ESSENTIALLY FINITE GENERATION OF VALUATION RINGS IN TERMS OF CLASSICAL INVARIANTS

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ABSTRACT. The main goal of this paper is to study some properties of an extension of valuations from classical invariants. More specifically, we consider a valued field (K, ν) and an extension ω of ν to a finite extension L of K. Then we study when the valuation ring of ω is essentially finitely generated over the valuation ring of ν. We present a necessary condition in terms of classic invariants of the extension by Hagen Knaf and show that in some particular cases, this condition is also sufficient. We also study when the corresponding extension of graded algebras is finitely generated. For this problem we present an equivalent condition (which is weaker than the one for the finite generation of the valuation rings).

1. INTRODUCTION

Let (K, ν) be a valued field, L a finite extension of K and ω an extension of ν to L. We denote the valuation rings of ω and ν by $\mathcal{O}_\omega$ and $\mathcal{O}_\nu$, respectively. We also define the graded algebras associated to ω and ν by

$$\mathrm{gr}_\omega(\mathcal{O}_\omega) = \bigoplus_{\gamma \in \Gamma_\omega} \{x \in \mathcal{O}_\omega \mid \omega(x) \geq \gamma\}/\{x \in \mathcal{O}_\omega \mid \omega(x) > \gamma\}$$

and

$$\mathrm{gr}_\nu(\mathcal{O}_\nu) = \bigoplus_{\gamma \in \Gamma_\nu} \{x \in \mathcal{O}_\nu \mid \nu(x) \geq \gamma\}/\{x \in \mathcal{O}_\nu \mid \nu(x) > \gamma\},$$

respectively, where Γ_ω and Γ_ν are the respective value groups of ω and ν. We have a natural inclusion of graded domains $\mathrm{gr}_\nu(\mathcal{O}_\nu) \to \mathrm{gr}_\omega(\mathcal{O}_\omega)$ which is an integral extension, so that $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ is a finitely generated $\mathrm{gr}_\nu(\mathcal{O}_\nu)$-algebra if and and only if $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ is a finitely generated $\mathrm{gr}_\nu(\mathcal{O}_\nu)$-module.

A ring extension $A \subseteq B$ is called essentially finitely generated if there exist $x_1, \ldots, x_r \in B$ such that

$$B = S^{-1}A[x_1, \ldots, x_r]$$

for some multiplicative set $S \subseteq A[x_1, \ldots, x_r]$. We also say that B is essentially of finite type over A if B is essentially finitely generated over A. This paper is devoted, mainly to the following question:

Question 1.1.   **(i):** When is $\mathcal{O}_\omega$ essentially finitely generated over $\mathcal{O}_\nu$?
          (ii): When is $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ a finitely generated $\mathrm{gr}_\nu(\mathcal{O}_\nu)$-algebra?

The main application that we have in mind is to the problem of local uniformization in positive characteristic. One program to solve this problem is by using *ramification theory*. In this direction, Knaf and Kuhlmann proved that every *Abhyankar valuation* admits local uniformization (see [7]) and that every valuation admits local uniformization in a suitable finite separable extension of the function field (see [8]). In order to prove local uniformization for the extension $(L|k, \nu)$ they proved that, in each situation, one can find another field K, $k \subseteq K \subseteq L$ such that $(K|k, \nu)$ admits local uniformization and that L is contained in the *absolute inertia field of* K. In the process, they use that if L is contained in the absolute inertia field of K, then the extension of the valuation rings is essentially finitely generated. Moreover, the property of having valuation rings essentially finitely generated seems to be necessary if one wants to lift local uniformization.

Another reason to study such problems is to understand the structure of valuation rings. Valuation rings, despite being almost never noetherian, seem to share important properties with some classes of noetherian local rings. For example, in [11] it is proved that the Frobenius map is always flat for valuation rings. This work is motivated by a famous theorem of Kunz (see [12]) that states that a noetherian local ring is regular if and only if the Frobenius map is flat.

It is known that if R is an excellent local domain with quotient field K, L is a finite extension of K and D is the integral closure of R in L, then D is a finitely generated R-module [6, Scholie IV.7.8.3 (vi)]. We are led to ask whether the same property holds for a valuation ring, i.e., if $\mathcal{O}$ is a valuation ring with quotient field K and L a finite extension of K, then is the integral closure of $\mathcal{O}$ in F a finitely generated $\mathcal{O}$-algebra? If that is the case, then every valuation ring of L extending ν is essentially finitely generated over $\mathcal{O}_\nu$ (because all such valuation rings are of the form $D_\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of the integral closure D of $\mathcal{O}$ in L).

Question 1.1 **(i)** has a positive answer in the case where L lies in the absolute inertia field of K (see [13]). In [13], it was studied also, in the case where L lies in the absolute inertia field of K, under which conditions the extension of valuation rings is finitely generated (i.e., when we do not need to localize at a prime ideal). In that paper it was also suggested that items **(i)** and **(ii)** of Question 1.1 are closely related. In this paper, we will give characterizations of when this statements are true in terms of classic invariants of extensions of valuation rings.

Throughout this paper we will fix the following notations and assumptions:

(1)
$$\begin{cases} L|K & \text{is a finite field extension,} \\ \omega & \text{is a valuation on } L \\ \nu & \text{is the restriction of } \omega \text{ to } K, \\ \mathcal{O}_\omega \text{ and } \mathcal{O}_\nu & \text{are the valuation rings of } \omega \text{ and } \nu, \text{ respectively,} \\ \Gamma_\omega \text{ and } \Gamma_\nu & \text{are the value groups of } \omega \text{ and } \nu, \text{ respectively,} \\ F\omega \text{ and } K\nu & \text{are the residue fields of } \omega \text{ and } \nu, \text{ respectively.} \end{cases}$$

The **ramification** and **inertia** indexes of the extension $\omega|\nu$ are defined as

$$e(\omega|\nu) = (\Gamma_\omega : \Gamma_\nu) \text{ and } f(\omega|\nu) = [F\omega : K\omega],$$

respectively. We also define the henselization of (K, ν) (for a fixed extension $\bar{\nu}$ of ν to an algebraic closure $\overline{K}$ of K) as the smallest field K^h such that $K \subseteq K^h \subseteq \overline{K}$ and that $\bar{\nu}$ is the only extension of $\bar{\nu}|_{K^h}$ to $\overline{K}$ ([4, Section 17]). One can prove (see Corollary 5.3.8 and discussion on page 75 of [5]) that

$$d(\omega|\nu) = \frac{\left[L^h : K^h\right]}{e(\omega|\nu) \cdot f(\omega|\nu)}$$

is a positive integer which we will call the **defect** of $\omega|\nu$ (in fact, one can prove that $d(\omega|\nu) = 1$ if $\text{char}(K\nu) = 0$ and $d(\omega|\nu) = p^n$ for some $n \in \mathbb{N}$ if $\text{char}(K\nu) = p > 0$).

For an extension $\Delta \subseteq \Gamma$ of ordered abelian groups, we define the **initial index** of Δ in Γ [4, page 138] as

$$\epsilon(\Gamma \mid \Delta) := |\{\gamma \in \Gamma_{\geq 0} \mid \gamma < \Delta_{>0}\}|,$$

where

$$\Gamma_{\geq 0} = \{\gamma \in \Gamma \mid \gamma \geq 0\} \text{ and } \Delta_{>0} = \{\delta \in \Delta \mid \delta > 0\}.$$

Hence, we can also define the initial index $\epsilon(\omega|\nu)$ of the extension $\omega|\nu$ as $\epsilon(\Gamma_\omega : \Gamma_\nu)$.

If S is a subsemigroup of an abelian semigroup T, we will say that T is a finitely generated S-module, or that S has finite index in T if there exist a finite number of elements $\gamma_1, \ldots, \gamma_t \in T$ such that

$$T = \bigcup_{i=1}^{t} (\gamma_i + S).$$

We now consider the following statements:

❶. $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.

❷. $\text{gr}_\omega(\mathcal{O}_\omega)$ is finitely generated over $\text{gr}_\nu(\mathcal{O}_\nu)$.

❸. $(\Gamma_\nu)_{\geq 0}$ has finite index in $(\Gamma_\omega)_{\geq 0}$.

❹. The integral closure D of $\mathcal{O}_\nu$ in L is a finitely generated $\mathcal{O}_\nu$-algebra.

❺. The integral closure D of $\mathcal{O}_\nu$ in L is a finite module over $\mathcal{O}_\nu$.

❻. $\epsilon(\omega|\nu) = e(\omega|\nu)$.

❼. $\epsilon(\omega|\nu) = e(\omega|\nu)$ and $d(\omega|\nu) = 1$.

❽. $\epsilon(\omega_i|\nu) = e(\omega_i|\nu)$ and $d(\omega_i|\nu) = 1$, where $\omega_1, \ldots, w_r$ are all the extensions of ν to L.

One of the main pourposes of this paper is to prove the implications in the following diagram:

$$
\begin{array}{ccccc}
❹ & \Longleftrightarrow & ❺ & \Longleftarrow & ❽ \\
⇑\!\!\!\!\Downarrow & & & & ⇑\!\!\!\!\Downarrow \\
❶ & & \Longrightarrow & & ❼ \\
& & & & ⇑\!\!\!\!\Downarrow \\
❷ & \Longleftrightarrow & ❸ & \Longleftrightarrow & ❻
\end{array}
$$

Observe that

$$❽ \Longrightarrow ❼ \Longrightarrow ❻ \text{ and } ❺ \Longrightarrow ❹$$

are trivial and $❹ \Longrightarrow ❶$ follows from the fact that $\mathcal{O}_\omega = D_{D \cap \mathfrak{m}\omega}$. Also, the counter-examples for $❶ \not\Longrightarrow ❹$, $❻ \not\Longrightarrow ❼$ and $❼ \not\Longrightarrow ❽$ are not difficult and will be presented in Section 2.

The equivalence $❽ \Longleftrightarrow ❺$ is Theorem 18.6 of [4] and we will show $❹ \Longrightarrow ❺$ in Section 2 (Proposition 2.6). The equivalences

$$❷ \Longleftrightarrow ❸ \Longleftrightarrow ❻$$

are the main subject of Section 3 and the implication $❶ \Longrightarrow ❼$ is the main subject of Section 4. We present in Section 4 a proof of this implication by Hagen Knaf.

It is proven in [13] that if L lies in the absolute inertia field of (K, ν) (this is equivalent to $e(\omega|\nu) = 1$, $d(\omega|\nu) = 1$ and that $L\omega|K\nu$ is separable), then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$. Also, the condition $\epsilon(\omega|\nu) = e(\omega|\nu)$ and $d(\omega|\nu) = 1$ implies that $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$ in the cases where $L|K$ is normal and when ω is the only extension of ν to L (see Corollary 2.2). Hence, we are led to believe that this holds in general, i.e., that the following conjecture is true.

Conjecture 1.2 ($❼ \Longrightarrow ❶$). Assume that we are in situation

$$
(2) \qquad\qquad \begin{cases} \text{We are in situation (1); and} \\ d(\omega|\nu) = 1 \text{ and } \epsilon(\omega \mid \nu) = e(\omega \mid \nu) \end{cases}
$$

Then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.

In Section 5 we will show that in order to prove the conjecture above, we can assume that $L|K$ is separable, i.e., that it is enough to show the following conjecture.

Conjecture 1.3. Assume that we are in situation (2) and that $L|K$ is separable. Then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.

One of the main results of this paper is that Conjecture 1.2 is satisfied for the case where ν is centered in a quasi-excellent local domain of dimension two, or if ν is an Abhyankar valuation.

If R is a local domain which is contained in K, then we say that ν is centered in R is $R \subset \mathcal{O}_\nu$ and $m_\nu \cap R = m_R$.

