Defects of Algebraic Function Fields, Completion Defects and Defect Quotients

Franz-Viktor Kuhlmann, Asim Naseem*

Abstract. The *defect* (also called *ramification deficiency*) of valued field extensions is a major stumbling block in deep open problems of valuation theory in positive characteristic. For a detailed analysis, we define and investigate two weaker notions of defect: the *completion defect* and the *defect quotient*. We define all three defects for finite valued field extensions as well as for certain valued function fields (those with *Abhyankar valuations* that are allowed to be nontrivial on the ground field). These defects of valued function fields have played an important role in genus reduction formulas that were presented by several authors. We prove the most general known form of the Finiteness and Independence Theorem for the defect of valued function fields. Further, we investigate the completion defect and the defect quotient in detail and present analogues of the results that hold for the usual defect.

1. Introduction

For a valued field (K, v) we denote its value group by vK and its residue field by Kv; for $a \in K$, va denotes its value, and av its residue. We denote the algebraic closure of K by $\tilde{K}$ and the perfect hull by K^{1/p^∞}. By $(L|K, v)$ we mean an extension of valued fields where v is a valuation on L and its subfield K is endowed with the restriction of v. Throughout, **function field** will always mean an *algebraic function field*.

In what follows, we fix an extension of the valuation v from K to its algebraic closure. All algebraic extensions of K will be endowed with the restriction of this valuation. All of these valuations will again be denoted by v. This also determines uniquely the henselizations of all algebraic extensions of K (cf. Section 2.1). An algebraic extension $(L|K, v)$ is called **h-finite** if $(L^h|K^h, v)$ is finite, where K^h is the unique henselization of K inside the henselization L^h of L. The **(henselian)**

defect of an h-finite extension $(L|K, v)$ is the natural number

$$d(L|K, v) \; := \; \frac{[L^h : K^h]}{(vL^h : vK^h)[L^h v : K^h v]} \; = \; \frac{[L^h : K^h]}{(vL : vK)[Lv : Kv]} \; ; \qquad (1)$$

the second equation holds since henselizations are immediate extensions (see Section 2.1). By the Lemma of Ostrowski (see Section 2.2), this quotient is always 1 if Kv has characteristic 0, and it is a non-negative power of p if Kv has characteristic $p > 0$.

1.1. The defect of valued function fields. Matignon [M] and later Green, Matignon and Pop [GMP] defined a "vector space defect" of certain valued function fields and used it for the formulation and the proof of important genus reduction inequalities. These inequalities connect the genus of a function field $F|K$ of transcendence degree 1 (where K is the exact constant field) with the genera of the reduced function fields

$$Fv_i \,|\, Kv \; (1 \le i \le s)$$

where the valuations v_i are distinct extensions of a given valuation v on K such that all extensions $Fv_i \,|\, Kv$ are transcendental, hence finitely generated of transcendence degree 1.

Also for function fields of higher transcendence degree, the studies arising from the questions of genus reduction and good reduction assume that the transcendence degree of the function field is equal to the transcendence degree of the residue field extension (which by Corollary 2.18 implies that the residue field extension is also an algebraic function field). Problems like local uniformization and the investigation of the model theoretic properties of valued fields force us to consider a more general situation. A valued field extension $(F|K, v)$ of finite transcendence degree is called **without transcendence defect** if equality holds in the **Abhyankar Inequality**

$$\text{trdeg}\, F|K \; \ge \; \text{trdeg}\, Fv|Kv + \text{rr}\, vF/vK \; . \qquad (2)$$

Here, for any ordered abelian group G, rr $G := \dim_{\mathbb{Q}} G \otimes \mathbb{Q}$ denotes the maximal number of rationally independent elements in G; this is called the **rational rank** of G. In particular in the case where v is trivial on K, valuations without transcendence defect are also called **Abhyankar valuations**.

Every extension $(F|K, v)$ without transcendence defect admits a **standard valuation transcendence basis**, that is, a transcendence basis $\{x_i, y_j \mid i \in I, j \in J\}$ such that

$$\left.\begin{array}{l} \text{the values } vx_i \, , \, i \in I, \text{ are rationally independent over } vK, \\ \text{the residues } y_j v, \, j \in J, \text{ are algebraically independent over } Kv. \end{array}\right\} \quad (3)$$

(The second condition implicitly says that $vy_j = 0$ for all $j \in J$.) Indeed, if the elements x_i are chosen such that their values form a maximal set of elements in vF rationally independent modulo vK, and the elements y_j are chosen such that their

residues form a transcendence basis of $Fv|Kv$, then by Lemma 2.17, the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K. If equality holds in the Abhyankar Inequality, then their number equals the transcendence degree, and so they form a standard valuation transcendence basis.

As for h-finite extensions, a notion of defect can be defined for every valued function field $(F|K, v)$ without transcendence defect. More generally, we consider **subhenselian function fields**, that is, extensions $(F|K, v)$ for which $(F, v)^h$ is the henselization of some valued function field $(F_0|K, v)$. Note that in this case, $F|K(T)$ is an h-finite extension for every transcendence basis T of $F|K$ and hence we may consider the defect $d(F|K(T), v)$. For subhenselian function fields without transcendence defect, the defect can be introduced as:

$$d(F|K, v) := \sup_T \, d(F|K(T), v) \tag{4}$$

where the supremum is taken over all transcendence bases of $(F|K, v)$. In the case of a henselian ground field (K, v) and $\operatorname{trdeg} F|K = \operatorname{trdeg} Fv|Kv$, this ("henselian") defect coincides with the "vector space defect" defined by Green, Matignon and Pop (see Section 2 of [GMP] for details).

The following theorem shows the **finiteness** of the defect, as it is equal to $d(F|K(T), v)$ for a standard valuation transcendence basis T, and its **independence** of the choice of this standard valuation transcendence basis.

Theorem 1.1 (Finiteness and Independence Theorem). *Take a subhenselian function field $(F|K, v)$ without transcendence defect. Then for every standard valuation transcendence basis T of $F|K$,*

$$d(F|K, v) \, = \, d(F|K(T), v) \, < \, \infty \, . \tag{5}$$

Moreover, there exists a finite extension K' of K such that for every algebraic extension L of K containing K' we have:

(1) *for every standard valuation transcendence basis T of $(F|K, v)$, the extension $(L.F|L(T), v)$ is defectless,*

(2) $d(F|K, v) \, = \, \dfrac{d(L|K, v)}{d(L.F|F, v)} \, = \, \max_{N|K \text{ finite}} \, \dfrac{d(N|K, v)}{d(N.F|F, v)} \, .$

The proof of this theorem is given in Section 2.5. It heavily depends on the following result proved in [K1] and [K4]:

Theorem 1.2 (Generalized Stability Theorem). *Let $(F|K, v)$ be a valued function field without transcendence defect. If (K, v) is a defectless field, then also (F, v) is a defectless field. The same holds for "inseparably defectless" in place of "defectless". If vK is cofinal in vF, then it also holds for "separably defectless" in place of "defectless".*

This theorem is a generalization of a result of Grauert and Remmert [G-R] which is restricted to the case of algebraically closed complete ground fields of rank

1 (i.e., with archimedean value group, meaning that it is a subgroup of the reals).
A first generalization of their result was given by Gruson [GRU]; an improved
presentation of it can be found in the book [BGR] of Bosch, Güntzer and Remmert.
Further generalizations are due to M. Matignon and J. Ohm; see also [GMP]. In
[O], Ohm arrived at a version of Theorem 1.2 with the already discussed restriction
that $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$. The work of all authors mentioned above is based
on methods of nonarchimedean analysis. In contrast, the proofs given in [K1] and
[K4] are purely valuation theoretic.

Theorem 1.1 was independently obtained by Ohm [O] in the case of $\mathrm{trdeg}\, F|K =$
$\mathrm{trdeg}\, Fv|Kv$ by using his version of the stability theorem. The name *Independence
Theorem* was coined by him. Another special case was proved by Sudesh Khanduja
in [Kh]. She considered simple transcendental extensions $(K(x)|K, v)$ satisfying the
condition $\mathrm{rr}\, vK(x)|vK = 1 = \mathrm{trdeg}\, K(x)|K$.

1.2. Completion defect and defect quotient. Matignon and Ohm (cf. [M],
[O]) used a *completion defect* which measures the defect, for valuations of rank
1, using completions instead of henselizations. The notion of completion defect
played a key role in the proof of Matignon's genus reduction inequality for valued
function fields. This inequality was first proved by Matignon in [M] for valued
function fields of transcendence degree 1 and rank 1. It was extended in [GMP]
to an arbitrary finite family of valuations coinciding on the constant field, now
passing from the completion defect to the henselian defect or vector space defect
for the case of higher rank. The proof still depends on the rank 1 case with its use
of the completion defect.

Although in [GMP] a completion defect was neither introduced nor used in
the case of higher rank, it is worthwile to study its generalization to arbitrary
rank. This allows us to explore the defect in more precise detail. The process
of passing to the completion cuts out a special type of defect which appears to
be less malicious than the remaining completion defect. The fact that even for
valuations of higher rank, completions play a role in the study of the defect is
indicated by results such as Theorems 1.11, 5.1 and 5.2 and Corollary 6.8 in [K5];
the assertion of Corollary 6.8 was originally presented by F. Delon in [D]. The
assertion of Theorem 5.1 will reappear in this paper in Theorem 1.6, in a different
formulation and with a different proof.

Since for valuations of arbitrary rank, the completion is in general not henselian,
we measure the defect over the completion of the henselization. This defect then
coincides with Matignon's and Ohm's completion defect for valuations of rank 1.

Let K^{hc} denote the completion of (K^h, v). Take any h-finite extension $(L|K, v)$.
We define the **completion defect** $d_c(L|K, v)$ as follows:

$$d_c(L|K, v) := \frac{[L^{hc} : K^{hc}]}{(vL : vK)[Lv : Kv]} = d(L^{hc}|K^{hc}, v) ;$$

this is well-defined and the second equation holds because completions are im-
mediate extensions and $(L^{hc}|K^{hc}, v)$ is a finite extension of henselian fields (see

Section 2.1, in particular Lemma 2.3 and Lemma 2.4). This is why we have chosen to work with the completion of the henselization. Instead, we could have chosen to work with

$$d(L^c|K^c, v) \; = \; \frac{[L^{ch} : K^{ch}]}{(vL^c : vK^c)[L^cv : K^cv]} \; = \; \frac{[L^{ch} : K^{ch}]}{(vL : vK)[Lv : Kv]},$$

where K^{ch} is the henselization of the completion of K (which is not necessarily complete). In fact, we will prove:

Proposition 1.3. *For every h-finite extension* $(L|K, v)$,

$$d(L^c|K^c, v) \; = \; d_c(L|K, v)\,,$$

and for every h-finite separable extension $(L|K, v)$,

$$d_c(L|K, v) \; = \; d(L|K, v)\,.$$

In order to characterize those extensions for which the completion defect is equal to the ordinary defect we compute the **defect quotient**:

$$d_q(L|K, v) \; := \; \frac{d(L|K, v)}{d_c(L|K, v)} \; = \; \frac{[L^h : K^h]}{[L^{hc} : K^{hc}]}\,.$$

We denote by $[L : K]_{\mathrm{insep}}$ the **inseparable degree** of a finite extension $L|K$, that is, the degree of $L|K$ divided by the degree of the maximal separable subextension $L_s|K$.

