u_\rho \asymp 1,$$

and $v(t_\rho)$ is strictly increasing as a function of $\rho > \rho_0$. Then for indices $\sigma > \rho > \rho_0$ we have $R[u_\rho] \subseteq R[u_\sigma]$, and thus

$$R\langle a\rangle \;\subseteq\; R\langle u_\rho\rangle \;\subseteq\; R\langle u_\sigma\rangle.$$

This yields an A-closed subring $R_a := \bigcup_{\rho>\rho_0} R\langle u_\rho\rangle$ of S. Note that R_a does not change upon increasing ρ_0, and the next proposition shows more: as the notation suggests, R_a depends only on R and a, not on (a_ρ).

Proposition 7.1. *The subring R_a of S has the following properties:*

(i) *the valued subfield $K_a := \operatorname{Frac}(R_a)$ of L is an immediate extension of K;*
(ii) *R_a is the least A-closed subring of S, with respect to inclusion, that contains $R[a]$ and is a valuation ring dominated by S;*

Proof. Let $P \in K[Y] \setminus K$ where $n = 1$, so $Y = Y_1$. Let I be the set of i in $\{1, \dots, \deg P\}$ with $P_{(i)}(Y) \neq 0$. Then $I \neq \emptyset$ and for all $\rho > \rho_0$,

$$P(a) \;=\; P(a_\rho) + \sum_{i \in I} P_{(i)}(a_\rho)(a - a_\rho)^i \;=\; P(a_\rho) + \sum_{i \in I} t_\rho^i P_{(i)}(a_\rho) u_\rho^i.$$

The proof of [13, Proposition 4.7] gives $i_0 \in I$ such that, eventually,

$$\begin{aligned} \text{for all } i \in I \setminus \{i_0\}, \quad t_\rho^{i_0} P_{(i_0)}(a_\rho) \;&\succ\; t_\rho^i P_{(i)}(a_\rho), \\ P(a) - P(a_\rho) \;&\sim\; t_\rho^{i_0} P_{(i_0)}(a_\rho), \end{aligned}$$

and $v\big(t_\rho^{i_0} P_{(i_0)}(a_\rho)\big) = v\big(P(a) - P(a_\rho)\big)$ is eventually strictly increasing. Now (a_ρ) is of transcendental type over K, so $v\big(P(a_\rho)\big)$ is eventually constant, and thus $P(a_\rho) \succ P(a) - P(a_\rho)$, eventually. Thus eventually,

$$P(a) \;=\; P(a_\rho) \cdot \big(1 + \sum_{i \in I} \frac{t_\rho^i P_{(i)}(a_\rho)}{P(a_\rho)} u_\rho^i\big) \;\in\; P(a_\rho) \cdot (1 + tR\langle u_\rho \rangle).$$

Now suppose $Q(Y) \in K[Y]^{\neq}$. Then likewise we have for $j = 1, \dots, \deg Q$ that eventually $0 \neq Q(a_\rho) \succ t_\rho^j Q_{(j)}(a_\rho)$, so eventually

$$Q(a) \;=\; Q(a_\rho) \cdot \big(1 + \sum_{j=1}^{\deg Q} \frac{t_\rho^j Q_{(j)}(a_\rho)}{Q(a_\rho)} u_\rho^j\big) \;\in\; Q(a_\rho) \cdot (1 + tR\langle u_\rho \rangle).$$

Therefore, if $P(a) \preccurlyeq Q(a)$, then eventually $\frac{P(a_\rho)}{Q(a_\rho)} \in R$, and so eventually

$$\frac{P(a)}{Q(a)} \;\in\; \frac{P(a_\rho)}{Q(a_\rho)} \cdot (1 + tR\langle u_\rho \rangle) \;\subseteq\; R \cdot (1 + tR\langle u_\rho \rangle) \;\subseteq\; R\langle u_\rho \rangle.$$

Thus the valuation ring of the valued subfield $K(a)$ of L is contained in R_a. Now we use the reduction to polynomials from Corollary 6.8(ii) to the effect that for g, h in $R\langle Y \rangle$ with $h \neq 0$, if $g(a) \preccurlyeq h(a)$, then $g(a)/h(a) \in R_a$. Thus the valuation ring of the valued subfield $\operatorname{Frac}(R\langle a \rangle)$ of L is contained in R_a, and it also follows from the last display that $\operatorname{Frac}(R\langle a \rangle)$ is an immediate extension of K.

Next, fix $\rho > \rho_0$ and note that for $\sigma > \rho$ we have

$$u_\rho \;=\; a_{\sigma\rho} + t_{\sigma\rho} u_\sigma, \qquad a_{\sigma\rho} \;:=\; \frac{a_\sigma - a_\rho}{t_\rho} \in R, \qquad t_{\sigma\rho} \;:=\; \frac{t_\sigma}{t_\rho},$$

and $(a_{\sigma\rho})_{\sigma > \rho}$ is a pc-sequence in K and of transcendental type over K such that $a_{\sigma\rho} \rightsquigarrow \frac{a - a_\rho}{t_\rho} = u_\rho$. Hence the above arguments applied to u_ρ instead of a show that $\operatorname{Frac}(R\langle u_\rho \rangle)$ as a valued subfield of L is an immediate extension of K, and that the valuation ring of $\operatorname{Frac}(R\langle u_\rho \rangle)$ is contained in $\bigcup_{\sigma > \rho} R\langle u_\sigma \rangle = R_a$. Taking the union over all $\rho > \rho_0$ and using $R_a \subseteq S$ yields that R_a is the valuation ring of the valued subfield $K_a := \operatorname{Frac}(R_a)$ of L, and that K_a is an immediate extension of K. This proves (i) and also shows that S dominates R_a.

As to (ii), let R^* be any A-closed subring of S containing $R[a]$ such that R^* is a valuation ring dominated by S. Then clearly $u_\rho \in R^*$ for all $\rho > \rho_0$, and thus $R_a \subseteq R^*$. □

We keep (a_ρ) for now, and show that K_a is essentially unique:

Corollary 7.2. *Let L' be an A-extension of K with valuation A-ring S'. Suppose $a_\rho \rightsquigarrow a' \in S'$, thus giving rise to $R_{a'} \subseteq K_{a'} \subseteq L'$. Then there is a unique isomorphism $R_a \to R_{a'}$ of A-rings that is the identity on R and sends a to a'. It extends to a valued field isomorphism $K_a \to K_{a'}$.*

Proof. Using notations from the proof of Proposition 7.1 we have $a' = a_\rho + t_\rho u'_\rho$ with $u'_\rho \in K(a')$, $u'_\rho \asymp 1$ for $\rho > \rho_0$. That same proof and Corollary 6.9 yields for all $\rho > \rho_0$ a unique isomorphism $R\langle u_\rho\rangle \to R\langle u'_\rho\rangle$ of A-rings that is the identity on R and sends u_ρ to u'_ρ. Moreover, for $\sigma > \rho > \rho_0$ we have

$$u_\rho \;=\; a_{\sigma\rho} + t_{\sigma\rho} u_\sigma, \qquad u'_\rho \;=\; a_{\sigma\rho} + t_{\sigma\rho} u'_\sigma,$$

and so the above isomorphism $R\langle u_\sigma\rangle \to R\langle u'_\sigma\rangle$ extends the above isomorphism $R\langle u_\rho\rangle \to R\langle u'_\rho\rangle$. Taking the union over all $\rho > \rho_0$ yields an isomorphism $R_a \to R_{a'}$ of A-rings that is the identity on R and sends a to a'. Any such isomorphism sends u_ρ to u'_ρ for $\rho > \rho_0$, and this gives uniqueness. Now R_a and $R_{a'}$ are the valuation rings of K_a and $K_{a'}$, so this isomorphism $R_a \to R_{a'}$ extends to an isomorphism $K_a \to K_{a'}$ of valued fields. □

Uniqueness of maximal immediate extensions over A. The results in this subsection about maximal immediate A-extensions will not be used later, but are included for their intrinsic interest. So far we did not restrict the characteristic of $\boldsymbol{k}$ or K, but now we also assume:

Either $\operatorname{char}(\boldsymbol{k}) = 0$ (*the equicharacteristic* 0 *case*), *or K as a valued field is finitely ramified of mixed characteristic.*

This is a well-known sufficient condition for an ordinary valued field to have an essentially unique maximal immediate extension; see [13, 4.29]. We now adapt this to our A-setting. A first consequence of the present assumptions is that K has no proper algebraic immediate A-extension, by [13, Corollary 4.22]. Note that any immediate A-extension of K inherits all the conditions we imposed so far on K. By a *maximal immediate A-extension of K* we mean an immediate A-extension L of K such that L has no proper immediate A-extension. The previous subsection, the nonexistence of proper algebraic immediate A-extensions of K, and [13, Section 4] yield for an immediate A-extension L of K that the following are equivalent:

(1) L is a maximal immediate A-extension of K,
(2) L is maximal as a valued field,
(3) L is spherically complete.

Corollary 7.3. *K has a maximal immediate A-extension, and such an extension is unique up to $\mathcal{L}^A_{\preccurlyeq}$-isomorphism over K.*

Proof. This goes along the same lines as the proof for ordinary valued fields: First, existence of a maximal immediate A-extension of K follows by Zorn and Krull's cardinality bound, like [13, Corollary 4.14]. As to uniqueness, using Corollary 7.2 this goes as in the proof of [13, Corollary 4.29]. □

Using Corollary 7.2 we obtain in the same way:

Corollary 7.4. *Any maximal immediate A-extension of K can be embedded, as an $\mathcal{L}^A_{\preccurlyeq}$-structure, into any $|\Gamma|^+$-saturated A-extension of K.*

8. Truncation

The aim of this section is to prove an A-version of Kaplansky's embedding theorem from [23] "with truncation". This section is not needed for the later AKE-results, but is included for its independent interest. Returning to the Hahn field example from the beginning of Section 4 we are given:

(1) a ring A_0 with $1 \neq 0$,
(2) $A = A_0[[t]]$ with the norm specified there,
(3) a ring morphism $\iota : A_0 \to \boldsymbol{k}$ into a field $\boldsymbol{k}$,
(4) an ordered abelian group Γ with a distinguished element $1 > 0$ (allowing the possibility that there are $\gamma \in \Gamma$ with $0 < \gamma < 1$).

As in the example mentioned we use this to make the valuation ring $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ of the Hahn field $K := \boldsymbol{k}((t^{\Gamma}))$ into an A-ring.

Preserving truncation closedness. We refer to [14] for notations and terminology concerning truncation in K. The A-closed subrings $\iota_0(A) = \iota(A_0)[[t]]$ and $\boldsymbol{k}[[t]]$ of $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ are also truncation closed. This subsection uses the "Hahn field example" from Section 4, but little else from the present paper.

Lemma 8.1. *Let E be a truncation closed subring of $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$. Then the A-closure R of E in $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ is also truncation closed.*

Proof. We have $\iota_0(A) = \iota(A_0)[[t]] \subseteq R$, so $E[\iota_0(A)] \subseteq R$. Now $\iota_0(A)$ is truncation closed, hence $E \cup \iota_0(A)$ is as well, and so is $E[\iota_0(A)]$ by [14, Corollary 2.5]. Thus replacing E by $E[\iota_0(A)]$ we arrange $\iota_0(A) \subseteq E$. Let F be a truncation closed subring of R containing E such that $F \neq R$; in view of [14, Corollary 2.6] and Zorn it suffices to show that then some element of $R \setminus F$ has all its proper truncations in F. Let n be minimal such that there are $y \in F^n$ and $f \in A\langle Y\rangle$ with $f(y) \notin F$. Because of $\iota_0(A) \subseteq F$ we have $n \geqslant 1$. With the lexicographic ordering on n-tuples $(\lambda_1, \dots, \lambda_n)$ of ordinals we take $y \in F^n$ with minimal $\big(o(y_1), \dots, o(y_n)\big)$ such that $f(y) \notin F$ for some $f \in A\langle Y\rangle$. Fix such f; it suffices to show that then all proper truncations of $f(y)$ lie in F. Minimality of n gives $y_1, \dots, y_n \neq 0$, so $o(y_1), \dots, o(y_n) \geqslant 1$.

Let c be a proper truncation of $f(y)$. Take $\gamma \in \{1\} \cup \operatorname{supp} y_1 \cup \dots \cup \operatorname{supp} y_n$ and a positive integer N such that $N\gamma > \operatorname{supp} c$. We first consider the case $\gamma = 1$. Then $f(Y) = P(Y) + t^N Q(Y)$ with $P(Y) \in A[Y]$ and $Q(Y) \in A\langle Y\rangle$. Hence $f(y) = P(y) + t^N Q(y)$ with $v\big(t^N Q(y)\big) \geqslant N\gamma$, so c is a truncation of $P(y)$, and as $P(y) \in F$ and F is truncation closed, this gives $c \in F$.