Theorem 1.4. *Assume that R is an excellent two-dimensional local domain with quotient field K. Suppose that ν is a valuation of K centered at R. Assume that L is a finite separable extension of K and that ω is an extension of ν to L. If $d(\omega \mid \nu) = 1$ and $\epsilon(\omega \mid \nu) = e(\omega \mid \nu)$, then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.*

Section 6 is devoted to the proof of Theorem 1.4.

Let K be an algebraic function field over a field k. A valuation ν on K (which is trivial on k) satisfies the fundamental inequality ([1, Lemma 1])

$$\operatorname{trdeg}_k K \geq \operatorname{trdeg}_k \mathcal{O}_\nu/\mathfrak{m}_\nu + \operatorname{ratrk}(\nu).$$

Here the rational rank $\operatorname{ratrk}(\nu)$ of ν is the $\mathbb{Q}$-dimension of the tensor product of the value group Γ_ν of ν with $\mathbb{Q}$. We say that ν is an Abhyankar valuation if equality holds in this equation.

Theorem 1.5. *Let K be an algebraic function field over a field k, and let ν be an Abhyankar valuation on K. Assume that L is a finite extension of K and that ω is an extension of ν to L. Suppose that $\mathcal{O}_\omega/\mathfrak{m}_\omega$ is separable over k. If $d(\omega|\nu) = 1$ and $\epsilon(\omega|\nu) = e(\omega|\nu)$, then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.*

We prove Theorem 1.5 in Section 7.

2. Preliminaries

We will start by stating some known results related to this paper. For a subring R of $\mathcal{O}_\nu$ (and S of $\mathcal{O}_\omega$) we will denote $R_\nu = R_{\mathfrak{m}_\nu \cap R}$ ($S_\omega = S_{\mathfrak{m}_\omega \cap S}$).

Theorem 2.1. *([4, Theorem 18.6]) Assume that we are in situation (1) and let $\omega = \omega_1, \ldots, \omega_r$ be all the extensions of ν to L. Let D be the integral closure of $\mathcal{O}_\nu$ in L. Then D is a finite free module over $\mathcal{O}_\nu$ if and only if $(L|K, \nu)$ is defectless and $\epsilon(\omega_i|\nu) = e(\omega_i|\nu)$ for every $i = 1, \ldots, r$.*

Corollary 2.2. *Assume that we are in one of the following case:*
(i) ω is the only extension of ν to L;
(ii) $L|K$ is a normal extension.
Then D is a finite $\mathcal{O}_\nu$-module if and only if $d(\omega|\nu) = 1$ and $\epsilon(\omega|\nu) = e(\omega|\nu)$.

Proof. Item **(i)** is trivial and item **(ii)** follows from the fact that if $L|K$ is normal, any every extension ω_i of ν to L is conjugate to ω and hence $d(\omega_i|\nu) = d(\omega|\nu)$, $\epsilon(\omega_i|\nu) = \epsilon(\omega|\nu)$ and $e(\omega_i|\nu) = e(\omega|\nu)$. $\square$

Theorem 2.3. *([13, Theorem 1.3]) Assume that we are in situation (1). If $e(\omega|\nu) = d(\omega|\nu) = 1$ and $L\omega|K\nu$ is separable, then*

$$\mathcal{O}_\omega = \mathcal{O}_\nu[\eta]_\omega \text{ for some } \eta \in \mathcal{O}_\omega.$$

In particular, $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.

We will present now the examples that show that the "up arrows" in our diagram are satisfied.

Example 2.4. (❻ $\nRightarrow$ ❼) For this, it is enough to present an immediate extension with non trivial defect. Example 3.1 of [14] is a simple example of that situation.

Example 2.5. (❶ $\nRightarrow$ ❹ and ❼ $\nRightarrow$ ❽) Let (K, ν) be a valued field with group of values $\frac{1}{3^\infty}\mathbb{Z}$. Let L be the extension

$$K(\eta) = K[X]/(X(X-1)^2 - a)$$

where $a \in K$ is such that $\nu(a) = 1$. Then ν admits two extensions ω_1 and ω_2 to L defined by

$$\omega_1(\eta) = 1 \text{ and } \omega_2(\eta - 1) = \frac{1}{2}.$$

Since

$$e(\omega_1|\nu) \geq 1 \text{ and } e(\omega_2|\nu) \geq 2,$$

we have by the fundamental inequality, that these are the only extensions of ν to L. Moreover, we have that

$$e(\omega_1|\nu) = 1, e(\omega_2|\nu) = 2 \text{ and } d(\omega_i|\nu) = f(\omega_i|\nu) = 1 \text{ for } i = 1, 2.$$

Since the value group of ν is not discrete, we have that $\epsilon(\omega_i|\nu) = 1$ for $i = 1, 2$. Hence, the extension ω_1 satisfies ❼ but not ❽, since $\epsilon(\omega_2|\nu) < e(\omega_2|\nu)$.

Moreover, since $\epsilon(\omega_2|\nu) < e(\omega_2|\nu)$, by Theorem 2.1 we conclude that ❹ is not satisfied. By Theorem 2.3, we have that ❶ is satisfied. In fact, one can show that

$$\mathcal{O}_{\omega_1} = \mathcal{O}_\nu[\eta]_{\omega_1}.$$

Proposition 2.6 (❹ $\implies$ ❺). *Let A be a normal domain (integrally closed in $K = \mathrm{Quot}(A)$) and L an algebraic extension extension of K. Let $I_L(A)$ be the integral closure of A in L. If $I_L(A)$ is a finitely generated A-algebra, then it is a finite A-module.*

Proof. Take any $x \in I_L(A)$. Since x is integral over A, there exists a relation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0, \text{ for } a_0, \ldots, a_{n-1} \in A.$$

This means that

$$x^n = -a_{n-1}x^{n-1} - \ldots - a_0 \in A_x := Ax^{n-1} + \ldots + Ax + A.$$

Assume now that for $k \geq n$, we have $x^i \in A_x$ for every $1 \leq i \leq k$. Then

$$
\begin{aligned}
x^{k+1} &= xx^k = x(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} \ldots + b_0) \\
&= b_{n-1}x^n + b_{n-2}a^{n-1} \ldots + b_0x \\
&= (b_{n-2} - b_{n-1}a_{n-1})x_i^{n-1} + \ldots + (b_0 - b_{n-1}a_1)x - b_{n-1}a_0 \in A_x.
\end{aligned}
$$

Therefore, $A[x] = A_x$.

If $I_L(A) = A[x_1, \ldots, x_r]$, then we choose $n_1, \ldots, n_r$ such that $x_i^{n_i} \in A_{x_i}$. Then we can prove, by induction on s, $1 \leq s \leq r$, that $A[x_1, \ldots, x_s]$ is generated as an

A-module by $x_1^{j_1} \ldots x_s^{j_s}$ where $0 \le j_i < n_i$ and $1 \le i \le s$. Therefore, $I_L(A)$ is a finite A-module. $\square$

3. The initial index

In this section we prove a few basic results about the initial index of a subgroup Δ of finite index in an ordered group Γ.

Lemma 3.1. *Let Γ_0 be the first convex subgroup of Γ. If $1 < \epsilon(\Gamma \mid \Delta) < \infty$, then $\Gamma_0 \simeq \mathbb{Z}$.*

Proof. Assume, towards a contradiction, that $\Gamma_0 \not\simeq \mathbb{Z}$. Then Γ does not admit a smallest positive element. Since $\epsilon > 1$, there exists $\gamma \in \Gamma_{>0}$ such that $\gamma < \Delta_{>0}$. Since Γ does not admit a smallest positive element, there would exist infinitely many positive elements in Γ smaller than $\Delta_{>0}$, and this is a contradiction to $\epsilon < \infty$. $\square$

Proposition 3.2. *Assume that $\epsilon := \epsilon(\Gamma \mid \Delta) > 1$ and denote*

$$0 = \gamma_0 < \gamma_1 < \ldots < \gamma_{\epsilon-1}$$

for all the elements in $\Gamma_{\ge 0}$ which are smaller than $\Delta_{>0}$. Then

(3) $$\gamma_k = k\gamma_1 \text{ for every } k, \ 1 \le k \le \epsilon - 1 \text{ and } \epsilon\gamma_1 \in \Delta.$$

Proof. We will prove it by induction on k. For $k = 1$, the first assertion of (3) is trivially satisfied. Assume that for a given k, $1 \le k \le \epsilon - 2$, we have $\gamma_k = k\gamma_1$. We will show that $\gamma_{k+1} = (k+1)\gamma_1$. Since $\gamma_k = k\gamma_1$, we have

$$(k+1)\gamma_1 = k\gamma_1 + \gamma_1 = \gamma_k + \gamma_1,$$

hence we have to show that $\gamma_{k+1} = \gamma_k + \gamma_1$.

Since $\gamma_k < \gamma_{k+1}$ we have $0 < \gamma_{k+1} - \gamma_k$ and because γ_1 is the smallest positive element of Γ, we obtain that $\gamma_1 \le \gamma_{k+1} - \gamma_k$. Hence, $\gamma_k + \gamma_1 \le \gamma_{k+1}$. Since $\gamma_k + \gamma_1 \le \gamma_{k+1} < \Delta_{>0}$, there exists j, $1 \le j \le \epsilon - 1$, such that $\gamma_k + \gamma_1 = \gamma_j$. On the other hand, since $\gamma_k < \gamma_j$, by our assumption on γ_i's, $\gamma_{k+1} \le \gamma_j = \gamma_k + \gamma_1$. Therefore, $\gamma_{k+1} = \gamma_k + \gamma_1$, which is what we wanted to prove.

Since $\gamma_{\epsilon-1}$ is the largest element in $\Gamma_{\ge 0}$ which is smaller than every element $\delta \in \Delta_{>0}$, there exists $\delta \in \Delta$ such that

$$\gamma_{\epsilon-1} < \delta \le \gamma_{\epsilon-1} + \gamma_1 = (\epsilon - 1)\gamma_1 + \gamma_1 = \epsilon\gamma_1.$$

If $\delta < \epsilon\gamma_1$, we would have $0 < \delta - \gamma_{\epsilon-1} < \gamma_1$, which is a contradiction. Therefore, $\delta = \epsilon\gamma_1$, which completes our proof. $\square$

Proposition 3.3 (❸ ⟺ ❺). *Let Γ be an ordered abelian group and Δ a subgroup of Γ of finite index. We have:*

  (i): $\epsilon(\Gamma \mid \Delta) \le (\Gamma : \Delta)$*; and*

(ii): $\epsilon(\Gamma \mid \Delta) = (\Gamma : \Delta)$ *if and only if* $\Delta_{\geq 0}$ *has finite index in* $\Gamma_{\geq 0}$, *i.e., if there exist* $\gamma_0, \ldots, \gamma_{n-1} \in \Gamma_{\geq 0}$ *($\gamma_0 = 0$) such that*

$$\Gamma_{\geq 0} = \bigcup_{i=0}^{n-1} \left(\gamma_i + \Delta_{\geq 0} \right).$$

Proof. Set $\epsilon := \epsilon(\Gamma \mid \Delta)$ and choose $\gamma_0, \ldots, \gamma_{\epsilon-1} \in \Gamma_{\geq 0}$ such that

$$0 = \gamma_0 < \ldots < \gamma_{\epsilon-1} < \Delta_{>0}.$$

We claim that the cosets of the γ_i's modulo Δ are disjoint, i.e, that

$$(\gamma_i + \Delta) \cap (\gamma_j + \Delta) = \emptyset \text{ if } i \neq j.$$

Assume otherwise. Then, there would exist i and j, $0 \leq i < j \leq \epsilon - 1$ such that

$$\gamma_j - \gamma_i = \delta \in \Delta.$$

Then

$$0 < \gamma_j - \gamma_i = \delta < \gamma_j,$$

which is a contradiction to our assumption on the γ_i's.