Proposition 1.4. *For every finite extension* $(L|K, v)$,

$$d_q(L|K) = \frac{[L : K]_{\mathrm{insep}}}{[L^c : K^c]_{\mathrm{insep}}}\,.$$

An h-finite extension $(L|K, v)$ is called **c-defectless** if $d_c(L|K, v) = 1$ and **q-defectless** if $d_q(L|K, v) = 1$. A valued field (K, v) will be called a **c-defectless field** or **q-defectless field** if every finite extension $(L|K, v)$ is c-defectless or q-defectless, respectively. The properties of being "c-defectless" or "q-defectless" are weaker than "defectless" (cf. Section 2.2 for the latter notion). Another pair of weaker properties are inseparably defectless and separably defectless (again, see Section 2.2 for these notions).

In Section 3.1, we prove the following characterizations:

Theorem 1.5. *A valued field* (K, v) *is a c-defectless field if and only if it is a separably defectless field.*

Theorem 1.6. *A valued field* (K, v) *is q-defectless if and only if its completion is a separable extension (that is, linearly disjoint from the perfect hull* K^{1/p^∞} *over* K). *In particular, every complete field and every valued field of characteristic 0 is q-defectless.*

The completion defect and defect quotient of valued function fields without transcendence defect may be defined similarly as it was done for the defect. For a subhenselian function field $(F|K,v)$ without transcendence defect we set

$$d_c(F|K,v) := \sup_T d_c(F|K(T),v) \quad \text{and} \quad d_q(F|K,v) := \sup_T d_q(F|K(T),v) \,,$$

where the supremum is taken over all transcendence bases of $(F|K,v)$. The same finiteness and independence as for the defect (Theorem 1.1) also hold for the completion defect and defect quotient:

Theorem 1.7. *Take a subhenselian function field $(F|K,v)$ without transcendence defect over K. Then for every standard valuation transcendence basis T of $F|K$,*

$$d_c(F|K,v) \,=\, d_c(F|K(T),v) \quad \text{and} \quad d_q(F|K,v) \,=\, d_q(F|K(T),v) \,. \quad (6)$$

Further,

$$d(F|K,v) \,=\, d_c(F|K,v) \cdot d_q(F|K,v) \,. \quad (7)$$

Assume in addition that vK is cofinal in vF. If K' is chosen as in the assertion of Theorem 1.1, then for every finite extension L of K containing K',

$$d_c(F|K,v) \;=\; \frac{d_c(L|K,v)}{d_c(L.F|F,v)} \;=\; \max_{N|K \text{ finite}} \frac{d_c(N|K,v)}{d_c(N.F|F,v)} \quad (8)$$

$$d_q(F|K,v) \;=\; \frac{d_q(L|K,v)}{d_q(L.F|F,v)} \;=\; \max_{N|K \text{ finite}} \frac{d_q(N|K,v)}{d_q(N.F|F,v)} \;. \quad (9)$$

We will prove Theorem 1.7 in Section 3.2, together with the following "q-defectless and c-defectless versions" of Theorem 1.2.

Theorem 1.8. *Take a subhenselian function field $(F|K,v)$ without transcendence defect.*

a) If (K,v) is a q-defectless field or vK is not cofinal in vF, then $d_q(F|K,v) = 1$ and (F,v) is a q-defectless field.

b) If vK is cofinal in vF and (K,v) is a c-defectless field, then $d_c(F|K,v) = 1$ and (F,v) is a c-defectless field.

We will use $d(F|K)$, $d_c(F|K)$ and $d_q(F|K)$ for defect, completion defect and quotient defect of $(F|K,v)$, respectively, when there is no ambiguity for the valuation v.

2. Valuation theoretic preliminaries

For the basic facts of valuation theory, we refer the reader to [E], [EP], [R], [W] and [Z-S].

2.1. Henselization and completion. Every finite extension L of (K, v) satisfies the **fundamental inequality** (cf. [E]):

$$[L : K] \geq \sum_{i=1}^{g} \mathrm{e}_i \mathrm{f}_i, \tag{10}$$

where $v_1, \ldots, v_g$ are the distinct extensions of v from K to L, $\mathrm{e}_i = (v_i L : vK)$ are the respective ramification indices and $\mathrm{f}_i = [Lv_i : Kv]$ are the respective inertia degrees. If $g = 1$ for every finite extension $L|K$ then (K, v) is called **henselian**. This means that (K, v) is henselian if and only if v extends uniquely to each algebraic extension of K. Therefore, every algebraically closed valued field is trivially henselian.

Every valued field (K, v) admits a **henselization**, that is, a separable-algebraic extension field which is henselian and has the universal property that it admits a unique embedding in every henselian extension field of (K, v). In particular, if (L, w) is a henselian extension field of (K, v), then (K, v) has a unique henselization in (L, w), which we will denote by $K^{h(w)}$. Further, all henselizations of (K, v) are isomorphic over K, so we often talk of *the* henselization of (K, v) and just write K^h. We can also fix the henselizations by consistently working inside a large algebraically closed field. A valued field is henselian if and only if it is equal to any (and thus all) of its henselizations.

An extension $(L|K, v)$ is **immediate** if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto, which we also express by the less precise assertion that $vK = vL$ and $Kv = Lv$. The henselization is an immediate extension.

If L is a finite extension of K and $v_1, \ldots, v_g$ are the distinct extensions of v from K to L, then (K, v) has a henselization $K^{h(v_i)}$ in each henselization $L^{h(v_i)}$ of (L, v_i), and

$$[L : K] = \sum_{1 \leq i \leq g} [L^{h(v_i)} : K^{h(v_i)}]. \tag{11}$$

We have:

Lemma 2.1. *An algebraic extension of a henselian field is again henselian. If $(L|K, v)$ is algebraic, then $(L.K^h, v)$ is the henselization of (L, v).*

Let (K, v) be any valued field. A valuation w on K is a **coarsening** of v if its valuation ring $\mathcal{O}_w$ contains the valuation ring $\mathcal{O}_v$ of v. If H is a convex subgroup of vK, then it gives rise to a coarsening w with valuation ring $\mathcal{O}_w := \{x \in K \mid \exists \alpha \in H, \ \alpha \leq vx\}$. Then v induces a valuation $\overline{w}$ on Kw with valuation ring $\mathcal{O}_{\overline{w}} := \{xw \mid x \in \mathcal{O}_v\}$, and there are canonical isomorphisms $wK \cong vK/H$ and $\overline{w}(Kw) \cong H$. If (K, w) is any valued field and if w' is any valuation on the residue field Kw, then $w \circ w'$, called the **composition of w and w'**, will denote the valuation whose valuation ring is the subring of the valuation ring of w consisting of all elements whose w-residues lie in the valuation ring of w'. In our above situation, v is the composition of w and $\overline{w}$.

The following fact is well known:

Lemma 2.2. *Take a valued field (K, v) and a composition $v = w \circ \overline{w}$. Then (K, v) is henselian if and only if (K, w) and $(Kw, \overline{w})$ are henselian.*

We conclude this section with a few results about the completion K^c of K. Like the henselization, also the completion is an immediate extension.

Lemma 2.3. *If $(L|K, v)$ is finite, then the extension of v from K^c to $L.K^c$ is unique, and $(L.K^c, v)$ is the completion of (L, v).*

Proof. A finite extension of a complete valued field is again complete, so $(L.K^c, v)$ is complete for each extension of v from K^c to $L.K^c$. On the other hand, since $(L|K, v)$ is finite, the value group vK is cofinal in vL, which implies that the completion of (L, v) must contain the completion of (K, v). As it also contains L, it contains $L.K^c$. Thus, $(L.K^c, v)$ is the completion of (L, v), which also implies that the extension of v from K^c to $L.K^c$ is unique. $\square$

A proof of the following theorem can be found in [W] (Theorem 32.19):

Lemma 2.4. *The completion of a henselian field is henselian too. Consequently,*

$$(K^{hc})^{hc} = K^{hc}.$$

Moreover, a henselian field is separable-algebraically closed in its completion.

2.2. Defect and defectless fields. Assume that $(L|K, v)$ is a finite extension such that v extends uniquely from K to L. Then the Lemma of Ostrowski (cf. [EN], [R]) says that

$$[L : K] = p^{\nu}(vL : vK)[Lv : Kv] \, , \text{ for some integer } \nu \geq 0$$

where $p = \operatorname{char} Lv$ if it is positive and $p = 1$ otherwise. The factor $d(L|K, v) := p^{\nu}$ is called the **defect** of the extension $(L|K, v)$. If $d(L|K, v) = 1$, then $L|K$ is called a **defectless extension**. More generally (i.e., for $g \geq 1$), a finite extension $(L|K, v)$ is called defectless if equality holds in (10). A valued field (K, v) is said to be a **defectless**, **separably defectless** or **inseparably defectless field** if every finite, finite separable or finite purely inseparable, respectively, extension of K satisfies equality in the fundamental inequality (10). One can trace this back to the case of unique extensions of the valuation; for the proof of the following theorem, see [K2] (a partial proof was already given in [E]):

Lemma 2.5. *A valued field is defectless if and only if its henselization is defectless. The same holds for "separably defectless" and "inseparably defectless" in place of "defectless".*

Therefore, the Lemma of Ostrowski shows that:

Corollary 2.6. *Every valued field (K, v) with $\operatorname{char} Kv = 0$ is a defectless field.*

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of a field extension and of ramification index and inertia degree.

Lemma 2.7. *Let $K \subset L \subset M$ be fields and v extends uniquely from K to M. Then*

$$d(M|K,v) = d(M|L,v) \cdot d(L|K,v).$$

In particular, $(M|K,v)$ is defectless if and only if $(M|L,v)$ and $(L|K,v)$ are defectless.

Using this lemma together with Lemma 2.5, one easily shows:

Lemma 2.8. *Every finite extension of a defectless field is again a defectless field.*

The following theorem is proved in [K5], where it is stated with the additional hypothesis "char $K = p > 0$". For our purpose in this paper, we state it in general and include the proof.