Next assume $\gamma \in \operatorname{supp} y_n$. (For any $j \in \{1, \dots, n-1\}$ the case $\gamma \in \operatorname{supp} y_j$ is similar.) Then $y_n = y_{n0} + z$ with $y_{n0}, z \in F$ and $\operatorname{supp} y_{n0} < \gamma$, $v(z) = \gamma$. We have $f_0, \dots, f_{N-1} \in A\langle Y\rangle$, $g(Y,Z) \in A\langle Y, Z\rangle$ such that

$$f(Y_1, \dots, Y_{n-1}, Y_n + Z) \;=\; \sum_{i<N} f_i(Y) Z^i + g(Y,Z) Z^N, \text{ so}$$

$$f(y) \;=\; \sum_{i<N} f_i(y_1, \dots, y_{n-1}, y_{n0}) z^i + \varepsilon, \quad v(\varepsilon) \geqslant N\gamma.$$

Thus c is a truncation of $d := \sum_{i<N} f_i(y_1, \dots, y_{n-1}, y_{n0}) z^i$. Since $o(y_{n0}) < o(y_n)$, the minimality of $\big(o(y_1), \dots, o(y_n)\big)$ gives $f_i(y_1, \dots, y_{n-1}, y_{n0}) \in F$ for all $i < N$, so $d \in F$. Since F is truncation closed, this gives $c \in F$. $\square$

Corollary 8.2. *Let R be an A-closed subring of $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ that is also truncation closed, and let $B \subseteq \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ be such that all proper truncations of all $b \in B$ lie in R. Then the A-closure $R\langle B\rangle$ of $R \cup B$ in $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ is truncation closed.*

Proof. By [14, Corollary 2.6] the subring $R[B]$ is truncation closed, so Lemma 8.1 applied to $E := R[B]$ gives the desired result. □

A-embedding with truncation. In addition to (1)–(4) above we now assume:

(5) the ring A_0 is noetherian;
(6) $\operatorname{char} \boldsymbol{k} = 0$;
(7) there is no $\gamma \in \Gamma$ with $0 < \gamma < 1$.

By (5) and [26, Theorem 3.3]) the ring $A = A_0[[t]]$ is noetherian. The assumptions on A and K made in Section 7 are thus satisfied, with $t := t^1$. In Section 6 we considered A-extensions of the "base" structure K, but below K plays the opposite role of an ambient structure.

In the next lemma and corollary $E \supseteq \boldsymbol{k}$ is a truncation closed valued subfield of K whose valuation ring R_E is A-closed in $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$. Thus $\boldsymbol{k}[t] \subseteq E$, and E is an $\mathcal{L}^A_{\preccurlyeq}$-substructure of K. Note also that $t^{\Delta} \subseteq E$ with $\Delta := v(E^{\times}) \subseteq \Gamma$.

Let (a_ρ) be a divergent pc-sequence in E with all $a_\rho \in R_E$. Since $\operatorname{char}(\boldsymbol{k}) = 0$, E is algebraically maximal by [13, 4.22] and Lemma 4.3. Hence (a_ρ) is of transcendental type over E. Since K is spherically complete, $a_\rho \rightsquigarrow a$ for some $a \in \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$; we choose such a so that $o(a)$ is minimal. With E, K in the role of K, L earlier in this section we obtain the valued subfield $E_a \supseteq E(a)$ of K whose valuation ring $R_{E_a} = (R_E)_a$ is uniquely determined by R_E and a within $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$, as described in Proposition 7.1(ii). Recall also that E_a is an immediate extension of E.

Lemma 8.3. *E_a is truncation closed.*

Proof. We first show that all proper truncations of a lie in E. We have $a = \sum_\lambda c_\lambda t^{\gamma_\lambda}$ where λ ranges over all ordinals $< o(a)$, all $c_\lambda \in \boldsymbol{k}^{\times}$ and (γ_λ) is a strictly increasing enumeration of $\operatorname{supp} a$. Consider a proper truncation $a|_\gamma$ of a, with $\gamma \in \operatorname{supp} a$. Then $\gamma = v(a - a|_\gamma) < v(a - a_\rho)$ for some ρ, so $a|_\gamma$ is a truncation of such a_ρ, and therefore $a|_\gamma \in R_E$, as claimed.

It follows that $\operatorname{supp} a$ has no largest element: if $\gamma = \gamma_\mu$ were the largest element, then $a|_\gamma \in E$ and $a - a|_\gamma = c_\mu t^\gamma \in E_a$, so $\gamma \in v(E_a^{\times}) = v(E^{\times})$, hence $c_\mu t^\gamma \in E$, contradicting $a \notin E$. This yields a divergent pc-sequence $(a|_{\gamma_\lambda})_\lambda$ in E with pseudolimit a, and we now use it instead of (a_ρ) to describe E_a (which after all does not depend on the particular approximating pc-sequence). We have

$$a \;=\; a|_{\gamma_\lambda} + t^{\gamma_\lambda} u_\lambda, \quad u_\lambda \;:=\; \sum_{\mu \geqslant \lambda} c_\mu t^{\gamma_\mu - \gamma_\lambda}.$$

All proper truncations of all u_λ lie clearly in R_E, so all $R_E\langle u_\lambda\rangle$ are truncation closed by Corollary 8.2, hence so is $R_{E_a} = \bigcup_\lambda R_E\langle u_\lambda\rangle$, and thus E_a as well. □

Corollary 8.4. *Let F be an immediate A-extension of E. Then there exists an $\mathcal{L}^A_{\preccurlyeq}$-embedding $F \to K$ over E with truncation closed image.*

Proof. Let R_F be the valuation ring of F and $f \in R_F \setminus R_E$. Take a divergent pc-sequence (a_ρ) in E with all $a_\rho \in R_E$ such that $a_\rho \rightsquigarrow f$. Then (a_ρ) is of transcendental type over E. Thus with E, F, f in the role of K, L, a earlier in this section we obtain the valued subfield $E_f \supseteq E(f)$ of F whose valuation ring R_{E_f} is A-closed in R_F. Using Zorn it suffices to show that then there is an $\mathcal{L}^A_{\preccurlyeq}$-embedding $E_f \to K$ over E with truncation closed image.

Lemma 8.3 and the remarks preceding it provide a pseudolimit $a \in \boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$ of (a_ρ) and a truncation closed valued subfield $E_a \supseteq E(a)$ of K whose valuation ring R_{E_a} is A-closed in $\boldsymbol{k}[[t^{\Gamma^{\geqslant}}]]$. By Corollary 7.2 this gives a unique isomorphism $R_{E_f} \to R_{E_a}$ of A-rings that is the identity on R_E sending f to a. It extends to an $\mathcal{L}^A_{\preccurlyeq}$-embedding $F \to K$ over E with truncation closed image E_a. □

9. Quantifier-free 1-types

Functions given by one-variable terms in the language $\mathcal{L}^{A,K}_{\preccurlyeq,D}$ are piecewise given by analytic functions on annuli. A precise statement of this is Proposition 9.10, which is essential for all that follows. The requisite notions of "separated A-analytic structure" and "annulus" come from [9, 7] from which we also borrow results.

We keep the assumptions from Section 6 on A, R, t, K, so R is a viable A-valuation ring, $t \in \iota_0(A)$, $\mathcal{o}(R) = tR$. For now we fix an algebraically closed A-extension K^{a} of K with A-valuation ring R^{a}. Thus R, K, R^{a}, K^{a} are $\mathcal{L}^A_{\preccurlyeq,D}$-structures as specified earlier. Let K_{alg} be the algebraic closure of K in K^{a}. With Corollary 6.5 we make K_{alg} an A-extension of K by taking as its A-valuation ring the integral closure R_{alg} of R in K_{alg}; note that $R_{\mathrm{alg}} = R^{\mathrm{a}} \cap K_{\mathrm{alg}}$.

Recall the language $\mathcal{L}^{A,K}_{\preccurlyeq,D}$ introduced in Section 6 and the sublanguage $\mathcal{L}_{\preccurlyeq}$ (for valued fields) of $\mathcal{L}^A_{\preccurlyeq,D}$; augmenting $\mathcal{L}_{\preccurlyeq}$ with names for the elements of K gives the sublanguage $\mathcal{L}^K_{\preccurlyeq}$ of $\mathcal{L}^{A,K}_{\preccurlyeq,D}$. The $\mathcal{L}_{\preccurlyeq}$-theory of algebraically closed valued fields with nontrivial valuation has quantifier elimination [13, Theorem 3.29], so

$$(\star) \qquad K_{\mathrm{alg}} \text{ is an elementary } \mathcal{L}_{\preccurlyeq}\text{-substructure of } K^{\mathrm{a}}.$$

Separated A-analytic structures. A *separated A-analytic structure on R* is a family $(\iota_{m,n})$ of ring morphisms

$$\iota_{m,n} \;:\; A\langle X_1,\dots,X_m\rangle[[Y_1,\dots,Y_n]] \to \text{ ring of } R\text{-valued functions on } R^m \times \mathcal{o}(R)^n$$

indexed by the pairs $(m,n) \in \mathbb{N} \times \mathbb{N}$, such that:

(S1) for $(x_1,\dots,x_m,y_1,\dots,y_n) \in R^m \times \mathcal{o}(R)^n$,
$$\iota_{m,n}(X_k)(x_1,\dots,x_m,y_1,\dots,y_n) = x_k \qquad \text{for} \qquad k = 1,\dots,m,$$
$$\iota_{m,n}(Y_l)(x_1,\dots,x_m,y_1,\dots,y_n) = y_l \qquad \text{for} \qquad l = 1,\dots,n;$$

(S2) for $f \in A\langle X_1,\dots,X_m\rangle[[Y_1,\dots,Y_n]] \subseteq A\langle X_1,\dots,X_m,X_{m+1}\rangle[[Y_1,\dots,Y_n]]$ and $(x_1,\dots,x_m,x_{m+1},y_1,\dots,y_n) \in R^{m+1} \times \mathcal{o}(R)^n$ we have
$$\iota_{m,n}(f)(x_1,\dots,x_m,y_1,\dots,y_n) = \iota_{m+1,n}(f)(x_1,\dots,x_m,x_{m+1},y_1,\dots,y_n),$$
and similarly with the Y-variables;

(S3) for $m \geqslant 1$, $f \in A\langle X_1,\dots,X_m\rangle[[Y_1,\dots,Y_n]]$,
$$g := f(X_{m+1},\dots,X_{2m},Y_1,\dots,Y_n) \in A\langle X_1,\dots,X_{2m}\rangle[[Y_1,\dots,Y_n]],$$
and $(x_1,\dots,x_{2m},y_1,\dots,y_n) \in R^{2m} \times \mathcal{o}(R)^n$ we have:
$$\iota_{m,n}(f)(x_{m+1},\dots,x_{2m},y_1,\dots,y_n) = \iota_{2m,n}(g)(x_1,\dots,x_{2m},y_1,\dots,y_n),$$
and similarly with the Y-variables.

Remark. Since R is viable, a separated A-analytic structure on R gives a *separated analytic A-structure on K* in the sense of [9, Definition 2.7]. Compared to [9], we include an extra axiom (S3), parallel to (A3) from Section 4. We believe (S3) is needed for a proof of [9, Proposition 2.8], just as (A3) in proving Lemma 4.1.

Annuli and the corresponding rings of analytic functions. The A-analytic structure of R induces a separated A-analytic structure on R as follows. For $f(X,Y) \in A\langle X\rangle[[Y]]$ we have

$$\tilde{f}(X,Y) \;:=\; f(X,tY) \in A\langle X,Y\rangle,$$

and we associate to the series f the function $f : R^m \times \mathcal{O}(R)^n \to R$ given by

$$f(x,ty) \;:=\; \tilde{f}(x,y) \quad (x \in R^m, y \in R^n).$$

It is straightforward to check that the axioms (S1), (S2), (S3) are satisfied. Hence by the remark above, [9, 7] applies to our setting; see [7, 4.4(1)].

In the rest of this section we borrow terminology and results from [9, 7]. At the start of [7, Section 5] the authors impose a condition that in our setting would correspond to $\ker(\iota_0) = \{0\}$. Nevertheless, we can use their work by replacing A with $A/\ker(\iota_o)$ in view of Corollary 5.3 for $I = \ker(\iota_0)$. A more important difference is that [7] takes $K^{\mathrm{a}} := K_{\mathrm{alg}}$ as the ambient structure, whereas for later model-theoretic use we allow K^{a} to be any algebraically closed A-extension of K. Fortunately, the results we need from [7] about K_{alg} will readily transfer to our K^{a}: it helps that K^{a} as a valued field is an elementary extension of K_{alg}. In the rest of this section we fix an indeterminate Z, also to be used as a syntactic variable in $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-formulas.

Adopting [7, Definition 5.1.1], an *R-annulus in K^{a}*, or just *R-annulus* if K^{a} is clear from the context, is a set $F \subseteq K^{\mathrm{a}}$ given by monic polynomials $p_0,\dots,p_n$ in $R[Z]$, irreducible in $K[Z]$, $l_0,\dots,l_n \in \mathbb{N}^{\geqslant 1}$, and $\pi_0,\dots,\pi_n \in R\setminus\{0\}$, as follows:

$$F \;=\; \{z \in K^{\mathrm{a}} : \; p_0^{l_0}(z) \preccurlyeq \pi_0,\; p_1^{l_1}(z) \succcurlyeq \pi_1 \dots, p_n^{l_n}(z) \succcurlyeq \pi_n\}$$

where the "holes" $\{z \in K^{\mathrm{a}} : \; p_i^{l_i}(z) \prec \pi_i\}$, $1 \leqslant i \leqslant n$, are pairwise disjoint and contained in $\{z \in K^{\mathrm{a}} : \; p_0^{l_0}(z) \preccurlyeq \pi_0\}$. Such F is said to be *given by* $(p_0^{l_0},\dots,p_n^{l_n};\pi_0,\dots,\pi_n)$. Thus R^{a} is an R-annulus given by $(Z;1)$ (with $n=0$). With respect to the valuation topology on K^{a}, every R-annulus is a nonempty open-and-closed subset of R^{a}, and so infinite without any isolated point. Note also that every R-annulus is defined in K^{a} by a quantifier-free $\mathcal{L}^{K}_{\preccurlyeq}$-formula $\phi(Z)$.

Remark. In [7, Definition 5.1.1], with $K^{\mathrm{a}} = K_{\mathrm{alg}}$, the above notion of R-annulus is less general than that of K-annulus; there our R-annuli, for $K^{\mathrm{a}} = K_{\mathrm{alg}}$, would be among *closed* K-annuli.