Since the cosets of the γ_i's are disjoint, we have **(i)**. To prove **(ii)** we assume first that $\epsilon = (\Gamma : \Delta)$. This, together with the fact that the γ_i's have distinct cosets modulo Δ gives us that

$$\Gamma = \bigcup_{i=0}^{\epsilon-1} \left(\gamma_i + \Delta \right).$$

Now take any element $\gamma \in (\Gamma)_{\geq 0}$. Since $\gamma \in \Gamma$, there exist i, $0 \leq i \leq \epsilon - 1$, and $\delta \in \Delta$ such that $\gamma = \gamma_i + \delta$. We claim that $\delta \geq 0$. Indeed, if $\delta < 0$, then

$$0 \leq \gamma = \gamma_i + \delta < \gamma_i,$$

and consequently $\gamma = \gamma_j$ for some j. This implies that

$$(\gamma_i + \Delta) \cap (\gamma_j + \Delta) \neq \emptyset \text{ and } \gamma_i \neq \gamma_i,$$

and this is a contradiction to what we proved in the previous paragraph. Therefore,

$$(\Gamma)_{\geq 0} = \bigcup_{i=0}^{\epsilon-1} \left(\gamma_i + (\Delta)_{\geq 0} \right).$$

Assume now that

$$(4) \qquad\qquad (\Gamma)_{\geq 0} = \bigcup_{i=0}^{n-1} \left(\gamma_i + (\Delta)_{\geq 0} \right),$$

for some $\gamma_0, \ldots, \gamma_{n-1} \in \Gamma_{\geq 0}$. We claim that we can choose the γ_i's such that

$$(\gamma_i + \Delta_{\geq 0}) \cap (\gamma_j + \Delta_{\geq 0}) = \emptyset, \text{ for } 0 \leq i < j \leq n - 1.$$

Indeed, if for some i and j, $0 \leq i, j \leq n - 1$, we have

$$(\gamma_i + \Delta_{\geq 0}) \cap (\gamma_j + \Delta_{\geq 0}) \neq \emptyset,$$

then there exist $\delta_i, \delta_j \in \Delta_{\geq 0}$ such that $\gamma_i + \delta_i = \gamma_j + \delta_j$. Assume, without loss of generality, that $\gamma_j > \gamma_i$. Then $\delta_i - \delta_j = \gamma_j - \gamma_i \in \Delta_{\geq 0}$ and consequently

$$\gamma_j = \gamma_i + \delta_i - \delta_j \in \gamma_i + \Delta_{\geq 0}.$$

Hence, $\gamma_j + \Delta_{\geq 0} \subseteq \gamma_i + \Delta_{\geq 0}$ and we can remove $\gamma_j + \Delta_{\geq 0}$ in (4). Since there are only finitely many γ_i's, we proceed as above until we reach disjoint cosets modulo $\Delta_{\geq 0}$. Assume, without loss of generality, that

$$0 = \gamma_0 < \gamma_1 < \ldots < \gamma_{n-1}.$$

We will show that

$$\Gamma = \bigcup_{i=0}^{n-1} (\gamma_i + \Delta),$$

that $(\gamma_i + \Delta) \cap (\gamma_j + \Delta) = \emptyset$ when $i \neq j$ and that $\gamma_{n-1} < \delta$ for every $\delta \in \Delta_{>0}$. This will imply that $(\Gamma : \Delta) = n \leq \epsilon$, and since $\epsilon \leq (\Gamma : \Delta)$ we conclude that $\epsilon = (\Gamma : \Delta)$.

Take $\gamma \in \Gamma$. If $\gamma \geq 0$, then by our assumption

$$\gamma \in \gamma_i + \Delta_{\geq 0} \subseteq \gamma_i + \Delta \text{ for some } i, 0 \leq i \leq n - 1$$

and if $\gamma \in \Delta$, then

$$\gamma \in \gamma_0 + \Delta,$$

because $\gamma_0 = 0$. Assume now that $\gamma < 0$ and that $\gamma \in \Gamma \setminus \Delta$. Since Δ has finite index in Γ, there exist $\delta \in \Delta$ and $r \in \mathbb{N}$, $r > 1$, such that $r\gamma = \delta$. Since $\gamma < 0$, we have $\delta = r\gamma < \gamma$ and hence $\gamma - \delta \in \Gamma_{\geq 0}$. By our assumption, there exists γ_i and $\delta' \in \Delta_{\geq 0}$ such that $\gamma - \delta = \gamma_i + \delta'$ and consequently $\gamma \in \gamma_i + \Delta$. Therefore,

$$(5) \qquad \Gamma = \bigcup_{i=0}^{n-1} (\gamma_i + \Delta).$$

Assume now that $(\gamma_i + \Delta) \cap (\gamma_j + \Delta) \neq \emptyset$ for some $i \leq j$. This means that there exist $\delta_i, \delta_j \in \Delta$ such that $\gamma_i + \delta_i = \gamma_j + \delta_j$. Since $i \leq j$, this means that $\delta_i - \delta_j = \gamma_j - \gamma_i \in \Delta_{\geq 0}$ and since we assumed that the cosets in (4) are disjoint, we must have that $i = j$. This implies that the cosets in (5) are disjoint and therefore $n = (\Gamma : \Delta)$.

It remains to show that $n \leq \epsilon$, i.e., that $\gamma_{n-1} < \delta$ for every $\delta \in \Delta_{>0}$. If this were not the case, then there would exist $\delta \in \Delta$ such that $0 < \delta \leq \gamma_{n-1}$ and consequently $\gamma_{n-1} - \delta \in \Gamma_{\geq 0}$. Since $\delta > 0$ we have that $\gamma_{n-1} - \delta < \gamma_{n-1}$ and hence there exists $i, 0 \leq i < n - 1$ such that $\gamma_{n-1} - \delta = \gamma_i + \delta'$. Therefore, $\gamma_{n-1} - \gamma_i \in \Delta$, which is a contradiction to the fact that the cosets in (5) are disjoint. This concludes our proof. $\qquad \square$

Remark 3.4. (i) We observe from the proof of the above proposition, that if $\epsilon := \epsilon(\Gamma : \Delta) = (\Gamma : \Delta)$, then

$$\Gamma = \bigcup_{i=0}^{\epsilon-1} (\gamma_i + \Delta) \text{ and } \Gamma_{\geq 0} = \bigcup_{i=0}^{\epsilon-1} (\gamma_i + \Delta_{\geq 0}),$$

where $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_{\epsilon-1}$ are all the elements in $\Gamma_{\geq 0}$ smaller than $\Delta_{>0}$.

(ii) From the above propositions we conclude that if $\epsilon(\Gamma|\Delta) = (\Gamma : \Delta)$, then

$$\Gamma/\Delta \simeq \mathbb{Z}/\epsilon\mathbb{Z}.$$

Corollary 3.5. *Assume that* $\Gamma = \Gamma_1 \oplus_{\mathrm{lex}} \mathbb{Z}$. *If* $\epsilon := \epsilon(\Gamma \mid \Delta) = (\Gamma : \Delta) > 1$, *then for every* $\gamma_1 \in (\Gamma_1)_{>0}$ *there exists* $a \in \mathbb{Z}$ *such that* $(\gamma_1, a) \in \Delta$.

Proof. Since $(0, 1)$ is the smallest element of $\Gamma_{>0}$, Proposition 3.2 gives us that $(0, i) \notin \Delta$ for every i, $1 \leq i < \epsilon$, and $(1, \epsilon) \in \Delta$. Since $\epsilon(\Gamma|\Delta) = (\Gamma : \Delta)$, Proposition 3.3 (and the remark above) gives us

$$\Gamma_{\geq 0} = \bigcup_{i=0}^{\epsilon-1} (0, i) + \Delta_{\geq 0}.$$

Since $(\gamma_1, 0) \in \Gamma_{\geq 0}$, there exists $(b, c) \in \Delta_{\geq 0}$ and i, $0 \leq i \leq \epsilon - 1$ such that $(\gamma_1, 0) = (b, c) + (0, i)$. Hence, $(\gamma_1, -i) = (b, c) \in \Delta_{\geq 0}$. $\square$

As a consequence of the previous corollary, if $\Gamma = \mathbb{Z}^n$ (with the lexicographic ordering) and $\Delta \subseteq \Gamma$ is such that $\epsilon(\Gamma|\Delta) = (\Gamma : \Delta)$, then there exist $a_{i,j} \in \mathbb{Z}$, for $1 \leq i \leq n-1$ and $i + 1 \leq j \leq n$, such that

$$(1, a_{12}, \ldots, a_{1n}), \ldots, (0, \ldots, 0, 1, a_{n-1n}) \in \Delta.$$

Indeed, setting $\Gamma_1 = \mathbb{Z}^{n-1}$, the previous corollary says that for every $\gamma_1 \in \mathbb{Z}^{n-1}$, $\gamma_1 > 0$, there exists $a \in \mathbb{Z}$ such that $(\gamma_1, a) \in \Delta$. Hence, if we take

$$\gamma_1 = (1, a_{12}, \ldots, a_{1n-1}), \ldots, \gamma_{n-1} = (0, \ldots, 0, 1) \in \Gamma_1$$

(we could choose $a_{ij} = 0$ if we wanted), there exist $a_{1n}, \ldots, a_{n-1n} \in \mathbb{Z}$ such that

$$(1, a_{12}, \ldots, a_{1n}), \ldots, (0, \ldots, 0, 1, a_{n-1n}) \in \Delta.$$

We will now prove the converse of the previous statement.

Proposition 3.6. *Assume that* $\Gamma = \mathbb{Z}^n$ (*with lexicographic ordering*) *and that* Δ *is a subgroup of* Γ *of finite order. If there exist* $a_{i,j} \in \mathbb{Z}$ *with* $1 \leq i \leq n-1$ *and* $i + 1 \leq j \leq n$ *such that*

$$(1, a_{12}, \ldots, a_{1n}), \ldots, (0, \ldots, 0, 1, a_{n-1n}) \in \Delta,$$

then $\epsilon(\Gamma : \Delta) = (\Gamma : \Delta)$.

Proof. First we observe that if there exist $a_{i,j} \in \mathbb{Z}$, $1 < i \leq j \leq n$ such that

$$(1, a_{12}, \ldots, a_{1n}), \ldots, (0, \ldots, 0, 1, a_{n-1n}) \in \Delta,$$

then (using "Gaussian elimination") there exist $a_1, \ldots, a_{n-1} \in \mathbb{Z}$ such that

$$\alpha_1 := (1, 0, \ldots, 0, a_1), \ldots, \alpha_{n-1} := (0, \ldots, 0, 1, a_{n-1}) \in \Delta.$$

If $\epsilon(\Gamma|\Delta) = 1$, then $(0, \ldots, 0, 1) \in \Delta$ and we can use again Gaussian elimination to obtain that $(1, \ldots, 0, 0), \ldots, (0, \ldots, 0, 1) \in \Delta$ and hence $\Gamma = \Delta$.

If $\epsilon(\Gamma : \Delta) > 1$, then by Lemma 3.2, we have that $(0, \ldots, 0, 1), \ldots, (0, \ldots, 0, \epsilon - 1) \notin \Delta$ and $(0, \ldots, 0, \epsilon) \in \Delta$. We will show that

$$\Gamma_{\geq 0} = \bigcup_{i=0}^{\epsilon-1} (0, \ldots, 0, i) + \Delta_{\geq 0}$$

and the result will follow from Proposition 3.3 **(ii)**. Take any $\gamma = (b_1, \ldots, b_n) \in \Gamma_{\geq 0}$. If $b_1 = b_2 = \ldots = b_{n-1} = 0$, then $b_n \geq 0$ and there exist $c' \in \mathbb{N}$ and i, $0 \leq i \leq \epsilon - 1$, such that $b_n = i + c'\epsilon$. Hence, $\gamma = (0, \ldots, 0, i) + \delta$ where $\delta = c'(0, \ldots, 0, \epsilon) \in \Delta_{\geq 0}$. If $b_j \neq 0$ for some j, $1 \leq j \leq n-1$, then there exists i, $1 \leq i \leq n-1$, such that $b_j = 0$ for $j < i$ and $b_i > 0$. Then, we can write

$$\gamma = b_i \alpha_i + b_{i+1} \alpha_{i+1} + \ldots + b_{n-1} \alpha_{n-1} + (0, \ldots, 0, c)$$

for some $c \in \mathbb{Z}$. Then there exists $c' \in \mathbb{Z}$ and i, $1 \leq i \leq \epsilon - 1$, such that

$$c = i + c'\epsilon.$$

Therefore, $\gamma = (0, \ldots, 0, i) + \delta$, where

$$\delta = b_i \alpha_i + b_{i+1} \alpha_{i+1} + \ldots + b_{n-1} \alpha_{n-1} + c'(0, \ldots, 0, \epsilon) \in \Delta_{\geq 0}.$$

This concludes our proof. $\square$

The next lemma shows that ϵ is multiplicative.

