THE DEFECT

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Abstract. We give an introduction to the valuation theoretical phenomenon of "defect", also known as "ramification deficiency". We describe the role it plays in deep open problems in positive characteristic: local uniformization (the local form of resolution of singularities), the model theory of valued fields, the structure theory of valued function fields. We give several examples of algebraic extensions with non-trivial defect. We indicate why Artin-Schreier defect extensions play a central role and describe a way to classify them. Further, we give an overview of various results about the defect that help to tame or avoid it, in particular "stability" theorems and theorems on "henselian rationality", and show how they are applied. Finally, we include a list of open problems.

Contents

1. Valued fields

Historically, there are three main origins of valued fields:

1) Number theory: Kurt Hensel introduced the fields $\mathbb{Q}_p$ of p-adic numbers and proved the famous Hensel's Lemma (see below) for them. They are defined as the

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completions of $\mathbb{Q}$ with respect to the (ultra)metric induced by the p-adic valuations of $\mathbb{Q}$, similarly as the field of reals, $\mathbb{R}$, is the completion of $\mathbb{Q}$ with respect to the usual metric induced by the ordering on $\mathbb{Q}$.

2) Ordered fields: $\mathbb{R}$ is the maximal archimedean ordered field; any ordered field properly containing $\mathbb{R}$ will have infinite elements, that is, elements larger than all reals. Their inverses are infinitesimals. The classes of magnitude, called archimedean classes, give rise to a natural valuation. These valuations are important in the theory of ordered fields and in real algebraic geometry.

In connection with ordered fields and their classes of magnitude, Hans Hahn ([27]) introduced an important class of valued fields, the **(generalized) power series fields**. Take any field K and any ordered abelian group G. Let $K((G))$ (also denoted by $K((t^G))$) be the set of all maps μ from G to K with well-ordered **support** $\{g \in G \mid \mu(g) \neq 0\}$. One can visualize the elements of $K((G))$ as formal power series $\sum_{g\in G} c_g t^g$ for which the support $\{g \in G \mid c_g \neq 0\}$ is well-ordered. Using this condition one shows that $K((G))$ is a field. Also, one uses it to introduce the valuation:

$$v_t \sum_{g\in G} c_g t^g \;=\; \min\{g \in G \mid c_g \neq 0\} \tag{1.1}$$

(the minimum exists because the support is well-ordered). This valuation is called the **canonical valuation** or **t-adic valuation** of $K((G))$, and sometimes called the **minimum support valuation**. Note that $v_t t = 1$. For $G = \mathbb{Z}$, one obtains the field of formal Laurent series $K((t))$.

3) Function fields: if K is any field and X an indeterminate, then the rational function field $K(X)$ has a $p(X)$-adic valuation for every irreducible polynomial $p(X) \in K[X]$, plus the $1/X$-adic valuation. These valuations are trivial on K. As a valuation can be extended to every extension field, these valuations together with the p-adic valuations mentioned in 1) yield that a field admits a non-trivial valuation as soon as it is not algebraic over a finite field. In particular, all algebraic function fields over K (i.e., finitely generated field extensions of K of transcendence degree ≥ 1) admit non-trivial valuations that are trivial on K. Such valued function fields play a role in several areas of algebra and number theory, some of which we will mention in this paper. Throughout, **function field** will always mean *algebraic function field*.

If K is a field with a valuation v, then we will denote its value group by vK and its residue field by Kv. For $a \in K$, its value is va, and its residue is av. An extension of valued fields is written as $(L'|L, v)$, meaning that $L'|L$ is a field extension, v is a valuation on L' and L is equipped with the restriction of this valuation. Then there is a natural embedding of the value group vL in the value group vL', and a natural embedding of the residue field Lv in the residue field $L'v$. If both embeddings are onto (which we just express by writing $vL = vL'$ and $Lv = L'v$), then the extension

$(L'|L,v)$ is called **immediate**. For $a \in L'$ we set $v(a-L) := \{v(a-c) \mid c \in L\}$. The easy proof of the following lemma is left to the reader:

Lemma 1.1. *The extension $(L'|L,v)$ is immediate if and only if for all $a \in L'$ there is $c \in L$ such that $v(a-c) > va$. If the extension $(L'|L,v)$ is immediate, then $v(a-L)$ has no maximal element and is an initial segment of vL, that is, if $\alpha \in v(a-L)$ and $\alpha > \beta \in vL$, then $\beta \in v(a-L)$.*

If for each $a \in L'$ and every $\alpha \in vL'$ there is $c \in L$ such that $v(a-c) > \alpha$, then we say that (L,v) is **dense in** (L',v). If this holds, then the extension $(L'|L,v)$ is immediate. The maximal extension in which (L,v) is dense is its **completion** $(L,v)^c$, which is unique up to isomorphism.

Every finite extension L' of a valued field (L,v) satisfies the **fundamental inequality** (cf. (17.5) of [18] or Theorem 19 on p. 55 of [67]):

$$n \geq \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{1.2}$$

where $n = [L' : L]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from L to L', $\mathrm{e}_i = (v_i L' : vL)$ are the respective **ramification indices** and $\mathrm{f}_i = [L'v_i : Lv]$ are the respective **inertia degrees**. If $\mathrm{g} = 1$ for every finite extension $L'|L$ then (L,v) is called **henselian**. This holds if and only if (L,v) satisfies **Hensel's Lemma**, that is, if f is a polynomial with coefficients in the valuation ring $\mathcal{O}$ of (L,v) and there is $b \in \mathcal{O}$ such that $vf(b) > 0$ and $vf'(b) = 0$, then there is $a \in \mathcal{O}$ such that $f(a) = 0$ and $v(b-a) > 0$.

Every valued field (L,v) admits a **henselization**, that is, a minimal algebraic extension which is henselian (see Section 4 below). All henselizations are isomorphic over L, so we will frequently talk of *the* henselization of (L,v), denoted by $(L,v)^h$. The henselization becomes unique in absolute terms once we fix an extension of the valuation v from L to its algebraic closure. All henselizations are immediate separable-algebraic extensions. If (L',v) is a henselian extension field of (L,v), then a henselization of (L,v) can be found inside of (L',v).

For the basic facts of valuation theory, we refer the reader to [5, Appendix], [18], [19], [59], [65] and [67]. For ramification theory, we recommend [18], [19] and [55]. For basic facts of model theory, see [11].

For a field K, $\tilde{K}$ will denote its algebraic closure and K^{sep} will denote its separable-algebraic closure. If $\operatorname{char} K = p$, then K^{1/p^∞} will denote its perfect hull. If we have two subfields K, L of a field M (in our cases, we will usually have the situation that $L \subset \tilde{K}$) then $K.L$ will denote the smallest subfield of M which contains both K and L; it is called the **field compositum of K and L**.

2. Two problems

Let us look at two important problems that will lead us to considering the phenomenon of defect:

2.1. **Elimination of ramification.** Given a valued function field $(F|K,v)$, we want to find nice generators of F over K. For instance, if $F|K$ is separable then it is separably generated, that is, there is a transcendence basis T such that $F|K(T)$ is a finite separable extension, hence simple. So we can write $F = K(T,a)$ with a separable-algebraic over $K(T)$.

In the presence of the valuation v, we may want to ask for more. The problem of **smooth local uniformization** is to find generators $x_1,\dots,x_n$ of $F|K$ in the valuation ring $\mathcal{O}$ of v on F such that the point $x_1v,\dots,x_nv$ is smooth, that is, the Implicit Function Theorem holds in this point. We say that $(F|K,v)$ is **inertially generated** if there is a transcendence basis T such that F lies in the **absolute inertia field** $K(T)^i$ (see Section 4 for its definition). A connection between both notions is given by Theorem 1.6 of [35]:

Theorem 2.1. *If $(F|K,v)$ admits smooth local uniformization, then it is inertially generated.*

If $(F|K,v)$ is inertially generated by the transcendence basis T, then $vF = vK(T)$, and $Fv|K(T)v$ is separable. If this were not true, we would say that $(F|K(T),v)$ is **ramified**. Let us consider an example.

Example 2.2. Suppose that v is a discrete valuation on F which is trivial on K and such that $Fv|K$ is algebraic. So there is an element $t\in F$ such that $vF = \mathbb{Z}vt = vK(t)$. Take the henselization F^h of F with respect to some fixed extension of v to the algebraic closure of F.

Assume that $\operatorname{trdeg} F|K = 1$; then $F|K(t)$ is finite. Take $K(t)^h$ to be the henselization of $K(t)$ within F^h. Then $F^h|K(t)^h$ is again finite since $F^h = F.K(t)^h$ (cf. Theorem 4.14 below). If $\operatorname{trdeg} F|K > 1$, we can take T to be a transcendence basis of $F|K$ which contains t. Then again, $F^h|K(T)^h$ is finite, and $vF = vK(T)$. But does that prove that $F|K$ is inertially generated? Well, if for instance K is algebraically closed, then it follows that $Fv = K = K(T)v$, so $Fv|K(T)v$ is separable. But "inertially generated" asks for more. In order to show that $F|K$ is inertially generated in this particular case, we would have to find T such that $F\subseteq K(T)^h$, that is, the extension $F^h|K(T)^h$ is trivial (see Section 4).

Since $K(T)^h$ is henselian, there is only one extension of the given valuation from $K(T)^h$ to F^h. By our choice of T, we have $\mathrm{e} = (vF^h : vK(T)^h) = (vF : vK(T)) = 1$, and if K is algebraically closed, also $\mathrm{f} = (F^hv : K(T)^hv) = (Fv : K(T))v = 1$. Hence equality holds in the fundamental inequality (1.2) if and only if $F^h|K(T)^h$ is trivial. $\diamondsuit$

This example shows that it is important to know when the fundamental inequality (1.2) is in fact an equality, or more precisely, what the quotient n/ef is. A first and important answer is given by the **Lemma of Ostrowski**. Assume that $(L'|L,v)$ is a finite extension and the extension of v from L to L' is unique. Then the Lemma of Ostrowski says that

$$[L':L] \;=\; p^{\nu}\cdot(vL':vL)\cdot[L'v:Lv] \quad \text{with } \nu\geq 0 \tag{2.1}$$

where p is the **characteristic exponent** of Lv, that is, $p=\operatorname{char} Lv$ if this is positive, and $p=1$ otherwise. The Lemma of Ostrowski can be proved using Tschirnhausen transformations (cf. [59, Theoreme 2, p. 236]). But it can also be deduced from ramification theory, as we will point out in Section 4 (see also [67, Corollary to Theorem 25, p. 78]).

The factor $\mathrm{d}=\mathrm{d}(L'|L,v)=p^{\nu}$ is called the **defect** (or **ramification deficiency** as in [67, p. 58]) of the extension $(L'|L,v)$. If $\mathrm{d}=1$, then we call $(L'|L,v)$ a **defectless extension**; otherwise, we call it a **defect extension**. Note that $(L'|L,v)$ is always defectless if $\operatorname{char} Kv=0$.