In what follows we deviate from the convention that $Y = (Y_1,\dots,Y_n)$, and instead use $Y = (Y_0,\dots,Y_n)$ with an extra indeterminate Y_0. Let the R-annulus F be given by $(p_0^{l_0},\dots,p_n^{l_n};\pi_0,\dots,\pi_n)$ as above, and consider the ideal

$$I(F) \;:=\; \big(p_0^{l_0}(Z) - \pi_0 Y_0,\; p_1^{l_1}(Z)Y_1 - \pi_1,\dots,p_n^{l_n}(Z)Y_n - \pi_n\big)$$

of $K\langle Z,Y\rangle$. Define $\psi : F \to (R^{\mathrm{a}})^{2+n}$ by

$$\psi(z) \;:=\; \Big(z,\; \frac{p_0^{l_0}(z)}{\pi_0},\; \frac{\pi_1}{p_1^{l_1}(z)},\; \dots,\; \frac{\pi_n}{p_n^{l_n}(z)}\Big).$$

It is routine to verify that

$$\psi(F) \ = \ \mathrm{Z}\big(I(F)\big) \ := \ \{(z,y) \in (R^{\mathrm{a}})^{2+n} : \ g(z,y) = 0 \text{ for all } g \in I(F)\}.$$

Let $\mathcal{R}(F)$ be the K-algebra of K^{a}-valued functions on F. Then

$$\psi^* \ : \ K\langle Z, Y\rangle \to \mathcal{R}(F), \qquad \psi^*(g)(z) \ := \ g\big(\psi(z)\big) \text{ for } g \in K\langle Z, Y\rangle, \ z \in F,$$

is a K-algebra morphism. We set $\mathcal{O}(F) := \psi^*\big(K\langle Z, Y\rangle\big)$, a K-subalgebra of $\mathcal{R}(F)$. We refer to $\mathcal{O}(F)$ as *the ring of analytic functions on* F. Clearly, $I(F) \subseteq \ker(\psi^*)$, and if $K^{\mathrm{a}} = K_{\mathrm{alg}}$, then $I(F) = \ker(\psi^*)$ by [7, Corollary 5.6.6], so

$$K\langle Z, Y\rangle / I(F) \ \cong \ \mathcal{O}(F) \ \text{ as } K\text{-algebras.}$$

(For $K^{\mathrm{a}} = K_{\mathrm{alg}}$ the K-algebras $K\langle Z, Y\rangle / I(F)$ and $\mathcal{O}(F)$ are denoted by $\mathcal{O}_K^{\dagger}(F)$ and $\mathcal{O}_K^{\sigma}(F)$ in [7, Definition 5.1.4].)

Of course, $I(F)$ depends on how F is given. However, $\mathcal{O}(F)$ is independent of the choice of the tuple $(p_0^{l_0}, \dots, p_L^{l_L}; \pi_0, \dots, \pi_L)$ that gives F, by [7, 5.3.3]. Strictly speaking, we don't need this rather subtle fact, since any R-annulus F below is assumed to come with a tuple that gives F, with $\mathcal{O}(F)$ defined accordingly.

Example. For the R-annulus $F = R^{\mathrm{a}}$ given by $(Z; 1)$ we have $\mathcal{O}(F) = K\langle Z\rangle$ where $g \in K\langle Z\rangle$ is identified with the function $z \mapsto g(z) : R^{\mathrm{a}} \to K^{\mathrm{a}}$.

Univariate functions given by terms involving restricted division. For our AKE-theory for valuation A-rings we need to understand what data about $z \in K^{\mathrm{a}}$ determine the isomorphism type of the A-extension K_z over K. Section 7 basically settles this issue for the case when K_z is an immediate A-extension. For the general case we exploit below results from [7]. Let F be an R-annulus given by $(p_0^{l_0}, \dots, p_n^{l_n}; \pi_0, \dots, \pi_n)$ with corresponding map $\psi : F \to (R^{\mathrm{a}})^{2+n}$. We can represent any function $f \in \mathcal{O}(F)$ by an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$: Let $f = \psi^*(g)$ with $g \in K\langle Z, Y\rangle$. Take $c \in K^{\times}$, $x \in R^m$, and $G \in A\langle X, Z, Y\rangle$, $X = (X_1, \dots, X_m)$, such that $g = g(Z, Y) = c \cdot G(x, Z, Y)$, and thus $g(z, y) = c \cdot G(x, z, y)$ for all $(z, y) \in (R^{\mathrm{a}})^{2+n}$. Then for all $z \in F$,

$$f(z) \ = \ g\big(\psi(z)\big) \ = \ c \cdot G\Big(x, z, D\big(p_0^{l_0}(z), \pi_0\big), D\big(\pi_1, p_1^{l_1}(z)\big), \dots, D\big(\pi_L, p_n^{l_n}(z)\big)\Big).$$

Lemma 9.1. *For each polynomial $p(Z) \in K[Z]$ the function $z \mapsto p(z) : F \to K^{\mathrm{a}}$ belongs to $\mathcal{O}(F)$. If $f \in \mathcal{O}(F)$ and $f(z) \neq 0$ for all $z \in F$, then f is a unit in the ring $\mathcal{O}(F)$. If $r(Z) \in K(Z)$ has no pole in F, then the function*

$$z \mapsto r(z) : F \to K^{\mathrm{a}}$$

belongs to $\mathcal{O}(F)$.

Proof. The first claim follows from $K[Z] \subseteq K\langle Z\rangle$. For the second claim, suppose $f \in \mathcal{O}(F)$ and $f(z) \neq 0$ for all $z \in F$. Take $g \in K\langle Z, Y\rangle$ such that $f = \psi^*(g)$. Then g has no zero on $\psi(F) = \mathrm{Z}(I(F))$. Hence $1 \in gK\langle Z, Y\rangle + I(F)$ by Theorem 6.16(iii). Take $h \in K\langle Z, Y\rangle$ with $1 \in gh + I(F)$. Then $f\psi^*(h) = 1$ in $\mathcal{O}(F)$. The claim about $r(Z) \in K(Z)$ follows from the first claim and the second claim. $\square$

Next we transfer facts from [7], where K_{alg} is the ambient A-extension, to our K^{a}. For $\mathcal{L}^{K}_{\preccurlyeq,D}$-definable $P \subseteq K^{\mathrm{a}}$, let $P_{\mathrm{alg}} \subseteq K_{\mathrm{alg}}$ be the corresponding definable subset of K_{alg}: any $\mathcal{L}^{K}_{\preccurlyeq,D}$-formula defining P in K^{a} defines P_{alg} in K_{alg} (and $P \cap K_{\mathrm{alg}} = P_{\mathrm{alg}}$). To apply this to annuli, note that F_{alg} is the R-annulus in K_{alg} given by the same tuple $(p_0^{l_0}, \dots, p_n^{l_n}; \pi_0, \dots, \pi_n)$, with $I(F) = I(F_{\mathrm{alg}})$, and the corresponding map $\psi_{\text{alg}} : F_{\text{alg}} \to R_{\text{alg}}^{2+n}$ is the restriction of ψ to F_{alg}. This also yields the K-algebra $\mathcal{O}(F_{\text{alg}})$ of analytic functions on F_{alg} (in the sense of K_{alg} as the ambient A-extension of K).

Lemma 9.2. *For $f \in \mathcal{O}(F)$, let $f_{\text{alg}} : F_{\text{alg}} \to K_{\text{alg}}$ be defined by $f_{\text{alg}}(z) = f(z)$ for $z \in F_{\text{alg}}$. This yields an isomorphism of K-algebras:*

$$f \mapsto f_{\text{alg}} \ : \ \mathcal{O}(F) \to \mathcal{O}(F_{\text{alg}}).$$

Proof. Let $f \in \mathcal{O}(F)$ and take $g \in K\langle Z, Y\rangle$ such that $f = \psi^*(g)$. Then for $z \in F_{\text{alg}}$ we have $f_{\text{alg}}(z) = f(z) = g(\psi(z)) = g(\psi_{\text{alg}}(z))$, so $f_{\text{alg}} = \psi_{\text{alg}}^*(g) \in \mathcal{O}(F_{\text{alg}})$. This also shows surjectivity of $f \mapsto f_{\text{alg}} \ : \ \mathcal{O}(F) \to \mathcal{O}(F_{\text{alg}})$. With $f = \psi^*(g)$ as above, if $f_{\text{alg}} = 0$, then $\psi_{\text{alg}}^*(g) = 0$, so $g \in \ker(\psi_{\text{alg}}^*) = I(F) \subseteq \ker(\psi)$, hence $f = 0$. $\square$

The proof of this lemma yields $I(F) = \ker(\psi^*)$ (without assuming $K^{\text{a}} = K_{\text{alg}}$).

Lemma 9.3. *For $g \in K\langle Z, Y\rangle$ and $f = \psi^*(g) \in \mathcal{O}(F)$ the following are equivalent:*

(i) *$f(z) \preccurlyeq 1$ for all $z \in F$;*
(ii) *$f(z) \preccurlyeq 1$ for all $z \in F_{\text{alg}}$;*
(iii) *for some $d \in \mathbb{N}^{\geqslant 1}$ and $h_1, \dots, h_d \in R\langle Z, Y\rangle$ we have*

$$g^d + h_1 g^{d-1} + h_2 g^{d-2} + \cdots + h_d \ \in \ I(F).$$

Proof. The implications (i)⇒(ii) and (iii)⇒(i) are clear. As to (ii)⇒(iii), this holds if $K^{\text{a}} = K_{\text{alg}}$ by Proposition 6.16(ii) and [7, Proposition 5.2.12(b)]. Hence it holds for our K^{a} by Lemma 9.2. $\square$

By a *very strong unit on F* we mean a function $f \in \mathcal{O}(F)$ such that $f(z) \sim 1$ for all $z \in F$; cf. [7, Definition 5.1.4].

Lemma 9.4. *Let $f \in \mathcal{O}(F)$. Then there is a very strong unit u on F and a rational function $r \in K(Z)$ without pole in F such that $f(z) = u(z)r(z)$ for all $z \in F$. In particular, if $f \neq 0$, then f has only finitely many zeros in F.*

Proof. If $K^{\text{a}} = K_{\text{alg}}$, this is a consequence of the Mittag-Leffler Decomposition from [7, Theorem 5.5.2]. By Lemma 9.2 this yields $u \in \mathcal{O}(F)$ and $r \in K(Z)$ without pole in F such that u_{alg} is a very strong unit on F_{alg} and $f(z) = u(z)r(z)$ for all $z \in F_{\text{alg}}$, and thus for all $z \in F$ by Lemma 9.1. It remains to show that u is a very strong unit on F. Applying the above to $u - 1$ instead of f we have $u^* \in \mathcal{O}(F)$ and $r^* \in K(Z)$ without pole in F such that u_{alg}^* is a very strong unit on F_{alg} and $u_{\text{alg}}(z) - 1 = u_{\text{alg}}^*(z)r^*(z)$ for all $z \in F_{\text{alg}}$. Then $r^*(z) \prec 1$ for all $z \in F_{\text{alg}}$, hence for all $z \in F$. Lemma 9.2 gives $u(z) - 1 = u^*(z)r^*(z)$ for all $z \in F$, and Lemma 9.3 yields $u^*(z) \preccurlyeq 1$ for all $z \in F$, hence $u(z) \sim 1$ for all $z \in F$. $\square$

The proof above also yields another useful fact:

Corollary 9.5. *For $u \in \mathcal{O}(F)$ we have the following equivalence:*

$$u \text{ is a very strong unit on } F \iff u_{\text{alg}} \text{ is a very strong unit on } F_{\text{alg}}.$$

Lemma 9.6. *Let $f_1, \dots, f_m \in \mathcal{O}(F)$ be such that $f_1(z), \dots, f_m(z) \preccurlyeq 1$ for all $z \in F$. Then for $G \in R\langle X\rangle$, $X = (X_1, \dots, X_m)$, the function*

$$z \mapsto G\big(f_1(z), \dots, f_m(z)\big) \ : \ F \to R^{\text{a}}$$

belongs to $\mathcal{O}(F)$.