Lemma 2.9. *Let (K,v) be a henselian valued field. Then K is separably defectless if and only if K^c is defectless.*

Proof. In view of Corollary 2.6, we may assume that char $Kv = p > 0$. Since K is a henselian field, the same holds for K^c (Lemma 2.4). The field K^c is defectless if and only if it is separably defectless, indeed, this is trivially true when char $K = 0$, and in the case of positive characteristic it is implied by Theorem 5.1 of [K5]. Thus it suffices to prove that K^c is a separably defectless field if and only if K is.

Let $L|K$ be an arbitrary finite separable extension. The henselian field K is separable-algebraically closed in K^c (Lemma 2.4). Consequently, every finite separable extension of K is linearly disjoint from K^c over K, whence

$$[L.K^c : K^c] = [L : K].$$

On the other hand, $L.K^c = L^c$ by Lemma 2.3. Consequently,

$$(v(L.K^c) : vK^c)[(L.K^c)v : K^c v] = (vL : vK)[Lv : Kv].$$

Assume that K^c is a separably defectless field. Then $(L.K^c|K^c,v)$ is defectless, i.e., $[L.K^c : K^c] = (v(L.K^c) : vK^c)[(L.K^c)v : K^c v]$. Hence, $[L : K] = (vL : vK)[Lv : Kv]$, showing that $L|K$ is defectless. We have shown that K is separably defectless if K^c is.

Now assume that K^c is not a separably defectless field. Then there exists a finite Galois extension $L'|K^c$ with nontrivial defect. In view of Lemma 2.7, we may assume that the extension is Galois (after passing to the normal hull if necessary). We take an irreducible polynomial $f = X^n + c_{n-1}X^{n-1} + \cdots + c_0 \in K^c[X]$ of which L' is the splitting field. For every $\alpha \in vK$ there are $d_{n-1}, \ldots, d_0 \in K$ such that $v(c_i - d_i) \geq \alpha$. If α is large enough, then by Theorem 32.20 of [W], the splitting fields of f and $g = X^n + d_{n-1}X^{n-1} + \cdots + d_0$ over the henselian field K^c are the same. Consequently, if L denotes the splitting field of g over K, then $L' = L.K^c = L^c$. We obtain

$$\begin{aligned}
[L : K] &\geq [L.K^c : K^c] = [L' : K^c] \\
&> (vL' : vK^c)[L'v : K^c v] = (vL^c : vK^c)[L^c v : K^c v] \\
&= (vL : vK)[Lv : Kv].
\end{aligned}$$

That is, the separable extension $L|K$ is not defectless. Hence, K is not a separably defectless field. □

The following lemma describes the behaviour of the defect under composition of valuations:

Lemma 2.10. *Take a finite extension $(L|K,v)$ of henselian fields and a coarsening w of v on L. Then*

$$d(L|K,v) \;=\; d(L|K,w) \cdot d(Lw|Kw,\overline{w}) \,.$$

In particular, if $d(L|K,v) = 1$, then $d(L|K,w) = 1$ for every coarsening w of v.

Proof. Since (L,v) and (K,v) are henselian by assumption, also (L,w), (K,w), $(Lw,\overline{w})$ and $(Kw,\overline{w})$ are henselian by Lemma 2.2. Therefore, we can compute:

$$
\begin{aligned}
d(L|K,v) \;&=\; \frac{[L:K]}{(vL:vK)[Lv:Kv]} \\[2mm]
&=\; \frac{[L:K]}{(wL:wK)(\overline{w}(Lw):\overline{w}(Kw))[(Lw)\overline{w}:(Kw)\overline{w}]} \\[2mm]
&=\; \frac{[L:K]}{(wL:wK)[Lw:Kw]} \cdot \frac{[Lw:Kw]}{(\overline{w}(Lw):\overline{w}(Kw))[(Lw)\overline{w}:(Kw)\overline{w}]} \\[2mm]
&=\; d(L|K,v) \cdot d(Lw|Kw,\overline{w}) \,.
\end{aligned}
$$

□

In the next lemma, the relation between immediate and defectless extensions is studied.

Lemma 2.11. *Take an arbitrary immediate extension $(F|K,v)$ of valued fields, and $(L|K,v)$ a finite extension such that $[L:K] = (vL:vK)[Lv:Kv]$. Then $F|K$ and $L|K$ are linearly disjoint, the extension of v from F to $L.F$ is unique, $(L.F|F,v)$ is defectless, and $(L.F|L,v)$ is immediate. Moreover,*

$$[L.F : F] = [L : K] \,,$$

i.e., F is linearly disjoint from L over K.

Proof. $v(L.F)$ contains vL and $(L.F)v$ contains Lv. On the other hand, we have $vF = vK$ and $Fv = Kv$ by hypothesis. Therefore,

$$
\begin{aligned}
[L.F : F] \;&\geq\; (v(L.F):vF) \cdot [(L.F)v:Fv] \\
&\geq\; (vL:vK) \cdot [Lv:Kv] = [L:K] \;\geq\; [L.F:F]
\end{aligned}
$$

hence equality holds everywhere. This shows that $[L.F : F] = [L : K]$ and that $L.F|F$ is defectless with unique extension of the valuation. Furthermore, it follows that $v(L.F) = vL$ and $(L.F)v = Lv$, i.e., $L.F|L$ is immediate. □

The reader should note that if the finite extension $L|K$ is not normal and there are more than one extension of v from K to L, then $d(L|K,v) = 1$ does not imply that equality holds in (10). It may happen that for one extension of v the henselian defect is 1 while for another extension it is > 1. In this case, the henselian defect depends on the chosen extension of v from K to L. On the other hand, this will not happen when $L|K$ is normal.

Applying the lemma to purely inseparable extensions $L|K$, we obtain:

Corollary 2.12. *Every immediate extension of an inseparably defectless field is separable.*

Proof. If (K,v) is an inseparably defectless field and $(F|K,v)$ an immediate extension, then every finite purely inseparable extension $(L|K,v)$ satisfies $[L : K] = (vL : vK)[Lv : Kv]$, and L is therefore linearly disjoint from F over K by the previous lemma. It follows that also K^{1/p^∞} is linearly disjoint from F over K. □

In the following we give two basic examples for extensions with defect > 1 (one can find more nasty examples in [K5] and [K7]). The following is due to F. K. Schmidt.

Example 2.13. We consider $\mathbb{F}_p((t))$ with its canonical valuation $v = v_t$. Since $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose some element $s \in \mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. Since $(\mathbb{F}_p((t))|\mathbb{F}_p(t),v)$ is an immediate extension, the same holds for the extension $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t),v)$ and thus also for $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p),v)$. The latter extension is purely inseparable of degree p (since s,t are algebraically independent over $\mathbb{F}_p$, the extension $\mathbb{F}_p(s)|\mathbb{F}_p(s^p)$ is linearly disjoint from $\mathbb{F}_p(t,s^p)|\mathbb{F}_p(s^p)$). Hence, there is only one extension of the valuation v from $\mathbb{F}_p(t,s^p)$ to $\mathbb{F}_p(t,s)$. So we have $e = f = g = 1$ for this extension and consequently, its defect is p.

A defect can appear "out of nothing" when a finite extension is lifted through another finite extension:

Example 2.14. In the foregoing example, we can choose s such that $vs > 1 = vt$. Now we consider the extensions

$$(\mathbb{F}_p(t,s^p)|\mathbb{F}_p(t^p,s^p),v) \quad \text{and} \quad (\mathbb{F}_p(t+s,s^p)|\mathbb{F}_p(t^p,s^p),v)$$

of degree p. Both are defectless: since $v\mathbb{F}_p(t^p,s^p) = p\mathbb{Z}$ and $v(t+s) = vt = 1$, the index of $v\mathbb{F}_p(t^p,s^p)$ in $v\mathbb{F}_p(t,s^p)$ and in $v\mathbb{F}_p(t+s,s^p)$ must be (at least) p. But $\mathbb{F}_p(t,s^p).\mathbb{F}_p(t+s,s^p) = \mathbb{F}_p(t,s)$, which shows that the defectless extension $(\mathbb{F}_p(t,s^p)|\mathbb{F}_p(t^p,s^p),v)$ does not remain defectless if lifted up to $\mathbb{F}_p(t+s,s^p)$ (and vice versa).

2.3. Defect of h-finite extensions. For a finite extension $(L|K,v)$ such that v extends uniquely from K to L, the defect measures how far the fundamental inequality (10) is from being an equality. More generally, this can be done for every algebraic extension $(L|K,v)$ such that $(L^h|K^h,v)$ is finite, i.e., $(L|K,v)$ is an

h-finite extension. This requires that we work with a fixed extension of v to the algebraic closure $\tilde{K}$, which in turn determines the henselizations of K and all its algebraic extensions. We can then define the defect to be that of the extension of the respective henselizations, as we hve done if (1).

In general, the defect can increase or decrease if an h-finite extension is lifted up through another extension. A defectless extension may turn into an extension with nontrivial defect after lifting up through an algebraic extension (as seen in Example 2.14). On the other hand, every h-finite extension with nontrivial defect of a valued field (K, v) becomes trivial and thus defectless if lifted up to the algebraic closure $\tilde{K}$. At least we can show that if the defect decreases, then there is no further descent after a suitable finitely generated extension.

As a preparation, we need the following fact which at first glance may appear to be obvious. But a closer look reveals that proving it is more difficult than expected. In order to get a feeling for the hidden difficulties, the reader should note that if $K(x)|K$ is a simple transcendental extension and we take an element c in the henselization of $K(x)$ which is algebraic over K, it may not lie in the henselization of K (cf. Theorem 1.3 of [K3]).

Lemma 2.15. *Take an arbitrary extension $(L|K, v)$ and elements $c_1, \ldots, c_m \in L^h$. Then there exist elements $d_1, \ldots, d_n \in L$ such that $c_1, \ldots, c_m \in K(d_1, \ldots, d_n)^h$.*

This lemma is proved in [K8]. Now we are ready to prove the following result.

Lemma 2.16. *Let $(L|K, v)$ and $(F|K, v)$ be subextensions of a valued field extension $(\Omega|K, v)$ such that $F|K$ is finitely generated and $L.F|L$ is h-finite. Then there exists a finitely generated subextension $L_0|K$ of $L|K$ such that for every subfield L_1 of L containing L_0, the following holds:*

(1) $[(L.F)^h : L^h] = [(L_1.F)^h : L_1^h]$,

(2) $(v(L.F) : vL) \leq (v(L_1.F) : vL_1)$,

(3) $[(L.F)v : Lv] \leq [(L_1.F)v : L_1v]$,

(4) $d(L.F|L, v) \geq d(L_1.F|L_1, v)$.