Lemma 3.7. *Let $\Delta \subseteq \Sigma \subseteq \Gamma$ be ordered abelian groups (with $\Delta \subseteq \Sigma$, $\Sigma \subseteq \Gamma$ and $\Delta \subseteq \Gamma$ of finite index). Then*

$$\epsilon(\Gamma \mid \Sigma) \cdot \epsilon(\Sigma \mid \Delta) = \epsilon(\Gamma \mid \Delta).$$

In particular, $\epsilon(\Gamma \mid \Delta) = (\Gamma : \Delta)$ if and only if

$$\epsilon(\Gamma \mid \Sigma) = (\Gamma : \Sigma) \text{ and } \epsilon(\Sigma \mid \Delta) = (\Sigma : \Delta).$$

Proof. Let $r = \epsilon(\Sigma \mid \Delta)$ and $s = \epsilon(\Gamma \mid \Sigma)$ and choose elements $\sigma_0, \ldots, \sigma_{r-1} \in \Sigma_{\geq 0}$ and $\gamma_0, \ldots, \gamma_{s-1} \in \Gamma_{\geq 0}$ such that

$$0 = \sigma_0 < \ldots < \sigma_{r-1} < \Delta_{>0} \text{ and } 0 = \gamma_0 < \ldots < \gamma_{s-1} < \Sigma_{>0}.$$

If $\delta \in \Delta_{>0}$, then $\sigma_{r-1} < \delta$ and since $\Delta \subseteq \Sigma$ we have $\delta - \sigma_{r-1} \in \Sigma_{>0}$. Hence, $\gamma_{s-1} < \delta - \sigma_{r-1}$ which implies that

$$\sigma_i + \gamma_j \leq \sigma_{r-1} + \gamma_{s-1} < \delta, \text{ for every } i \text{ and } j, 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq s-1.$$

Therefore,

$$\epsilon(\Gamma \mid \Sigma) \cdot \epsilon(\Sigma \mid \Delta) = r \cdot s \leq \epsilon(\Gamma \mid \Delta).$$

Take now $\gamma \in \Gamma_{\geq 0}$ such that $\gamma < \Delta_{>0}$ and let j, $0 \leq j \leq r-1$ be the largest index for which $\sigma_j \leq \gamma$, i.e.,

$$j = r - 1 \text{ and } \sigma_{r-1} \leq \gamma \text{ or } j < r - 1 \text{ and } \sigma_j \leq \gamma < \sigma_{j+1}.$$

We claim that $\gamma - \sigma_j < \Sigma_{>0}$ and consequently $\gamma - \sigma_j = \gamma_i$ (i.e., $\gamma = \gamma_i + \sigma_j$) for some i, $0 \leq i \leq s - 1$. Indeed, if this were not the case, then there would exist $\sigma \in \Sigma_{>0}$ such that

$$0 < \sigma \leq \gamma - \sigma_j \leq \gamma < \Delta_{>0}$$

and then $\sigma = \sigma_{j'}$ for some j', $0 \leq j' \leq r - 1$. This would imply that

$$\sigma_j < \sigma_j + \sigma_{j'} \leq \gamma < \Delta_{>0}.$$

Since $\sigma_j + \sigma_{j'} \in \Sigma_{>0}$ and $\sigma_j + \sigma_{j'} \leq \gamma < \Delta_{>0}$, this implies that $\sigma_j + \sigma_{j'} = \sigma_{j''}$ for some j'', $j < j'' \leq r - 1$. This is a contradiction to the fact that j was chosen as the largest index for which $\sigma_j \leq \gamma$. We have proved that for every $\gamma \in \Gamma_{\geq 0}$, if $\gamma < \Delta_{>0}$, then there exist i and j, $0 \leq i \leq s - 1$ and $0 \leq j \leq r - 1$, such that $\gamma = \sigma_j + \gamma_i$. Therefore,

$$\epsilon(\Gamma \mid \Delta) \leq r \cdot s = \epsilon(\Gamma \mid \Sigma) \cdot \epsilon(\Sigma \mid \Delta).$$

This concludes the proof that $\epsilon(\Gamma \mid \Delta) = \epsilon(\Gamma \mid \Sigma) \cdot \epsilon(\Sigma \mid \Delta)$.

Since $\epsilon(\Gamma \mid \Sigma) \leq (\Gamma : \Sigma)$, $\epsilon(\Sigma : \Delta) \leq (\Sigma : \Delta)$ and $\epsilon(\Gamma : \Delta) \leq (\Gamma : \Delta)$, we have

$$\epsilon(\Gamma : \Delta) = \epsilon(\Gamma \mid \Sigma) \cdot \epsilon(\Sigma \mid \Delta) \leq (\Gamma : \Sigma) \cdot (\Sigma : \Delta) = (\Gamma : \Delta) \leq \epsilon(\Gamma : \Delta).$$

If the equality holds in one of the above inequalities, then equality holds everywhere. Therefore, $\epsilon(\Gamma \mid \Delta) = (\Gamma : \Delta)$ if and only if $\epsilon(\Gamma \mid \Sigma) = (\Gamma : \Sigma)$ and $\epsilon(\Sigma \mid \Delta) = (\Sigma : \Delta)$. $\qquad\square$

Theorem 3.8 (❷ $\Longleftrightarrow$ ❸). *We have that $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ is finitely generated over $\mathrm{gr}_\nu(\mathcal{O}_\nu)$ if and only if $(\Gamma_\omega)_{\geq 0}$ is finitely generated over $(\Gamma_\nu)_{\geq 0}$.*

Proof. Assume first that $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ is finitely generated over $\mathrm{gr}_\nu(\mathcal{O}_\nu)$. This implies that there exist $f_1, \ldots, f_n \in \mathcal{O}_\omega$ such that $\mathrm{in}_\omega(f_1), \ldots, \mathrm{in}_\omega(f_n)$ generate $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ over $\mathrm{gr}_\nu(\mathcal{O}_\nu)$. For $\gamma \in (\Gamma_\omega)_{\geq 0}$ there exists $x \in \mathcal{O}_\omega$ such that $\omega(x) = \gamma$. Since $\mathrm{in}_\omega(f_1), \ldots, \mathrm{in}_\omega(f_n)$ generate $\mathrm{gr}_\omega(\mathcal{O}_\omega)$, there exist $g_{11}, \ldots, g_{1r_1}, \ldots, g_{1n}, \ldots, g_{nr_n} \in \mathcal{O}_\nu$, such that

$$
\begin{aligned}
\mathrm{in}_\omega(x) &= \left(\sum_{j=1}^{r_1} \mathrm{in}_\nu(g_{1j}) \right) \mathrm{in}_\omega(f_1) + \ldots + \left(\sum_{j=1}^{r_n} \mathrm{in}_\nu(g_{nj}) \right) \mathrm{in}_\omega(f_n) \\
&= \left(\sum_{j=1}^{r_1} \mathrm{in}_\nu(g_{1j}) \mathrm{in}_\omega(f_1) \right) + \ldots + \left(\sum_{j=1}^{r_n} \mathrm{in}_\nu(g_{nj}) \mathrm{in}_\omega(f_n) \right)
\end{aligned}
$$

By the definition of the graded algebra, this means that we just need to consider the homogeneous components of degree γ in the equation above, hence there exists some f_i and g_{ij} such that

$$\gamma = \omega(x) = \omega(g_{ij} f_i) = \nu(g_{ij}) + \omega(f_i).$$

Hence

$$(\Gamma_\omega)_{\geq 0} = \bigcup_{i=1}^{n} \left(\omega(f_i) + (\Gamma_\nu)_{\geq 0} \right).$$

For the converse, assume that $(\Gamma_\omega)_{\geq 0}$ has finite index over $(\Gamma_\nu)_{\geq 0}$. Thus there exist $f_1, \ldots, f_n \in \mathcal{O}_\omega$ such that

$$(\Gamma_\omega)_{\geq 0} = \bigcup_{i=1}^{n} \left(\omega(f_i) + (\Gamma_\nu)_{\geq 0} \right).$$

Moreover, since $F\omega$ is a finite $K\nu$-vector space, there exists $h_1, \ldots, h_r \in \mathcal{O}_\omega$ such that $h_1\omega, \ldots, h_r\omega$ is a $K\nu$-basis of $L\omega$. For $x \in \mathcal{O}_\omega$ there exists $i_0 \in \{1, \ldots, n\}$ such that $\omega(x) = \nu(g_{i_0}) + \omega(f_{i_0})$ for some $g_{i_0} \in \mathcal{O}_\nu$. This means that $\omega\left(\frac{x}{g_{i_0} f_{i_0}}\right) = 0$. Then, there exists $a_1, \ldots, a_r \in \mathcal{O}_\nu$ such that

$$\left(\frac{x}{g_{i_0} f_{i_0}} \right) \omega = a_1\nu \cdot h_1\omega + \ldots + a_r\nu \cdot h_r\omega.$$

Hence

$$\omega(x - a_1 h_1 g_{i_0} f_{i_0} - \cdots - a_r h_r g_{i_0} f_{i_0}) > \omega(x).$$

Consequently,

$$\mathrm{in}_\omega(x) = \sum_{j \in J} \mathrm{in}_\nu(a_j g_{i_0}) \mathrm{in}_\omega(h_j f_{i_0}),$$

where

$$J = \{j \in \{1, \ldots, r\} \mid \omega(a_1 h_1 g_{i_0} f_{i_0}) = \omega(x)\}.$$

Therefore, $\mathrm{gr}_\omega(\mathcal{O}_\omega)$ is generated over $\mathrm{gr}_\nu(\mathcal{O}_\nu)$ by

$$\{\mathrm{in}_\omega(h_j f_i) \mid 1 \leq j \leq r \text{ and } 1 \leq i \leq n\}.$$

$\square$

4. Proof of the necessity condition

The statement and proof of Theorem 4.1 are by Hagen Knaf.

Theorem 4.1. (❶ $\implies$ ❼)(Knaf) *Assume that we are in condition (1). Then for the local ring extension $\mathcal{O}_\nu \subseteq \mathcal{O}_\omega$ to be essentially finitely generated the following conditions are necessary:*

$$e(\omega|\nu) = \epsilon(\omega|\nu) \text{ and } d(\omega|\nu) = 1.$$

The proof given here is based on three results, which are interesting for their own sake. Comparing the construction of the henselization K^h of a valued field (K, ν) in [4, Section 17] and the construction of the henselization of a normal local ring in [10, Section 43], we see that the valuation ring of the valued field K^h is the henselization $\mathcal{O}_\nu^h$ of $\mathcal{O}_\nu$.

Lemma 4.2. *Assume that we are in condition (1). Then the ring $\mathcal{O}_\omega/\mathfrak{m}_\nu\mathcal{O}_\omega$ is a $K\nu$-vector space of dimension $\epsilon(\omega|\nu)f(\omega|\nu)$.*

The proof can be found in [4] (18.5).