We call (L,v) a **defectless field**, **separably defectless field** or **inseparably defectless field** if equality holds in the fundamental inequality (1.2) for every finite, finite separable or finite purely inseparable, respectively, extension L' of L. One can trace this back to the case of unique extensions of the valuation; for the proof of the following theorem, see [38] (a partial proof was already given in Theorem 18.2 of [18]):

Theorem 2.3. *A valued field (L,v) is a defectless field if and only if its henselization is. The same holds for "separably defectless" and "inseparably defectless".*

Therefore, the Lemma of Ostrowski shows:

Corollary 2.4. *Every valued field (L,v) with* $\operatorname{char} Lv=0$ *is a defectless field.*

The defect is multiplicative in the following sense. Let $(L|K,v)$ and $(M|L,v)$ be finite extensions. Assume that the extension of v from K to M is unique. Then the defect satisfies the following product formula

$$\mathrm{d}(M|K,v)=\mathrm{d}(M|L,v)\cdot\mathrm{d}(L|K,v) \tag{2.2}$$

which is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. This formula implies:

Lemma 2.5. *$(M|K,v)$ is defectless if and only if $(M|L,v)$ and $(L|K,v)$ are defectless.*

Corollary 2.6. *If (L,v) is a defectless field and (L',v) is a finite extension of (L,v), then (L',v) is also a defectless field. Conversely, if there exists a finite extension (L',v) of (L,v) such that (L',v) is a defectless field, the extension of v from L to L' is unique, and the extension $(L'|L,v)$ is defectless, then (L,v) is a defectless field. The same holds for "separably defectless" in the place of "defectless" if $L'|L$ is separable, and for "inseparably defectless" if $L'|L$ is purely inseparable.*

The situation of our Example 2.2 becomes more complicated when the valuations are not discrete:

Example 2.7. There are valued function fields $(F|K,v)$ of transcendence degree 2 with v trivial on K such that vF is not finitely generated. Already on a rational function field $K(x,y)$, the value group of a valuation trivial on K can be any subgroup of the rationals $\mathbb{Q}$ (see Theorem 1.1 of [42] and the references given in that paper). In such cases, if F is not a rational function field, it is not easy to find a transcendence basis T such that $vF = vK(T)$. But even if we find such a T, what do we know then about the extension $(F^h|K(T)^h,v)$? For example, is it defectless? $\diamondsuit$

An extension $(L'|L,v)$ of henselian fields is called **unramified** if $vL' = vL$, $L'v|Lv$ is separable and every finite subextension of $(L'|L,v)$ is defectless. Hence if $\operatorname{char} Lv = 0$, then $(L'|L,v)$ is unramified already if $vL' = vL$. Note that our definition of "unramified" is stronger than the definition in [18, \$22] which does not require "defectless".

For a valued function field $(F|K,v)$, **elimination of ramification** means to find a transcendence basis T such that $(F^h|K(T)^h,v)$ is unramified. According to Theorem 4.18 in Section 4 below, this is equivalent to F lying in the absolute inertia field $K(T)^i$. Hence, $(F|K,v)$ admits elimination of ramification if and only if it is inertially generated.

If $\operatorname{char} K = 0$ and v is trivial on K, then $(F|K,v)$ is always inertially generated; this follows from Zariski's local uniformization ([66]) by Theorem 2.1. Since then $\operatorname{char} Fv = \operatorname{char} Kv = \operatorname{char} K = 0$, Zariski did not have to deal with inseparable residue field extensions and with defect. But if $\operatorname{char} K > 0$, then the existence of defect makes the problem of local uniformization much harder. This becomes visible in the approach to local uniformization that is used in the papers [34] and [35]. Local uniformization can be proved for Abhyankar places in positive characteristic because the defect does not appear ([34]); we will discuss this in more detail below. For other places ([35]), the defect has to be "killed" by a finite extension of the function field ("alteration").

2.2. **Classification of valued fields up to elementary equivalence.** Value group and residue field are invariants of a valued field, that is, two isomorphic valued fields have isomorphic value groups and isomorphic residue fields. But two valued fields with the same value groups and residue fields need not at all be isomorphic. For example, the valued field $(\mathbb{F}_p(t),v_t)$ and $(\mathbb{F}_p((t)),v_t)$ both have value group $\mathbb{Z}$ and residue field $\mathbb{F}_p$, but they are not isomorphic since $\mathbb{F}_p(t)$ is countable and $\mathbb{F}_p((t))$ is not.

In situations where classification up to isomorphism fails, classification up to elementary equivalence may still be possible. Two algebraic structures are **elementarily equivalent** if they satisfy the same elementary (first order) sentences. For example, Abraham Robinson proved that all algebraically closed valued fields of fixed characteristic are elementarily equivalent (cf. [60, Theorem 4.3.12]). James

Ax and Simon Kochen and, independently, Yuri Ershov proved that two henselian valued fields are elementarily equivalent if their value groups are elementarily equivalent and their residue fields are elementarily equivalent and of characteristic 0 (cf. [6] and [11, Theorem 5.4.12])). They also proved that all p-adically closed fields are elementarily equivalent (cf. [7, Theorem 2]). Likewise, Alfred Tarski proved that all real closed fields are elementarily equivalent (cf. [60, Theorem 4.3.3] or [11, Theorem 5.4.4]). This remains true if we consider non-archimedean real closed fields together with their natural valuations ([12]). These facts (and the corresponding model completeness results) all have important applications in algebra (for instance, Nullstellensätze, Hilbert's 17th Problem, cf. [28, Chap. A 4, $2]). So we would like to know when classification up to elementary equivalence is possible for more general classes of valued fields.

Two elementarily equivalent valued fields have elementarily equivalent value groups and elementarily equivalent residue fields. When does the converse hold? We mentioned already that the henselization is an immediate extension. So the elementary properties of value group and residue field do not determine whether a field is henselian or not. But being henselian is an elementary property, expressed by a scheme of elementary sentences, one each for all polynomials of degree n, where n runs through all natural numbers. In our above example, $(\mathbb{F}_p((t)), v_t)$ is henselian, but $(\mathbb{F}_p(t), v_t)$ is not, so they are not elementarily equivalent. We see that in order to have classification up to elementary equivalence relative to value groups and residue fields, our fields need to be (at least) henselian. But if the characteristic of the residue fields is positive, then we may have proper immediate algebraic extensions of henselian valued fields, as we will see in the next section. So our fields need to be (at least) **algebraically maximal**, that is, not admitting any proper immediate algebraic extensions.

Our fields even have to be defectless. Indeed, every valued field (K, v) admits a maximal immediate extension (M, v). Then (M, v) is maximal and therefore henselian and defectless. Since $vK = vM$ and $Kv = Mv$, we want that $(K, v) \equiv (M, v)$. The property "henselian and defectless field" is elementary (cf. [16, 1.33], [38] or the background information in [46]), so (K, v) should be a henselian defectless field.

If $\mathcal{L}$ is an elementary language and $\mathcal{A} \subset \mathcal{B}$ are $\mathcal{L}$-structures, then we will say that $\mathcal{A}$ is **existentially closed in** $\mathcal{B}$ and write $\mathcal{A} \prec_\exists \mathcal{B}$ if every existential sentence with parameters from $\mathcal{A}$ that holds in $\mathcal{B}$ also holds in $\mathcal{A}$. When we talk of fields, then we use the language of rings ($\{+, -, \cdot, 0, 1\}$) or fields (adding the unary function symbol "$\cdot^{-1}$"). When we talk of valued fields, we augment this language by a unary relation symbol for the valuation ring or a binary relation symbol for valuation divisibility ("$vx \leq vy$"). For ordered abelian groups, we use the language of groups augmented by a binary predicate ("$x < y$") for the ordering. For the meaning of "existentially closed in" in the settings of fields, valued fields and ordered abelian groups, see [51, p. 183].

By model theoretical tools such as Robinson's Test, the classification problem can be transformed into the problem of finding conditions which ensure that the following **Ax–Kochen–Ershov Principle** holds:

$$(K,v)\subseteq(L,v)\;\wedge\;vK\prec_\exists vL\;\wedge\;Kv\prec_\exists Lv\quad\Longrightarrow\quad(K,v)\prec_\exists(L,v)\;. \tag{2.3}$$

In order to prove that $(K,v)\prec_\exists(L,v)$, we first note that existential sentences in L only talk about finitely many elements of L, and these generate a function field over K. So it suffices to show $(K,v)\prec_\exists(F,v)$ for every function field F over K contained in L. One tool to show that $(K,v)\prec_\exists(F,v)$ is to prove an embedding lemma: we wish to construct an embedding of (F,v) over K in some "big" (highly saturated elementary) extension (K^*,v^*) of (K,v). Existential sentences are preserved by embeddings and will then hold in (K^*,v^*) from where they can be pulled down to (K,v). In order to construct the embedding, we need to understand the algebraic structure of (F,v).

Example 2.8. Assume that (K,v) is henselian (the same will then be true for (K^*,v^*)) and that $(F|K,v)$ is an immediate extension of transcendence degree 1. Pick an element $x\in F$ transcendental over K. Even if we know how to embed $(K(x),v)$ in (K^*,v^*), how can we extend this embedding to (F,v)? Practically the only tool we have for such extensions is Hensel's Lemma. So if $F\subset K(x)^h$, we can use the universal property of henselizations (Theorem 4.11 below) to extend the embedding to $K(x)^h$ and thus to F. If F is not a subfield of $K(x)^h$, we do not know what to do. $\diamondsuit$

More generally, we have to deal with extensions which are not immediate, but for which the conditions "$vK\prec_\exists vL$" and "$Kv\prec_\exists Lv$" hold. By the saturation of (K^*,v^*), they actually provide us with an embedding of vF over vK in v^*K^* and an embedding of Fv over Kv in K^*v^*. Using Hensel's Lemma, they can be lifted to an embedding of (F,v) in (K^*,v^*) if (F,v) is inertially generated with a transcendence basis T such that $(K(T),v)$ can be embedded, as we will discuss at the end of Section 5.1. If (F,v) is not inertially generated, we are lost again. So we see that both of our problems share the important approach of elimination of ramification.

Before we discuss the stated problems further, let us give several examples of defect extensions, in order to meet the enemy we are dealing with.

3. Examples for non-trivial defect

In this section, we shall give examples for extensions with defect >1. There is one basic example which is quick at hand. It is due to F. K. Schmidt.

Example 3.1. We consider $\mathbb{F}_p((t))$ with its canonical valuation $v=v_t$. Since $\mathbb{F}_p((t))|\mathbb{F}_p(t)$ has infinite transcendence degree, we can choose some element $s\in\mathbb{F}_p((t))$ which is transcendental over $\mathbb{F}_p(t)$. Since $(\mathbb{F}_p((t))|\mathbb{F}_p(t),v)$ is an immediate extension, the same holds for $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t),v)$ and thus also for $(\mathbb{F}_p(t,s)|\mathbb{F}_p(t,s^p),v)$.

The latter extension is purely inseparable of degree p (since s, t are algebraically independent over $\mathbb{F}_p$, the extension $\mathbb{F}_p(s)|\mathbb{F}_p(s^p)$ is linearly disjoint from $\mathbb{F}_p(t, s^p)|\mathbb{F}_p(s^p)$). Hence, Theorem 4.1 shows that there is only one extension of the valuation v from $\mathbb{F}_p(t, s^p)$ to $\mathbb{F}_p(t, s)$. So we have $\mathrm{e} = \mathrm{f} = \mathrm{g} = 1$ for this extension and consequently, its defect is p. $\diamondsuit$

Remark 3.2. This example is the easiest one used in commutative algebra to show that the integral closure of a noetherian ring of dimension 1 in a finite extension of its quotient field need not be finitely generated.

In some sense, the field $\mathbb{F}_p(t, s^p)$ is the smallest possible admitting a defect extension. Indeed, a function field of transcendence degree 1 over its prime field $\mathbb{F}_p$ is defectless under every valuation. More generally, a valued function field of transcendence degree 1 over a subfield on which the valuation is trivial is always a defectless field; this follows from Theorem 5.1 below.