Proof. Since $[(L.F)^h : L^h]$ is finite, the fundamental inequality (10) shows that $(v(L.F) : vL)$ and $[(L.F)v : Lv]$ are finite too. Hence there exist $a_1, \ldots, a_r \in L.F$ such that

$$v(L.F) = vL + \mathbb{Z}va_1 + \ldots + \mathbb{Z}va_r ,$$

and there exist $b_1, \ldots, b_s \in L.F$ such that

$$(L.F)v = Lv(b_1v, \ldots, b_sv) .$$

In order to write out $a_1, \ldots, a_r, b_1, \ldots, b_s$ as elements of the compositum $L.F$, we need finitely many elements $a'_1, \ldots, a'_k, b'_1, \ldots, b'_\ell \in L$. Whenever $L_1 \subseteq L$ is an

extension of K which contains these elements, then $a_1, \ldots, a_r, b_1, \ldots, b_s \in L_1.F$ and it follows that

$$
\begin{align}
(v(L_1.F) : vL_1) &\geq (v(L.F) : vL) \,, \tag{12}\\
[(L_1.F)v : L_1 v] &\geq [(L.F)v : Lv] \,, \tag{13}
\end{align}
$$

the left hand sides not necessarily being finite.

Now if L_1 satisfies assertion 1 of our lemma, then the left hand sides of (12) and (13) have to be finite, and we will have that

$$
\begin{align}
d(L_1.F | L_1, v) &= \frac{[(L_1.F)^h : L_1^h]}{(v(L_1.F) : vL_1) \cdot [(L_1.F)v : L_1 v]}\\
&\leq \frac{[(L.F)^h : L^h]}{(v(L.F) : vL) \cdot [(L.F)v : Lv]} = d(L.F | L, v) \,.
\end{align}
$$

Since $F|K$ is finitely generated by assumption, we can write $F = K(z_1, \ldots, z_t)$. Then $z_1, \ldots, z_t$ are algebraic over L^h, and we take $c_1, \ldots, c_m \in L^h$ to be all of the coefficients appearing in their minimal polynomials. By Lemma 2.15 there exist elements $d_1, \ldots, d_n \in L$ such that $c_1, \ldots, c_m \in K(d_1, \ldots, d_n)^h$. Hence as soon as $d_1, \ldots, d_n \in L_1$, $z_1, \ldots, z_t$ are algebraic over L_1^h with $[L_1^h(z_1, \ldots, z_t) : L_1^h] = [L^h(z_1, \ldots, z_t) : L^h]$. It then follows that $F|L_1$ is algebraic, so that $(L_1.F)^h = L_1^h.F$ by Lemma 2.1 (where we take $L = L_1.F$ and $K = L_1$). This yields the equalities

$$
\begin{align}
[(L_1.F)^h : L_1^h] &= [L_1.F^h : L_1^h] = [L_1^h(z_1, \ldots, z_t) : L_1^h] = [L^h(z_1, \ldots, z_t) : L^h]\\
&= [L^h.F : L^h] = [(L.F)^h : L^h] \,,
\end{align}
$$

so that assertion 1 is satisfied. Hence if we set

$$
L_0 := K(a_1', \ldots, a_k', b_1', \ldots, b_\ell', d_1, \ldots, d_n) \,,
$$

then all assertions of our lemma will be satisfied by every subfield $L_1 \subseteq L$ that contains L_0. $\qquad\square$

2.4. Transcendence bases of valued function fields. For the easy proof of the following lemma, see [B], chapter VI, §10.3, Theorem 1.

Lemma 2.17. *Let $(L|K, v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, if we write

$$
f = \sum_k c_k \prod_{i \in I} x_i^{\mu_{k,i}} \prod_{j \in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]
$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$
vf = \min_k v\, c_k \prod_{i \in I} x_i^{\mu_{k,i}} \prod_{j \in J} y_j^{\nu_{k,j}} = \min_k vc_k + \sum_{i \in I} \mu_{k,i} vx_i \,. \tag{14}
$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$vK(x_i, y_j \mid i \in I, j \in J) \;=\; vK \oplus \bigoplus_{i \in I} \mathbb{Z}vx_i$$

$$K(x_i, y_j \mid i \in I, j \in J)v \;=\; Kv\,(y_j v \mid j \in J)\,.$$

Moreover, the valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues $y_j v$.

Conversely, if (K,v) is any valued field and we assign to the elements vx_i any values in an ordered abelian group extension of vK which are rationally independent, then (14) defines a valuation on F, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv.

As a consequence of the above lemma and the fundamental inequality (10), we have:

Corollary 2.18. *Let $(F|K,v)$ be an extension of valued fields of finite transcendence degree. Then the Abhyankar Inequality (2) holds. If in addition $F|K$ is a function field and if equality holds in (2), then the extensions $vF|vK$ and $Fv|Kv$ are finitely generated.*

The following is Lemma 2.8 of [K4]:

Lemma 2.19. *Let $(F|K,v)$ be a valued function field without transcendence defect and $v = w \circ \overline{w}$, then $(F|K,w)$ and $(Fw|Kw,\overline{w})$ are valued function fields without transcendence defect.*

A transcendence basis T of an extension $(L|K,v)$ is called **valuation transcendence basis**, if for every choice of finitely many distinct elements $t_1, \ldots, t_n \in T$, the value of every polynomial f in $K[t_1, \ldots, t_n]$ is equal to the value of a summand of f of minimal value, i.e.,

$$v(\sum_{\underline{\nu}} c_{\underline{\nu}} t_1^{\nu_1} \cdots t_n^{\nu_n}) \;=\; \min_{\underline{\nu}} v(c_{\underline{\nu}} t_1^{\nu_1} \cdots t_n^{\nu_n}). \tag{15}$$

By Lemma 2.17, every standard valuation transcendence basis is a valuation transcendence basis.

Lemma 2.20. *Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following assertions are equivalent:*

(1) *$(L|K,v)$ is an extension without transcendence defect.*

(2) *$(L|K,v)$ admits a standard valuation transcendence basis,*

(3) *$(L|K,v)$ admits a valuation transcendence basis.*

Proof. 1.$\Rightarrow$ 2. was shown in the Introduction.

2.$\Rightarrow$ 3. follows from our remark preceding the lemma.

3.$\Rightarrow$ 1.: Let $T = \{t_1, \ldots, t_n\}$ be a valuation transcendence basis of $(L|K, v)$. Hence $n = \operatorname{trdeg} L|K$. We can assume that the numbering is such that for some $r \geq 0$, the values $vt_1, \ldots, vt_r$ are rationally independent over vK and the values of every $r + 1$ elements in T are rationally dependent over vK. That is, for every j such that $0 < j \leq s := n - r$, there are integers $\nu_j > 0$ and ν_{ij}, $1 \leq i \leq r$, and a constant $c_j \in K$ such that the element

$$t'_j := c_j t_{r+j}^{\nu_j} \prod_{i=1}^{r} t_i^{\nu_{ij}}$$

has value 0. Observe that $r \leq \operatorname{rr} vL/vK$ and that $r + s = \operatorname{trdeg} L|K$. Now assume that $(L|K, v)$ has nontrivial transcendence defect. Then

$$s = \operatorname{trdeg} L|K - r \geq \operatorname{trdeg} L|K - \operatorname{rr} vL|vK > \operatorname{trdeg} Lv|Kv.$$

This yields that the residues $t'_1 v, \ldots, t'_s v$ are not Kv-algebraically independent. Hence, there is a nontrivial polynomial $g(X_1, \ldots, X_s) \in \mathcal{O}_K[X_1, \ldots, X_s]$ such that $g(t'_1, \ldots, t'_s)v = 0$. Hence $vg(t'_1, \ldots, t'_s) > 0$. After multiplying with sufficiently high powers of every element $t_i, 1 \leq i \leq r$, we obtain a polynomial f in $t_1, \ldots, t_n$ which violates (15). But this contradicts our assumption that T be a valuation transcendence basis. Consequently, $(L|K, v)$ can not have a nontrivial transcendence defect. $\square$

2.5. The defect of valued function fields. Let $(F|K, v)$ be a subhenselian function field without transcendence defect. In the following we will show that the defect $d(F|K, v)$, defined in (4), is finite and equal to the henselian defect $d(F|K(T), v)$ for every standard valuation transcendence basis T of $(F|K, v)$.

Lemma 2.21. *Take an extension $(K(T)|K, v)$, where $T = \{x_i, y_j \mid i \in I, j \in J\}$ satisfies (3). Let $v_1, \ldots, v_g$ be the extensions of v from $K(T)$ to $L(T)$. Then $v_1, \ldots, v_g$ are uniquely determined by their restrictions to L, and these restrictions are precisely the extensions of v from K to L. Moreover, for $1 \leq i \leq g$,*

$$d(L(T)|K(T), v_i) \quad = \quad d(L|K, v_i), \tag{16}$$

$$e(L(T)|K(T), v_i) \quad = \quad e(L|K, v_i), \tag{17}$$

$$f(L(T)|K(T), v_i) \quad = \quad f(L|K, v_i), \tag{18}$$

$$v_i L(T) = v_i L + vK(T) \quad and \quad L(T)v_i = Lv_i \,.\, K(T)v \,. \tag{19}$$

Proof. The first two assertions follow from Lemma 2.17, which also shows that $vK(T) = vK \oplus \bigoplus_{i \in I} \mathbb{Z}vx_i$ and $v_i L(T) = v_i L \oplus \bigoplus_{i \in I} \mathbb{Z}vx_i$. Hence, $v_i L(T)/vK(T)$ is isomorphic to $v_i L/vK$, which proves equation (17). Again by Lemma 2.17, $K(T)v = Kv(y_j v \mid j \in J)$ and $L(T)v_i = Lv_i(y_j v \mid j \in J)$. Since the elements $y_j v$ are algebraically independent over Kv and $Lv_i|Kv$ is algebraic, $Kv(y_j v \mid j \in J)|Kv$

is linearly disjoint from $Lv_i|Kv$, which yields (18). Also, (19) follows immediately from the above described form of the value groups and residue fields.