Proof. For $i = 1, \ldots, m$, take $g_i \in K\langle Z, Y\rangle$ such that $f_i(z) = g_i\big(\psi(z)\big)$ for all $z \in F$. Lemma 9.3 gives for $i = 1, \ldots, m$ a polynomial $P_i = P_i(Z, Y, X_i) \in R\langle Z, Y\rangle[X_i]$ over $R\langle Z, Y\rangle$, monic and of degree $d_i \geqslant 1$ in X_i, such that $P_i\big(Z, Y, g_i(Z, Y)\big) \in I(F)$, and thus $P_i\big(z, y, g_i(z, y)\big) = 0$ for all $(z, y) \in \psi(F)$. Since elements of $R\langle Z, Y, X\rangle$ are specializations of restricted power series over A, Lemma 3.5 gives in $R\langle Z, Y, X\rangle$ an equality

$$G(X) \;=\; \sum_{i=1}^{m} q_i P_i + r, \quad \text{with } q_1, \ldots, q_m \in R\langle Z, Y, X\rangle,\ r \in R\langle Z, Y\rangle[X].$$

Let $z \in F$; so $\psi(z) = (z, y) \in \psi(F)$, $y \in (R^{\mathrm{a}})^{1+n}$. Substituting in the equality above (z, y) for (Z, Y) and $f_i(z)$ for X_i and using $P_i\big(z, y, g_i(z, y)\big) = 0$ we obtain

$$G\big(f_1(z), \ldots, f_m(z)\big) \;=\; r\big(z, y, f_1(z), \ldots, f_m(z)\big).$$

Now $r = \sum_j r_j(Z, Y)X^j$ with $j = (j_1, \ldots, j_m)$ ranging over a finite subset of $\mathbb{N}^m$ and all $r_j \in R\langle Z, Y\rangle$. So $r\big(z, y, f_1(z), \ldots, f_m(z)\big) = \sum_j \psi^*(r_j)(z) f_1(z)^{j_1} \cdots f_m(z)^{j_m}$, which exhibits this function of $z \in F$ as being in $\mathcal{O}(F)$. □

Lemma 9.7. *For any rational function $r(Z) \in K(Z)$ the set $\{z \in R^{\mathrm{a}} : \ r(z) \preccurlyeq 1\}$ is a finite union of R-annuli. (Here $r(z) \preccurlyeq 1$ includes z not being a pole of r.)*

Proof. If $K^{\mathrm{a}} = K_{\mathrm{alg}}$, this can be shown along the lines of the proof of [9, Lemma 3.16]. It then holds for our K^{a} by $(\star)$. □

Lemma 9.8. *If F' is an R-annulus and $F' \cap F \neq \emptyset$, then $F' \cap F$ is an R-annulus.*

Proof. If $K^{\mathrm{a}} = K_{\mathrm{alg}}$, this holds by [7, Lemma 5.1.2 (iv)]. It then holds for our K^{a} by $(\star)$. □

Lemma 9.9. *Let F_1 be an R-annulus, $F_1 \subseteq F$, and $f \in \mathcal{O}(F)$. Then $f|_{F_1} \in \mathcal{O}(F_1)$.*

Proof. Let F_1 be given by $(q_0^{e_0}, \ldots, q_m^{e_m}; \rho_0, \ldots, \rho_m)$ with corresponding map

$$\psi_1 \ : \ F_1 \to (R^{\mathrm{a}})^{2+m}, \quad \psi_1(z) \ := \ \Big(z, \frac{q_0^{e_0}(z)}{\rho_0}, \frac{\rho_1}{q_1^{e_1}(z)}, \ldots, \frac{\rho_m}{q_m^{e_m}(z)}\Big).$$

This also yields the corresponding ideal $I(F_1)$ of $K\langle Z, V\rangle$ where we use a tuple $V = (V_0, \ldots, V_m)$ of new indeterminates $V_0, \ldots, V_m$. Accordingly, ψ_1 yields the surjective K-algebra morphism $\psi_1^* : K\langle Z, V\rangle \to \mathcal{O}(F_1)$ with kernel $I(F_1)$. We now define I to be the ideal of $K\langle Z, V, Y\rangle$ generated by $I(F_1) \subseteq K\langle Z, V\rangle$ and $I(F) \subseteq K\langle Z, Y\rangle$. This yields the K-algebra morphism

$$\iota \ : \ K\langle Z, V\rangle/I(F_1) \to K\langle Z, V, Y\rangle/I, \quad g + I(F_1) \mapsto g + I \qquad (g \in K\langle Z, V\rangle).$$

The proof of [CL, Proposition 5.3.2] shows that ι is an isomorphism. For s in $K\langle Z, V, Y\rangle$ we define $s^* : F_1 \to (R^{\mathrm{a}})^{3+m+n}$ by

$$s^*(z) \ := \ s\Big(z, \frac{q_0^{e_0}(z)}{\rho_0}, \frac{\rho_1}{q_1^{e_1}(z)}, \ldots, \frac{\rho_m}{q_m^{e_m}(z)}, \ \frac{p_0^{l_0}(z)}{\pi_0}, \ \frac{\pi_1}{p_1^{l_1}(z)}, \ \ldots, \ \frac{\pi_n}{p_n^{l_n}(z)}\Big),$$

so $s^*(z)$ is s evaluated at a combination of $\psi_1(z)$ and $\psi(z)$. Using $I(F_1) \subseteq \ker \psi_1^*$ and $I(F) \subseteq \ker \psi^*$ we see that for $s \in I$ we have $s^*(z) = 0$ for all $z \in F_1$.

Take $g \in K\langle Z, Y\rangle$ with $f = \psi^*(g)$. Surjectivity of ι gives $h \in K\langle Z, V\rangle$ with $g - h \in I$. It follows that for $z \in F_1$ we have $(g - h)^*(z) = g^*(z) - h^*(z) = 0$. For $z \in F_1$ we have $g^*(z) = \psi^*(g)(z)$ and $h^*(z) = \psi_1^*(h)(z)$, so $f(z) = \psi^*(g)(z) = \psi_1^*(h)(z)$. Thus $f|_{F_1} = \psi_1^*(h) \in \mathcal{O}(F_1)$. □

The next result for $K^{\mathrm{a}} := K_{\mathrm{alg}}$ is close to [7, Theorem 5.5.3]; see also [8, A.1.10]. We give a complete proof because we require R-annuli where [loc. cit.] allows more general annuli, and because the details are used to obtain Corollary 9.12.

Proposition 9.10. *Let $\tau(Z)$ be an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. Then there are quantifier-free $\mathcal{L}^{K}_{\preccurlyeq}$-formulas $\phi_1(Z),\dots,\phi_n(Z)$, R-annuli $F_1,\dots,F_n$, and $f_1\in\mathcal{O}(F_1),\dots,f_n\in\mathcal{O}(F_n)$, such that:*

(i) $R^{\mathrm{a}}=\phi_1(R^{\mathrm{a}})\cup\cdots\cup\phi_n(R^{\mathrm{a}})$;
(ii) $\phi_j(R^{\mathrm{a}})\subseteq F_j$ *and* $\tau(z)=f_j(z)$ *for all* $z\in\phi_j(R^{\mathrm{a}})$, *for* $j=1,\dots,n$.

Proof. By induction on the complexity of $\tau=\tau(Z)$. For τ the name of an element of K or just the variable Z one can take $n=1$, and make the obvious choices of ϕ_1,F_1,f_1. Next, given $\phi_1,\dots,\phi_n,F_1,\dots,F_n,f_1,\dots,f_n$ as in the proposition, we only need to replace each f_j by $-f_j$ to make it work for $-\tau$ instead of τ.

Suppose $\tau=\tau_1+\tau_2$. The inductive assumption gives quantifier-free $\mathcal{L}^{K}_{\preccurlyeq}$-formulas $\phi_{11}(Z),\dots,\phi_{1n_1}(Z)$ and $\phi_{21}(Z),\dots,\phi_{2n_2}(Z)$, R-annuli

$$F_{11},\dots,F_{1n_1},F_{21},\dots,F_{2n_2},$$

and $f_{ij}\in\mathcal{O}(F_{ij})$ for $i=1,2$ and $j=1,\dots,n_i$ such that

- $R^{\mathrm{a}}=\phi_{11}(R^{\mathrm{a}})\cup\cdots\cup\phi_{1n_1}(R^{\mathrm{a}})=\phi_{21}(R^{\mathrm{a}})\cup\cdots\cup\phi_{2n_2}(R^{\mathrm{a}})$;
- $\phi_{ij}(R^{\mathrm{a}})\subseteq F_{ij}$ and $\tau_i(z)=f_{ij}(z)$ for all $z\in\phi_{ij}(R^{\mathrm{a}})$.

Let $1\leqslant j_1\leqslant n_1$ and $1\leqslant j_2\leqslant n_2$ and set $\phi_{j_1j_2}:=\phi_{1j_1}\wedge\phi_{2j_2}$ and $F_{j_1j_2}:=F_{1j_1}\cap F_{2j_2}$. Then $\phi_{j_1j_2}(R^{\mathrm{a}})\subseteq F_{j_1j_2}$ and $\tau(z)=f_{1j_1}(z)+f_{2j_2}(z)$ for $z\in\phi_{j_1j_2}(R^{\mathrm{a}})$. Thus listing the nonempty $F_{j_1j_2}$ as $F_1,\dots,F_n$, the corresponding $f_{1j_1}|_{F_{j_1j_2}}+f_{2j_2}|_{F_{j_1j_2}}$ as $f_1,\dots,f_n$, and the corresponding $\phi_{j_1j_2}$ as $\phi_1,\dots,\phi_n$ yields (i) and (ii). (This uses Lemmas 9.8 and 9.9.) The case $\tau=\tau_1\cdot\tau_2$ is handled in the same way.

Next, suppose $\tau=D(\tau_1,\tau_2)$, and let the ϕ_{ij},F_{ij},f_{ij} be as before and also define $\phi_{j_1j_2}$ and $F_{j_1j_2}$ as before. Consider one such pair $j=(j_1,j_2)$ with $F_{j_1j_2}\neq\emptyset$ and set $\phi_j=\phi_{j_1j_2}$ and $F_j=F_{j_1j_2}$. If $f_{1j_1}=0$ or $f_{2j_2}=0$, then $D(\tau_1,\tau_2)(z)=0$ for all $z\in F_j$, a trivial case. Assume $f_{1j_1}\neq 0$ and $f_{2j_2}\neq 0$. Then Lemma 9.4 yields very strong units u_1,u_2 on F_j and rational functions $r_1,r_2\in K(Z)^{\times}$ without pole in F_j such that $f_{1j_1}(z)=u_1(z)r_1(z)$ and $f_{2j_2}(z)=u_2(z)r_2(z)$ for all $z\in F_j$. Set $r=r_1/r_2\in K(Z)^{\times}$. If $z\in F_j$ and $r_2(z)\neq 0$, this gives $f_{2j_2}(z)\neq 0$ and $f_{1j_1}(z)/f_{2j_2}\asymp r(z)$. Hence by Lemma 9.7 we have $N\in\mathbb{N}$ such that

$$\{z\in F_j:\ f_{1j_1}(z)\preccurlyeq f_{2j_2}(z)\neq 0\}\ =\ \big(F^{j,1}\cup\cdots\cup F^{j,N}\big)\setminus E,$$

with R-annuli $F^{j,1},\dots,F^{j,N}\subseteq F_j$, and finite $E=\{z\in F_j:r_2(z)=0\}$. Let $1\leqslant\nu\leqslant N$. Then r has no pole in $F^{j,\nu}$: if $z\in F^{j,\nu}$ were a pole, then there would be $z'\in F^{j,\nu}\setminus E$ arbitrarily close to z with $r(z')\succ 1$, a contradiction. Thus by setting $f^{j,\nu}(z):=\frac{u_1(z)}{u_2(z)}r(z)$ for $z\in F^{j,\nu}$ we obtain $f^{j,\nu}\in\mathcal{O}(F^{j,\nu})$ with $D\big(f_{1j_1}(z),f_{2j_2}(z)\big)=f^{j,\nu}(z)$ for all $z\in F^{j,\nu}\setminus E$. Take a quantifier-free $\mathcal{L}^{K}_{\preccurlyeq}$-formula $\phi^{j,\nu}(Z)$ such that $\phi^{j,\nu}(R^{\mathrm{a}})=F^{j,\nu}\setminus E$. Then for $z\in(\phi_j\wedge\phi^{j,\nu})(R^{\mathrm{a}})$ we have $\tau(z)=f^{j,\nu}(z)$. Lemma 9.4 also gives a quantifier-free $\mathcal{L}^{K}_{\preccurlyeq}$-formula $\theta_j(Z)$ such that for all $z\in R^{\mathrm{a}}$,

$$R^{\mathrm{a}}\models\theta_j(z)\iff z\in F_j\text{ and }\big(f_{1j_1}(z)\succ f_{2j_2}(z)\text{ or }f_{2j_2}(z)=0\big).$$

Thus $\theta_j(R^{\mathrm{a}})\subseteq F_j$, and for $z\in(\phi_j\wedge\theta_j)(R^{\mathrm{a}})$ we have $\tau(z)=0$.

Finally, suppose $\tau = G(\tau_1, \dots, \tau_m)$, where $G \in A\langle X\rangle$. The inductive assumption gives for $i = 1, \dots, m$ quantifier-free $\mathcal{L}^K_{\preccurlyeq}$-formulas $\phi_{i1}(Z), \dots, \phi_{in_i}(Z)$, R-annuli $F_{i1}, \dots, F_{in_i}$, and functions $f_{i1} \in \mathcal{O}(F_{i1}), \dots, f_{in_i} \in \mathcal{O}(F_{in_i})$, such that

- $R^{\mathrm{a}} = \phi_{i1}(R^{\mathrm{a}}) \cup \dots \cup \phi_{in_i}(R^{\mathrm{a}})$;
- $\phi_{ij}(R^{\mathrm{a}}) \subseteq F_{ij}$ and $\tau_i(z) = f_{ij}(z)$, for all $z \in \phi_{ij}(R^{\mathrm{a}})$ and $j = 1, \dots, n_i$.

Let $j = (j_1, \dots, j_m)$ with $1 \leqslant j_1 \leqslant n_1, \dots, 1 \leqslant j_m \leqslant n_m$ and set

$$\phi_j \ := \ \phi_{1j_1} \wedge \dots \wedge \phi_{mj_m}, \qquad F_j \ := \ F_{1j_1} \cap \dots \cap F_{mj_m}.$$

Using Lemmas 9.4, 9.7, and 9.8 we get $N \in \mathbb{N}$ such that

$$\{z \in F_j :\ f_{1j_1}(z) \preccurlyeq 1, \dots, f_{mj_m}(z) \preccurlyeq 1\} \ = \ F^{j,1} \cup \dots \cup F^{j,N}$$

where $F^{j,1}, \dots, F^{j,N}$ are R-annuli. Let $1 \leqslant \nu \leqslant N$. Take a quantifier-free $\mathcal{L}^K_{\preccurlyeq}$-formula $\phi^{j,\nu}(Z)$ such that $\phi^{j,\nu}(R^{\mathrm{a}}) = F^{j,\nu}$. For $i = 1, \dots, m$, set $f_i^{j,\nu} := f_{ij_i}|_{F^{j,\nu}}$, a function in $\mathcal{O}(F^{j,\nu})$ with $|f_i^{j,\nu}(z)| \leqslant 1$ for all $z \in F^{j,\nu}$, so by Lemma 9.6,

$$f^{j,\nu} \ := \ G\big(f_1^{j,\nu}, \dots, f_m^{j,\nu}\big) \in \mathcal{O}(F^{j,\nu}).$$

Then $\tau(z) = f^{j,\nu}(z)$ for all $z \in (\phi_j \wedge \phi^{j,\nu})(R^{\mathrm{a}})$. It remains to note that $\tau(z) = 0$ for all $z \in (\phi_j \wedge \neg\phi^{j,1} \wedge \neg\phi^{j,2} \wedge \dots \wedge \neg\phi^{j,N})(R^{\mathrm{a}})$. $\square$

Corollary 9.11. *Let $z \in K^{\mathrm{a}}$. Then K_z is an immediate extension of $K(z)$:*

$$\operatorname{res} K_z \ = \ \operatorname{res} K(z) \ \subseteq \ \operatorname{res} K^{\mathrm{a}}, \qquad v(K_z^{\times}) \ = \ v\big(K(z)^{\times}\big) \subseteq \ v\big((K^{\mathrm{a}})^{\times}\big).$$

As a consequence, $\Gamma = v(K^{\times})$ and $v(K_z^{\times})$ have the same cardinality, and if $\operatorname{res} K$ is infinite, then $\operatorname{res} K$ and $\operatorname{res} K_z$ have the same cardinality.