With respect to defects, discrete valuations are not too bad. The following is easy to prove (cf. [38]):

Theorem 3.3. *Let (K, v) be a discretely valued field, that is, with value group $vK \simeq \mathbb{Z}$. Then every finite separable extension is defectless. If in addition* $\mathrm{char}\, K = 0$*, then (K, v) is a defectless field.*

A defect can appear "out of nothing" when a finite extension is lifted through another finite extension:

Example 3.4. In the foregoing example, we can choose s such that $vs > 1 = vt$. Now we consider the extensions $(\mathbb{F}_p(t, s^p)|\mathbb{F}_p(t^p, s^p), v)$ and $(\mathbb{F}_p(t+s, s^p)|\mathbb{F}_p(t^p, s^p), v)$ of degree p. Both are defectless: since $v\mathbb{F}_p(t^p, s^p) = p\mathbb{Z}$ and $v(t + s) = vt = 1$, the index of $v\mathbb{F}_p(t^p, s^p)$ in $v\mathbb{F}_p(t, s^p)$ and in $v\mathbb{F}_p(t + s, s^p)$ must be (at least) p. But $\mathbb{F}_p(t, s^p).\mathbb{F}_p(t + s, s^p) = \mathbb{F}_p(t, s)$, which shows that the defectless extension $(\mathbb{F}_p(t, s^p)|\mathbb{F}_p(t^p, s^p), v)$ does not remain defectless if lifted up to $\mathbb{F}_p(t+s, s^p)$ (and vice versa). $\diamondsuit$

We can derive from Example 3.1 an example of a defect extension of henselian fields.

Example 3.5. We consider again the immediate extension $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), v)$ of Example 3.1. We take the henselization $(\mathbb{F}_p(t, s), v)^h$ of $(\mathbb{F}_p(t, s), v)$ in $\mathbb{F}_p((t))$ and the henselization $(\mathbb{F}_p(t, s^p), v)^h$ of $(\mathbb{F}_p(t, s^p), v)$ in $(\mathbb{F}_p(t, s), v)^h$. We find that $(\mathbb{F}_p(t, s), v)^h|(\mathbb{F}_p(t, s^p), v)^h$ is again a purely inseparable extension of degree p. Indeed, the purely inseparable extension $\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p)$ is linearly disjoint from the separable extension $\mathbb{F}_p(t, s^p)^h|\mathbb{F}_p(t, s^p)$, and by virtue of Theorem 4.14, $\mathbb{F}_p(t, s)^h = \mathbb{F}_p(t, s).\mathbb{F}_p(t, s^p)^h$. Also for this extension we have that $\mathrm{e} = \mathrm{f} = \mathrm{g} = 1$ and again, the defect is p. Note that by Theorem 3.3, a proper immediate extension over a field like $(\mathbb{F}_p(t, s^p), v)^h$ can only be purely inseparable. $\diamondsuit$

The next example is easily found by considering the purely inseparable extension $\tilde{K}|K^{\rm sep}$. In comparison to the last example, the involved fields are "much bigger", for instance, they do not have value group $\mathbb{Z}$ anymore.

Example 3.6. Let K be a field which is not perfect. Then the extension $\tilde{K}|K^{\rm sep}$ is non-trivial. For every non-trivial valuation v on $\tilde{K}$, the value groups $v\tilde{K}$ and $vK^{\rm sep}$ are both equal to the divisible hull $\widetilde{vK}$ of vK, and the residue fields $\tilde{K}v$ and $K^{\rm sep}v$ are both equal to the algebraic closure of Kv (cf. Lemma 2.16 of [42]). Consequently, $(\tilde{K}|K^{\rm sep},v)$ is an immediate extension. Since the extension of v from $K^{\rm sep}$ to $\tilde{K}$ is unique (cf. Theorem 4.1 below), we find that the defect of every finite subextension is equal to its degree. $\diamondsuit$

Note that the separable-algebraically closed field $K^{\rm sep}$ is henselian for every valuation. Hence, our example shows:

Theorem 3.7. *There are henselian valued fields of positive characteristic which admit proper purely inseparable immediate extensions. Hence, the property "henselian" does not imply the property "algebraically maximal".*

We can refine the previous example as follows. Let $p>0$ be the characteristic of the residue field Kv. An **Artin-Schreier extension** of K is an extension of degree p generated over K by a root of a polynomial X^p-X-c with $c\in K$. An extension of degree p of a field of characteristic p is a Galois extension if and only if it is an Artin-Schreier extension. A field K is **Artin-Schreier closed** if it does not admit Artin-Schreier extensions.

Example 3.8. In order that every purely inseparable extension of the valued field (K,v) be immediate, it suffices that vK be p-divisible and Kv be perfect. But these conditions are already satisfied for every non-trivially valued Artin-Schreier closed field K (see Corollary 2.17 of [42]). Hence, the perfect hull of every non-trivially valued Artin-Schreier closed field is an immediate extension. $\diamondsuit$

Until now, we have only presented purely inseparable defect extensions. But our last example can give an idea of how to produce a separable defect extension by interchanging the role of purely inseparable extensions and Artin-Schreier extensions.

Example 3.9. Let (K,v) be a valued field of characteristic $p>0$ whose value group is not p-divisible. Let $c\in K$ such that $vc<0$ is not divisible by p. Let a be a root of the Artin-Schreier polynomial X^p-X-c. Then $va=vc/p$ and $[K(a):K]=p=(vK(a):vK)$. The fundamental inequality shows that $K(a)v=Kv$ and that the extension of v from K to $K(a)$ is unique. By Theorem 4.1 below the further extension to $K(a)^{1/p^\infty}=K^{1/p^\infty}(a)$ is unique. It follows that the extension of v from K^{1/p^∞} to $K^{1/p^\infty}(a)$ is unique. On the other hand, $[K^{1/p^\infty}(a):K^{1/p^\infty}]=p$ since the separable extension $K(a)|K$ is linearly disjoint from $K^{1/p^\infty}|K$. The value group $vK^{1/p^\infty}(a)$ is the p-divisible hull of $vK(a)=vK+\mathbb{Z}va$. Since $pva\in vK$, this is the same as the p-divisible hull of vK, which in turn is equal to vK^{1/p^∞}. The residue field of $K^{1/p^\infty}(a)$ is the perfect hull of $K(a)v = Kv$. Hence it is equal to the residue field of K^{1/p^∞}. It follows that the extension $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$ is immediate and that its defect is p, like its degree.

Similarly, one can start with a valued field (K, v) of characteristic $p > 0$ whose residue field is not perfect. In this case, the Artin-Schreier extension $K(a)|K$ is constructed as in the proof of Lemma 2.13 of [42]. We leave the details to the reader. $\diamondsuit$

In the previous example, we can always choose (K, v) to be henselian (since passing to the henselization does not change value group and residue field). Then all constructed extensions of (K, v) are also henselian, since they are algebraic extensions (cf. Theorem 4.14 below). Hence, our example shows:

Theorem 3.10. *There are henselian valued fields of positive characteristic which admit immediate Artin-Schreier defect extensions.*

If the perfect hull of a given valued field (K, v) is not an immediate extension, then vK is not p-divisible or Kv is not perfect, and we can apply the procedure of our above example. This shows:

Theorem 3.11. *If the perfect hull of a given valued field of positive characteristic is not an immediate extension, then it admits an immediate Artin-Schreier extension.*

An important special case of Example 3.9 is the following:

Example 3.12. We choose (K, v) to be $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{F}_p((t)), v_t)$ or any intermediate field, and set $L := K(t^{1/p^i} \mid i \in \mathbb{N})$, the perfect hull of K. By Theorem 4.1 below, $v = v_t$ has a unique extension to L. In all cases, L can be viewed as a subfield of the power series field $\mathbb{F}_p((\mathbb{Q}))$. The power series

$$\vartheta \;:=\; \sum_{i=1}^{\infty} t^{-1/p^i} \;\in\; \mathbb{F}_p((\mathbb{Q})) \tag{3.1}$$

is a root of the Artin-Schreier polynomial

$$X^p - X - \frac{1}{t}$$

because

$$\begin{aligned}\vartheta^p - \vartheta - \frac{1}{t} \;&=\; \sum_{i=1}^{\infty} t^{-1/p^{i-1}} - \sum_{i=1}^{\infty} t^{-1/p^i} \;-\; t^{-1}\\ &=\; \sum_{i=0}^{\infty} t^{-1/p^i} - \sum_{i=1}^{\infty} t^{-1/p^i} \;-\; t^{-1} \;=\; 0\;.\end{aligned}$$

By Example 3.9, the extension $L(\vartheta)|L$ is an immediate Artin-Schreier defect extension. The above power series expansion for ϑ was presented by Shreeram Abhyankar in [1]. It became famous since it shows that there are elements algebraic over $\mathbb{F}_p(t)$

with a power series expansion in which the exponents do not have a common denominator. This in turn shows that Puiseux series fields in positive characteristic are in general not algebraically closed (see also [30, 40]). With $p = 2$, the above was also used by Irving Kaplansky in [29, Section 5] for the construction of an example that shows that if his "hypothesis A" (see [29, Section 3]) is violated, then the maximal immediate extension of a valued field may not be unique up to isomorphism. See also [49] for more information on this subject.

Let us compute $v(\vartheta - L)$. For the partial sums

$$\vartheta_k := \sum_{i=1}^{k} t^{-1/p^i} \in L \tag{3.2}$$

we see that $v(\vartheta-\vartheta_k) = -1/p^{k+1} < 0$. Assume that there is $c \in L$ such that $v(\vartheta-c) > -1/p^k$ for all k. Then $v(c-\vartheta_k) = \min\{v(\vartheta - c), v(\vartheta - \vartheta_k)\} = -1/p^{k+1}$ for all k. On the other hand, there is some k such that $c \in K(t^{-1/p}, \ldots, t^{-1/p^k}) = K(t^{-1/p^k})$. But this contradicts the fact that $v(c - t^{-1/p} - \ldots - t^{-1/p^k}) = v(c - \vartheta_k) = -1/p^{k+1} \notin vK(t^{-1/p^k})$. This proves that the values $-1/p^k$ are cofinal in $v(\vartheta - L)$. Since vL is a subgroup of the rationals, this shows that the least upper bound of $v(\vartheta - L)$ in vL is the element 0. As $v(\vartheta - L)$ is an initial segment of vL by Lemma 1.1, we conclude that $v(\vartheta - L) = (vL)^{<0}$. It follows that $(L(\vartheta)|L, v)$ is immediate without (L, v) being dense in $(L(\vartheta), v)$. ♢

A version of this example with $(K, v) = (\widetilde{\mathbb{F}_p}((t)), v_t)$ was given by S. K. Khanduja in [31] as a counterexample to Proposition $2'$ on p. 425 of [4]. That proposition states that if (K, v) is a perfect henselian valued field of rank 1 and $a \in \tilde{K} \setminus K$, then there is $c \in K$ such that

$$v(a - c) \geq \min\{v(a - a') \mid a' \neq a \text{ conjugate to } a \text{ over } K\} .$$

But for $a = \vartheta$ in the previous example, we have that $a - a' \in \mathbb{F}_p$ so that the right hand side is 0, whereas $v(\vartheta-c) < 0$ for all c in the perfect hull L of $\mathbb{F}_p((t))$. The same holds if we take L to be the perfect hull of $K = \widetilde{\mathbb{F}_p}((t))$. In fact, it is Corollary 2 to Lemma 6 on p. 424 in [4] which is in error; it is stated without proof in the paper.