Since the elements of T are algebraically independent over K, the extension $K(T)|K$ is linearly disjoint from $\tilde{K}|K$ and thus, $[L(T) : K(T)] = [L : K]$. In view of Lemma 2.1, we have that

$$
\begin{aligned}
[L(T)^{h(v_i)} : K(T)^{h(v_i)}] &= [L^{h(v_i)}.K(T)^{h(v_i)} : K^{h(v_i)}.K(T)^{h(v_i)}] \\
&\leq [L^{h(v_i)} : K^{h(v_i)}] \, .
\end{aligned}
\tag{20}
$$

But from (11) we obtain that

$$
\sum_{1 \leq i \leq g} [L(T)^{h(v_i)} : K(T)^{h(v_i)}] = [L(T) : K(T)] = [L : K] = \sum_{1 \leq i \leq g} [L^{h(v_i)} : K^{h(v_i)}]
$$

which shows that equality must hold in (20). Now (16) follows from the definition of the henselian defect.   □

To facilitate notation, we will from now on assume that all valued field extensions of (K, v) are contained in a large algebraically closed valued field extension of (K, v) and their henselizations are taken within this extension. This enables us to suppress the mentioning of the valuation.

Lemma 2.22. *Let $F|K$ be a subhenselian function field without transcendence defect. Then for every standard valuation transcendence basis T of $F|K$ there exists a finite extension K_T of K such that for every algebraic extension L of K containing K_T, the following holds:*

(1) *the extension $L.F|L(T)$ is defectless*

(2) *if $L|K$ is h-finite, then $d(L.F|K(T)) = d(L(T)|K(T)) = d(L|K)$.*

Proof. Assume that $L|K$ is an h-finite extension such that $L.F|L(T)$ is defectless. Then

$$
d(L.F|K(T)) = d(L.F|L(T)) \cdot d(L(T)|K(T)) = d(L(T)|K(T)) = d(L|K) \, ,
$$

where the last equation holds by Lemma 2.21. Hence we may restrict our attention to the fulfillment of assertion 1.

The extension $\tilde{K}.F|\tilde{K}(T)$ is defectless by Theorem 1.2 and Lemma 2.5. By Lemma 2.16 there exists a finitely generated subextension $L_0|K(T)$ of $\tilde{K}(T)|K(T)$ such that

$$
d(L_1.F|L_1) \leq d(\tilde{K}.F|\tilde{K}(T)) = 1
$$

whenever $L_0 \subset L_1 \subset \tilde{K}(T)$. Let K_T be a finitely generated algebraic (and hence finite) extension of K such that $L_0 \subset K_T(T)$. Then $d(L.F|L(T)) \leq d(\tilde{K}.F|\tilde{K}(T)) = 1$ for every algebraic extension L of K which contains K_T.   □

Proof of Theorem 1.1:
Take any transcendence basis T_0 of $F|K$. Then also $K(T_0)|K$ is without transcendence defect because it has the same transcendence degree as $F|K$, $vF/vK(T_0)$ is a torsion group, and $Fv|K(T_0)v$ is algebraic. Hence $K(T_0)$ admits a standard valuation transcendence basis T over K. We compute:

$$d(F|K(T_0)) \leq d(F|K(T_0)) \cdot d(K(T_0)|K(T)) = d(F|K(T)) .$$

This shows that

$$d(F|K) = \sup_T \, d(F|K(T)) ,$$

where T runs over standard valuation transcendence bases only. Since F is a subhenselian function field, every $d(F|K(T))$ is a finite number. It remains to show that for any two standard valuation transcendence bases T_1 and T_2,

$$d(F|K(T_1)) = d(F|K(T_2)) .$$

We choose finite extensions K_{T_1} and K_{T_2} according to Lemma 2.22. Putting $L_0 = K_{T_1}.K_{T_2}$ we get by Lemma 2.22:

$$d(L_0.F|K(T_1)) = d(L_0|K) = d(L_0.F|K(T_2))$$

and from this we deduce

$$d(F|K(T_1)) = \frac{d(L_0.F|K(T_1))}{d(L_0.F|F)} = \frac{d(L_0.F|K(T_2))}{d(L_0.F|F)} = d(F|K(T_2)) .$$

This proves (5) and that the defect is independent of the chosen standard valuation transcendence basis.

Furthermore, using Lemmas 2.21 and 2.22, for any finite extension L of K containing K_{T_1} we observe the following:

$$
\begin{aligned}
d(L(T_2)|K(T_2)) &= d(L|K) = d(L.F|K(T_1)) = d(L.F|F) \cdot d(F|K(T_1)) \\
&= d(L.F|F) \cdot d(F|K(T_2)) = d(L.F|K(T_2))
\end{aligned}
$$

showing that

$$d(L.F|L(T_2)) = d(L.F|K(T_2))/d(L(T_2)|K(T_2)) = 1 .$$

Hence every algebraic extension L of K_{T_1} satisfies assertion 1 and also the first part of assertion 2, because

$$d(F|K) = d(F|K(T_1)) = \frac{d(L.F|K(T_1))}{d(L.F|F)} = \frac{d(L|K)}{d(L.F|F)}$$

where the last equation holds by Lemma 2.22. The second part of assertion 2 follows from

$$
\begin{aligned}
d(N|K) &= d(N(T_1)|K(T_1)) \\
&\leq d(N.F|K(T_1)) = d(N.F|F) \cdot d(F|K(T_1)) = d(N.F|F) \cdot d(F|K)
\end{aligned}
$$

and the fact that equality holds for the finite extension K_{T_1} of K. $\qquad\square$

3. Completion Defect and Defect Quotient

This section contains our results on completion defect and defect quotients for finite (and more generally, h-finite) extensions as well as for valued algebraic function fields (and more generally, subhenselian function fields).

3.1. The case of h-finite extensions. The following observations are immediate from the definitions. We have that

$$d(L|K) = d_c(L|K) \cdot d_q(L|K) \ . \tag{21}$$

Thus for h-finite extensions of q-defectless fields, the completion defect equals the ordinary defect. Every h-finite extension $L|K$ satisfies:

$$d_c(L|K) = d_c(L^h|K^h) \quad \text{and} \quad d_q(L|K) = d_q(L^h|K^h) \ .$$

Hence, K is a c-defectless or q-defectless field if and only if its henselization K^h is a c-defectless or q-defectless field, respectively. If also $M|L$ is h-finite, then we have that $[M^{hc} : K^{hc}] = [M^{hc} : L^{hc}] \cdot [L^{hc} : K^{hc}]$, and from the multiplicativity of ramification index and inertia degree we obtain the following analogue of Lemma 2.7:

$$d_c(M|K) = d_c(M|L) \cdot d_c(L|K) \quad \text{and} \quad d_q(M|K) = d_q(M|L) \cdot d_q(L|K) \ . \tag{22}$$

From this multiplicativity, one derives:

Lemma 3.1. *Let $(L|K, v)$ be an h-finite extension. Then (K, v) is a q-defectless field if and only if $(L|K, v)$ is q-defectless and (L, v) is a q-defectless field. The same holds for "c-defectless" instead of "q-defectless".*

In passing, we make the following observation, which we will not need further in this paper.

Lemma 3.2. *If $L|K$ is a c-defectless and immediate h-finite extension, then L^h is a purely inseparable extension of K^h included in its completion K^{hc}.*

Proof. If $L|K$ is an immediate h-finite extension, then $(vL : vK)[Lv : Kv] = 1$ and therefore, $d_c(L|K) = [L^{hc} : K^{hc}]$. If in addition $L|K$ is c-defectless, then $K^{hc} = L^{hc} = L^h.K^{hc}$, which implies that $L^h \subseteq K^{hc}$. Since K^h is separable-algebraically closed in its completion K^{hc} by Lemma 2.4, the extension $L^h|K^h$ must be purely inseparable. $\square$

The completion defect $d_c(L|K)$ and the defect quotient $d_q(L|K)$ are integers dividing $d(L|K)$ and hence are powers of p. To see this, we use that $[L^{hc} : K^{hc}] = [L.K^{hc} : K^{hc}] \leq [L.K^h : K^h] = [L^h : K^h]$. This gives:

$$d_c(L|K) = \frac{[L^{hc} : K^{hc}]}{(vL : vK) \cdot [Lv : Kv]} \leq \frac{[L^h : K^h]}{(vL : vK) \cdot [Lv : Kv]} = d(L|K) \ . \tag{23}$$

Since on the other hand, $d_c(L|K)$ is the defect of the extension $L^{hc}|K^{hc}$, it is a power of p and consequently a divisor of $d(L|K)$. This yields that also $d_q(L|K) = d(L|K)d_c(L|K)^{-1}$ is an integer dividing $d(L|K)$ and a power of p.

In (23), equality holds if and only if

$$[L^{hc} : K^{hc}] = [L^h : K^h] \, , \tag{24}$$

which in view of $L^{hc} = L^h.K^{hc}$ means that L^h is linearly disjoint from K^{hc} over K^h. Since the henselian field K^h is relatively separable-algebraically closed in its completion, equation (24) holds whenever $L^h|K^h$ is separable-algebraic; hence it holds for every h-finite separable extension $L|K$. This proves:

Lemma 3.3. *Every h-finite separable extension is q-defectless. In general, an h-finite extension $L|K$ is q-defectless if and only if Equation (24) holds.*

We deduce:

Proof of Theorem 1.5:
Let K be a c-defectless field. By Lemma 3.3, we know that every h-finite separable extension of K is q-defectless, i.e., its completion defect equals the ordinary defect. Thus every finite separable extension of K is defectless and consequently, K is a separably defectless field.

For the converse, assume that K is separably defectless. Then by Lemma 2.5, also its henselization is separably defectless. Now Lemma 2.9 shows that K^{hc} is defectless. By virtue of the definition of the completion defect, this proves K to be c-defectless. $\qquad\square$

We will need the following theorem from [K6]:

Theorem 3.4. *Take $z \in \tilde{K} \setminus K$ such that*

$$v(a - z) \; > \; \{v(a - c) \mid c \in K\}$$

for some $a \in K^h$. Then K^h and $K(z)$ are not linearly disjoint over K, that is,

$$[K^h(z) : K^h] < [K(z) : K]$$

and in particular, $K(z)|K$ is not purely inseparable.

With this theorem, we are able to prove:

Lemma 3.5. *For every finite purely inseparable extension $L|K$, L^c is linearly disjoint from K^{hc} over K^c, and*

$$[L^{hc} : K^{hc}] \; = \; [L^{ch} : K^{ch}] \; = \; [L^c : K^c] \, .$$

The first equation holds more generally whenever $L|K$ is h-finite.

Proof. Take a finite purely inseparable extension $L|K$ and assume that L^c is not linearly disjoint from K^{hc} over K^c. Then there exists an intermediate field N between L and K and an element $z \in L \setminus N$, $z^p \in N$, such that

$$z \notin N.K^c \text{ but } z \in N.K^{hc} .$$

Since $z \notin N.K^c = N^c$, the set $\{z - c \mid c \in N\}$ is bounded from above. Since $z \in N.K^{hc} = N^{hc}$, there exists an element $a \in N^h$ such that $v(a - z) = v(z - a) > v(z - c)$ for all $c \in N$. This implies $v(a - c) = \min\{v(a - z), v(z - c)\} = v(z - c)$, so that $v(a - z) > \{v(a - c) \mid c \in N\}$. Now Theorem 3.4, with N in place of K, proves that N^h and $N(z)$ are not linearly disjoint over N, which is a contradiction since $N(z)|N$ is purely inseparable, while the henselization of a valued field is a separable extension.