Lemma 4.3. *Assume that we are in condition (1). If $\mathcal{O}_\nu \subseteq \mathcal{O}_\omega$ is a finite ring extension, then $e(\omega|\nu) = \epsilon(\omega|\nu)$ and $d(\omega|\nu) = 1$ hold.*

Proof. The finitely generated, torsion-free $\mathcal{O}_\nu$-module $\mathcal{O}_\omega$ is free of rank $[L:K]$ by [4, (18.6)]. By Nakayama's lemma, lemma 4.2 and [4, (18.3)] one gets

$$[L:K] = \epsilon(\omega|\nu)f(\omega|\nu) \leq d(\omega|\nu)e(\omega|\nu)f(\omega|\nu) = [L^h:K^h] \leq [L:K]$$

and thus the assertion. $\square$

Lemma 4.4. *Assume that we are in condition (1). Then $\mathcal{O}_\omega^h = \mathcal{O}_\omega \cdot \mathcal{O}_\nu^h$, where the ring compositum is formed within the field L^h.*

Proof. Recall that a local ring R is called henselian, if Hensel's lemma holds for that ring. The henselization R^h of a local ring is henselian [10], (43.3), or just for valuation domains [4], Theorem 17.11. The latter suffices for the current proof. The extension $\mathcal{O}_\nu^h \subseteq \mathcal{O}_\omega \cdot \mathcal{O}_\nu^h$ is integral and local, since $\mathcal{O}_\omega^h$ is the integral closure of $\mathcal{O}_\nu^h$ in L^h. Therefore by [10], Corollary 43.13 the ring $\mathcal{O}_\omega \cdot \mathcal{O}_\nu^h$ is henselian, which by [10], Theorem 43.5 yields the existence of an injective $\mathcal{O}_\omega$-algebra homomorphism $f : \mathcal{O}_\omega^h \longrightarrow \mathcal{O}_\omega \cdot \mathcal{O}_\nu^h$. In particular, $\mathcal{O}_\omega \cdot \mathcal{O}_\nu^h$ is an overring of the valuation ring $f(O_\omega^h)$ within L^h and is thus a valuation ring itself. Since $\mathcal{O}_\omega \cdot \mathcal{O}_\nu^h \subseteq \mathcal{O}_\omega^h$ is integral the assertion follows. $\square$

Proof of Theorem 4.1. By assumption there exists a finitely generated $\mathcal{O}_\nu$-algebra $A \subseteq \mathcal{O}_\omega$ and a multiplicative subset $S \subseteq A$ such that $\mathcal{O}_\omega = S^{-1}A$. The prime $\mathfrak{q} = \mathfrak{m}_\omega \cap A$ has the property $\mathfrak{q} \cap S = \emptyset$, hence $A_\mathfrak{q} = \mathcal{O}_\omega$. The ring compositum $A \cdot \mathcal{O}_\nu^h$ formed within L^h is local, because it lies between $\mathcal{O}_\nu^h$ and $\mathcal{O}_\omega^h$. Its maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m} \cap A = \mathfrak{q}$, because on the one hand $\mathfrak{m} = \mathfrak{m}_\omega \cap (A \cdot \mathcal{O}_\nu^h)$ and on the other hand $\mathfrak{m}_\omega^h \cap A = \mathfrak{q}$. Consequently $A_\mathfrak{q} \subseteq A \cdot \mathcal{O}_\nu^h$, which by assumption and lemma 4.4 implies $A \cdot \mathcal{O}_\nu^h = \mathcal{O}_\omega^h$. In particular, by Proposition 2.6, $\mathcal{O}_\nu^h \subseteq \mathcal{O}_\omega^h$ is a finite ring extension, which by Lemma 4.3 implies $e(\omega^h|\nu^h) = \epsilon(\omega^h|\nu^h)$ and $d(\omega^h|\nu^h) = 1$. Since the henselization is an immediate extension, we have that $e(\omega^h|\nu^h) = e(\omega|\nu)$ and $\epsilon(\omega^h|\nu^h) = \epsilon(\omega|\nu)$. Moreover by definition $d(\omega^h|\nu^h) = d(\omega|\nu)$. $\square$

5. Reduction of Conjecture 1.2 to the separable case

Lemma 5.1. *Let $K \subseteq F \subseteq L$ be finite extensions of fields and let ω be a valuation on L (whose restrictions to F and K we will denote by μ and ν, respectively). If $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\mu$ and $\mathcal{O}_\mu$ is essentially finitely generated over $\mathcal{O}_\nu$, then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.*

Proof. First we observe that if for some $R \subseteq \mathcal{O}_\omega$ there exists a prime ideal $\mathfrak{q}$ of R such that $\mathcal{O}_\omega = R_\mathfrak{q}$, then $\mathcal{O}_\omega = R_\omega$. Indeed, since $R_\mathfrak{q} = \mathcal{O}_\omega$ we have that $R \setminus \mathfrak{q} \subseteq R \setminus \mathfrak{m}_\omega$. Therefore $\mathcal{O}_\omega = R_\omega$.

Assume now that $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\mu$ and that $\mathcal{O}_\mu$ is essentially finitely generated over $\mathcal{O}_\nu$. Then there exist $x_1, \ldots, x_r \in \mathcal{O}_\omega$ and $y_1, \ldots, y_s \in \mathcal{O}_\mu$ such that

$$\mathcal{O}_\omega = \mathcal{O}_\mu[x_1, \ldots, x_r]_\omega \text{ and } \mathcal{O}_\mu = \mathcal{O}_\nu[y_1, \ldots, y_s]_\mu.$$

Since $\mathcal{O}_\nu[y_1, \ldots, y_s] \cap \mathfrak{m}_\mu = \mathcal{O}_\nu[y_1, \ldots, y_s] \cap \mathfrak{m}_\omega$ this implies that

$$\mathcal{O}_\omega = \mathcal{O}_\nu[x_1, \ldots, x_r, y_1, \ldots, y_s]_\omega.$$

Therefore, $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$. $\qquad\square$

Proposition 5.2. *Assume that we are in situation (1). Let L' be a subfield of L containing K and set $\omega' = \omega|_{L'}$. Assume that ω is the unique extension of ω' to L. If $\mathcal{O}_{\omega'}$ is essentially finitely generated over $\mathcal{O}_\nu$, $d(\omega|\omega') = 1$ and $\epsilon(\omega|\omega') = e(\omega|\omega')$, then $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$.*

Proof. Since ω is the only extension of ω' to L, $d(\omega|\omega') = 1$ and $\epsilon(\omega|\omega') = e(\omega|\omega')$, Corollary 2.2 guarantees that the integral closure D' of $\mathcal{O}_{\omega'}$ in L is a finite $\mathcal{O}_{\omega'}$-module. Since $O_\omega = D'_\omega$, this implies that $\mathcal{O}_\omega$ is finitely generated over $\mathcal{O}_{\omega'}$. By our assumption and Lemma 5.1 we conclude that $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$. $\qquad\square$

Corollary 5.3. *If Conjecture 1.3 is true, then Conjecture 1.2 is also true.*

Proof. Assume now that we are in situation (2) and set L' to be the separable closure of K in L and $\omega' = \omega|_{L'}$. Since $L|L'$ is purely inseparable, ω is the unique extension of ω' to L.

Since $(L|K, \nu)$ is defectless, so are $(L|L', \omega')$ and $(L'|K, \nu)$. Since ϵ is multiplicative (Lemma 3.7) and $\epsilon(\omega|\nu) = e(\omega|\nu)$ then also $\epsilon(\omega|\omega') = e(\omega|\omega')$ and $\epsilon(\omega'|\nu) = e(\omega'|\nu)$. If Conjecture 1.3 is true, then we are in the situation of Proposition 5.2, and therefore $\mathcal{O}_\omega$ is essentially finitely generated over $\mathcal{O}_\nu$. $\qquad\square$

6. Proof of Theorem 1.4

We first review some notation that will be used in the proof. If R is a local ring, we will denote its maximal ideal by $\mathfrak{m}_R$. The maximal ideal of a valuation ring $\mathcal{O}_\nu$ will be denoted by $\mathfrak{m}_\nu$. If a local domain R is a subring of a local domain S and $\mathfrak{m}_S \cap R = \mathfrak{m}_R$ we will say that S dominates R and that $R \to S$ is dominating. We will say that ν dominates R if $\mathcal{O}_\nu$ dominates R. Recall that if R is a subring of a valuation ring $\mathcal{O}_\nu$, then R_ν is the localization $R_\nu = R_{\mathfrak{m}_\nu \cap R}$.

If a local domain S dominates a local domain R, S is essentially of finite type over R and R and S have the same quotient field, then $R \to S$ will be called a birational extension. If R is a regular local ring, then a quadratic transform of R is a dominating map $R \to S$ such that S is a local ring of the blowup of the maximal ideal of R. A quadratic transform $R \to S$ is a quadratic transform along a valuation ν if ν dominates S. An iterated quadratic transform is a finite sequence of quadratic transforms.

We now present the proof of Theorem 1.4.

After possibly replacing R with a birational extension of R along ν, we may assume that R is normal and $\mathcal{O}_\nu/\mathfrak{m}_\nu$ is algebraic over $R/\mathfrak{m}_R$. If $\dim R = 1$, then R is a valuation ring, so $R = \mathcal{O}_\nu$. Since R is excellent, the integral closure of $\mathcal{O}_\nu$ in L is a finite R-module. Thus $\mathcal{O}_\omega$ is the localization of a finite $\mathcal{O}_\nu$-module.

From now on, assume that $\dim R = 2$.

Let S be the local ring of the integral closure of R in L which is dominated by ω. The ring S is essentially of finite type over R since R is excellent.

First assume that the rank $\mathrm{rk}(\nu) > 1$. By Abhyankar's inequality [1, Theorem 1], this implies that $\mathrm{rk}(\nu) = 2$ and

$$\Gamma_\omega \simeq (\mathbb{Z}^2)_{\mathrm{lex}}.$$

By Theorem 3.7 of [3], our assumption that $\mathrm{rk}(\nu) = 2$ implies the assumption of our theorem that $d(\omega|\nu) = 1$, and that there exists a diagram

$$\begin{array}{ccc} R_0 & \longrightarrow & S_0 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that R_0 and S_0 are two-dimensional regular local rings, ω dominates S_0, all the arrows are dominating and the vertical arrows are birational extensions (i.e., localizations of blow-ups at ideals) and regular parameter $x_1(0), x_2(0)$ in R_0 and $y_1(0), y_2(0)$ in S_0 such that there exist units $\alpha, \beta \in S_0$ and $a, b, c, d \in \mathbb{N}$ such that $|ad - bc| = e$ and

$$x_1(0) = \alpha \cdot y_1(0)^a y_2(0)^b \text{ and } x_2(0) = \beta \cdot y_1(0)^c y_2(0)^d.$$

Further, the forms of these equations are stable under further quadratic transforms of ω along ν. We have that S_0 is a local ring of a finitely generated R_0-algebra since S is a localization of a finitely generated R-algebra and $S \longrightarrow S_0$ is a birational extension. That is, there exists $z_1, \ldots, z_r \in S_0$ such that $S_0 = T_\omega$ where $T = R_0[z_1, \ldots, z_r]$.

Since Γ_ν has finite index in Γ_ω, there exist $u, v \in K$ such that $\nu(u) > m\nu(v) > 0$ for all positive integers m. Replacing R_0 and S_0 with suitable iterated quadratic transforms along ν and ω respectively, we may assume that $u, v \in R_0$ and

$$u = \phi x_1(0)^{s_1} x_2(0)^{s_2}, \ v = \psi x_1(0)^{t_1} x_2(0)^{t_2}$$

for some $s_1, s_2, t_1, t_2 \in \mathbb{N}$ and units $\phi, \psi \in R_0$. After possibly interchanging $x_1(0)$ and $x_2(0)$, we then have that $\nu(x_1(0)) > m\nu(x_2(0)) > 0$ for all positive integers m. Similarly, we can assume that $\omega(y_1(0)) > m\omega(y_2(0)) > 0$ for all positive integers m.