Proof. Replacing z by z^{-1} if $z \succ 1$, we arrange $z \in R^{\mathrm{a}}$. Consider a nonzero element $\tau(z)$ of K_z, where $\tau(Z)$ is an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. Let $\phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$ be as in Proposition 9.10. Take $j \in \{1, \dots, n\}$ with $z \in \phi_j(R^{\mathrm{a}})$. Then Lemma 9.4 applied to F_j, f_j in the role of F, f yields $r(Z) \in K(Z)$ without pole in F such that $\tau(z) \sim r(z)$. Thus K_z is an immediate extension of $K(z)$. The rest now follows from [13, Corollary 5.19]. $\square$

Uniformity with respect to K^{a}. So far we we kept K^{a} fixed, but in the rest of this section K^{a} ranges over arbitrary algebraically closed A-extensions of K. Let $\tau(Z)$ be an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term. To enable model-theoretic arguments we need to show that

$$\phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$$

in Proposition 9.10 is "independent" of K^{a}. To make sense of this we consider tuples $\big(\phi_j, \Phi_j\big)_{j=1}^n$ of quantifier-free $\mathcal{L}^K_{\preccurlyeq}$-formulas $\phi_j(Z)$ and $\Phi_j(Z)$, $j = 1, \dots, n$. Call such a tuple $\big(\phi_j, \Phi_j\big)_{j=1}^n$ *good for τ in K^{a}* if the following hold:

(1) $F_j := \Phi_j(K^{\mathrm{a}})$ is an R-annulus in K^{a} for $j = 1, \dots, n$;
(2) $z \mapsto \tau(z) : F_j \to K^{\mathrm{a}}$ is a function $f_j \in \mathcal{O}(F_j)$ for $j = 1, \dots, n$;
(3) $\phi_1, \dots, \phi_n, F_1, \dots, F_n, f_1, \dots, f_n$ satisfy (i) and (ii) in Proposition 9.10.

Of course, $\mathcal{O}(F_j)$ in (2) is meant in the sense of K^{a}.

Corollary 9.12. *Some tuple $\big(\phi_j, \Phi_j\big)_{j=1}^n$ is good for τ in all K^{a}.*

Proof. All K^{a} that are algebraic over K are isomorphic as A-extensions of K, so any tuple that is good for one is good for all. Now, let any K^{a} be given, and let K_{alg} be the algebraic closure of K in K^{a}, with K_{alg} as $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-substructure of K^{a}. Following the steps and recursive constructions in the proof of Proposition 9.10 yields a tuple that is good for τ in K_{alg}, as well as in K^{a}: to see this, use $(\star)$, Lemmas 9.2, 9.3, and Corollary 9.5 to pass from K_{alg} to K^{a}. □

The quantifier-free type of an element over K. For a valued field E and an element z in a valued field extension L of E, the *quantifier-free $\mathcal{L}_{\preccurlyeq}$-type of z over E* is the set $\mathrm{qftp}_{\preccurlyeq}(z|E)$ of all quantifier-free $\mathcal{L}^{E}_{\preccurlyeq}$-formulas $\theta(Z)$ such that $L \models \theta(z)$. Likewise, for an element z in an A-extension L of K, the *quantifier-free $\mathcal{L}^{A}_{\preccurlyeq,D}$-type of z over K* is the set $\mathrm{qftp}^{A}_{\preccurlyeq,D}(z|K)$ of all quantifier-free $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-formulas $\theta(Z)$ such that $L \models \theta(z)$.

Proposition 9.13. *Let K_1 and K_2 be A-extensions of K, and suppose $z_1 \in K_1$ and $z_2 \in K_2$ are such that* $\mathrm{qftp}_{\preccurlyeq}(z_1|K) = \mathrm{qftp}_{\preccurlyeq}(z_2|K)$. *Then*

$$\mathrm{qftp}^{A}_{\preccurlyeq,D}(z_1|K) \;=\; \mathrm{qftp}^{A}_{\preccurlyeq,D}(z_2|K).$$

Proof. Our assumption gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $i : K(z_1) \to K(z_2)$ over K that sends z_1 to z_2. If z_1 is algebraic over K, then so is z_2 and $K(z_1)$ and $K(z_2)$ underly the $\mathcal{L}^{A}_{\preccurlyeq,D}$-substructures K_{z_1} and K_{z_2} of K_1 and K_2, respectively, by Corollary 6.6, so i is an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism over K by Corollary 4.9. Hence z_1 and z_2 have the same quantifier-free $\mathcal{L}^{A}_{\preccurlyeq,D}$-type over K.

For the rest of the proof, assume that z_1 and z_2 are both transcendental over K. Replacing z_1, z_2 by their reciprocals if necessary, we arrange $z_1, z_2 \preccurlyeq 1$. We claim that for every $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term $\tau(Z)$,

$$\tau(z_1) = 0 \iff \tau(z_2) = 0.$$

For c, d in any A-extension of K, $c \preccurlyeq d$ if and only if $c = 0$ or $D(c, d) \neq 0$. Hence, in light of Corollary 6.3, our claim yields an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism $K_{z_1} \to K_{z_2}$ over K given by $\tau(z_1) \mapsto \tau(z_2)$, where $\tau(Z)$ ranges over $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-terms. Thus a proof of the claim will complete the proof of the proposition.

By passing to algebraic closures we arrange that K_1 and K_2 are algebraically closed, with respective A-valuation rings R_1 and R_2. Let $\tau(Z)$ be an $\mathcal{L}^{A,K}_{\preccurlyeq,D}$-term such that $\tau(z_1) = 0$. Take a tuple

$$\big(\phi_j, \Phi_j\big)_{j=1}^{n}$$

as in Corollary 9.12. This gives $j \in \{1, \dots, n\}$ with $z_1 \in \phi_j(R_1)$ and $z_2 \in \phi_j(R_2)$. Set $F_{j1} := \Phi_j(K_1) \subseteq R_1$, and let $f_{j1} \in \mathcal{O}(F_{j1})$ be given by $f_{j1}(z) = \tau_1(z)$, and define $F_{j2} \subseteq R_2$ and $f_{j2} \in \mathcal{O}(F_{j2})$ in the same way.

Then $\tau(z_1) = f_{j1}(z_1) = 0$, so $f_{j1} = 0$ by Corollary 9.4 and z_1 being transcendental over K. By descending to the algebraic closures of K in K_1 and K_2 and using that these algebraic closures are isomorphic A-extensions of K we obtain $f_{j2} = 0$ by Lemma 9.2. Hence $\tau(z_2) = f_{j2}(z_2) = 0$. This proves the forward direction of our claim. The backward direction follows in the same way. □

10. Analytic AKE-type Equivalence and Induced Structure

We begin with some terminology and conventions. A valued field will be construed as an $\mathcal{L}_{\preccurlyeq}$-structure in the usual way.

Let K be a valued field. We denote its valuation ring by R (by R_F if we are dealing with a valued field F instead). Let $\smallo(R)$ be the maximal ideal of R and $\boldsymbol{k} := R/\smallo(R)$ the residue field of K. We also let $v : K^\times \to \Gamma$ with $\Gamma = v(K^\times)$ be a valuation on the field K such that $R = \{z \in K : v(z) \geqslant 0\}$ (and if we are dealing instead with a valued field F, we have likewise the residue field $\boldsymbol{k}_F$ and a valuation $v_F : F^\times \to \Gamma_F$).

A *coefficient field of* K is a lift of $\boldsymbol{k}$, that is, a subfield C of K such that $C \subseteq R$ and C maps bijectively onto $\boldsymbol{k}$ under the residue map $R \to \boldsymbol{k}$, equivalently, a subfield C of K such that $R = C + \smallo(R)$. Likewise, a *monomial group* of K is a lift of Γ, that is, a subgroup G of $K^\times$ that is mapped bijectively onto Γ by $v : K^\times \to \Gamma$. If K is henselian (by which we mean that the local ring R is henselian) and $\boldsymbol{k}$ has characteristic 0, then K has a coefficient field; see for example [13, Lemma 2.9]. If K is algebraically closed or $\aleph_1$-saturated, then K has a monomial group; see for example [1, Lemmas 3.3.32, 3.3.39].

Throughout A is as in Section 6: A is noetherian with an ideal $\smallo(A) \neq A$, such that $\bigcap_e \smallo(A)^e = \{0\}$ and A is $\smallo(A)$-adically complete.

By an *A*cg*-field* we mean an expansion $\mathcal{F} = (F, C, G)$ of an A-field F where C is (the underlying set of) a coefficient field of F and G is (the underlying set of) a monomial group G of F. Let $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$ be the language $\mathcal{L}^{A}_{\preccurlyeq,D}$ augmented by unary predicate symbols C and G. We construe an Acg-field as an $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-structure in the obvious way.

Example to keep in mind: $\mathcal{F} = \big(F, C, t^{\mathbb{Z}}\big)$, where C is any field, F is the Laurent series field $C(\!(t)\!)$ with valuation ring $C[[t]]$, and $A = C[[t]]$, $\smallo(A) = tC[[t]]$, with the natural A-analytic structure on $C[[t]]$. To simplify notation we denote this Acg-field $\mathcal{F}$ by $\big(C(\!(t)\!), C, t^{\mathbb{Z}}\big)$.

In the rest of this section $\mathcal{K} = (K, C_{\mathcal{K}}, G_{\mathcal{K}})$ is an Acg-field such that the valuation A-ring R of K is viable; $\boldsymbol{k}$ is the residue field of K and $\Gamma := v(K^\times)$ its value group.

Good substructures and good maps. Our aim is to establish an analogue of the Equivalence Theorem [13, 5.21] in our *analytic* setting *with coefficient field and monomial group*, and we follow the general setup and proof strategy there.

A *good substructure* of $\mathcal{K}$ is an $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-substructure $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ of $\mathcal{K}$ which is also an Acg-field. Note that then E is an $\mathcal{L}^{A}_{\preccurlyeq,D}$-substructure of K, and

$$C_{\mathcal{E}} \ = \ C_{\mathcal{K}} \cap E, \qquad G_{\mathcal{E}} \ = \ G_{\mathcal{K}} \cap E.$$

Below, $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ is a good substructure of $\mathcal{K}$. By Lemma 6.4 the valuation A-ring R_E is viable. For $a \in K$, set $\mathcal{E}_a := \big(E_a, C_{\mathcal{K}} \cap E_a, G_{\mathcal{K}} \cap E_a\big) \subseteq \mathcal{K}$.

Lemma 10.1. *We consider four cases for an element $a \in C_{\mathcal{K}} \cup G_{\mathcal{K}}$:*

(i) $a \in C_{\mathcal{K}}$ is algebraic over E. *Then a is algebraic over $C_{\mathcal{E}}$, $E[a]$ is the underlying field of E_a, $C_{\mathcal{K}} \cap E_a = C_{\mathcal{E}}[a]$, and $G_{\mathcal{K}} \cap E_a = G_{\mathcal{E}}$;*

(ii) $a \in C_{\mathcal{K}}$ is transcendental over E. *Then $C_{\mathcal{K}} \cap E_a = C_{\mathcal{E}}(a)$, $G_{\mathcal{K}} \cap E_a = G_{\mathcal{E}}$;*

(iii) $a \in G_{\mathcal{K}}$ is algebraic over E. *Then $a^d \in G_{\mathcal{E}}$ for some $d \geqslant 1$, $E[a]$ is the underlying field of E_a, $C_{\mathcal{K}} \cap E_a = C_{\mathcal{E}}$, and $G_{\mathcal{K}} \cap E_a = G_{\mathcal{E}} \cdot a^{\mathbb{Z}}$;*

(iv) $a \in G_{\mathcal{K}}$ is transcendental over E. *Then* $C_{\mathcal{K}} \cap E_a = C_{\mathcal{E}}$, $G_{\mathcal{K}} \cap E_a = G_{\mathcal{E}} \cdot a^{\mathbb{Z}}$.

In each of these four cases, $\mathcal{E}_a$ *is a good substructure of* $\mathcal{K}$.

Proof. If $a \in C_{\mathcal{K}} \cup G_{\mathcal{K}}$ is algebraic over E, this follows from Corollary 6.6. In the transcendental case, use Corollary 9.11. □

In this subsection, $\mathcal{K}' = (K', C_{\mathcal{K}'}, G_{\mathcal{K}'})$ is an Acg-field like $\mathcal{K}$: its valuation A-ring R' is viable. We also let $\mathcal{E}' = (E', C_{\mathcal{E}'}, G_{\mathcal{E}'})$ be a good substructure of $\mathcal{K}'$, and for $b \in K'$ we set $\mathcal{E}'_b := \big(E'_b, C_{\mathcal{K}'} \cap E'_b, G_{\mathcal{K}'} \cap E'_b\big)$.