We have proved that L^c is linearly disjoint from K^{hc} over K^c. Since $L^c \subseteq L^{hc}$ and $L.K^{hc} = L^{hc}$, we also have that $L^c.K^{hc} = L^{hc}$. This together with the fact we have just proved implies that $[L^{hc} : K^{hc}] = [L^c : K^c]$.

Now we observe that $K^c \subseteq K^{ch} \subseteq K^{ch}$ and $L^c.K^{ch} = L^{ch}$, which yields that

$$[L^{hc} : K^{hc}] \leq [L^{ch} : K^{ch}] \leq [L^c : K^c] = [L^{hc} : K^{hc}] ,$$

so equality holds everywhere.

Now take an arbitrary h-finite extension $L|K$, and take $L_s|K$ to be its maximal separable subextension. By what we have seen earlier, the finite extension L_s^h is linearly disjoint from K^{hc} over K^h. Since $L|K$ is h-finite, the subextension $L_s^h|K^h$ is finite, and since $L_s^h.K^{hc} = L_s^{hc}$, we obtain that $[L_s^{hc} : K^{hc}] = [L_s^h : K^h]$. Since $K^h \subseteq K^{ch} \subseteq K^{ch}$ and $L^h.K^{ch} = L^{ch}$, we find that

$$[L_s^{hc} : K^{hc}] \leq [L_s^{ch} : K^{ch}] \leq [L_s^h : K^h] = [L_s^{hc} : K^{hc}] ,$$

so equality holds everywhere.

We observe that $[L_s : L]$ must be finite since $L^h|K^h$ is finite by assumption and the purely inseparable extension L is linearly disjoint from the separable extension L_s^h over L_s. Thus, applying what we have shown in the first part of the proof, with L_s in place of K, we obtain that $[L^{hc} : L_s^{hc}] = [L^{ch} : L_s^{ch}]$. Therefore, $[L^{hc} : K^{hc}] = [L^{hc} : L_s^{hc}] \cdot [L_s^{hc} : K^{hc}] = [L^{ch} : L_s^{ch}] \cdot [L_s^{ch} : K^{ch}] = [L^{ch} : K^{ch}]$. $\quad\square$

Proof of Proposition 1.3:
Take an h-finite extension $L|K$. We have:

$$
\begin{aligned}
d(L^c|K^c) &= \frac{[L^{ch} : K^{ch}]}{(vL^c : vK^c) \cdot [L^c v : K^c v]} = \frac{[L^{ch} : K^{ch}]}{(vL : vK) \cdot [Lv : Kv]} \\
&= \frac{[L^{hc} : K^{hc}]}{(vL : vK) \cdot [Lv : Kv]} = d_c(L|K) ,
\end{aligned}
$$

where the second equality holds since the completion is an immediate extension, and the third equality is taken from the previous lemma.

The second assertion of the proposition has been proven in Lemma 3.3. $\quad\square$

Proof of Proposition 1.4:
Take a finite extension $L|K$, and $L_s|K$ its the maximal separable subextension of $L|K$. Then $L|L_s$ is purely inseparable, and as we have see in the proof of Lemma 3.5, it must be finite. By Lemma 3.3, $d_q(L_s|K) = 1$ and

$$d_q(L|K) \,=\, d_q(L|L_s) \cdot d_q(L_s|K) \,=\, d_q(L|L_s) \,=\, \frac{[L^h : L_s^h]}{[L^{hc} : L_s^{hc}]} \;.$$

Since L is linearly disjoint from L_s^h over L_s , we find that

$$[L^h : L_s^h] \,=\, [L.L_s^h : L_s^h] \,=\, [L : L_s] \,=\, [L : K]_{\text{insep}} \;.$$

Further, $L^c = L.L_s^c$ is a purely inseparable extension of L_s^c and therefore linearly disjoint from L_s^{ch} over L_s^c. Using in addition the first equation from Lemma 3.5 and the fact that $L_s^c = L_s.K^c$ is a separable extension, we obtain that

$$[L^{hc} : L_s^{hc}] \,=\, [L^{ch} : L_s^{ch}] \,=\, [L^c.L_s^{ch} : L_s^{ch}] \,=\, [L^c : L_s^c] \,=\, [L^c : K^c]_{\text{insep}} \;.$$

This proves the proposition. $\quad\square$

Proof of Theorem 1.6:
K is q-defectless if and only if every finite extension $L|K$ is q-defectless. In view of the multiplicativity (22) of the defect quotient and the fact that every finite extension is contained in a finite normal extension, it follows that K is q-defectless if and only if every finite normal extension $L|K$ is q-defectless. Again by multiplicativity, and by the fact that a normal extension $L|K$ admits an intermediate field N such that $N|K$ is purely inseparable and $L|N$ is separable and thus q-defectless (Lemma 3.3), it follows that K is q-defectless if and only if every finite purely inseparable extension $L|K$ is q-defectless. By Proposition 1.4, this is the case if and only if $[L : K] = [L^c : K^c] = [L.K^c : K^c]$, i.e., L is linearly disjoint from K^c over K^c, for every finite purely inseparable extension $L|K$. This holds if and only if $K^c|K$ is separable. $\quad\square$

We will now consider the behaviour of the defects under coarsenings of the valuation.

Lemma 3.6. *Take a finite extension $(L|K, v)$ and a decomposition $v = w \circ \overline{w}$ of v with nontrivial w. Then*

$$d_q(L|K, v) \,=\, d_q(L|K, w) \tag{25}$$
$$d_c(L|K, v) \,=\, d_c(L|K, w) \cdot d(Lw|Kw, \overline{w}) \;. \tag{26}$$

Proof. To prove the equation for the defect quotient, we use Proposition 1.4 together with the fact that for every nontrivial coarsening w of the valuation v, the completion $K^{c(w)}$ of K with respect to w coincides with the completion K^c with respect to v. We obtain:

$$d_q(L|K, v) \,=\, \frac{[L : K]}{[L^{c(v)} : K^{c(v)}]} \,=\, \frac{[L : K]}{[L^{c(w)} : K^{c(w)}]} \,=\, d_q(L|K, w) \;.$$

This proves equation (25). Using this result, we compute:

$$d_c(L|K,v) = \frac{d(L|K,v)}{d_q(L|K,v)} = \frac{d(L|K,w) \cdot d(Lw|Kw,\overline{w})}{d_q(L|K,w)} = d_c(L|K,w) \cdot d(Lw|Kw,\overline{w}),$$

proving equation (26). □

Lemma 3.7. *Let $(L|K,v)$ be a finite extension of henselian fields and assume that v admits no coarsest nontrivial coarsening. Then there is a nontrivial coarsening w such that*

$$d_c(L|K,w) = 1.$$

If in addition the extension $L|K$ is separable, this means that

$$d(L|K,w) = 1 \quad and \quad d(L|K,v) = d(Lw|Kw,\overline{w}),$$

where $\overline{w}$ is the valuation induced by v on Lw.

Proof. First, we note that if $L|K$ is a finite separable extension with $d_c(L|K,w) = 1$, then $d(L|K,w) = 1$ by Proposition 1.3; in view of Lemma 2.10, this implies that $d(L|K,v) = d(Lw|Kw,\overline{w})$.

Next, we prove the first assertion of our lemma in two special cases:

<u>Case 1:</u> $L = K(a)$ is separable. Let $f(X) \in K[X]$ be the minimal polynomial of a over K and let c_i, $0 \le i \le n$ be the coefficients of f. Then by our hypothesis on the rank of v there exists a nontrivial coarsening w of v such that w is trivial on $k(c_0,\ldots,c_n)$, where k denotes the prime field of K. This shows that fw is a separable polynomial over Kw of the same degree as f; moreover it is irreducible since if it were reducible, then the same would follow for f by Hensel's Lemma (as (K,w) is henselian by our hypothesis on (K,v) and Lemma 2.2). Hence in this case, $[L : K] = [Lw : Kw]$ and consequently $d(L|K,w) = 1$ and $d_c(L|K,w) = 1$.

<u>Case 2:</u> $L|K$ is a purely inseparable extension of degree p. If $d_q(L|K,v) = p$, we are done because then $d_c(L|K,v) = 1$ and we take $w = v$. So we assume that $d_q(L|K,v) = 1$. Then by Proposition 1.4, $[L : K]_{\mathrm{insep}} = [L^c : K^c]_{\mathrm{insep}}$, showing that a cannot be an element of K^c. Consequently, there is an element $\alpha \in vK$ such that $v(a - b) < \alpha$ for all $b \in K$. By our hypothesis on the coarsenings of v there exists a nontrivial coarsening w of v such that $w(a - b) = 0$, hence $aw \ne bw$ for all $b \in K$ (this is satisfied if the coarsening corresponds to any proper convex subgroup of vK which includes α). This shows $aw \notin Kw$ and thus $[Lw : Kw] = p$, which yields that $d(L|K,w) = 1$ and $d_c(L|K,w) = 1$.

It remains to treat the case where the extension $L|K$ is not simple. We then write $L = K(a_1,\ldots,a_n)$ with $n > 1$ and such that for every $i < n$, the extension $K(a_1,\ldots,a_{i+1})|K(a_1,\ldots,a_i)$ is of degree p if it is inseparable. Now we proceed by induction on n. Suppose that we have already found a nontrivial coarsening w' of w such that $d_c(K(a_1,\ldots,a_{n-1})|K,w') = 1$. Applying what we have proved above, with $K(a_1,\ldots,a_{n-1})$ in place of K and w' in place of v, we find a nontrivial coarsening w of w' such that $d_c(L|K(a_1,\ldots,a_{n-1}),w) = 1$. From Lemma 3.6 we

know that also $d_c(K(a_1, \ldots, a_{n-1})|K, w) = 1$ since w is a coarsening of w'. By the multiplicativity of the completion defect, we obtain that $d_c(L|K, w) = 1$. This completes the proof of our Lemma. $\qquad\square$

3.2. The case of subhenselian function fields. Let us introduce some useful notions. Take an element z in some valued field extension of (K, v). Then z is called **value transcendental over** (K, v) if vz is not a torsion element modulo vK, and it is called **residue transcendental over** (K, v) if $vz = 0$ and zv is transcendental. If either is the case, we call z **valuation transcendental over** (K, v). For example, the elements x_i from (3) are value transcendental, and the elements y_j are residue transcendental over (K, v).