Since $\epsilon = e$, there exists $a \in \mathbb{Z}$ such that $(1, a) \in \Gamma_\nu$ (by Corollary 3.5). Thus, after possibly performing some further iterated quadratic transforms along ν and ω, we can assume that there exists $f \in R_0$ such that $\omega(f) = (1, a)$ for some $a \in \mathbb{Z}$ and that $f = \gamma \cdot x_1(0)^m x_2(0)^n$ for some $m, n \in \mathbb{N}$, where $\gamma \in R_0$ is a unit. Then $m = 1$ and $\nu(x_1(0)) = (1, l)$ for some $l \in \mathbb{Z}$. Consequently, $\omega(y_1(0)) = (1, m)$ for some $m \in \mathbb{Z}$ and $a = 1$. Further, $c = 0$. So

$$x_1(0) = \alpha \cdot y_1(0) y_2(0)^b \text{ and } x_2(0) = \beta \cdot y_2(0)^e.$$

Let $S_0 \longrightarrow S_1$ be the iterated quadratic transform of S_0 along ω with

$$S_1 = S_0[y_1(1)]_\omega$$

where

$$y_1(0) = y_1(1)y_2(0)^r \text{ and } y_2(0) = y_2(1)$$

and $r \in \mathbb{Z}_+$ is chosen so that $b + r = se$ for some $s \in \mathbb{Z}_+$. Let

$$R_0 \longrightarrow R_1 = R_0[x_1(1)]_\nu$$

be the iterated quadratic transform along ν defined by

$$x_1(0) = x_1(1)x_2(0)^s \text{ and } x_2(0) = x_2(1).$$

We have that

$$x_1(1) = \frac{x_1(0)}{x_2(0)^s} = \alpha \cdot \beta^{-s} y_1(1) \text{ and } x_2(1) = \beta \cdot y_2(1)^e.$$

In particular, S_1 dominates R_1. Further, S_1 is a localization of $R_1[z_1, \ldots, z_r]$.

Iterating this construction, we produce an infinite commutative diagram of regular local rings dominated by ω

$$
\begin{array}{ccc}
\vdots & & \vdots \\
R_n & \longrightarrow & S_n \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
R_1 & \longrightarrow & S_1 \\
\uparrow & & \uparrow \\
R_0 & \longrightarrow & S_0
\end{array}
$$

such that S_i is a localization of $R_i[z_1, \ldots, z_r]$ for every $i \in \mathbb{N}$ and the vertical arrows are iterated quadratic transforms. Since R and S are two-dimensional local domains,

$$\mathcal{O}_\nu = \bigcup_{i=0}^\infty R_i \text{ and } \mathcal{O}_\omega = \bigcup_{i=0}^\infty S_i$$

by [1, Lemma 12]. Therefore, $\mathcal{O}_\omega$ is a localization of $\mathcal{O}_\nu[z_1, \ldots, z_r]$.

Now we assume that $\mathrm{rk}(\nu) = 1$. Since $d(\omega|\nu) = 1$, by Theorem 3.3 and Theorem 3.7 of [3], there exists a commutative diagram

$$
\begin{array}{ccc}
R_0 & \longrightarrow & S_0 \\
\uparrow & & \uparrow \\
R & \longrightarrow & S
\end{array}
$$

such that R_0 and S_0 are two-dimensional regular local rings, ω dominates S_0, all the arrows are dominating and the vertical arrows are birational extensions and

there exist regular parameters $x_1(0), x_2(0)$ in R_0 and $y_1(0), y_2(0)$ in S_0 such that, if the rational rank $\operatorname{ratrk}(\nu) = 1$, then there exists a unit $\alpha \in S_0$ such that

(6) $x_1(0) = \alpha \cdot y_1(0)^e$ and $x_2(0) = y_2(0),$

and if $\operatorname{ratrk}(\nu) = 2$, then there exist units γ and τ in S_0 such that

(7) $x_1(0) = \gamma y_1(0)^a y_2(0)^b, \ x_2(0) = \tau y_1(0)^c y_2(0)^d$

where $|ad - bc| = e$. In both cases, we have that $\mathcal{O}_\omega/\mathfrak{m}_\omega$ is the join of $(\mathcal{O}_\nu/\mathfrak{m}_\nu)$ and $S_0/\mathfrak{m}_{S_0}$ and

(8) $f(\omega|\nu) = [S_0/\mathfrak{m}_{S_0} : R_0/\mathfrak{m}_{R_0}] = [\mathcal{O}_\omega/\mathfrak{m}_\omega : \mathcal{O}_\nu/\mathfrak{m}_\nu].$

Further, these equations are stable under suitable iterated quadratic transforms along ν and ω.

Also, S_0 is a localization of a finitely generated R-algebra, since $S \longrightarrow S_0$ is a birational extension. Thus there exist $z_1, \ldots, z_r \in S_0$ such that $S_0 = T_\omega$ where $T = R_0[z_1, \ldots, z_r]$.

We will treat the cases $e = 1$ and $e > 1$ separately. Assume first that $e = 1$. If $\operatorname{ratrk}(\nu) = 1$, then replacing $y_1(0)$ with $\alpha \cdot y_1(0)$, we have that

(9) $x_1(0) = y_1(0)$ and $x_2(0) = y_2(0).$

If $\operatorname{ratrk}(\nu) = 2$, then from $|ad - bc| = 1$, we have that there exists an iterated quadratic transform $R_0 \to R'$ along ν such that S_0 dominates R' and R' has regular parameters x_1', x_2' such that $x_1' = \gamma' y_1(0)$ and $x_2' = \tau' y_2(0)$ where $\gamma', \tau' \in S_0$ are units. After replacing R_0 with R', we then have that (9) holds.

After possibly interchanging $x_1(0)$ and $x_2(0)$ (and $y_1(0)$ and $y_2(0)$) we can assume that $\nu(x_2(0)) \geq \nu(x_1(0))$. Then the quadratic transform of R_0 along ν is

$$R_0 \longrightarrow R_1 = R_0 \left[\frac{x_2(0)}{x_1(0)} \right]_\nu.$$

Let $x_1(1) = x_1(0)$. Then

$$R_1/(x_1(1)) \simeq R_0/\mathfrak{m}_{R_0} \left[\frac{x_2(0)}{x_1(0)} \right]_\mathfrak{n},$$

where $\mathfrak{n}$ is a suitable maximal ideal and $\frac{x_2(0)}{x_1(0)}$ is transcendental over $R_0/\mathfrak{m}_{R_0}$. Thus, the maximal ideal of $R_1/(x_1(0))$ is generated by an element

$$\overline{f} = \left(\frac{x_2(0)}{x_1(0)} \right)^d + \overline{a_1} \left(\frac{x_2(0)}{x_1(0)} \right)^{d-1} + \ldots + \overline{a_d}$$

for some d and $\overline{a_i} \in R_0/\mathfrak{m}_{R_0}$. Then $R_1/\mathfrak{m}_{R_1} = R_0/\mathfrak{m}_{R_0}[z]$ where z is the class of $\frac{x_2(0)}{x_1(0)}$ in $R_1/\mathfrak{m}_{R_1} \subseteq \mathcal{O}_\nu/\mathfrak{m}_\nu \subseteq \mathcal{O}_\omega/\mathfrak{m}_\omega$ and $\overline{f}$ is the minimal polynomial of z over $R_0/\mathfrak{m}_{R_0}$. We have that $d = [R_1/\mathfrak{m}_{R_1} : R_0/\mathfrak{m}_{R_0}]$. The quadratic transform of S_0 along ω is

$$S_0 \longrightarrow S_1 = S_0 \left[\frac{y_2(0)}{y_1(0)} \right]_\omega, \quad \text{where } \frac{y_2(0)}{y_1(0)} = \frac{x_2(0)}{x_1(0)}.$$

Let $y_1(1) = y_1(0)$. Now, as in the inclusion of $R_0 \longrightarrow R_1$ we have

$$S_1/\mathfrak{m}_{S_1} \simeq S_0/\mathfrak{m}_{S_0}[z]$$

and the minimal polynomial of z in $S_0/\mathfrak{m}_{S_0} \left[\frac{x_2(0)}{x_1(0)} \right]$ divides the minimal polynomial $\overline{f}$ of z in $R_0/\mathfrak{m}_{S_0} \left[\frac{x_2(0)}{x_1(0)} \right]$. But by (8), we have

$$[S_1/\mathfrak{m}_{S_1} : R_1/\mathfrak{m}_{R_1}] = [S_0/\mathfrak{m}_{S_0} : R_0/\mathfrak{m}_{R_0}] = f(\omega|\nu),$$

so from

$$\begin{aligned}
[S_1/\mathfrak{m}_{S_1} : R_0/\mathfrak{m}_{R_0}] &= [S_1/\mathfrak{m}_{S_1} : S_0/\mathfrak{m}_{S_0}][S_0/\mathfrak{m}_{S_0} : R_0/\mathfrak{m}_{R_0}] \\
&= [S_1/\mathfrak{m}_{S_1} : R_1/\mathfrak{m}_{R_1}][R_1/\mathfrak{m}_{R_1} : R_0/\mathfrak{m}_{R_0}]
\end{aligned}$$

we have that $[S_1/\mathfrak{m}_{S_1} : S_0/\mathfrak{m}_{S_0}] = d$ and so $\overline{f}$ is the minimal polynomial of z over $S_0/\mathfrak{m}_{S_0}$. Now letting $x_2(1) = f$ where f is a lifting of $\overline{f}$ to R_1, we have that $x_1(1), x_2(1)$ are regular parameters in R_1 and in S_1, giving an expresion like in (9) in $R_1 \longrightarrow S_1$.

Further, S_1 is a localization of $R_1[z_1, \ldots, z_r]$. Iterating this construction, we have as in the case $\mathrm{rk}(\nu) = 2$ that $\mathcal{O}_\nu = \cup_{i=0}^\infty R_i$ and $\mathcal{O}_\omega = \cup_{i=0}^\infty S_i$ are unions of iterated quadratic transforms of R_0 and S_0 along ν and ω respectively, and S_i is a localization of $R_i[z_1, \ldots, z_r]$ for all i. Thus $\mathcal{O}_\omega$ is a localization of $\mathcal{O}_\nu[z_1, \ldots, z_r]$.

It remains to prove our theorem in the case when $\mathrm{rk}(\nu) = 1$ and $\epsilon = e > 1$. We then have that ν is discrete of rank one by [4, (18.4) b)]. In this case we have

$$\Gamma_\nu = e\mathbb{Z} \subseteq \mathbb{Z} = \Gamma_\omega.$$

Then, there exists $g \in K$ such that $\nu(g) = e$ and we may assume (after possibly performing iterated quadratic transforms of R_0 and S_0 along ν and ω) that $g \in R_0$ amd $g = \gamma \cdot x_1(0)^a$ for some unit $\gamma \in R_0$ and some $a \in \mathbb{Z}_{>0}$. Thus $a = 1$ and $\nu(x_1(0)) = e$, and so $\omega(y_1(0)) = 1$. Now $e \mid \omega(x_2(0)) = \omega(y_2(0))$, so $\omega(x_2(0)) = \omega(y_2(0)) = es$ for some $s \in \mathbb{Z}_{>0}$. Thus

$$\omega\left(\frac{y_2(0)}{y_1(0)}\right) = \omega\left(\frac{x_2(0)}{x_1(0)}\right) \geq 0.$$

Let $R_1 = R_0 \left[\frac{x_2(0)}{x_1(0)} \right]_\nu$ and $S_1 = S_0 \left[\frac{y_2(0)}{y_1(0)} \right]_\omega$. Now $R_0 \longrightarrow R_1$ is a quadratic transform along ν and $S_0 \longrightarrow S_1$ is a quadratic transform along ω. As in the previous case, S_1 dominates R_1 and there exists $x_2(1) \in R_1$ such that $x_1(1) = x_1(0)$ and $x_2(1)$ are a regular system of parameters in R_1 such that $y_1(1) = y_1(0)$ and $y_2(1) = x_2(1)$ are regular system of parameters in S_1 satisfying $x_1(1) = \alpha \cdot y_1(1)^e$ and $x_2(1) = y_2(1)$ by (6). In particular, S_1 is a localization of $R_1[z_1, \ldots, z_r]$.