Let $\mathcal{L}_{\mathrm{r}} := \{0, 1, +, -, \cdot\}$ be the language of rings and $\mathcal{L}_{\mathrm{v}} := \{1, \cdot, \preccurlyeq\}$ the language of (multiplicative) ordered abelian groups, taken as sublanguages of $\mathcal{L}^{A}_{\preccurlyeq,D}$; we construe $C_{\mathcal{K}}, C_{\mathcal{K}'}$ as $\mathcal{L}_{\mathrm{r}}$-structures and $G_{\mathcal{K}}, G_{\mathcal{K}'}$ as $\mathcal{L}_{\mathrm{v}}$-structures accordingly.

A *good map* $f : \mathcal{E} \to \mathcal{E}'$ is an $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-isomorphism $\mathcal{E} \to \mathcal{E}'$ such that:

(r) the $\mathcal{L}_{\mathrm{r}}$-isomorphism $f|_{C_{\mathcal{E}}} : C_{\mathcal{E}} \to C_{\mathcal{E}'}$ is a partial elementary map from the field $C_{\mathcal{K}}$ to the field $C_{\mathcal{K}'}$;

(v) the $\mathcal{L}_{\mathrm{v}}$-isomorphism $f|_{G_{\mathcal{E}}} : G_{\mathcal{E}} \to G_{\mathcal{E}'}$ is a partial elementary map from the ordered group $G_{\mathcal{K}}$ to the ordered group $G_{\mathcal{K}'}$.

Theorem 10.2. *Suppose* $\operatorname{char} \boldsymbol{k} = 0$. *Let* $f : \mathcal{E} \to \mathcal{E}'$ *be a good map. Then* f *is a partial elementary map from* $\mathcal{K}$ *to* $\mathcal{K}'$.

We need $\operatorname{char} \boldsymbol{k} = 0$ only towards the end of the proof below to guarantee that a certain pc-sequence (a_ρ) introduced there is of transcendental type.

Proof. By passing to suitable elementary extensions we arrange that the underlying valued fields of $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal greater than the cardinalities of $C_{\mathcal{E}}$ and $G_{\mathcal{E}}$. A good substructure

$$\mathcal{E}_1 \;=\; (E_1, C_{\mathcal{E}_1}, G_{\mathcal{E}_1})$$

of $\mathcal{K}$ is termed *small* if κ is greater than the cardinalities of $C_{\mathcal{E}_1}$ and $G_{\mathcal{E}_1}$. We shall prove that for any $a \in \mathcal{K}$ we can extend f to a good map with small domain $\mathcal{F} \supseteq \mathcal{E}$ such that $a \in \mathcal{F}$. By the properties of "back-and-forth" this suffices. In addition to Corollary 7.2, we will need the extension procedures in (1)–(4) below.

In (1) and (2) we assume $a \in C_{\mathcal{K}}$ and extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{F}$ of $\mathcal{K}$ and $\mathcal{F}'$ of $\mathcal{K}'$ and our good map f to a good map $g : \mathcal{F} \to \mathcal{F}'$ such that $a \in C_{\mathcal{F}}$ and $G_{\mathcal{E}} = G_{\mathcal{F}}$. In (3) and (4) we assume that $a \in G_{\mathcal{K}}$, and extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{F}$ of $\mathcal{K}$ and $\mathcal{F}'$ of $\mathcal{K}'$ and our good map f to a good map $g : \mathcal{F} \to \mathcal{F}'$ such that $a \in G_{\mathcal{F}}$ and $C_{\mathcal{E}} = C_{\mathcal{F}}$.

(1) *The case that* $a \in C_{\mathcal{K}}$ *is algebraic over* E. Then κ-saturation of $\mathcal{K}'$ gives $b \in C_{\mathcal{K}'}$ and an $\mathcal{L}_{\mathrm{r}}$-isomorphism $g_{\mathrm{r}} : C_{\mathcal{E}}[a] \to C_{\mathcal{E}'}[b]$ extending $f|_{C_{\mathcal{E}}}$ and sending a to b such that g_{r} is a partial elementary map from $C_{\mathcal{K}}$ to $C_{\mathcal{K}'}$.

Now [13, Lemma 3.21] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $g : E[a] \to E'[b]$ extending both f and g_{r}. Then g is an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism $E_a \to E'_b$ by Corollary 6.6 and Proposition 9.13. By Lemma 10.1 (i), $\mathcal{E}_a$ and $\mathcal{E}'_b$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

(2) *The case that* $a \in C_{\mathcal{K}}$ *is transcendental over* E. As in (1) we have $b \in C_{\mathcal{K}'}$ and an $\mathcal{L}_{\mathrm{r}}$-isomorphism $g_{\mathrm{r}} : C_{\mathcal{E}}(a) \to C_{\mathcal{E}'}(b)$ extending $f|_{C_{\mathcal{E}}}$ and sending a to b such that g_{r} is a partial elementary map from $C_{\mathcal{K}}$ to $C_{\mathcal{K}'}$.

Then [13, Lemma 3.22] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $E(a) \to E'(b)$ extending both f and g_{r}. This $\mathcal{L}_{\preccurlyeq}$-isomorphism extends to an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism $g: E_a \to E'_b$ by Proposition 9.13. Lemma 10.1(ii) gives that $\mathcal{E}_a$ and $\mathcal{E}'_b$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and that g is a good map.

(3) *The case that $a \in G_{\mathcal{K}} \setminus G_{\mathcal{E}}$ and $a^p \in G_{\mathcal{E}}$, where p is a prime number.* As before we get $b \in G_{\mathcal{K}'}$ and an $\mathcal{L}_{\mathrm{v}}$-isomorphism $g_{\mathrm{v}} : G_{\mathcal{E}} \cdot a^{\mathbb{Z}} \to G_{\mathcal{E}'} \cdot b^{\mathbb{Z}}$ extending $f|_{G_{\mathcal{E}}}$ and sending a to b such that g_{v} is a partial elementary map from $G_{\mathcal{K}}$ to $G_{\mathcal{K}'}$. Now [13, Lemma 5.6] gives an $L_{\preccurlyeq}$-isomorphism $g : E(a) \to E'(b)$ extending both f and g_{v}. Then g is an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism $E_a \to E'_b$ by Corollary 6.6 and Proposition 9.13. By Lemma 10.1(iii), $\mathcal{E}_a$ and $\mathcal{E}'_b$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

(4) *The case that $a \in G_{\mathcal{K}}$ and $a^d \notin G_{\mathcal{E}}$ for all $d \geqslant 1$.* As before we get $b \in G_{\mathcal{K}'}$ and an $\mathcal{L}_{\mathrm{v}}$-isomorphism

$$g_{\mathrm{v}} : G_{\mathcal{E}} \cdot a^{\mathbb{Z}} \to G_{\mathcal{E}'} \cdot b^{\mathbb{Z}}$$

extending $f|_{G_{\mathcal{E}}}$ and sending a to b such that g_{v} is a partial elementary map from $G_{\mathcal{K}}$ to $G_{\mathcal{K}'}$. Note that a is transcendental over E by [13, Proposition 3.19]; likewise, b is transcendental over E'.

Now [13, Lemma 3.23] gives an $\mathcal{L}_{\preccurlyeq}$-isomorphism $E(a) \to E'(b)$ extending both f and g_{v}. This $\mathcal{L}_{\preccurlyeq}$-isomorphism extends to an $\mathcal{L}^{A}_{\preccurlyeq,D}$-isomorphism $g: E_a \to E'_b$ by Proposition 9.13. By Lemma 10.1(iv), $\mathcal{E}_a$ and $\mathcal{E}'_b$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$ respectively, and g is a good map.

Let now any $a \in K$ be given. Let C_1 be the subfield of $C_{\mathcal{K}}$ such that $\operatorname{res} C_1 = \operatorname{res} E_a$, and let G_1 be the subgroup of $G_{\mathcal{K}}$ such that $v(G_1) = v(E_a^{\times})$. We do not guarantee that $C_1 \subseteq E_a$ or $G_1 \subseteq E_a^{\times}$, but $C_{\mathcal{E}}$ and C_1 have the same cardinality, and so do $G_{\mathcal{E}}$ and G_1, by Corollary 9.11. Thus by iterating (1)–(4), we extend $\mathcal{E}$ and $\mathcal{E}'$ to small good substructures $\mathcal{E}_1 = (E_1, C_1, G_1)$ of $\mathcal{K}$ and $\mathcal{E}'_1 = (E'_1, C'_1, G'_1)$ of $\mathcal{K}'$, and extend f to a good map $f_1 : \mathcal{E}_1 \to \mathcal{E}'_1$. Next, let C_2 be the subfield of $C_{\mathcal{K}}$ such that $\operatorname{res} C_2 = \operatorname{res} E_{1,a}$, and let G_2 be the subgroup of $G_{\mathcal{K}}$ such that $v(G_2) = v(E_{1,a}^{\times})$, and obtain likewise $\mathcal{E}_2 = (E_2, C_2, G_2)$ with $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{K}$, and an extension of f_1 to a good map $f_2 : \mathcal{E}_2 \to \mathcal{E}'_2$, with $\mathcal{E}'_1 \subseteq \mathcal{E}'_2 \subseteq \mathcal{K}'$. Continuing this way we obtain for each n small good substructures

$$\mathcal{E}_n \;=\; (E_n, C_n, G_n) \;\subseteq\; \mathcal{E}_{n+1} \;=\; (E_{n+1}, C_{n+1}, G_{n+1})$$

of $\mathcal{K}$ such that $\operatorname{res} C_{n+1} = \operatorname{res} E_{n,a}$ and $v(G_{n+1}) = v(E_{n,a}^{\times})$, and small good substructures $\mathcal{E}'_n \subseteq \mathcal{E}'_{n+1}$ of $\mathcal{K}'$, and good maps

$$f_n \;:\; \mathcal{E}_n \to \mathcal{E}'_n, \qquad f_{n+1} \;:\; \mathcal{E}_{n+1} \to \mathcal{E}'_{n+1}$$

such that f_{n+1} extends f_n; here $\mathcal{E}_0 := \mathcal{E}$, $\mathcal{E}'_0 := \mathcal{E}'$ and $f_0 := f$. Then

$$\mathcal{E}_{\infty} \;:=\; \bigcup_n \mathcal{E}_n \;=\; (E_{\infty}, C_{\infty}, G_{\infty})$$

is a small good substructure of $\mathcal{K}$, and $\mathcal{E}'_{\infty} := \bigcup_n \mathcal{E}'_n = (E'_{\infty}, C'_{\infty}, G'_{\infty})$ is a small good substructure of $\mathcal{K}'$, and we have a good map $f_{\infty} : \mathcal{E}_{\infty} \to \mathcal{E}'_{\infty}$ extending each f_n. Using $E_{\infty,a} = \bigcup_n E_{n,a}$ we see that $E_{\infty,a}$ is an immediate extension of E_{∞}.

If $a \in E_{\infty}$ we have achieved our goal of extending f to a good map with small domain containing a, so assume $a \notin E_{\infty}$. Replacing a by a^{-1} if necessary we arrange $a \preccurlyeq 1$. Take a divergent pc-sequence (a_{ρ}) in E_{∞} such that all $a_{\rho} \preccurlyeq 1$ and $a_{\rho} \rightsquigarrow a$. Then $(b_{\rho}) := \big(f_{\infty}(a_{\rho})\big)$ is a divergent pc-sequence in E'_{∞}. Since the underlying valued field of $\mathcal{K}'$ is κ-saturated and the cardinality of the value group of E'_∞ is less than κ we have $b \in K'$ such that $b_\rho \rightsquigarrow b$. Note that (a_ρ) is of transcendental type over E_∞, by [13, 4.22, 4.16]. Hence (b_ρ) is of transcendental type over E'_∞, and so $E'_{\infty,b}$ is an immediate extension of E'_∞ by Proposition 7.1. This yields the (small) good substructures $\mathcal{E}_{\infty,a} := \big(E_{\infty,a}, C_\infty, G_\infty\big)$ of $\mathcal{K}$ and $\mathcal{E}'_{\infty,b} := \big(E'_{\infty,b}, C'_\infty, G'_\infty\big)$ of $\mathcal{K}'$. Moreover, f_∞ extends by Corollary 7.2 to a good map $\mathcal{E}_{\infty,a} \to \mathcal{E}'_{\infty,b}$, and we have achieved our goal. $\square$

Corollary 10.3. *Suppose* $\operatorname{char} \boldsymbol{k} = 0$, $C_{\mathcal{E}} \preccurlyeq C$ *as* $\mathcal{L}_{\mathrm{r}}$*-structures, and* $G_{\mathcal{E}} \preccurlyeq G$ *as* $\mathcal{L}_{\mathrm{v}}$*-structures. Then* $\mathcal{E} \preccurlyeq \mathcal{K}$.