In a slightly different form, the previous example was already given by Alexander Ostrowski in [57], Section 57:

Example 3.13. Ostrowski takes $(K, v) = (\mathbb{F}_p(t), v_t)$, but works with the polynomial $X^p - tX - 1$ in the place of the Artin-Schreier polynomial $X^p - X - 1/t$. After an extension of K of degree $p - 1$, it also can be transformed into an Artin-Schreier polynomial. Indeed, if we take b to be an element which satisfies $b^{p-1} = t$, then replacing X by bX and dividing by b^p will transform $X^p - tX - 1$ into the polynomial $X^p - X - 1/b^p$. Now we replace X by $X + 1/b$. Since we are working in characteristic p, this transforms $X^p - X - 1/b^p$ into $X^p - X - 1/b$. (This sort of transformation plays a crucial role in the proofs of Theorem 5.1 and Theorem 5.10 as well as in

Abhyankar's and Epp's work.). Now we see that the Artin-Schreier polynomial $X^p - X - 1/b$ plays the same role as $X^p - X - 1/t$. Indeed, $vb = \frac{1}{p-1}$ and it follows that $(v\mathbb{F}_p(b) : v\mathbb{F}_p(t)) = p - 1 = [\mathbb{F}_p(b) : \mathbb{F}_p(t)]$, so that $v\mathbb{F}_p(b) = \mathbb{Z}\frac{1}{p-1}$. In this value group, vb is not divisible by p. $\diamondsuit$

Interchanging the role of purely inseparable and Artin-Schreier extensions in Example 3.12, we obtain:

Example 3.14. We proceed as in Example 3.12, but replace t^{-1/p^i} by a_i, where we define a_1 to be a root of the Artin-Schreier polynomial $X^p - X - 1/t$ and a_{i+1} to be a root of the Artin-Schreier polynomial $X^p - X + a_i$. Now we choose η such that $\eta^p = 1/t$. Note that also in this case, $a_1, \ldots, a_i \in K(a_i)$ for every i, because $a_i = a_{i+1}^p - a_{i+1}$ for every i. By induction on i, we again deduce that $va_1 = -1/p$ and $va_i = -1/p^i$ for every i. We set $L := K(a_i \mid i \in \mathbb{N})$, that is, $L|K$ is an infinite tower of Artin-Schreier extensions. By our construction, vL is p-divisible and $Lv = \mathbb{F}_p$ is perfect. On the other hand, for every purely inseparable extension $L'|L$ the group vL'/vL is a p-group and the extension $L'v|Lv$ is purely inseparable. This fact shows that $(L(\eta)|L, v)$ is an immediate extension.

In order to compute $v(\eta - L)$, we set

$$\eta_k := \sum_{i=1}^{k} a_i \in L . \tag{3.3}$$

Bearing in mind that $a_{i+1}^p = a_{i+1} - a_i$ and $a_1^p = a_1 + 1/t$ for $i \geq 1$, we compute

$$\begin{aligned} (\eta - \eta_k)^p &= \eta^p - \eta_k^p = \frac{1}{t} - \sum_{i=1}^{k} a_i^p = \frac{1}{t} - \left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{k-1} a_i + \frac{1}{t} \right) \\ &= a_k . \end{aligned}$$

It follows that $v(\eta - \eta_k) = \frac{va_k}{p} = -1/p^{k+1}$. The same argument as in Example 3.12 now shows that again, $v(\eta - L) = (vL)^{<0}$. $\diamondsuit$

We can develop Examples 3.12 and 3.14 a bit further in order to treat complete fields.

Example 3.15. Take one of the immediate extensions $(L(\vartheta)|L, v)$ of Example 3.12 and set $\zeta = \vartheta$, or take one of the immediate extensions $(L(\eta)|L, v)$ of Example 3.14 and set $\zeta = \eta$. Consider the completion $(L, v)^c = (L^c, v)$ of (L, v). Since every finite extension of a complete valued field is again complete, $(L^c(\zeta), v) = (L(\zeta).L^c, v)$ is the completion of $(L(\zeta), v)$ for every extension of the valuation v from (L^c, v) to $L(\zeta).L^c$. Consequently, the extension $(L^c(\zeta)|L(\zeta), v)$ and thus also the extension $(L^c(\zeta)|L, v)$ is immediate. It follows that $(L^c(\zeta)|L^c, v)$ is immediate. On the other hand, this extension is non-trivial since $v(\zeta - L) = (vL)^{<0}$ shows that $\zeta \notin L^c$. $\diamondsuit$

A valued field is **maximal** if it does not admit any proper immediate extension. All power series fields are maximal. A valued field is said to have **rank 1** if its value group is archimedean, i.e., a subgroup of the reals. Every complete discretely valued field of rank 1 is maximal. Every complete valued field of rank 1 is henselian (but this is not true in general in higher ranks). The previous example proves:

Theorem 3.16. *There are complete fields of rank 1 which admit immediate separable-algebraic and immediate purely inseparable extensions. Consequently, not every complete field of rank 1 is maximal.*

In Example 3.14 we constructed an immediate purely inseparable extension not contained in the completion of the field. Such extensions can be transformed into immediate Artin-Schreier defect extensions:

Example 3.17. In the situation of Example 3.14, extend v from $L(\eta)$ to $\tilde{L}$. Take $d \in L$ with $vd \geq 1/p$, and ϑ_0 a root of the polynomial $X^p - dX - 1/t$. It follows that

$$-1 \;=\; v\frac{1}{t} \;=\; v(\vartheta_0^p - d\vartheta_0) \;\geq\; \min\{v\vartheta_0^p, vd\vartheta_0\} \;=\; \min\{pv\vartheta_0, vd + v\vartheta_0\}$$

which shows that we must have $v\vartheta_0 < 0$. But then

$$pv\vartheta_0 \;<\; v\vartheta_0 \;<\; vd + v\vartheta_0 \;,$$

so

$$v\vartheta_0 \;=\; -\frac{1}{p}\;.$$

We compute:

$$pv(\vartheta_0 - \eta) \;=\; v(\vartheta_0 - \eta)^p \;=\; v(\vartheta_0^p - \eta^p) \;=\; v(d\vartheta_0 + 1/t - 1/t) \;=\; vd + v\vartheta_0 \;\geq\; 0\;.$$

Hence $v(\vartheta_0 - \eta) \geq 0$, and thus for all $c \in L$,

$$v(\vartheta_0 - c) \;=\; \min\{v(\vartheta_0 - \eta), v(\eta - c)\} \;=\; v(\eta - c)\;.$$

In particular, $v(\vartheta_0 - L) = v(\eta - L) = (vL)^{<0}$. The extension $(L(\vartheta_0)|L, v)$ is immediate and has defect p; however, this is not quite as easy to show as it has been before. To make things easier, we choose (K, v) to be henselian, so that also (L, v), being an algebraic extension, is henselian. So there is only one extension of v from L to $L(\vartheta_0)$. Since $v(\vartheta_0 - c) < 0$ for all $c \in L$, we have that $\vartheta_0 \notin L$. We also choose $d = b^{p-1}$ for some $b \in L$. Then we will see below that $L(\vartheta_0)|L$ is an Artin-Schreier extension. If it were not immediate, then $\mathrm{e} = p$ or $\mathrm{f} = p$. In the first case, we can choose some $a \in L(\vartheta_0)$ such that $0, va, \ldots, (p-1)va$ are representatives of the distinct cosets of $vL(\vartheta_0)$ modulo vL. Then $1, a, \ldots, a^{p-1}$ are L-linearly independent and thus form an L-basis of $L(\vartheta_0)$. Writing $\vartheta_0 = c_0 + c_1 a + \ldots + c_{p-1}a^{p-1}$, we find that $v(\eta - c_0) = v(\vartheta_0 - c_0) = \min\{vc_1 + va, \ldots, vc_{p-1} + (p-1)va\} \notin vL$ as the values $vc_1 + va, \ldots, vc_{p-1} + (p-1)va$ lie in distinct cosets modulo vL. But this is a contradiction. In the second case, $\mathrm{f} = p$, one chooses $a \in L(\vartheta_0)$ such that $1, av, \ldots, (av)^{p-1}$ form a basis of $L(\vartheta_0)v|Lv$, and derives a contradiction in a similar way. (Using this method one actually proves that an extension $(L(\zeta)|L, v)$ of degree

p with unique extension of the valuation is immediate if and only if $v(\zeta - L)$ has no maximal element.)

Now consider the polynomial $X^p - dX - 1/t = X^p - b^{p-1}X - 1/t$ and set $X = bY$. Then $X^p - dX - 1/t = b^pY^p - b^pY - 1/t$, and dividing by b^p we obtain the polynomial $Y^p - Y - 1/b^pt$ which admits ϑ_0/b as a root. So we see that $(L(\vartheta_0)|L, v)$ is in fact an immediate Artin-Schreier defect extension. But in comparison with Example 3.14, something is different:

$$\begin{aligned} v(\frac{\vartheta_0}{b} - L) &= \{v(\frac{\vartheta_0}{b} - c) \mid c \in L\} = \{v(\frac{\vartheta_0}{b} - \frac{c}{b}) \mid c \in L\} \\ &= \{v(\vartheta_0 - c) - vb \mid c \in L\} = \{\alpha \in vL \mid \alpha < vb\}\,, \end{aligned}$$

where $vb > 0$. $\diamondsuit$

A similar idea can be used to turn the defect extension of Example 3.1 into a separable extension. However, in the previous example we made use of the fact that η was not an element of the completion of (L, v), that is, $v(\eta - L)$ was bounded from above. We use a "dirty trick" to first transform the extension of Example 3.1 to an extension whose generator does not lie in the completion of the base field.

Example 3.18. Taking the extension $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), v)$ as in Example 3.1, we adjoin a new transcendental element z to $\mathbb{F}_p(t, s)$ and extend the valuation v in such a way that $vs \gg vt$, that is, $v\mathbb{F}_p(t, s, z)$ is the lexicographic product $\mathbb{Z} \times \mathbb{Z}$. The extension $(\mathbb{F}_p(t, s, z)|\mathbb{F}_p(t, s^p, z), v)$ is still purely inseparable and immediate, but now s does not lie anymore in the completion $\mathbb{F}_p(t, s^p)((z))$ of $\mathbb{F}_p(t, s^p, z)$. In fact, $v(s - \mathbb{F}_p(t, s^p, z)) = \{\alpha \in v\mathbb{F}_p(t, s^p, z) \mid \exists n \in \mathbb{N} : nvt \geq \alpha\}$ is bounded from above by vz.

Taking ϑ_0 to be a root of the polynomial $X^p - z^{p-1}X - s^p$ we obtain that $v(\vartheta_0 - c) = v(s - c)$ for all $c \in \mathbb{F}_p(t, s^p, z)$ and that the Artin-Schreier extension $(\mathbb{F}_p(t, \vartheta_0, z)|\mathbb{F}_p(t, s^p, z), v)$ is immediate with defect p. We leave the proof as an exercise to the reader. Note that one can pass to the henselizations of all fields involved, cf. Example 3.5. $\diamondsuit$

The interplay of Artin-Schreier extensions and radical extensions that we have used in the last examples can also be transferred to the mixed characteristic case. There are infinite algebraic extensions of $\mathbb{Q}_p$ which admit immediate Artin-Schreier defect extensions. To present an example, we need a lemma which shows that there is some **quasi-additivity** in the mixed characteristic case.