Iterating this construction, as in the cases of $\mathrm{rk}(\nu) = 2$ and $\mathrm{rk}(\nu) = 1$ with $e = 1$, we have that $\mathcal{O}_\nu = \cup_{i=0}^\infty R_i$ and $\mathcal{O}_\omega = \cup_{i=0}^\infty S_i$ are unions of iterated quadratic transforms of R_0 and S_0 along ν and ω respectively., and S_i is a localization of $R_i[z_1, \ldots, z_r]$ for all i. Thus $\mathcal{O}_\omega$ is a localization of $\mathcal{O}_\nu[z_1, \ldots, z_r]$.

7. Proof of Theorem 1.5

Proposition 7.1. *Suppose that (K, ν) is a valued field, L is a finite extension field of K and ω is an extension of ν to L such that*

$$1 < \epsilon(\omega|\nu) = e(\omega|\nu).$$

Let $\Gamma_{\nu,1}$ be the first convex subgroup of Γ_ν and $\Gamma_{\omega,1}$ be the first convex subgroup of Γ_ω. Then $\Gamma_{\omega,1} \cong \mathbb{Z}$ and in the short exact sequence of groups

(10) $$0 \to \Gamma_{\omega,1}/\Gamma_{\nu,1} \to \Gamma_\omega/\Gamma_\nu \to (\Gamma_\omega/\Gamma_{\omega,1})/(\Gamma_\nu/\Gamma_{\nu,1}) \to 0$$

we have that

$$(\Gamma_\omega/\Gamma_{\omega,1})/(\Gamma_\nu/\Gamma_{\nu,1}) = 0$$

and

$$\Gamma_\omega/\Gamma_\nu \cong \Gamma_{\omega,1}/\Gamma_{\nu,1} \cong \mathbb{Z}_e.$$

Proof. We have that $\Gamma_{\omega,1} \cong \mathbb{Z}$ by Lemma 3.1. The proposition now follows from Proposition 3.2, Remark 3.4 and (10). $\qquad\square$

Suppose that K is an algebraic function field over a field k. An algebraic local ring of K is a local domain R such that R is essentially of finite type over k and the quotient field of R is K.

Theorem 7.2. *([7, Theorem 1.1]) Let K be an algebraic function field over a field k, and let ν be an Abhyankar valuation on K. Suppose that $\mathcal{O}_\nu/\mathfrak{m}_\nu$ is separable over k and $Z \subset \mathcal{O}_\nu$ is a finite set. Then there exists an algebraic regular local ring R of K such that ν dominates R and $\dim R = \mathrm{ratrk}(\nu)$. Further, there exists a regular system of parameters in R such that each element of Z is a monomial in the regular system of parameters times a unit in R.*

7.1. **Abhyankar valuations on algebraic function fields.** In this subsection, suppose that K is an algebraic function field over a field k and ν is an Abhyankar valuation on K. Then Γ_ν is a finitely generated (torsion free) abelian group by [1, Lemma 1]. Let $n = \mathrm{ratrk}(\nu)$, and

$$0 = \Gamma_{\nu,0} \subset \Gamma_{\nu,1} \subset \cdots \subset \Gamma_{\nu,r} = \Gamma_\nu$$

be the chain of convex subgroups of Γ_ν. Each quotient $\Gamma_{\nu_i}/\Gamma_{\nu,i+1}$ is a torsion free abelian group.

Proposition 7.3. *In the conclusions of Theorem 7.2, we can choose R so that R has a regular system of parameters $\{z_{i,j}\}$ such that for each fixed i with $1 \leq i \leq r$, we have that $\{\nu(z_{i,j})\}$ is a $\mathbb{Z}$-basis of $\Gamma_{\nu,i}/\Gamma_{\nu,i-1}$.*

Proof. There exists a $\mathbb{Z}$-basis $\{\gamma_{i,j}\}$ of Γ_ν such that $\gamma_{i,j} \in (\Gamma_\nu)_{\geq 0}$ for all i, j and for each fixed i with $1 \leq i \leq r$, the images of $\gamma_{i,j}$ in $\Gamma_{\nu,i}/\Gamma_{\nu,i-1}$ form a $\mathbb{Z}$-basis of $\Gamma_{\nu,i}/\Gamma_{\nu,i-1}$.

There exist $f_{i,j} \in \mathcal{O}_\nu$ such that $\nu(f_{i,j}) = \gamma_{i,j}$ for all i, j. Reindex the $f_{i,j}$ as $f_1, \ldots, f_n$ so that the $\nu(f_i)$ are increasing. By Theorem 7.2, there exists an

algebraic regular local ring R of K such that $\dim R = n$ and there exists a regular system of parameters $z_1, \ldots, z_n$ in R and units $\lambda_i \in R$ such that

$$f_i = \lambda_i z_1^{a_1,i} \cdots z_n^{a_n,i} \text{ for } 1 \leq i \leq n.$$

Since $\nu(f_1), \ldots, \nu(f_n)$ is a $\mathbb{Z}$-basis of Γ_ν, we have that $\nu(z_1), \ldots, \nu(z_n)$ is a $\mathbb{Z}$-basis of Γ_ν. Further, we can reindex the z_i as $z_{i,j}$ so that for all fixed i with $1 \leq i \leq r$, $\{\nu(z_{i,j})\}$ is a $\mathbb{Z}$-basis of $\Gamma_{\nu,i}/\Gamma_{\nu,i-1}$. $\qquad\square$

Suppose that R satisfies the conclusions of Proposition 7.3. Index the regular system of parameters $z_{i,j}$ as $x_1, \ldots, x_n$ so that $\nu(x_i) < \nu(x_j)$ if $i < j$.

We define a primitive monoidal transform (PMT) $R \to R_1$ along ν by $R_1 = R\left[\frac{x_j}{x_i}\right]_\nu$. We have that R_1 is a regular local ring with regular parameters $x_1(1), \ldots, x_n(1)$ defined by

$$x_k = x_k(1) \text{ if } k \neq j \text{ and } x_j = x_j(1)x_i(1).$$

Further, $\{\nu(x_k(1)) \mid 1 \leq k \leq n\}$ is a $\mathbb{Z}$-basis of Γ_ν, which satisfies the conclusions of Proposition 7.3 in R_1.

We will find the following proposition useful.

Proposition 7.4. *Suppose that R satisfies the conclusions of Proposition 7.3, with regular parameters $x_1, \ldots, x_n$ and $M_1 = x_1^{a_1} \cdots x_n^{a_n}$, $M_2 = x_1^{b_1} \cdots x_n^{b_n}$ are monomials such that $\nu(M_1) < \nu(M_2)$. Then there exists a sequence of PMTs along ν*

$$R \to R_1 \to \cdots \to R_s$$

such that M_1 divides M_s in R_s.

Proof. Consider the indexing $z_{i,j}$ of the regular parameters x_i of the statement of Proposition 7.3. Write

$$M_1 = \prod_{i,j} z_{i,j}^{a_{i,j}}, \ M_2 = \prod_{i,j} z_{i,j}^{b_{i,j}}.$$

There exists a largest index l such that $\prod_j z_{l,j}^{a_{l,j}} \neq \prod_j z_{l,j}^{b_{l,j}}$. Then $\nu(\prod_j z_{l,j}^{a_{l,j}}) < \nu(\prod_j z_{l,j}^{b_{l,j}})$. By [15, Theorem 2], there exists a sequence of PMTs $R \to R_s$ along ν in the variables $z_{l,j}(m)$ from the regular parameters of R_m as j varies, such that $\prod_j z_{l,j}^{a_{l,j}}$ divides $\prod_j z_{l,j}^{b_{l,j}}$ in R_s. Writing M_1 and M_2 in the regular parameters $z_{i,j}(s)$ of R_s as

$$M_1 = \prod z_{i,j}(s)^{a_{i,j}(s)} \text{ and } M_2 = \prod z_{i,j}(s)^{b_{i,j}(s)},$$

we have that

$$M_2 = \left(\prod_{i<l,j} z_{i,j}(s)^{b_{i,j}(s)}\right)\left(\prod_j z_{l,j}(s)^{b_{l,j}(s)-a_{l,j}(s)}\right)\left(\prod_{i>l,j} z_{l,j}(s)^{a_{i,j}(s)}\right)$$

with $b_{l,j}(s) - a_{l,j}(s) \geq 0$ for all j and for some j, $b_{l,j}(s) - a_{l,j}(s) > 0$. Without loss of generality, this occurs for $j = 1$.

Now perform a sequence of PMTs $R_s \to R_m$ along ν defined by $z_{l,1}(t+1) = z_{l,1}(t)z_{i,j}(t)$ for $i < l$ and j such that $b_{i,j}(t) > a_{i,j}(t)$ where

$$M_1 = \prod z_{i,j}(t)^{a_{i,j}(t)} \text{ and } M_2 = \prod z_{i,j}(t)^{b_{i,j}(t)}$$

to achieve that M_1 divides M_2 in R_m. $\qquad\qquad\qquad\qquad\qquad\square$

7.2. Abhyankar valuations in finite extensions.
We continue the notation of the previous section, and further suppose that L is a finite extension field of K and ω is an extension of ν to L, such that $\mathcal{O}_\omega/\mathfrak{m}_\omega$ is separable over k. We have that ω is also an Abhyankar valuation and $d(\omega|\nu) = 1$ by [9, Theorem 1].

We suppose that $\epsilon(\omega|\nu) = e(\omega|\nu)$.

Proposition 7.5. *There exist algebraic regular local rings R of K and S of L which are dominated by ω and ν respectively such that S dominates R and R has regular parameters $x_1, \ldots, x_n$ and S has regular parameters $y_1, \ldots, y_n$ satisfying the conclusions of Proposition 7.3 such that there is an expression*

$$x_1 = \gamma y_1^e \text{ and } x_i = y_i \text{ for } 2 \le i \le n$$

where γ is a unit in S.

Proof. By Theorem 7.2 and Proposition 7.3 there exist algebraic regular local rings R_0 of K and S_0 of L such that ω dominates S_0, S_0 dominates R_0, R_0 has regular parameters $x_1, \ldots, x_n$ and S_0 has regular parameters $y_1, \ldots, y_n$ satisfying the conclusions of Proposition 7.3 and there exist units $\gamma_i \in S$ such that

$$(11) \qquad\qquad x_i = \gamma_i y_1^{a_{i,1}} \cdots y_n^{a_{i,n}}$$

for $1 \le i \le n$. Since $\nu(x_1), \ldots, \nu(x_n)$ is a $\mathbb{Z}$-basis of Γ_ν and $\omega(y_1), \ldots, \omega(y_n)$ is a $\mathbb{Z}$-basis of Γ_ω, we have that

$$e = |\mathrm{Det}(A)|$$

where $A = (a_{ij})$ is the $n \times n$ matrix of exponents in (11).