Proof. Note that $\mathcal{E}$ is a good substructure of both $\mathcal{K}$ and $\mathcal{K}' := \mathcal{E}$, and the identity on $\mathcal{E}$ is a good map. Now apply Theorem 10.2. $\square$

Induced structure on coefficient field and monomial group. In this subsection we assume for our Acg-field $\mathcal{K} = (K, C_{\mathcal{K}}, G_{\mathcal{K}})$ that $\operatorname{char} \boldsymbol{k} = 0$. Our aim here is Corollary 10.5 on the structure that $\mathcal{K}$ induces on $C_{\mathcal{K}}$ and $G_{\mathcal{K}}$ combined. It will be derived in a familiar way from Theorem 10.2 and a fact implicit in its proof. To state that fact we let $\mathcal{E} = (E, C_{\mathcal{E}}, G_{\mathcal{E}})$ and $\mathcal{F} = (F, C_{\mathcal{F}}, G_{\mathcal{F}})$ be Acg-fields and $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-extensions of $\mathcal{K}$. For $a \in E^n$, let $\operatorname{tp}(a|K)$ be the $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-type of a over K, that is, the set of $\mathcal{L}^{A\mathrm{cg},K}_{\preccurlyeq,D}$-formulas $\phi(Y_1,\dots,Y_n)$ such that $\mathcal{E} \models \phi(a)$. Likewise, for $c \in C^n_{\mathcal{E}}$, let $\operatorname{tp}(c|C_{\mathcal{K}})$ be the $\mathcal{L}_{\mathrm{r}}$-type of c over $C_{\mathcal{K}}$, and for $g \in G^n_{\mathcal{E}}$, let $\operatorname{tp}(g|G_{\mathcal{K}})$ be the $\mathcal{L}_{\mathrm{v}}$-type of g over $G_{\mathcal{K}}$.

Lemma 10.4. *Suppose* $\mathcal{E}$ *and* $\mathcal{F}$ *are elementary extensions of* $\mathcal{K}$*. Let* $c_{\mathcal{E}} \in C^m_{\mathcal{E}}$, $g_{\mathcal{E}} \in G^n_{\mathcal{F}}$ *and* $c_{\mathcal{F}} \in C^m_F$, $g_{\mathcal{F}} \in G^n_{\mathcal{F}}$ *be such that*

$$\operatorname{tp}(c_{\mathcal{E}}|C_{\mathcal{K}}) \;=\; \operatorname{tp}(c_{\mathcal{F}}|C_{\mathcal{K}}), \qquad \operatorname{tp}(g_{\mathcal{E}}|G_{\mathcal{K}}) \;=\; \operatorname{tp}(g_{\mathcal{F}}|G_{\mathcal{K}}).$$

Then for the points $(c_{\mathcal{E}}, g_{\mathcal{E}}) \in E^{m+n}$ *and* $(c_{\mathcal{F}}, g_{\mathcal{F}}) \in F^{m+n}$ *we have*

$$\operatorname{tp}\big((c_{\mathcal{E}}, g_{\mathcal{E}})|K\big) \;=\; \operatorname{tp}\big((c_{\mathcal{F}}, g_{\mathcal{F}})|K\big).$$

Proof. By our assumptions $\mathcal{K}$ is a good substructure of both $\mathcal{E}$ and $\mathcal{F}$, and the identity on $\mathcal{K}$ is a good map. Using $\operatorname{tp}(c_{\mathcal{E}}|C) = \operatorname{tp}(c_{\mathcal{F}}|C)$ and the extension procedures (1) and (2) in the proof of Theorem 10.2 in conjunction with Lemma 10.1(i),(ii) we obtain a good map whose domain contains the elements of K and the components of $c_{\mathcal{E}}$ and that is the identity on $\mathcal{K}$ and sends $c_{\mathcal{E}}$ to $c_{\mathcal{F}}$, such that the monomial group of its domain is still $G_{\mathcal{E}}$. Next we use likewise the extension procedures from (3) and (4) in that proof to extend this good map further so that its domain now contains the components of $g_{\mathcal{E}}$ as well, and sends sends $g_{\mathcal{E}}$ to $g_{\mathcal{F}}$. It remains to use Theorem 10.2. $\square$

Corollary 10.5. *Each subset of* $C^m_{\mathcal{K}} \times G^n_{\mathcal{K}} \subseteq K^{m+n}$ *which is definable in* $\mathcal{K}$ *is a finite union of "rectangles"* $P \times Q$ *with* $P \subseteq C^m_{\mathcal{K}}$ *definable in the* $\mathcal{L}_{\mathrm{r}}$*-structure* $C_{\mathcal{K}}$ *and* $Q \subseteq G^n_{\mathcal{K}}$ *definable in the* $\mathcal{L}_{\mathrm{v}}$*-structure* $G_{\mathcal{K}}$.

Proof. Apply Lemma 10.4 in conjunction with [13, Lemmas 5.13, 5.14]. $\square$

Corollary 10.6. *If* $P \subseteq K^n$ *is definable in* $\mathcal{K}$*, then* $P \cap C^n_{\mathcal{K}}$ *is definable in the* $\mathcal{L}_{\mathrm{r}}$*-structure* $C_{\mathcal{K}}$*, and* $P \cap G^n_{\mathcal{K}}$ *is definable in the* $\mathcal{L}_{\mathrm{v}}$*-structure* $G_{\mathcal{K}}$.

In particular, the sets $C_{\mathcal{K}}, G_{\mathcal{K}} \subseteq K$ are stably embedded and orthogonal in $\mathcal{K}$. Next an application of Corollary 10.6.

Recovering the Binyamini-Cluckers-Novikov result. We construe $\mathbb{C}(\!(t)\!)$ below as an A-field in the usual way, with $A = \mathbb{C}[[t]], o(A) = tA$. Proposition 2 in [6] concerns the 3-sorted structure $\mathcal{M}$ consisting of the following:

$$\text{the } A\text{-field } \mathbb{C}(\!(t)\!), \qquad \text{the field } \mathbb{C}, \qquad \text{the ordered abelian group } \mathbb{Z},$$

(each a 1-sorted structure) and two functions relating the three sorts: the obvious t-adic valuation $v : \mathbb{C}(\!(t)\!)^\times \to \mathbb{Z}$, and the "reduced angular component map" $\overline{ac} : \mathbb{C}(\!(t)\!) \to \mathbb{C}$ that assigns to each nonzero Laurent series $f = \sum_{k\in\mathbb{Z}} c_k t^k$ (all $c_k \in \mathbb{C}$) its leading coefficient $c_{v(f)}$, with $\overline{ac}(0) := 0$ by convention.

This 3-sorted $\mathcal{M}$ should not be confused with the 1-sorted $\big(\mathbb{C}(\!(t)\!), \mathbb{C}, t^{\mathbb{Z}}\big)$ that is among the Acg-fields $\mathcal{K}$ considered in Section 10. We have a natural interpretation of $\mathcal{M}$ in $\big(\mathbb{C}(\!(t)\!), \mathbb{C}, t^{\mathbb{Z}}\big)$, which shows that if a set $P \subseteq \mathbb{C}(\!(t)\!)^n$ is definable in $\mathcal{M}$, then it is definable in $\big(\mathbb{C}(\!(t)\!), \mathbb{C}, t^{\mathbb{Z}}\big)$. The converse fails: the subsets $\mathbb{C}$ and $t^{\mathbb{Z}}$ of $\mathbb{C}(\!(t)\!)$ are definable in the latter, but not in the former by [12, Theorem 3.9]; thus $\big(\mathbb{C}(\!(t)\!), \mathbb{C}, t^{\mathbb{Z}}\big)$ is "richer" than $\mathcal{M}$.

For $d \geqslant 1$ we let $\mathbb{C}[t]_{<d}$ be the set of polynomials in $\mathbb{C}[t]$ of degree $< d$. Then $\mathbb{C}[t]_{<d}$ is a subset of $\mathbb{C}[[t]]$, and thus of $\mathbb{C}(\!(t)\!)$. We identify $\mathbb{C}[t]_{<d}$ with $\mathbb{C}^d$ via the bijection $c_0 + c_1 t + \cdots + c_{d-1}t^{d-1} \mapsto (c_0, \dots, c_{d-1})$ for $c_0, \dots, c_{d-1} \in \mathbb{C}$. For $P \subseteq \mathbb{C}(\!(t)\!)^n$ we set $P(d) := P \cap (\mathbb{C}[t]_{<d})^n$, which under the identification above becomes a subset of $\mathbb{C}^{dn}$. Now Proposition 2 in [6] says:

if $P \subseteq \mathbb{C}(\!(t)\!)^n$ is definable in $\mathcal{M}$, then for each $d \geqslant 1$ the set $P(d) \subseteq \mathbb{C}^{dn}$ is a constructible subset of the space $\mathbb{C}^{dn}$ with its Zariski topology.

By "Chevalley-Tarski" a subset of $\mathbb{C}^m$ is constructible iff it is definable in the field $\mathbb{C}$, so this proposition is for $\mathcal{K} = \big(\mathbb{C}(\!(t)\!), \mathbb{C}, t^{\mathbb{Z}}\big)$ a special case of Corollary 10.6.

NIP. The model-theoretic condition NIP forbids certain combinatorial configurations; there is a lot of information about it in [28]. Below a structure $\mathcal{M}$ is said to have NIP if its theory $\mathrm{Th}(\mathcal{M})$ has NIP. By Delon [10] and Gurevich & Schmitt [21], a henselian valued field of equicharacteristic 0 has NIP iff its residue field (as a ring) has NIP. Jahnke & Simon [22] extend this to a criterion that also applies to *expansions* of such valued fields. We apply their criterion now to our analytic setting:

Corollary 10.7. *Assume* $\operatorname{char} \boldsymbol{k} = 0$. *Then:*

$$\text{the } \mathcal{L}^{\mathrm{Acg}}_{\preccurlyeq,D}\text{-structure } \mathcal{K} \text{ has NIP} \iff \text{the ring } \boldsymbol{k} \text{ has NIP.}$$

Proof. The direction $\Rightarrow$ is obvious, and for $\Leftarrow$, assume that $\boldsymbol{k}$ has NIP as a ring. We refer to [22, Section 2] for definitions and notations used in this proof. Using [22, Theorem 2.3], it suffices to show that $\mathrm{Th}(\mathcal{K})$ satisfies the conditions (SE) and (Im). Condition (SE) requires that the residue field and the value group be stably embedded in $\mathcal{K}$. This condition is satisfied in our setting by Corollary 10.6.

In order that $\mathrm{Th}(\mathcal{K})$ satisfies condition (Im) it suffices to show the following: Let $\mathcal{E} = (E, \dots)$ be an elementary extension of $\mathcal{K}$ and $a \in E$ such that $K(a)$ is an immediate valued field extension of K; then the $\mathcal{L}^{\mathrm{Acg}}_{\preccurlyeq,D}$-type of a over K (in $\mathcal{E}$) is implied by instances of NIP formulas in this type. By the Delon-Gurevich-Schmitt result the valued field reduct of $\mathcal{E}$ has NIP. So let $\mathcal{F} = (F, \dots)$ also be an elementary extension of $\mathcal{K}$ and let $b \in F$ have the same $\mathcal{L}_{\preccurlyeq}$-type over K as a; it is enough to show that then a and b also have the same $\mathcal{L}^{\mathrm{Acg}}_{\preccurlyeq,D}$-type over K. Now $\mathcal{K}$ is a good substructure of both $\mathcal{E}$ and $\mathcal{F}$ and the identity on $\mathcal{K}$ is a good map. By Corollary 7.2

this gives a good map $\mathcal{K}_a \to \mathcal{K}_b$ that is the identity on $\mathcal{K}$ and sends a to b. Now apply Theorem 10.2. □

Elementary Equivalence. The AKE-results are often summarized suggestively as follows: For any henselian valued fields E and F of equicharacteristic 0,

$$E \equiv F \iff \operatorname{res} E \equiv \operatorname{res} F \text{ as fields, and } \Gamma_E \equiv \Gamma_F \text{ as ordered groups.}$$

For special A we now derive a similar result in our analytic setting. Let $A = C[[t]]$ where C is a field and take $o(A) = tA$. Recall that we construe A itself as an A-ring. In the beginning of this section we introduced the Acg-field $\big(C((t)), C, t^{\mathbb{Z}}\big)$. We try to embed it into our Acg-field $\mathcal{K} = (K, C_{\mathcal{K}}, G_{\mathcal{K}})$ (with viable valuation A-ring R). Lemma 4.2 yields the A-ring morphism $\iota_0 : A \to R$, which is injective: for $a \in A^{\neq}$ we have $a = ct^e(1+b)$ with $c \in C^\times$, $e \in \mathbb{N}$, and $b \in o(A)$, and so its image in R is nonzero, since viability of R gives $\iota_0(t) \neq 0$. Thus ι_0 extends to an embedding $C((t)) \to K$ of A-fields which we denote by ι_K to indicate its dependence on K. It is routine to verify the following:

Lemma 10.8. *The map $\iota_K : C((t)) \to K$ is an embedding $\big(C((t)), C, t^{\mathbb{Z}}\big) \to \mathcal{K}$ of Acg-fields if and only if $\iota_K(C) \subseteq C_{\mathcal{K}}$ and $\iota_K(t) \in G_{\mathcal{K}}$.*

The conditions $\iota_K(C) \subseteq C_{\mathcal{K}}$ and $\iota_K(t) \in G_{\mathcal{K}}$ are satisfied for $\mathcal{K} = \big(C((t)), C, t^{\mathbb{Z}}\big)$. These conditions are of a first-order nature, since for any $a \in A$ the constant symbol a of $\mathcal{L}^A$ names the element $\iota_K(a) \in K$. If these conditions are satisfied, let $C_{\mathcal{K},\star}$ be the expansion $\big(C_{\mathcal{K}}, (\iota_K(c))_{c\in C}\big)$ of the field $C_{\mathcal{K}}$, and let $G_{\mathcal{K},\star}$ be the expansion $\big(G_{\mathcal{K}}, \iota_K(t)\big)$ of the ordered group $G_{\mathcal{K}}$.