Lemma 3.19. *Let (K, v) be a valued field of characteristic 0 and residue characteristic $p > 0$, and with valuation ring $\mathcal{O}$. Further, let $c_1, \ldots, c_n$ be elements in K of value $\geq -\frac{vp}{p}$. Then*

$$(c_1 + \ldots + c_n)^p \equiv c_1^p + \ldots + c_n^p \pmod{\mathcal{O}}\,.$$

Proof. Every product of p many c_i's has value $\geq -vp$. In view of the fact that every binomial coefficient $\binom{p}{i}$ is divisible by p for $1 \leq i \leq p-1$, we find that $(c_1+c_2)^p \equiv c_1^p + c_2^p \pmod{\mathcal{O}}$. Now the assertion follows by induction on n. □

Example 3.20. We choose (K,v) to be $(\mathbb{Q},v_p)$ or $(\mathbb{Q}_p,v_p)$ or any intermediate field. Note that we write $vp=1$. We construct an algebraic extension (L,v) of (K,v) with a p-divisible value group as follows. By induction, we choose elements a_i in the algebraic closure of K such that $a_1^p = 1/p$ and $a_{i+1}^p = a_i$. Then $va_1 = -1/p$ and $va_i = -1/p^i$ for every i. Hence, the field $L := K(a_i \mid i \in \mathbb{N})$ must have p-divisible value group under any extension of v from K to L. Note that $a_1,\ldots,a_i \in K(a_i)$ for every i. Since $(vK(a_{i+1}) : vK(a_i)) = p$, the fundamental inequality shows that $K(a_{i+1})v = K(a_i)v$ and that the extension of v is unique, for every i. Hence, $Lv = \mathbb{Q}_p v = \mathbb{F}_p$ and the extension of v from K to L is unique.

Now we let ϑ be a root of $X^p - X - 1/p$. It follows that $v\vartheta = -1/p$. We define $b_i := \vartheta - a_1 - \ldots - a_i$. By construction, $va_i \geq -1/p$ for all i. It follows that also $vb_i \geq -1/p$ for all i. With the help of the foregoing lemma, and bearing in mind that $a_{i+1}^p = a_i$ and $a_1^p = 1/p$, we compute

$$\begin{aligned} 0 &= \vartheta^p - \vartheta - \frac{1}{p} = (b_i + a_1 + \ldots + a_i)^p - (b_i + a_1 + \ldots + a_i) - 1/p \\ &\equiv b_i^p - b_i + a_1^p + \ldots + a_i^p - a_1 - \ldots - a_i - 1/p = b_i^p - b_i - a_i \pmod{\mathcal{O}}\,. \end{aligned}$$

Since $va_i < 0$, we have that $vb_i = \frac{1}{p} va_i = -1/p^{i+1}$. Hence, $(vK(\vartheta,a_i) : vK(a_i)) = p = [K(\vartheta,a_i) : K(a_i)]$ and $K(\vartheta,a_i)v = K(a_i)v = \mathbb{F}_p$ for every i. If $[L(\vartheta) : L] < p$, then there would exist some i such that $[K(\vartheta,a_i) : K(a_i)] < p$. But we have just shown that this is not the case. Similarly, if $vL(\vartheta)$ would contain an element that does not lie in the p-divisible hull of $\mathbb{Z} = vK$, or if $L(\vartheta)v$ would be a proper extension of $\mathbb{F}_p$, then the same would already hold for $K(\vartheta,a_i)$ for some i. But we have shown that this is not the case. Hence, $(L(\vartheta)|L,v)$ is an Artin-Schreier defect extension.

For the partial sums $\vartheta_k = \sum_{i=1}^k a_i$ we obtain $v(\vartheta - \vartheta_k) = vb_k = -1/p^{k+1}$, and the same argument as in Example 3.12 shows again that $v(\vartheta - L) = (vL)^{<0}$. ♢

From this example we can derive a special case which was given by Ostrowski in [57, Section 39] (see also [8, Chap. VI \$8, exercise 2].

Example 3.21. In the last example, we take $K = \mathbb{Q}_2$. Then $(L(\sqrt{3})|L,v)$ is an immediate extension of degree 2. Indeed, this is nothing else than the Artin-Schreier extension that we have constructed. If one substitutes $Y = 1 - 2X$ in the minimal polynomial $Y^2 - 3$ of $\sqrt{3}$ and then divides by 4, one obtains the Artin-Schreier polynomial $X^p - X - 1/2$.

This is Ostrowski's original example. A slightly different version was presented by Paulo Ribenboim (cf. Exemple 2 of Chapter G, p. 246): The extension $(L(\sqrt{-1})|L,v)$ is immediate. Indeed, the minimal polynomial $Y^2 + 1$ corresponds to the Artin-Schreier polynomial $X^p - X + 1/2$ which does the same job as $X^p - X - 1/2$. ♢

As in the equal characteristic case, we can interchange the role of radical extensions and Artin-Schreier extensions:

Example 3.22. We proceed as in Example 3.20, with the only difference that we define a_1 to be a root of the Artin-Schreier polynomial $X^p - X - 1/p$ and a_{i+1} to be a root of the Artin-Schreier polynomial $X^p - X + a_i$, and that we choose η such that $\eta^p = 1/p$. Note that also in this case, $a_1, \ldots, a_i \in K(a_i)$ for every i, because $a_i = a_{i+1}^p - a_{i+1}$ for every i. By induction on i, we again deduce that $va_1 = -1/p$ and that $va_i = -1/p^i$ for every i. As before, we define $b_i := a - a_1 - \ldots - a_i$. Using Lemma 3.19 and bearing in mind that $a_{i+1}^p = a_{i+1} - a_i$ and $a_1^p = a_1 + 1/p$, we compute

$$\begin{aligned} 0 &= \vartheta^p - \frac{1}{p} = (b_i + a_1 + \ldots + a_i)^p - 1/p \\ &\equiv b_i^p + a_1^p + \ldots + a_i^p - 1/p = b_i^p + a_i \pmod{\mathcal{O}} . \end{aligned}$$

It follows that $v(b_i^p + a_i) \geq 0 > va_i$. Consequently, $vb_i^p = -1/p^{i+1}$, that is, $vb_i = \frac{1}{p} va_i = va_{i+1}$. As before, we set $L := K(a_i \mid i \in \mathbb{N})$. Now the same arguments as in Example 3.20 show that $(L(\vartheta)|L, v)$ is an immediate extension with $v(\vartheta - L) = (vL)^{<0}$. ♢

It can happen that it takes just a finite defect extension to make a field defectless and even maximal. The following example is due to Masuyoshi Nagata ([54, Appendix, Example (E3.1), pp. 206-207]):

Example 3.23. We take a field k of characteristic p and such that $[k : k^p]$ is infinite, e.g., $k = \mathbb{F}_p(t_i|i \in \mathbb{N})$ where the t_i are algebraically independent elements over $\mathbb{F}_p$. Taking t to be another transcendental element over k we consider the power series fields $k((t))$ and $k^p((t)) = k^p((t^p))(t) = k((t))^p(t)$. Since $[k : k^p]$ is not finite, we have that $k((t))|k^p((t)).k$ is a non-trivial immediate purely inseparable algebraic extension. In fact, a power series in $k((t))$ is an element of $k^p((t)).k$ if and only if its coefficients generate a finite extension of k^p. Since $k^p((t)).k$ contains $k((t))^p$, this extension is generated by a set $X = \{x_i \mid i \in I\} \subset k((t))$ such that $x_i^p \in k^p((t)).k$ for every $i \in I$. Assuming this set to be minimal, or in other words, the x_i to be p-independent over $k^p((t)).k$, we pick some element $x \in X$ and put $K := k^p((t)).k(X \setminus \{x\})$. Then $k((t))|K$ is a purely inseparable extension of degree p. Moreover, it is an immediate extension; in fact, $k((t))$ is the completion of K. As an algebraic extension of $k^p((t))$, K is henselian. ♢

This example proves:

Theorem 3.24. *There is a henselian discretely valued field (K, v) of characteristic p admitting a finite immediate purely inseparable extension $(L|K, v)$ of degree p such that (L, v) is complete, hence maximal and thus defectless.*

For the conclusion of this section, we shall give an example which is due to Francoise Delon (cf. [16], Exemple 1.51). It shows that an algebraically maximal field is not necessarily a defectless field, and that a finite extension of an algebraically maximal field is not necessarily again algebraically maximal.

Example 3.25. We consider $\mathbb{F}_p((t))$ with its t-adic valuation $v = v_t$. We choose elements $x, y \in \mathbb{F}_p((t))$ which are algebraically independent over $\mathbb{F}_p(t)$. We set $L := \mathbb{F}_p(t, x, y)$ and define

$$s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t, s)\ .$$

Then s is transcendental over $\mathbb{F}_p(t)$ and therefore, K has p-degree 2, that is, $[K : K^p] = p^2$. We take F to be the relative algebraic closure of K in $\mathbb{F}_p((t))$. Since the elements $1, t^{1/p}, \ldots, t^{(p-1)/p}$ are linearly independent over $\mathbb{F}_p((t))$, the same holds over F. Hence, the elements $1, t, \ldots, t^{p-1}$ are linearly independent over F^p. Now if F had p-degree 1, then s could be written in a unique way as an F^p-linear combination of $1, t, \ldots, t^{p-1}$. But this is not possible since $s = x^p + ty^p$ and x, y are transcendental over F. Hence, the p-degree of F is still 2 (as it cannot increase through algebraic extensions). On the other hand, $vF = v\mathbb{F}_p((t)) = \mathbb{Z}$ and $Fv = \mathbb{F}_p((t))v = \mathbb{F}_p$, hence $(vF : pvF) = p$ and $[Fv : Fv^p] = 1$. Now Theorem 6.3 shows that (F, v) is not inseparably defectless. Again from Theorem 6.3, we infer that $F^{1/p} = F(t^{1/p}, s^{1/p})$ must be an extension of F with non-trivial defect. So F is not a defectless field.

On the other hand, $\mathbb{F}_p((t))$ is the completion of F since it is already the completion of $\mathbb{F}_p(t) \subseteq F$. This shows that $\mathbb{F}_p((t))$ is the unique maximal immediate extension of F (up to valuation preserving isomorphism over F). If F would admit a proper immediate algebraic extension F', then a maximal immediate extension of F' would also be a maximal immediate extension of F and would thus be isomorphic over F to $\mathbb{F}_p((t))$. But we have chosen F to be relatively algebraically closed in $\mathbb{F}_p((t))$. This proves that (F, v) must be algebraically maximal.

As (F, v) is algebraically maximal, the extension $F^{1/p}|F$ cannot be immediate. Therefore, the defect of $F^{1/p}|F$ implies that both $F^{1/p}|F(s^{1/p})$ and $F^{1/p}|F(t^{1/p})$ must be non-trivial immediate extensions. Consequently, $F(s^{1/p})$ and $F(t^{1/p})$ are not algebraically maximal.