Lemma 3.8. *Let $K(T)|K$ be an extension of valued fields with standard valuation transcendence basis T. Then for every element $b \in K(T) \setminus K$, there exist elements $c', c \in K$ such that $c'(b - c)$ is valuation transcendental.*

Proof. Take $b = f/g \in K(T)$ with $f, g \in K[T]$. By Lemma 2.17, the value of the polynomials f, g is equal to the minimum of the values of the monomials in f resp. g, and these monomials are uniquely determined; we will call them f_0 and g_0. If f_0 differs from g_0 just by a constant factor $c \in K$, then we set $h = f - cg$ and observe that the monomial h_0 of least value in h will not anymore lie in Kg_0. If already $f_0 \notin Kg_0$, then we put $c = 0$, $h = f$ and $h_0 = f_0$. Note that $h \neq 0$ and thus $h_0 \neq 0$ since by hypothesis, $f/g \notin K$. We have that

$$b - c \;=\; \frac{f}{g} - c \;=\; \frac{h}{g} \ \text{ with } \ v\frac{h}{g} \;=\; v\frac{h_0}{g_0} \;,$$

and we know that in the quotient h_0/g_0, at least one element of T appears with a nonzero (integer) exponent. If at least one of these appearing elements from T is value transcendental, then h_0/g_0 and thus also $b - c$ is value transcendental over K. In this case, we set $c' = 1$.

In the remaining case, we write

$$\frac{h_0}{g_0} = d \cdot y_1^{e_1} \cdot \ldots \cdot y_s^{e_s} \;, \ \ e_1, \ldots, e_s \in \mathbb{Z} \;,$$
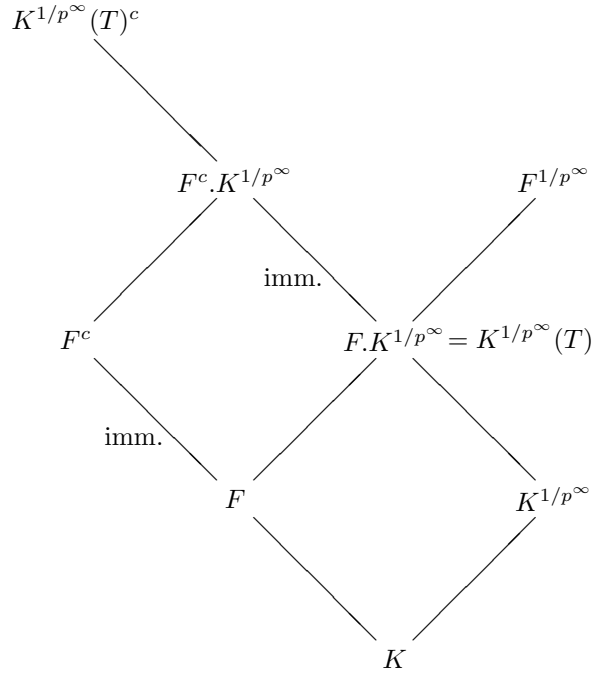
where $d \in K$, and $y_1, \ldots, y_s$ are different residue transcendental elements from T. Since the residues $y_1 v, \ldots, y_s v$ are algebraically independent over Kv, this shows that h_0/dg_0 and thus also h/dg are residue transcendental over K. Putting $c' = d^{-1}$, we obtain that $c'(b - c)$ is residue transcendental over K. $\qquad\square$

The following proposition is a part of the assertion of Theorem 1.8.

Proposition 3.9. *Take a subhenselian function field $F|K$ without transcendence defect. If K is a q-defectless field or vK is not cofinal in vF, then F is a q-defectless field.*

Proof. Take a standard valuation transcendence basis T of $F|K$. In view of Lemma 3.1 we only have to show that $K(T)$ is a q-defectless field (hence we may assume $F = K(T)$). This will follow from Theorem 1.6 if we can show that the completion of F is a separable extension.

As the first case, let us assume that K is a q-defectless field and that vK is cofinal in vF. Then the completion F^c of F contains the completion K^c of K. By our hypothesis on K and Theorem 1.6, K^{1/p^∞} is linearly disjoint from K^c over K. We want to show now that K^{1/p^∞} is even linearly disjoint from F^c over K. This will follow if we prove that $K^{1/p^\infty}.K^c$ is linearly disjoint from F^c over K^c.

$$K^{1/p^\infty}(T)^c$$

$$F^c.K^{1/p^\infty} \qquad\qquad F^{1/p^\infty}$$

$$\text{imm.}$$

$$F^c \qquad\qquad F.K^{1/p^\infty} = K^{1/p^\infty}(T)$$

$$\text{imm.}$$

$$F \qquad\qquad K^{1/p^\infty}$$

$$K$$

Assume the contrary. Then there is a finite purely inseparable extension N of K and an element $a \in K^{1/p^\infty}$ such that $a \in N.F^c \setminus N.K^c$. Since $a \notin N.K^c = N^c$, the set $v(a - N)$ must be bounded from above. Now $N.F^c = (N.F)^c$, hence there exists an element $b \in N.F = N(T)$ such that $v(a-b) > v(a-N)$. But according to the preceding Lemma, there exist elements $c, c' \in N$ such that $c'(b-c)$ is valuation transcendental. As a is algebraic over N, this yields that

$$v(c'(a - c) - c'(b - c)) \,=\, \min\{vc'(a - c), vc'(b - c)\} \,\leq\, vc'(a - c)$$

and consequently,

$$\begin{aligned} v(a - b) \;&=\; v(c'a - c'b) - vc' \;=\; v(c'(a - c) - c'(b - c)) - vc' \\ &\leq\; vc'(a - c) - vc' \;=\; v(a - c) \,, \end{aligned}$$

a contradiction as $v(a - b) > v(a - N)$. We have shown that K^{1/p^∞} is linearly disjoint from F^c over K. Consequently, $F.K^{1/p^\infty}$ is linearly disjoint from F^c over F.

By Theorem 1.2, $F.K^{1/p^\infty} = K^{1/p^\infty}(T)$ is an inseparably defectless field. On the other hand, the extension $F^c.K^{1/p^\infty}|F.K^{1/p^\infty}$ is immediate since $F^c.K^{1/p^\infty}$ is included in $K^{1/p^\infty}(T)^c$. Now Corollary 2.12 shows that $F^{1/p^\infty} = K(T)^{1/p^\infty}$ is linearly disjoint from $F^c.K^{1/p^\infty}$ over $F.K^{1/p^\infty}$. Putting this result together with what we have already proved, we see that F^{1/p^∞} is linearly disjoint from F^c over F. Hence by Theorem 1.6, F is q-defectless. This completes our proof in the first case.

In the remaining second case, vK is not cofinal in vF, i.e., the convex hull of vK in vF is a proper convex subgroup of vF. Consequently, there exists a nontrivial coarsening w of the valuation v on F which is trivial on K. Trivially, (K, w) is a defectless field, and so is (F, w) according to Theorem 1.2 since by Lemma 2.19 it is a function field without transcendence defect over (K, w). Thus any finite purely inseparable extension is defectless and thereby linearly disjoint from the w-completion $F^{c(w)}$ of F since this is an immediate extension of F. On the other hand, the topology induced by v equals the topology induced by any nontrivial coarsening of v, whence $F^{c(w)} = F^c$. Consequently, F^{1/p^∞} is linearly disjoint from F^c. By virtue of Theorem 1.6, this completes our proof. $\qquad\square$

On the basis of Proposition 3.9, we are able to prove the following lemma:

Lemma 3.10. *Let $K(T)|K$ be an extension of valued fields with standard valuation transcendence basis T. Let L be a finite extension of K. If vK is cofinal in $v(K(T))$, then*

$$\begin{aligned} d_c(L(T)|K(T)) &= d_c(L|K) \\ d_q(L(T)|K(T)) &= d_q(L|K) . \end{aligned}$$

If vK is not cofinal in $v(K(T))$, then

$$d_c(L(T)|K(T)) = d(L(T)|K(T)) = d(L|K)$$
$$d_q(L(T)|K(T)) = 1 .$$

Proof. If vK is not cofinal in $v(K(T))$, the assertion follows from Proposition 3.9 together with equations (16) and (21). Let us assume now that vK is cofinal in $v(K(T))$. Again by equations (16) and (21), it suffices to prove the first equality. Using Lemma 2.21 and that $L^{hc}(T) = (L.K^{hc})(T) = L.(K^{hc}(T))$, we obtain that

$$d_c(L|K) = d(L^{hc}|K^{hc}) = d(L^{hc}(T)|K^{hc}(T)) . \tag{27}$$

The complete field K^{hc} is q-defectless by Theorem 1.6. Hence by Proposition 3.9, $K^{hc}(T)$ is a q-defectless field too. Consequently,

$$\begin{aligned} d(L^{hc}(T)|K^{hc}(T)) &= d_c(L^{hc}(T)|K^{hc}(T)) = d((L^{hc}(T))^{hc}|(K^{hc}(T))^{hc}) \\ &= d(L(T)^{hc}|K(T)^{hc}) , \end{aligned}$$

$$L(T)^{hc} = L.K(T)^{hc} = (L^{hc}(T))^{hc}$$

$$(K^{hc}(T))^{hc} = K(T)^{hc}$$

$$L^{hc}(T) = L.K^{hc}(T)$$

$$K^{hc}(T)$$

$$L^{hc} = L.K^{hc}$$

$$K^{hc}$$

$$L$$

$$K$$

where the last equation holds since $(L^{hc}(T))^{hc} = L(T)^{hc}$ and $(K^{hc}(T))^{hc} = K(T)^{hc}$. Putting this together with equation (27), we obtain that

$$d_c(L|K) = d(L(T)^{hc}|K(T)^{hc}) = d_c(L(T)|K(T)) .$$

□

Proof of Theorem 1.7:

Take any transcendence basis T_0 of $F|K$. As in the proof of Theorem 1.7 it follows that $K(T_0)|K$ is without transcendence defect and admits a standard valuation transcendence basis T over K. We compute:

$$d_c(F|K(T_0)) \leq d_c(F|K(T_0)) \cdot d_c(K(T_0)|K(T)) = d_c(F|K(T)) ,$$
$$d_q(F|K(T_0)) \leq \leq d_q(F|K(T_0)) \cdot d_q(K(T_0)|K(T)) = d_q(F|K(T)) ,$$

showing that

$$d_c(F|K) = \sup_T d_c(F|K(T)) \quad \text{and} \quad d_q(F|K) = \sup_T d_q(F|K(T)) ,$$

where the supremum is only taken over all standard valuation transcendence bases of $F|K$. For the proof of equations (6) it suffices now to show that $d_c(F|K(T))$ is equal for all standard valuation transcendence basis T, and that the same holds for the defect quotient.