Corollary 10.9. *Assume* $\operatorname{char} C = 0$. *Suppose $\iota_K(C) \subseteq C_{\mathcal{K}}$ and $\iota_K(t) \in G_{\mathcal{K}}$, and likewise for $\mathcal{K}'$. Then we have the following equivalence:*

$$\mathcal{K} \equiv \mathcal{K}' \iff C_{\mathcal{K},\star} \equiv C_{\mathcal{K}',\star} \text{ and } G_{\mathcal{K},\star} \equiv G_{\mathcal{K}',\star}.$$

Proof. The direction $\Rightarrow$ is clear. For $\Leftarrow$, assume the right hand side. Then by Lemma 10.8 we have Acg-field embeddings

$$\iota_K : \big(C((t)), C, t^{\mathbb{Z}}\big) \to \mathcal{K}, \qquad \iota_{K'} : \big(C((t)), C, t^{\mathbb{Z}}\big) \to \mathcal{K}'.$$

Identifying $\big(C((t)), C, t^{\mathbb{Z}}\big)$ with its image in $\mathcal{K}$ and $\mathcal{K}'$ via these embeddings yields good substructures of $\mathcal{K}$ and $\mathcal{K}'$, with the identity on $\big(C((t)), C, t^{\mathbb{Z}}\big)$ as a good map. Now use Theorem 10.2. □

11. Separating Variables

For Acg-fields of equicharacteristic 0 with viable valuation A-ring we establish here a uniform reduction of any formula to a boolean combination of formulas whose quantifiers range only over C and formulas whose quantifiers range only over G. Towards this goal we extend the language by symbols for an absolute value map and for a coefficient map. We begin by introducing these maps. Throughout k and l range over $\mathbb{N}$ (as do d, m, n).

Let K be just a valued field and G a monomial group of K. Then we define the map $|\cdot|_G : K \to K$ as follows:

$$|0|_G := 0, \qquad |a|_G := g \text{ if } a \asymp g \in G.$$

This map is 0-definable in the expansion (K, G) of the valued field K, takes values in $G \cup \{0\}$, and is the identity on $G \cup \{0\}$. Moreover, $a \mapsto |a|_G : K^\times \to G$ is a group morphism, and for all $a, b \in K$: $|a|_G \preccurlyeq |b|_G \Leftrightarrow a \preccurlyeq b$.

Let in addition C be a coefficient field of K. Then we introduce the coefficient map $\mathrm{co} = \mathrm{co}_{C,G} : K \to K$ as follows:

$$\mathrm{co}(0) \ := \ 0, \qquad \mathrm{co}(a) \ := \ c \ \text{ if } a \neq 0 \text{ and } a/|a|_G \sim c \in C^\times.$$

This map is 0-definable in the expansion (K, C, G) of the valued field K, takes values in C, is the identity on C, and $\mathrm{co}(ab) = \mathrm{co}(a)\mathrm{co}(b)$ for all $a, b \in K$.

In the rest of this section A is noetherian with an ideal $\mathcal{o}(A) \neq A$ such that $\bigcap_e \mathcal{o}(A)^e = \{0\}$ and A is $\mathcal{o}(A)$-adically complete. We extend the language $\mathcal{L}^A_{\preccurlyeq,D}$ to $\mathcal{L}^{A,+}_{\preccurlyeq,D}$ by adding unary function symbols co and $|\cdot|$, and likewise we extend $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$ to $\mathcal{L}^{A\mathrm{cg},+}_{\preccurlyeq,D}$.

Below $\mathcal{K} = (K, C, G)$ ranges over Acg-fields whose valuation A-ring is viable. We expand $\mathcal{K}$ to an $\mathcal{L}^{A\mathrm{cg},+}_{\preccurlyeq,D}$-structure $\mathcal{K}^+$ by interpreting $|\cdot|$ as the function $|\cdot|_G$ and co as the corresponding coefficient map $\mathrm{co}_{C,G}$. Note that the good substructures of $\mathcal{K}$ are exactly the $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$-reducts of $\mathcal{L}^{A\mathrm{cg},+}_{\preccurlyeq,D}$-substructures of $\mathcal{K}^+$ whose underlying ring is a field. Now

$$\{\tau^{\mathcal{K}} : \ \tau \text{ is a variable-free } \mathcal{L}^{A,+}_{\preccurlyeq,D}\text{-term}\}$$

underlies an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of K, so by Lemma 6.1,

$$R_0 \ := \ \{\tau^{\mathcal{K}} : \ \tau \text{ is a variable-free } \mathcal{L}^{A,+}_{\preccurlyeq,D}\text{-term and } \tau^{\mathcal{K}} \preccurlyeq 1\}$$

is an A-subring of R and a valuation ring dominated by R. Hence

$$K_0 \ := \ \mathrm{Frac}(R_0)$$

underlies the smallest $\mathcal{L}^{A\mathrm{cg},+}_{\preccurlyeq,D}$-substructure of $\mathcal{K}^+$ whose underlying ring is a field. Thus $\mathcal{K}_0 := (K_0, C \cap K_0, G \cap K_0)$ is a good substructure of $\mathcal{K}$.

More generally, let $a = (a_1, \dots, a_k) \in K^k$, and take a tuple $u = (u_1, \dots, u_k)$ of distinct variables $u_1, \dots, u_k$. Then $\{\tau^{\mathcal{K}}(a) : \ \tau(u) \text{ is an } \mathcal{L}^{A,+}_{\preccurlyeq,D}\text{-term}\}$ underlies an $\mathcal{L}^A_{\preccurlyeq,D}$-substructure of K, so by Lemma 6.1,

$$R_{0|a} \ := \ \{\tau^{\mathcal{K}}(a) : \ \tau(u) \text{ is an } \mathcal{L}^{A,+}_{\preccurlyeq,D}\text{-term and } \tau^{\mathcal{K}}(a) \preccurlyeq 1\}$$

is an A-subring of R and a valuation ring dominated by R. Hence

$$K_{0|a} \ := \ \mathrm{Frac}(R_{0|a})$$

underlies the smallest $\mathcal{L}^{A\mathrm{cg},+}_{\preccurlyeq,D}$-substructure of $\mathcal{K}^+$ that contains $a_1, \dots, a_k$ and whose underlying ring is a field. Thus $\mathcal{K}_{0|a} := (K_{0|a}, C \cap K_{0|a}, G \cap K_{0|a})$ is a good substructure of $\mathcal{K}$.

Towards separating variables, let $\mathcal{L}^{\mathrm{c}}_{\mathrm{r}}$ be the language $\mathcal{L}_{\mathrm{r}}$ augmented by the unary relation symbol C, and let $\mathcal{L}^{\mathrm{g}}_{\mathrm{v}}$ be the language $\mathcal{L}_{\mathrm{v}}$ augmented by the unary relation symbol G, so $\mathcal{L}^{\mathrm{c}}_{\mathrm{r}}$ and $\mathcal{L}^{\mathrm{g}}_{\mathrm{v}}$ are sublanguages of $\mathcal{L}^{A\mathrm{cg}}_{\preccurlyeq,D}$. We define the c-*relative formulas* to be the $\mathcal{L}^{\mathrm{c}}_{\mathrm{r}}$-formulas obtained by applying the following recursive rules:

- quantifier-free $\mathcal{L}_{\mathrm{r}}$-formulas are c-relative formulas;
- if ϕ and ψ are c-relative formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$;
- if ϕ is a c-relative formula and u is a variable, then $\exists u\big(C(u) \wedge \phi\big)$ and $\forall u\big(C(u) \to \phi\big)$ are c-relative formulas.

So "c-relative" indicates that all quantifiers are relativized to C. Likewise, the g-*relative formulas* are the $\mathcal{L}_{\mathrm{v}}^{\mathrm{g}}$-formulas obtained by the following recursive rules:

- quantifier-free $\mathcal{L}_{\mathrm{v}}$-formulas are g-relative formulas;
- if ϕ and ψ are g-relative formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$;
- if ϕ is a g-relative formula and u is a variable, then $\exists u\big(G(u) \wedge \phi\big)$ and $\forall u\big(G(u) \to \phi\big)$ are g-relative formulas.

Let $x_1, \dots, x_m$ be distinct variables and $x = (x_1, \dots, x_m)$. A c-*formula* in x is by definition an $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg},+}$-formula

$$\psi(x) \ := \ \psi'\big(\mathrm{co}(\tau_1(x)), \dots, \mathrm{co}(\tau_k(x))\big)$$

with $\psi'(u_1, \dots, u_k)$ a c-relative formula and $\mathcal{L}_{\preccurlyeq,D}^{A,+}$-terms $\tau_1(x), \dots, \tau_k(x)$. Likewise, a g-*formula* in x is an $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg},+}$-formula

$$\theta(x) \ := \ \theta'\big(|\tau_1(x)|, \dots, |\tau_k(x)|\big)$$

with $\theta'(u_1, \dots, u_k)$ a g-relative formula and $\mathcal{L}_{\preccurlyeq,D}^{A,+}$-terms $\tau_1(x), \dots, \tau_k(x)$.

Let $\boldsymbol{k}$ be the residue field of the Acg-field $\mathcal{K} = (K, C, G)$, and Γ its value group. Let $a \in K^m$, let $\mathrm{tp}^{\mathcal{K}}(a)$ denote the $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg}}$-type realized by a in $\mathcal{K}$, that is, the set of $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg}}$-formulas $\phi(x)$ such that $\mathcal{K} \models \phi(a)$. Also,

$$\mathrm{tp}_{\mathrm{c}}^{\mathcal{K}}(a) \ := \ \{\psi(x) : \ \mathcal{K}^+ \models \psi(a)\}, \qquad \mathrm{tp}_{\mathrm{g}}^{\mathcal{K}}(a) \ := \ \{\theta(x) : \ \mathcal{K}^+ \models \theta(a)\},$$

where $\psi(x)$ ranges over c-formulas in x, and $\theta(x)$ over g-formulas in x.

Next, let $\mathcal{K}' = (K', C', G')$ also be an Acg-field whose valuation A-ring is viable, and let $a' \in K'^m$.

Lemma 11.1. *Suppose* $\mathrm{char}\,\boldsymbol{k} = 0$. *If* $\mathrm{tp}_{\mathrm{c}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{c}}^{\mathcal{K}'}(a')$ *and* $\mathrm{tp}_{\mathrm{g}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{g}}^{\mathcal{K}'}(a')$, *then* $\mathrm{tp}^{\mathcal{K}}(a) = \mathrm{tp}^{\mathcal{K}'}(a')$.

Proof. As at the beginning of this subsection we have the good substructures $\mathcal{K}_{0|a}$ of $\mathcal{K}$, and $\mathcal{K}'_{0|a'}$ of $\mathcal{K}'$. Assume $\mathrm{tp}_{\mathrm{c}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{c}}^{\mathcal{K}'}(a')$ and $\mathrm{tp}_{\mathrm{g}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{g}}^{\mathcal{K}'}(a')$. Then for any $\mathcal{L}_{\preccurlyeq,D}^{A,+}$-term $\tau(x)$, $\tau^{\mathcal{K}}(a) = 0 \Leftrightarrow \tau^{\mathcal{K}'}(a') = 0$, since it follows from $\mathrm{tp}_{\mathrm{c}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{c}}^{\mathcal{K}'}(a')$ that

$$\tau^{\mathcal{K}}(a) = 0 \ \Leftrightarrow \ \mathrm{co}(\tau)^{\mathcal{K}}(a) = 0 \ \Leftrightarrow \ \mathrm{co}(\tau)^{\mathcal{K}'}(a') = 0 \ \Leftrightarrow \ \tau^{\mathcal{K}'}(a') = 0.$$

We also have for $f, h \in K$ the equivalences

$$f \preccurlyeq h \ \Leftrightarrow \ f = hD(f,h), \quad f \in C \ \Leftrightarrow \ \mathrm{co}(f) = f, \quad f \in G \ \Leftrightarrow \ f \neq 0 \text{ and } |f| = f,$$

so Lemma 6.1(ii) gives a unique $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg}+}$-isomorphism $\sigma : \mathcal{K}_{0|a}^+ \to \mathcal{K}'^+_{0|a'}$ such that $\sigma(a_i) = a_i'$ for $i = 1, \dots, m$; in fact, $\sigma\big(\tau^{\mathcal{K}}(a)\big) = \tau^{\mathcal{K}'}(a')$ for every $\mathcal{L}_{\preccurlyeq,D}^{A,+}$-term $\tau(x)$. From $\mathrm{tp}_{\mathrm{c}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{c}}^{\mathcal{K}'}(a')$ and $\mathrm{tp}_{\mathrm{g}}^{\mathcal{K}}(a) = \mathrm{tp}_{\mathrm{g}}^{\mathcal{K}'}(a)$ it now follows that $\sigma : \mathcal{K}_{0|a} \to \mathcal{K}'_{0|a'}$ is a good map. Hence $\mathrm{tp}^{\mathcal{K}}(a) = \mathrm{tp}^{\mathcal{K}'}(a')$ by Theorem 10.2. $\square$

Let T_A be the $\mathcal{L}_{\preccurlyeq,D}^{A\mathrm{cg}}$-theory whose models are the Acg-fields of equicharacteristic 0 whose valuation A-ring is viable. (Recall that the viability of the valuation A-ring R of $\mathcal{K}$ means that $\mathcal{O}(R) = tR$ for some $t \in R^{\neq}$ with $t \in \mathcal{O}(A)R$; as $\mathcal{O}(A)$ is finitely generated, this is a first-order condition.) Let T_A^+ be the extension by definitions of T_A whose models are the expansions $\mathcal{K}^+$ of models $\mathcal{K}$ of T_A.

Corollary 11.2. *Every $\mathcal{L}^{\mathrm{Acg}}_{\preccurlyeq,D}$-formula $\phi(x)$ is T^+_A-equivalent to*

$$\big(\psi_1(x) \wedge \theta_1(x)\big) \vee \cdots \vee \big(\psi_N(x) \wedge \theta_N(x)\big)$$

for some $N \in \mathbb{N}$, c*-formulas $\psi_1(x), \ldots, \psi_N(x)$, and* g*-formulas $\theta_1(x), \ldots, \theta_N(x)$.*

Proof. Apply Lemma 11.1 in conjunction with [13, Lemmas 5.13, 5.14]. $\square$

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