Let us add to Delon's example by analyzing the situation in more detail and proving that F is the henselization of K and thus a separable extension of K. To this end, we first prove that K is relatively algebraically closed in L. Take $b \in L$ algebraic over K. The element b^p is algebraic over K and lies in $L^p = \mathbb{F}_p(t^p, x^p, y^p)$ and thus also in $K(x) = \mathbb{F}_p(t, x, y^p)$. Since x is transcendental over K, K is relatively algebraically closed in $K(x)$ and thus, $b^p \in K$. Consequently, $b \in K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p})$. Write

$$b \;=\; r_0 + r_1 s^{\frac{1}{p}} + \ldots + r_{p-1} s^{\frac{p-1}{p}} \qquad \text{with} \;\; r_i \in \mathbb{F}_p(t^{1/p}, s) = K(t^{1/p})\ .$$

By the definition of s,

$$b \;=\; r_0 \;+\; r_1 x + \ldots + r_{p-1} x^{p-1} \;+ \ldots +\; t^{1/p} r_1 y + \ldots + t^{(p-1)/p} r_{p-1} y^{p-1}$$

(in the middle, we have omitted the summands in which both x and y appear). Since x, y are algebraically independent over $\mathbb{F}_p$, the p-degree of $\mathbb{F}_p(x, y)$ is 2, and the elements x^iy^j, $0 \le i < p$, $0 \le j < p$, form a basis of $\mathbb{F}_p(x,y)|\mathbb{F}_p(x^p,y^p)$. Since t and $t^{1/p}$ are transcendental over $\mathbb{F}_p(x^p,y^p)$, we know that $\mathbb{F}_p(x,y)|\mathbb{F}_p(x^p,y^p)$ is linearly disjoint from $\mathbb{F}_p(t,x^p,y^p)|\mathbb{F}_p(x^p,y^p)$ and from $\mathbb{F}_p(t^{1/p},x^p,y^p)|\mathbb{F}_p(x^p,y^p)$. This shows that the elements x^iy^j also form a basis of $L|\mathbb{F}_p(t,x^p,y^p)$ and are still $\mathbb{F}_p(t^{1/p},x^p,y^p)$-linearly independent. Hence, b can also be written as a linear combination of these elements with coefficients in $\mathbb{F}_p(t,x^p,y^p)$, and this must coincide with the above $\mathbb{F}_p(t^{1/p},x^p,y^p)$-linear combination which represents b. That is, all coefficients r_i *and* $t^{i/p}r_i$, $1 \le i < p$, are in $\mathbb{F}_p(t,x^p,y^p)$. Since $t^{i/p} \notin \mathbb{F}_p(t,x^p,y^p)$, this is impossible unless they are zero. It follows that $b = r_0 \in K(t^{1/p})$. Assume that $b \notin K$. Then $[K(b):K] = p$ and thus, $K(b) = K(t^{1/p})$ since also $[K(t^{1/p}):K] = p$. But then $t^{1/p} \in K(b) \subset L$, a contradiction. This proves that K is relatively algebraically closed in L.

On the other hand, $t^{1/p} = y^{-1}(s^{1/p}-x) \in L(s^{1/p})$. Hence, $L.K^{1/p} = L(t^{1/p},s^{1/p}) = L(s^{1/p})$ and $[L.K^{1/p} : L] = [L(s^{1/p}) : L] \le p < p^2 = [K^{1/p} : K]$, that is, $L|K$ is not linearly disjoint from $K^{1/p}|K$ and thus not separable. Although being finitely generated, $L|K$ is consequently not separably generated; in particular, it is not a rational function field. At the same time, we have seen that $K(s^{1/p})$ admits a non-trivial purely inseparable algebraic extension in $L(s^{1/p})$ (namely, $K^{1/p}$). In contrast, $K(s^{1/p})$ and L are K-linearly disjoint because $s^{1/p} \notin L$.

Let us prove even more: if $K_1|K$ is any proper inseparable algebraic extension, then $t^{1/p} \in L.K_1$. Take such an extension $K_1|K$. Then there is some separable-algebraic subextension $K_2|K$ and an element $a \in K_1 \setminus K_2$ such that $a^p \in K_2$. Since $K_2|K$ is separable and K is relatively algebraically closed in L, we see that K_2 is relatively algebraically closed in $L_2 := L.K_2$. Hence, $a \notin L_2$ and therefore, $[L_2(a) : L_2] = p$. On the other hand, $K_2^{1/p} = K^{1/p}.K_2$ and thus, $L_2.K_2^{1/p} = L_2.K^{1/p} = L.K^{1/p}.K_2$. Consequently, $[L.K^{1/p} : L] = p$ implies that $[L_2.K_2^{1/p} : L_2] = [L.K^{1/p}.K_2 : L.K_2] \le p$. Since $a \in K_2^{1/p} \subset L_2.K_2^{1/p}$ and $[L_2(a) : L_2] = p$, it follows that $L_2.K_2^{1/p} = L_2(a)$. We obtain:

$$t^{1/p} \in K^{1/p} \subseteq K_2^{1/p} \subseteq L_2.K_2^{1/p} = L_2(a) \subseteq L.K_1 .$$

If $F|K$ were inseparable, then $t^{1/p} \in L.F$, which contradicts the fact that $L.F \subseteq \mathbb{F}_p((t))$. This proves that $F|K$ is separable. Since F is relatively closed in the henselian field $\mathbb{F}_p((t))$, it is itself henselian and thus contains the henselization K^h of K. Now $\mathbb{F}_p((t))$ is the completion of K^h since it is already the completion of $\mathbb{F}_p(t) \subseteq K^h$. Since a henselian field is relatively separable-algebraically closed in its completion (cf. [65], Theorem 32.19), it follows that $F = K^h$.

Note that the maximal immediate extension $\mathbb{F}_p((t))$ of F is not a separable extension since its subextension $L.F|F$ is not linearly disjoint from $K^{1/p}|K$. $\diamondsuit$

This example proves:

Theorem 3.26. *There are algebraically maximal fields which are not inseparably defectless. Hence, "algebraically maximal" does not imply "defectless". There are* *algebraically maximal fields admitting a finite purely inseparable extension which is not an algebraically maximal field.*

4. Absolute ramification theory

Assume that $L|K$ is an algebraic extension, not necessarily finite, and that v is a *non-trivial* valuation on K. Throuthout this section, we fix an arbitrary extension of v to the algebraic closure $\tilde{K}$ of K, which we again denote by v. Then for every $\sigma \in \mathrm{Aut}\,(\tilde{K}|K)$, the map

$$v\sigma = v \circ \sigma : \; L \ni a \;\mapsto\; v(\sigma a) \in v\tilde{K} \tag{4.1}$$

is a valuation of L which extends v. All extensions of v from K to L are conjugate:

Theorem 4.1. *The set of all extensions of v from K to L is*

$$\{v\sigma \mid \sigma \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\} \; .$$

In particular, a valuation on K has a unique extension to every purely inseparable field extension of K.

We will now give a quick introduction to **absolute ramification theory**, that is, the ramification theory of the extension $\tilde{K}|K$ with respect to a given valuation v on $\tilde{K}$ with valuation ring $\mathcal{O}_{\tilde{K}}$. For a corresponding quick introduction to general ramification theory, see [40].

We define distinguished subgroups of the **absolute Galois group** $G := \mathrm{Gal}\,K := \mathrm{Aut}\,(\tilde{K}|K) = \mathrm{Aut}\,(K^{\mathrm{sep}}|K)$ of K (with respect to the fixed extension of v to $\tilde{K}$). The subgroup

$$G^d := \{\sigma \in G \mid v\sigma = v \text{ on } \tilde{K}\} \tag{4.2}$$

is called the **absolute decomposition group of** (K,v) (w.r.t. $(\tilde{K},v)$). Further, the **absolute inertia group** (w.r.t. $(\tilde{K},v)$) is defined to be

$$G^i := \{\sigma \in G \mid \forall x \in \mathcal{O}_{\tilde{K}} : v(\sigma x - x) > 0\} \; , \tag{4.3}$$

and the **absolute ramification group** (w.r.t. $(\tilde{K},v)$) is

$$G^r := G^r(L|K,v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_{\tilde{K}} \setminus \{0\} : v(\sigma x - x) > vx\} \; . \tag{4.4}$$

The fixed fields K^d, K^i and K^r of G^d, G^i and G^r, respectively, in K^{sep} are called the **absolute decomposition field**, **absolute inertia field** and **absolute ramification field of** (K,v) (with respect to the given extension of v to $\tilde{K}$).

Remark 4.2. In contrast to the classical definition used by other authors, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the separable-algebraic closure of K*. The reason for this will become clear later.

By our definition, K^d, K^i and K^r are separable-algebraic extensions of K, and $K^{\mathrm{sep}}|K^r$, $K^{\mathrm{sep}}|K^i$, $K^{\mathrm{sep}}|K^d$ are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \quad \text{and thus,} \quad K^{\mathrm{sep}} \supset K^r \supset K^i \supset K^d \supset K \; . \tag{4.5}$$

(For the inclusion $G^i \subset G^d$ note that $vx \geq 0$ and $v(\sigma x - x) > 0$ implies that $v\sigma x \geq 0$.)

Theorem 4.3. *G^i and G^r are normal subgroups of G^d, and G^r is a normal subgroup of G^i. Therefore, $K^i|K^d$, $K^r|K^d$ and $K^r|K^i$ are (not necessarily finite) Galois extensions.*

First, we consider the decomposition field K^d. In some sense, it represents all extensions of v from K to $\tilde{K}$.

Theorem 4.4. *a) $v\sigma = v\tau$ on $\tilde{K}$ if and only if $\sigma\tau^{-1}$ is trivial on K^d.*
b) $v\sigma = v$ on K^d if and only if σ is trivial on K^d.
c) The extension of v from K^d to $\tilde{K}$ is unique.
d) The extension $(K^d|K, v)$ is immediate.

WARNING: It is in general not true that $v\sigma \neq v\tau$ holds already on K^d if it holds on $\tilde{K}$.

Assertions a) and b) are easy consequences of the definition of G^d. Part c) follows from b) by Theorem 4.1. For d), there is a simple proof using a trick mentioned by James Ax in [5, Appendix]; see also [5, Theorem 22, p. 70 and Theorem 23, p. 71] and [19].

Now we turn to the inertia field K^i. Let $\mathcal{M}_{\tilde{K}}$ denote the valuation ideal of v on $\tilde{K}$ (the unique maximal ideal of $\mathcal{O}_{\tilde{K}}$). For every $\sigma \in G^d$ we have that $\sigma\mathcal{O}_{\tilde{K}} = \mathcal{O}_{\tilde{K}}$, and it follows that $\sigma\mathcal{M}_{\tilde{K}} = \mathcal{M}_{\tilde{K}}$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_{\tilde{K}}/\mathcal{M}_{\tilde{K}} = \tilde{K}v = \widetilde{Kv}$ which satisfies $\overline{\sigma}(av) = (\sigma a)v$. Since σ fixes K, it follows that $\overline{\sigma}$ fixes Kv.

Lemma 4.5. *The map*

$$G^d \ni \sigma \;\mapsto\; \overline{\sigma} \in \mathrm{Gal}\, Kv \tag{4.6}$$

is a group homomorphism.

Theorem 4.6. *a) The homomorphism (4.6) is onto and induces an isomorphism*

$$\mathrm{Aut}\,(K^i|K^d) \;=\; G^d/G^i \;\simeq\; \mathrm{Aut}\,(K^iv|K^dv)\; . \tag{4.7}$$

b) For every finite subextension $F|K^d$ of $K^i|K^d$,

$$[F : K^d] \;=\; [Fv : K^dv]\; . \tag{4.8}$$

c) We have that $vK^i = vK^d = vK$. Further, K^iv is the separable closure of Kv, and therefore,

$$\mathrm{Aut}\,(K^iv|K^dv) \;=\; \mathrm{Gal}\, Kv\; . \tag{4.9}$$

If $F|K^d$ is normal, then b) is an easy consequence of a). From this, the general assertion of b) follows by passing from F to the normal hull of the extension $F|K^d$ and then using the multiplicativity of the extension degree. c) follows from b) by use of the fundamental inequality.

We set $p := \operatorname{char} Kv$ if this is positive, and $p := 1$ if $\operatorname{char} Kv = 0$. Given any abelian group Δ, the **p'-divisible hull of Δ** is defined to be the subgroup $\{\alpha \in \tilde{\Delta} \mid \exists n \in \mathbb{N} :\ (p,n) = 1 \wedge n\alpha \in \Delta\}$ of all elements in the divisible hull $\tilde{\Delta}$ of Δ whose order modulo Δ is prime to p.