If vK is not cofinal in vF, then by virtue of Proposition 3.9, $K(T)$ is q-defectless. Using Theorem 1.1, we obtain:

$$d_q(F|K(T)) = 1 \quad \text{and} \quad d_c(F|K(T)) = d(F|K(T)) = d(F|K) ,$$

independently of the standard valuation transcendence basis T.

If vK is cofinal in vF, then $K(T)^{hc}$ contains K^{hc}. By Proposition 3.9, $K^{hc}(T)$ is a q-defectless field. From this we deduce, using Theorem 1.1 again:

$$\begin{aligned} d_c(F|K(T)) &= d(F^{hc}|K(T)^{hc}) = d((F.K^{hc})^{hc}|(K^{hc}(T))^{hc}) \\ &= d_c(F.K^{hc}|K^{hc}(T)) = d(F.K^{hc}|K^{hc}(T)) = d(F.K^{hc}|K^{hc}) . \end{aligned}$$

For the defect quotient, this implies that

$$d_q(F|K(T)) \; = \; \frac{d(F|K(T))}{d_c(F|K(T))} \; = \; \frac{d(F|K)}{d(F.K^{hc}|K^{hc})} \; .$$

This completes the proof of equations (6).

To prove equation (7), we take an arbitrary standard valuation transcendence basis T and compute, using Theorem 1.1 together with what we have just proved,

$$d(F|K) \; = \; d(F|K(T)) \; = \; d_c(F|K(T)) \cdot d_q(F|K(T)) \; = \; d_c(F|K) \cdot d_q(F|K) \; .$$

For the remainder of the proof, we will assume that vK is cofinal in vF. Take K' as in the assertion of Theorem 1.1, and a finite extension L of K containing K'. Choosing any standard valuation transcendence basis T of $F|K$, we know by Theorem 1.1 that $d(L.F|L(T)) = 1$, whence

$$d_c(L.F|L(T)) = 1 \; .$$

Using this and equation (6) as well as the multiplicativity of the completion defect (see (22), we deduce:

$$
\begin{aligned}
d_c(L.F|F) \cdot d_c(F|K) & = & d_c(L.F|F) \cdot d_c(F|K(T)) \; = \; d_c(L.F|K(T)) \\
& = & d_c(L.F|L(T)) \cdot d_c(L(T)|K(T)) \; = \; d_c(L(T)|K(T)) \\
& = & d_c(L(T)|K(T)) \; ,
\end{aligned}
$$

where the last question holds by Lemma 3.10. This proves the first equation of (8). The proof of the first equation of (9) is obtained by just replacing the completion defect by the defect quotient in the above argument.

The second equations in (8) and (9) are shown as it was done for the defect in the proof of Theorem 1.1. □

As immediate consequences we get the following corollaries.

Corollary 3.11. *Assume F to be a subhenselian function field without transcendence defect over a q-defectless field K. Then $d_q(F|K)$ is trivial.*

Proof. □

Corollary 3.12. *Every subhenselian function field $F|K$ without transcendence defect satisfies*

$$
\begin{aligned}
d_c(F|K) & = & d_c(F^h|K) = d_c(F^h|K^h) \; , \\
d_q(F|K) & = & d_q(F^h|K) = d_q(F^h|K^h) \; .
\end{aligned}
$$

Proof. Any standard valuation transcendence basis T of $F|K$ is also a standard valuation transcendence basis of $F^h|K$ and of $F^h|K^h$. Hence, using Theorem 1.7,

$$
\begin{aligned}
d_c(F|K) & = & d_c(F|K(T)) = d_c(F^h|K(T)^h) = d_c(F^h|(K^h(T))^h) \\
& = & d_c((F^h)^h|(K^h(T))^h) = d_c(F^h|K^h(T)) = d_c(F^h|K^h)
\end{aligned}
$$

and

$$d_c(F^h|K(T)^h) \; = \; d_c((F^h)^h|K(T)^h) \; = \; d_c(F^h|K(T)) \; = \; d_c(F^h|K) \, .$$

The assertions for the defect quotient are shown by combining the above equations with the corresponding equations for the defect, and using Theorem 1.7. $\qquad\square$

Corollary 3.13. *Let E and F be subhenselian function fields over K. If $E|F$ is algebraic and $F|K$ has no transcendence defect, then $E|F$ is h-finite and the following multiplicativity holds for the completion defect and defect quotient:*

$$d_c(E|K) \; = \; d_c(E|F) \cdot d_c(F|K) \quad and \quad d_q(E|K) \; = \; d_q(E|F) \cdot d_q(F|K) \, .$$

Proof. First, we prove that $E|F$ is h-finite. As E and F are subhenselian function fields, E^h and F^h are the henselizations of valued function fields E_0 and F_0 over K. Since $E|F$ is algebraic, also $E^h|F^h$ and $E_0.F_0|F_0$ are algebraic. As $E_0.F_0$ is also a function field over K, $E_0.F_0|F_0$ is finite and the same holds for $(E_0.F_0)^h|F_0^h$. But $F_0^h = F^h$, and since $E^h = E_0^h$ contains F and thus also F_0, we see that $(E_0.F_0)^h = E^h$. We have proved that $E^h|F^h$ is finite.

Taking any standard valuation transcendence basis T of $F|K$ (which is also a standard valuation transcendence basis of $E|K$ since $E|F$ is algebraic), we compute

$$d_c(E|K) \; = \; d_c(E|K(T)) \; = \; d_c(E|F) \cdot d_c(F|K(T)) \; = \; d_c(E|F) \cdot d_c(F|K) \, ,$$

using Theorem 1.7 and the multiplicativity of the completion defect (see (22). The proof for the defect quotient is similar. $\qquad\square$

Corollary 3.14. *Let $F|K$ be a subhenselian function field without transcendence defect. If vK is cofinal in vF then there exists a finite extension K' of K such that*

$$d_c(K'.F|K) \; = \; d_c(K'|K) \quad and \quad d_q(K'.F|K) \; = \; d_q(K'|K) \, .$$

Proof. We take K' as in the assertion of Theorem 1.7 and apply Corollary 3.13 to the first equations in (8) and (9), where we set $L = K'$. $\qquad\square$

Now we are ready for the

Proof of Theorem 1.8:
a): The assertion that F is a q-defectless field has already been proven in Proposition 3.9. Now we take a standard valuation transcendence basis T of $F|K$. Again by Proposition 3.9, $K(T)$ is a q-defectless field, hence in view of Theorem 1.7,

$$d_q(F|K) \; = \; d_q(F|K(T)) \; = \; 1 \, .$$

b) We assume that vK is cofinal in vF and that K is a c-defectless field. We choose K' according to Corollary 3.14. Then we have:

$$d_c(K'.F|K) \; = \; d_c(K'|K) \; = \; 1 \, .$$

From Corollary 3.13, where we put $E = K'.F$, we get that $d_c(F|K) = 1$. On the other hand, if F' is an arbitrary finite extension of F, then it is also a subhenselian function field without transcendence defect over K and consequently, like F it satisfies $d_c(F'|K) = 1$. By Corollary 3.13, we conclude that

$$d_c(F'|F) = 1 .$$

This shows that F is a c-defectless field. □

The following theorem is a corollary to Theorems 1.5 and 1.8:

Theorem 3.15. *Let $F|K$ be a subhenselian function field without transcendence defect. If K is separably defectless and vK is cofinal in vF, then F is separably defectless.*

References

[B]   Bourbaki, N.: *Commutative algebra*, Paris, 1972

[BGR]   Bosch, S. – Güntzer, U. – Remmert, R.: *Non-Archimedean Analysis*, Grundlehren der Mathematischen Wissenschaften **261**. Springer-Verlag, Berlin, 1984

[D]   Delon, F.: *Quelques propriétés des corps valués en théories des modèles*, Thèse de doctorat d'état, Paris 7, 1982

[E]   Endler, O.: *Valuation theory*, Springer, Berlin, 1972

[EP]   Engler, A. – Prestel, A.: *Valued fields*, Springer-Verlag, Inc., New-York, 2005

[GMP]   Green, B. – Matignon, M. – Pop, F.: *On valued function fields I*, Manuscripta Math. **65** (1989), 357-376

[G-R]   Grauert, H. – Remmert, R.: *Über die Methode der diskret bewerteten Ringe in der nicht-archimedischen Analysis*, Invent. Math. **2** (1942), 87–133

[GRU]   Gruson, L.: *Fibrés vectoriels sur un polydisque ultramétrique*, Ann. Sci. Ec. Super., IV. Ser., **177** (1968), 45-89

[K1]   Kuhlmann, F.-V.: *Henselian function fields and tame fields*, preprint (extended version of Ph.D. thesis), Heidelberg, 1990

[K2]   Kuhlmann, F.-V.: *Valuation theory.* Book in preparation. Preliminary versions of several chapters are available on the web site

         http://math.usask.ca/~fvk/Fvkbook.htm

[K3]   Kuhlmann, F.-V.: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. 356 (2004), 4559-4600

[K4]   Kuhlmann, F.-V.: *Elimination of ramification I: The Generalized Stability Theorem*, Trans. Amer. Math. Soc. **362** (2010), 5697–5727

[K5]   Kuhlmann, F.-V.: *A classification of Artin-Schreier defect extensions and characterizations of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[K6]   Kuhlmann, F.-V.: *Approximation of elements in henselizations*, Manuscripta Math. **136** (2011), 461–474

[K7]   Kuhlmann, F.-V.: *The Defect*, in: Commutative Algebra – Noetherian and non-Noetherian perspectives. Marco Fontana, Salah-Eddine Kabbaj, Bruce Olberding and Irena Swanson (eds.), Springer-Verlag, 2011

[K8]   Kuhlmann, F.-V. – Novacoski, J.: *Henselian elements*, in preparation

[Kh]   Khanduja, S. K.: *An independence theorem in simple transcendental extensions of valued fields.*, J. Indian Math. Soc. **63** (1997) 243–248

[M]   Matignon, M.: *Genre et genre résiduel des corps de fonctions valués*, Manuscripta Math. **58** (1987), 179-214

[O]   Ohm, J.: *The Henselian defect for valued function fields*, Proc. Amer. Math. Soc., **107**(1989), no. 2, 299–308

[R]   Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal, 1968

[W]   Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam, 1989

[Z-S]   Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York 1960

Franz-Viktor Kuhlmann, Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6

E-mail: fvk@math.usask.ca

Asim Naseem, Department of Mathematics, GC University, Katchery Road, Lahore, Pakistan 54000

E-mail: asimroz@gmail.com