First suppose that $\epsilon(\omega|\nu) = e(\omega|\nu) > 1$. Then $\Gamma_{\omega,1} \cong \mathbb{Z}$ and $\Gamma_{\omega,1}/\Gamma_{\nu,1} \cong \mathbb{Z}_e$ by Proposition 7.1. Thus $\nu(x_1)$ is a $\mathbb{Z}$-basis of $\Gamma_{\nu,1}$ and $\nu(y_1)$ is a $\mathbb{Z}$-basis of $\Gamma_{\omega,1}$. We thus have that $a_{1,j} = 0$ for $j > 1$ and $a_{1,1}$ is a positive multiple of e. Thus from $|\mathrm{Det}(A)| = e$ we have that $a_{1,1} = e$ and $|\mathrm{Det}(\overline{A})| = 1$ where

$$\overline{A} = \begin{pmatrix} a_{2,2} & \cdots & a_{2,n} \\ & \vdots & \\ a_{n,2} & \cdots & a_{n,n} \end{pmatrix}.$$

Since $\mathrm{Det}(\overline{A}) = \pm 1$, there exist $r_2, \ldots, r_n \in \mathbb{Z}$ such that

$$\overline{A} \begin{pmatrix} r_2 \\ \vdots \\ r_n \end{pmatrix} = - \begin{pmatrix} a_{2,1} \\ \vdots \\ a_{n,1} \end{pmatrix}.$$

Define $s_2, \ldots, s_n \in \mathbb{Z}_{\geq 0}$ by

$$\begin{pmatrix} s_2 \\ \vdots \\ s_n \end{pmatrix} = \overline{A} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

There exists $r \in \mathbb{Z}_{>0}$ such that $r_i + te > 0$ for all i. Perform the sequence of PMTs $S_0 \to S_1$ along ω defined by

$$y_1 = y_1(1), y_i = y_i(1)y_1(1)^{r_i+te} \text{ for } 2 \leq i \leq n.$$

We have that $S_1 = S_0[y_1(1), \ldots, y_n(1)]_\omega$ is a regular local ring with regular parameters $y_1(1), \ldots, y_n(1)$ which dominates R_0 and there exist units $\gamma_i' \in S_1$ such that

$$x_1 = \gamma_1' y_1(1)^e \text{ and } x_i = \gamma_i' y_1(1)^{ets_i} y_2(1)^{a_{2,1}} \cdots y_n(1)^{a_{i,n}} \text{ for } 2 \leq i \leq n.$$

Now perform the sequence of PMTs $R_0 \to R_1$ along ν defined by

$$x_1 = x_1(1), x_i = x_1(1)^{ts_i} x_i(1) \text{ for } 2 \leq i \leq n.$$

We have that S_1 dominates R_1 and there exist units $\gamma_i(1) \in S_1$ such that

(12) $\qquad x_1(1) = \gamma_1(1)y_1(1)^e, x_i(1) = \gamma_i(1)y_2^{a_{i,2}} \cdots y_n(1)^{a_{i,n}} \text{ for } 2 \leq i \leq n.$

We continue to have $\mathrm{Det}(\overline{A}) = \pm 1$. Let $B = \overline{A}^{-1}$, Write

$$B = \begin{pmatrix} b_{2,2} & \cdots & b_{2,n} \\ & \vdots & \\ b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

with $b_{i,j} \in \mathbb{Z}$. We now replace the $y_i(1)$ with the product of the unit $\gamma_2(1)^{-b_{i,2}} \cdots \gamma_n(1)^{-b_{i,n}}$ and $y_i(1)$ for $2 \leq i \leq n$ to get $\gamma_i(1) = 1$ for $2 \leq i \leq n$ in (12).

Now define a birational transformation $R_1 \to R_2$ along ν by $R_2 = R_1[x_1(2), \ldots, x_n(2)]_\nu$ where

$$x_n(1) = x_n(2) \text{ and } x_i(1) = x_2(2)^{a_{i,2}} \cdots x_n(2)^{a_{i,n}} \text{ for } 2 \leq i \leq n.$$

The ring R_2 is a regular local ring with regular parameters $x_1(2), \ldots, x_n(2)$. We have that R_2 is dominated by S_1, and

$$x_1(2) = \gamma y_1(2)^e \text{ and } x_i(2) = y_i(1) \text{ for } 2 \leq i \leq n$$

where γ is a unit in S_1, giving the conclusions of the proposition.

Now suppose that $e = 1$. This case is much simpler. In (11) we then have that $\mathrm{Det}(A) = \pm 1$. Taking $B = A^{-1} = (b_{i,j})$, we can then make the change of variables in S_0 replacing the y_i with the product of the unit $\gamma_1^{-b_{i,1}} \cdots \gamma_n^{-b_{i,n}}$ times y_i for $1 \leq i \leq n$ to get $\gamma_i = 1$ for $1 \leq i \leq n$ in (11).

Now define a birational transformation $R_0 \to R_1$ along ν by $R_1 = R_0[x_1(1), \ldots, x_n(1)]_\nu$ where

$$x_i = x_1(1)^{a_{i,1}} \cdots x_n(1)^{a_{i,n}} \text{ for } 1 \leq i \leq n.$$

The ring R_1 is a regular local ring with regular parameters $x_1(1), \ldots, x_n(1)$. We have that R_1 is dominated by S, and

$$x_i(1) = y_i(1) \text{ for } 1 \le i \le n,$$

giving the conclusions of the proposition. $\qquad\square$

Proposition 7.6. *Suppose that $R \to S$ has the form of the conclusions of Proposition 7.5 and $1 \le i < j \le n$ (with $\nu(x_i) < \nu(x_j)$). Then there exist $z_1, \ldots, z_m \in \mathcal{O}_\omega$ such that $S_0 = R_0[z_1, \ldots, z_m]_\omega$.*

Let $R_0 \to R_1 = R_0[x_1(1), \ldots, x_n(1)]_\nu$ be the PMT along ν defined by

$$x_k = x_k(1) \text{ if } i \ne j \text{ and } x_j = x_j(1)x_i(1).$$

Then there exists a sequence of PMTs along ω, $S_0 \to S_1$ such that S_1 dominates R_1 and S_1 has regular parameters $y_1(1), \ldots, y_n(1)$ such that

$$x_1(1) = \gamma y_1(1)^e \text{ and } x_i(1) = y_i(1) \text{ for } 2 \le i \le n$$

and $S_1 = R_1[z_1, \ldots, z_m]_\omega$.

Proof. The expression $S_0 = R_0[z_1, \ldots, z_m]_\omega$ follows since S_0 is essentially of finite type over k.

If $i \ne 1$ we have that both $i = 1$ and $e = 1$, then S_1 is defined by

$$y_k = y_k(1) \text{ if } i \ne j \text{ and } y_j = y_j(1)y_i(1).$$

If $i = 1$ and $e > 1$, then define $S \to S_1$ by the sequence of PMTs along ω

$$y_k = y_k(1) \text{ if } k \ne j \text{ and } y_j = y_1(1)^e y_j(1).$$

$\qquad\square$

We now prove Theorem 1.5. Let $R_0 \to S_0$ have the form of the conclusions of Proposition 7.5, and write $S_0 = R_0[z_1, \ldots, z_m]_\omega$.

Suppose that $f \in \mathcal{O}_\omega$. Write $f = \frac{g}{h}$ with $g, h \in S_0$. Let κ be a coefficient field of the $\mathfrak{m}_{S_0}$-adic completion $\widehat{S_0}$ of S_0. We then have that $\widehat{S_0}$ is the power series ring $\widehat{S_0} = \kappa[[y_1, \ldots, y_n]]$. Expand $g = \sum \alpha_{i_1,\ldots,i_n} y_1^{i_1} \cdots y_n^{i_n}$ and $h = \sum \beta_{j_1,\ldots,j_n} y_1^{j_1} \cdots y_n^{j_n}$ with $\alpha_{i_1,\ldots,i_n}, \beta_{j_1,\ldots,j_n} \in \kappa$. We have that

$$\omega(g) = \min\{\omega(y_1^{i_1} \cdots y_n^{i_n}) \mid \alpha_{i_1,\ldots,i_n} \ne 0\}$$

and

$$\omega(h) = \min\{\omega(y_1^{j_1} \cdots y_n^{j_n}) \mid \beta_{j_1,\ldots,j_n} \ne 0\}.$$

Let U be the ideal $U = (y_1^{i_1} \cdots y_n^{i_n} \mid \alpha_{i_1,\ldots,i_n} \ne 0)$ in $\widehat{S_0}$ and V be the ideal $V = (y_1^{j_1} \cdots y_n^{j_n} \mid \beta_{j_1,\ldots,j_n} \ne 0)$ in $\widehat{S_0}$. Since $\widehat{S_0}$ is a noetherian ring, there exist monomials $M_1, \ldots, M_s, N_1, \ldots, N_t$ in $y_1, \ldots, y_n$ such that $U = (M_1, \ldots, M_s)$ and $V = (N_1, \ldots, N_t)$. We can further assume that $\omega(M_1) < \omega(M_i)$ for $i > 1$ and $\omega(N_1) < \omega(N_j)$ for $j > 1$. Since $\omega(f) \ge 0$ we have that $\omega(N_1) \le \omega(M_1)$. There exist units α_i and β_i in S_0 such that $\overline{M}_1 = \alpha_1 M_1^e, \ldots, \overline{M}_s = \alpha_s M_s^e, \overline{N}_1 = \beta_1 N_1^e, \ldots, \overline{N}_t = \beta_t N_t^e \in R_0$ are monomials in $x_1, \ldots, x_n$. By Proposition 7.4, there

exists a sequence of PMTs along ν $R_0 \to R_1$ such that $\overline{M}_1$ divides $\overline{M}_i$ in R_1 for all i, $\overline{N}_1$ divides $\overline{N}_j$ for all j in R_1 and $\overline{M}_1$ divides $\overline{N}_1$ in R_1. Let $S_0 \to S_1$ be the sequence of PMTs along ω obtained from applying Proposition 7.6 to the sequence $R_0 \to R_1$, so that $S_1 = R_1[z_1, \ldots, z_m]_\omega$, and R_1 has regular parameters $x_1(1), \ldots, x_n(1)$, S_1 has regular parameters $y_1(1), \ldots, y_n(1)$ such that there is a unit $\gamma \in S_1$ such that

$$x_1(1) = \gamma y_1(1)^e \text{ and } x_i(1) = y_i(1) \text{ for } 2 \leq i \leq n.$$

Now each of $\overline{M}_1, \ldots, \overline{M}_s, \overline{N}_1, \ldots, \overline{N}_t$ is a monomial in $y_1(1), \ldots, y_n(1)$ times a unit in S_1, so M_1^e divides M_i^e for all i in S_1, N_1^e divides N_j^e for all j in S_1 and N_1^e divides M_1^e in S_1. Thus M_1 divides M_i for all i in S_1, N_1 divides N_j for all j in S_1 and N_1 divides M_1 in S_1. Since $S_1/\mathfrak{m}_{S_1} = S_0/\mathfrak{m}_{S_0}$, we have that κ is a coefficient field for S_1, and $\widehat{S_1} = \kappa[[y_1(1), \ldots, y_n(1)]]$. Further, the induced homomorphism $\widehat{S_0} \to \widehat{S_1}$ is the natural κ-algebra homomorphism

$$\kappa[[y_1, \ldots, y_n]] \to \kappa[[y_1(1), \ldots, y_n(1)]]$$

defined by substitution of the y_i by suitable monomials in $y_1(1), \ldots, y_n(1)$. Write $g = u_1 M_1 + u_2 M_2 + \cdots + u_s M_s$ and $h = v_1 N_1 + v_2 N_2 + \cdots + v_t N_t$, with $u_1, \ldots, u_s, v_1, \ldots, v_t \in \widehat{S_0}$ and u_1, v_1 units. Now

$$g_1 = u_1 \frac{M_1}{N_1} + u_2 \frac{M_2}{N_1} + \cdots + u_s \frac{M_s}{N_1} \in \widehat{S_1}$$

and

$$h_1 = v_1 + v_2 \frac{N_2}{N_1} + \cdots + v_t \frac{N_t}{N_1} \in \widehat{S_1}.$$

so $\frac{g}{h} = g_1 h_1^{-1} \in \widehat{S_1}$. Thus by [2, Lemma 2],

$$f = \frac{g}{h} \in \widehat{S_1} \cap L = S_1 = R_1[z_1, \ldots, z_m]_\omega \subset \mathcal{O}_\nu[z_1, \ldots, z_m]_\omega.$$

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