Theorem 4.7. *a) There is an isomorphism*

$$\operatorname{Aut}(K^r|K^i) \;=\; G^i/G^r \;\simeq\; \operatorname{Hom}\left(vK^r/vK^i\,,\,(K^r v)^\times\right)\ , \tag{4.10}$$

where the character group on the right hand side is the full character group of the abelian group vK^r/vK^i. Since this group is abelian, $K^r|K^i$ is an abelian Galois extension.
b) For every finite subextension $F|K^i$ of $K^r|K^i$,

$$[F : K^i] \;=\; (vF : vK^i)\ . \tag{4.11}$$

c) $K^r v = K^i v$, and vK^r is the p'-divisible hull of vK.

Part b) follows from part a) since for a finite extension $F|K^i$, the group vF/vK^i is finite and thus there exists an isomorphism of vF/vK^i onto its full character group. The equality $K^r v = K^i v$ follows from b) by the fundamental inequality. The second assertion of part c) follows from the next theorem and the fact that the order of all elements in $(K^i v)^\times$ and thus also of all elements in $\operatorname{Hom}(vK^r/vK^i\,,\,(K^i v)^\times)$ is prime to p.

Theorem 4.8. *The ramification group G^r is a p-group, hence $K^{\mathrm{sep}}|K^r$ is a p-extension. Further, $v\tilde{K}/vK^r$ is a p-group, and the residue field extension $\tilde{K}v|K^r v$ is purely inseparable. If* $\operatorname{char} Kv = 0$*, then $K^r = K^{\mathrm{sep}} = \tilde{K}$.*

We note:

Lemma 4.9. *Every p-extension is a tower of Galois extensions of degree p. In characteristic p, all of them are Artin-Schreier extensions.*

From Theorem 4.8 it follows that there is a canonical isomorphism

$$\operatorname{Hom}\left(vK^r/vK^i\,,\,(K^i v)^\times\right) \;\simeq\; \operatorname{Hom}\left(v\tilde{K}/vK\,,\,(\tilde{K}v)^\times\right)\ . \tag{4.12}$$

The following theorem will be very useful for our purposes:

Theorem 4.10. *If $K'|K$ is algebraic, then the absolute decomposition field of (K',v) is $K^d.K'$, its absolute inertia field is $K^i.K'$, and its absolute ramification field is $K^r.K'$.*

From part c) of Theorem 4.4 we infer that the extension of v from K^d to $\tilde{K}$ is unique. On the other hand, if L is any extension field of K within K^d, then by Theorem 4.10, $K^d = L^d$. Thus, if $L \neq K^d$, then it follows from part b) of Theorem 4.4 that there are at least two distinct extensions of v from L to K^d and thus also to $\tilde{K} = \tilde{L}$. This proves that the absolute decomposition field K^d is a minimal algebraic extension of K admitting a unique extension of v to its algebraic closure. So it is the minimal algebraic extension of K which is henselian. We call it the **henselization of** (K,v) **in** $(\tilde{K},v)$. Instead of K^d, we also write K^h. A valued field is henselian if and only if it is equal to its henselization. Henselizations have the following universal property:

Theorem 4.11. *Let (K,v) be an arbitrary valued field and (L,v) a henselian extension field of (K,v). Then there is a unique embedding of (K^h,v) in (L,v) over K.*

From part d) of Theorem 4.4, we obtain another very important property of the henselization:

Theorem 4.12. *The henselization (K^h,v) is an immediate extension of (K,v).*

Corollary 4.13. *Every algebraically maximal and every maximal valued field is henselian. In particular, $(K((t)),v_t)$ is henselian.*

We employ Theorem 4.10 again to obtain:

Theorem 4.14. *If $K'|K$ is an algebraic extension, then the henselization of K' is $K'.K^h$. Every algebraic extension of a henselian field is again henselian.*

In conjunction with Theorems 4.6 and 4.7, Theorem 4.10 is also used to prove that there are no defects between K^h and K^r:

Theorem 4.15. *Take a finite extension $K_2|K_1$ such that $K^h \subseteq K_1 \subseteq K_2 \subseteq K^r$. Then $(K_2|K_1,v)$ is defectless.*

Proof. Since $K_3 := K_2 \cap K_1^i$ is a finite subextension of $K_1^i|K_1$, we have by parts b) and c) of Theorem 4.6 that $[K_3 : K_1] = [K_3v : K_1v]$ and $vK_3 = vK_1$. Since $K_1^i|K_1$ is Galois, K_2 is linearly disjoint from K_1^i over K_3. That is, $[K_2.K_1^i : K_1^i] = [K_2 : K_3]$. By Theorem 4.10, $K_2 \subseteq K^r = K_1.K^r = K_1^r$, so also $K_2.K_1^i|K_1^i$ is a finite subextension of $K_1^r|K_1^i$. By part b) of Theorem 4.7, we thus have $[K_2.K_1^i : K_1^i] = (v(K_2.K_1^i) : vK_1^i)$. By Theorem 4.10, $K_1^i = K_1^i.K_3 = K_3^i$ and $K_2.K_1^i = K_2^i$, so by part c) of Theorem 4.6, $vK_1^i = vK_3^i = vK_3$ and $v(K_2.K_1^i) = vK_2^i = vK_2$. Therefore,

$$[K_2 : K_3] \;=\; [K_2.K_1^i : K_1^i] \;=\; (v(K_2.K_1^i) : vK_1^i) \;=\; (vK_2 : vK_3) \;=\; (vK_2 : vK_1)$$

Putting everything together, we obtain

$$\begin{aligned} [K_2 : K_1] \;&=\; [K_2 : K_3][K_3 : K_1] \;=\; (vK_2 : vK_1)[K_3v : K_1v] \\ &\leq\; (vK_2 : vK_1)[K_2v : K_1v] \;\leq\; [K_2 : K_1]\,, \end{aligned}$$

so that equality must hold everywhere, which shows that $(K_2|K_1,v)$ is defectless. □

An algebraic extension of K^h is called **purely wild** if it is linearly disjoint from K^r over K^h. The following theorem has been proved by Matthias Pank (see Theorem 4.3 and Proposition 4.5 of [49]):

Theorem 4.16. *Every maximal purely wild extension W of K^h satisfies $W.K^r = \tilde{K}$ and hence is a field complement of K^r in $\tilde{K}$. Moreover, $W^r = \tilde{W}$, vW is the p-divisible hull of vK, and Wv is the perfect hull of Kv.*

Lemma 4.17. *If $(L|K^h, v)$ is a finite extension, then its defect is equal to the defect of $(L.K^r|K^r, v)$.*

Proof. We put $L_0 := L \cap K^r$. We have $L.K^r = L^r$ and $L_0^r = K^r$ by Theorem 4.10. Since $K^r|K^h$ is normal, L is linearly disjoint from $K^r = L_0^r$ over L_0, and $(L|L_0, v)$ is thus a purely wild extension.

As a finite subextension of $(K^r|K^h, v)$, the extension $(L_0|K^h, v)$ is defectless. Hence by the multiplicativity of the defect (2.2),

$$\mathrm{d}(L|K^h, v) = \mathrm{d}(L|L_0, v)\ . \tag{4.13}$$

It remains to show $\mathrm{d}(L|L_0, v) = \mathrm{d}(L.K^r|K^r, v)$. Since $L|L_0$ is linearly disjoint from $K^r|L_0$, we have

$$[L^r : L_0^r] \;=\; [L.K^r : L_0^r] \;=\; [L : L_0]\ . \tag{4.14}$$

Since $L|L_0$ is purely wild, vL/vL_0 is a p-group and $Lv|L_0v$ is purely inseparable. On the other hand, by Theorem 4.7,

vL^r is the p'-divisible hull of vL and $L^rv = (Lv)^{\mathrm{sep}}$,
vL_0^r is the p'-divisible hull of vL_0 and $L_0^rv = (L_0v)^{\mathrm{sep}}$.

It follows that

$$(vL^r : vL_0^r) \;=\; (vL : vL_0) \quad \text{and} \quad [L^rv : L_0^rv] \;=\; [Lv : L_0v]\ . \tag{4.15}$$

From (4.13), (4.14) and (4.15), keeping in mind that $L.K^r = L^r$ and $K^r = L_0^r$, we deduce

$$\begin{aligned}\mathrm{d}(L.K^r|K^r, v) \;&=\; \mathrm{d}(L^r|L_0^r, v) \;=\; \frac{[L^r : L_0^r]}{(vL^r : vL_0^r)[L^rv : L_0^rv]}\\ &=\; \frac{[L : L_0]}{(vL : vL_0)\cdot[Lv : L_0v]} \;=\; \mathrm{d}(L|L_0, v) \;=\; \mathrm{d}(L|K^h, v)\ .\end{aligned}$$

□

We can now describe the ramification theoretic proof for the lemma of Ostrowski (see also [67, Corollary to Theorem 25, p. 78]). Take a finite extension $(L'|L, v)$ of henselian fields. Then $L = L^h$. By the foregoing theorem, $\mathrm{d}(L'|L, v) = \mathrm{d}(L'.L^r|L^r, v)$. It follows from Theorem 4.7 that $[L'.L^r : L^r]$ is a power of p. Hence also $\mathrm{d}(L'.L^r|L^r, v)$, being a divisor of it, is a power of p.

We see that non-trivial defects can only appear between K^r and $\tilde{K}$, or equivalently, between K^h and W. These are the areas of **wild ramification**, whereas the extension from K^i to K^r is the area of **tame ramification**. Hence, local uniformization in characteristic 0 and the classification problem for valued fields of residue characteristic 0 only have to deal with tame ramification, while the two problems we described in Section 2 also have to fight the wild ramification.

An algebraic extension $(L'|L,v)$ of henselian fields is called **unramified** if every finite subextension is unramified. An algebraic extension $(L'|L,v)$ of henselian fields is called **tame** if every finite subextension $(L''|L,v)$ is defectless and such that $(L''v|Lv)$ is separable and p does not divide $(vL'':vL)$. A henselian field (L,v) is called a **tame field** if $(\tilde{L}|L,v)$ is a tame extension, and it is called a **separably tame field** if $(L^{\mathrm{sep}}|L,v)$ is a tame extension. The fields W of Theorem 4.16 are examples of tame fields.

The proof of the following theorem is given in [38].

Theorem 4.18. *The absolute inertia field is the unique maximal unramified extension of K^h in $(\tilde{K},v)$. The absolute ramification field is the unique maximal tame extension of K^h in $(\tilde{K},v)$.*

Note that an extension is tame if and only if it is defectless and "tamely ramified" in the sense of [18, $22]. As we have already mentioned, our notion of "unramified" is the same as "defectless" plus "unramified" in the sense of [18, $22] Hence for defectless valuations, the above theorem follows from [18, Corollary (22.9)].

We summarize our main results in the following table:

Galois group		**field**			**value group**		**residue field**	
		$\tilde{K}$			$\widetilde{vK}$		$\widetilde{Kv}$	
			purely inseparable			$=$		$=$
1		K^{sep}		separable-algebraic closure	$\widetilde{vK}$		$\widetilde{Kv}$	
			Galois p-extension			division by p		purely inseparable
G^r		K^r		absolute ramification field	$\frac{1}{p'^\infty}vK$		$(Kv)^{\mathrm{sep}}$	
	Char		abelian Galois p'-extension, defectless			division prime to p		$=$
G^i		K^i		absolute inertia field	vK		$(Kv)^{\mathrm{sep}}$	
	$\mathrm{Gal}\,Kv$		Galois, defectless			$=$		Galois
G^d		K^h		absolute decomposition field	vK		Kv	
			immediate			$=$		$=$
$\mathrm{Gal}\,K$		K			vK		Kv	

where $\frac{1}{p'^\infty}vK$ denotes the p'-divisible hull of vK and Char denotes the character group (4.12).

In algebraic geometry, the absolute inertia field is often called the **strict henselization**. Theorem 2.1 can be understood as saying that the Implicit Function Theorem, or equivalently, Hensel's Lemma, works within and only within the strict henselization. That the limit is the strict henselization and not the henselization becomes intuitively clear when one considers one of the equivalent forms of Hensel's Lemma which states that if f has coefficients in the valuation ring of a henselian field, then every simple root of the reduced polynomial fv (obtained by replacing the coefficients by their residues) can be lifted to a root of f. On the other hand, irreducible polynomials have only simple roots if and only if they are separable. Hence it is clear that Hensel's Lemma works as long as the residue field extensions are separable, which is the case between K^h and K^i.

5. Two theorems

5.1. **The Stability Theorem.** In this section we present two theorems about the defect which we have used for our results on local uniformization and in the model theory of valued fields in positive characteristic. The first one describes situations where no defect appears. The second one deals with with certain situation where defect may well appear, but shows that the defect can be eliminated.

Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following well known form of the "Abhyankar inequality" holds:

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{rr}\, vL/vK \,+\, \mathrm{trdeg}\, Lv|Kv\;, \tag{5.1}$$

where $\mathrm{rr}\, vL/vK := \dim_{\mathbb{Q}}(vL/vK)\otimes\mathbb{Q}$ is the **rational rank** of the abelian group vL/vK, i.e., the maximal number of rationally independent elements in vL/vK. This inequality is a consequence of Theorem 1 of [8, Chapter VI, §10.3], which states that if

$$\left\{\begin{array}{l} x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\in L \text{ such that}\\ vx_1,\ldots,vx_\rho \text{ are rationally independent over } vK\text{, and}\\ y_1v,\ldots,y_\tau v \text{ are algebraically independent over } Kv, \end{array}\right. \tag{5.2}$$

then $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau$ are algebraically independent over K. We will say that $(L|K,v)$ is **without transcendence defect** if equality holds in (5.1). In this case, every set $\{x_1,\ldots,x_\rho,y_1,\ldots,y_\tau\}$ satisfying (5.2) with $\rho=\mathrm{rr}\, vL/vK$ and $\tau=\mathrm{trdeg}\, Lv|Kv$ is a transcendence basis of $L|K$.

If $(F|K,v)$ is a valued function field without transcendence defect, then the extensions $vF|vK$ and $Fv|Kv$ are finitely generated (cf. [34, Corollary 2.2]).

Theorem 5.1. (Generalized Stability Theorem)
Let $(F|K,v)$ be a valued function field without transcendence defect. If (K,v) is a defectless field, then (F,v) is a defectless field. The same holds for "inseparably defectless" in the place of "defectless". If vK is cofinal in vF, then it also holds for "separably defectless" in the place of "defectless".

If the base field K is not a defectless field, we can say at least the following:

Corollary 5.2. *Let $(F|K,v)$ be a valued function field without transcendence defect, and $E|F$ a finite extension. Fix an extension of v from F to $\tilde{K}.F$. Then there is a finite extension $L_0|K$ such that for every algebraic extension L of K containing L_0, $(L.F,v)$ is defectless in $L.E$. If (K,v) is henselian, then $L_0|K$ can be chosen to be purely wild.*

Theorem 5.1 was stated and proved in [37]; the proof presented in [45] is an improved version.

The theorem has a long and interesting history. Hans Grauert and Reinhold Remmert [21, p. 119] first proved it in a very restricted case, where (K,v) is an algebraically closed complete discretely valued field and (F,v) is discrete too. A generalization was given by Laurent Gruson [26, Théorème 3, p. 66]. A good presentation of it can be found in the book on non-archimedean analysis by Siegfried Bosch, Ulrich Güntzer and Reinhold Remmert [9, §5.3.2, Theorem 1]. Further generalizations are due to Michel Matignon and Jack Ohm, and also follow from results in [23] and [24]. Ohm arrived independently of [37] at a general version of the Stability Theorem for the case of $\mathrm{trdeg}\, L|K = \mathrm{trdeg}\, Lv|Kv$ (see the second theorem on p. 306 of [56]). He deduces his theorem from Proposition 3 on p. 215 of [9], (more precisely, from a generalized version of this proposition which is proved but not stated in [9]).

All authors mentioned in the last paragraph use methods of non-archimedean analysis, and all results are restricted to the case of $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$. In this case we call the extension $(F|K,v)$ **residually transcendental**, and we call the valuation v a **constant reduction** of the algebraic function field $F|K$. The classical origin of such valuations is the study of curves over number fields and the idea to reduce them modulo a p-adic valuation. Certainly, the reduction should again render a curve, this time over a finite field. This is guaranteed by the condition $\mathrm{trdeg}\, F|K = \mathrm{trdeg}\, Fv|Kv$, where F is the function field of the curve and Fv will be the function field of its reduction. Naturally, one seeks to relate the genus of $F|K$ to that of $Fv|Kv$. Several authors proved **genus inequalities** (see, for example, [17, 53, 23] and the survey given in [22]). To illustrate the use of the defect, we will cite an inequality proved by Barry Green, Michel Matignon and Florian Pop [23, Theorem 3.1]. Let $F|K$ be a function field of transcendence degree 1, and v a constant reduction of $F|K$. We choose a henselization F^h of (F,v); all henselizations of subfields of F will be taken in F^h. We wish to define a defect of the extension $(F^h|K^h,v)$ even though this extension is not algebraic. The following result helps:

Theorem 5.3. (Independence Theorem)
The defect of the algebraic extension $(F^h|K(t)^h,v)$ is independent of the choice of the element $t\in F$, provided that tv is transcendental over Kv.

In [56], Ohm proves a more general version of this theorem for arbitrary transcendence degree, using his version of the Stability Theorem. The Stability Theorem tells us that in essence, the defect of a residually transcendental function field, and more generally, of a function field without transcendence defect, can only come from the base field. The following general Independence Theorem was proved in [37, Theorem 5.4 and Corollary 5.6]:

Theorem 5.4. *Take a valued function field $(F|K,v)$ without transcendence defect, and set $\rho = \mathrm{rr}\, vF/vK$ and $\tau = \mathrm{trdeg}\, Fv|Kv$. The defect of the extension $(F^h|K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)^h,v)$ is independent of the choice of the elements $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau$ as long as they satisfy (5.2). Moreover, there is a finite extension $K'|K$ such that $(F^h.K'|K'(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)^h,v)$ is defectless and*

$$\mathrm{d}(F^h.K'|K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)^h,v) \;=\; \mathrm{d}(K^h.K'|K^h,v)\;.$$

A special case for simple transcendental extensions $(K(x)|K,v)$ satisfying the condition $\operatorname{trdeg} K(x)|K = \operatorname{rr} vK(x)/vK$ was proved by Sudesh Khanduja in [33].

An interesting proof of Theorem 5.3 is given in [23], as it introduces another notion of defect. We take any valued field extension $(L|K,v)$ and a finite-dimensional K-vector space $V \subseteq L$. We choose a system $\mathcal{V}$ of representatives of the cosets $va + vK$, $0 \neq a \in V$. For every K-vector space $W \subseteq V$ and every $\gamma \in \mathcal{V}$ we set $W_\gamma := \{a \in W \mid va \geq 0\}$ and $W_\gamma^\circ := \{a \in W \mid va > 0\}$. The quotient $W_\gamma / W_\gamma^\circ$ is in a natural way a Kv-vector space. The **vector space defect** of $(V|K,v)$ is defined as

$$\mathrm{d}^{\mathrm{vs}}(V|K,v) \;:=\; \sup_{W\subseteq V} \frac{\dim_K W}{\sum_{\gamma\in\mathcal{V}} \dim_{Kv} W_\gamma/W_\gamma^\circ}\;,$$

where the supremum runs over all finite-dimensional subspaces W. For a finite extension $(L|K,v)$, by [23, Proposition 2.2],

$$\mathrm{d}^{\mathrm{vs}}(L|K,v) \;=\; \frac{[L:K]}{(vL:vK)[Lv:Kv]}$$

which is equal to the ordinary defect $\mathrm{d}(L|K,v)$ if the extension of v from K to L is unique.

Note that quotients of the form W_γ/W_γ° also appear in the definition of the graded ring of a subring in a valued field, then often written as "$\mathcal{P}_\gamma/\mathcal{P}_\gamma^+$" (see, for instance, [64, \$2]). Graded rings are used by Bernard Teissier in his program for a "characteristic blind" local uniformization, see [62].

The following result ([23, Theorem 2.13]) implies the Independence Theorem 5.3:

Theorem 5.5. *For every element $t \in F$ such that tv is transcendental over Kv,*

$$\mathrm{d}^{\mathrm{vs}}(F|K,v) \;=\; (F^h|K(t)^h,v)\;.$$

Now we are ready to cite the genus inequality for an algebraic function field $F|K$ with distinct constant reductions $v_1,\ldots,v_s$ which have a common restriction to K. We assume in addition that K coincides with the constant field of $F|K$ (the relative algebraic closure of K in F). Then:

$$1 - g_F \leq 1 - s + \sum_{i=1}^{s} \mathrm{d}_i \mathrm{e}_i r_i (1-g_i) \tag{5.3}$$

where g_F is the genus of $F|K$ and g_i the genus of $Fv_i|Kv_i$, r_i is the degree of the constant field of $Fv_i|Kv_i$ over Kv_i, $\mathrm{d}_i = \mathrm{d}^{\mathrm{vs}}(F|K,v_i)$, and $\mathrm{e}_i = (v_iF : v_iK)$ (which is always finite in the constant reduction case, see, for instance, [42, Corollary 2.7]). It follows that constant reductions v_1, v_2 with common restriction to K and $g_1 = g_2 = g_F \geq 1$ must be equal. In other words, for a fixed valuation on K there is at most one extension v to F which is a **good reduction**, that is, (i) $g_F = g_{Fv}$, (ii) there exists $f \in F$ such that $vf = 0$ and $[F:K(f)] = [Fv:Kv(fv)]$, (iii) Kv is the constant field of $Fv|Kv$. An element f as in (ii) is called a **regular function**.

More generally, f is said to have the **uniqueness property** if fv is transcendental over Kv and the restriction of v to $K(f)$ has a unique extension to F. In this case, $[F : K(f)] = \mathrm{d} \cdot \mathrm{e} \cdot [Fv : Kv(fv)]$ where d is the defect of $(F^h|K^h, v)$ and $\mathrm{e} = (vF : vK(f)) = (vF : vK)$. If K is algebraically closed, then $\mathrm{e} = 1$, and it follows from the Stability Theorem that $\mathrm{d} = 1$; hence in this case, every element with the uniqueness property is regular.

It was proved in [24, Theorem 3.1] that F has an element with the uniqueness property already if the restriction of v to K is henselian. The proof uses Abraham Robinson's model completeness result for algebraically closed valued fields, and ultraproducts of function fields. Elements with the uniqueness property also exist if vF is a subgroup of $\mathbb{Q}$ and Kv is algebraic over a finite field. This follows from work in [25] where the uniqueness property is related to the **local Skolem property** which gives a criterion for the existence of algebraic v-adic integral solutions on geometrically integral varieties. This result is a special case of a theorem proved in [32] which states that elements with the uniqueness property exist if and only if the completion of (K, v) is henselian.

As an application to rigid analytic spaces, the Stability Theorem is used to prove that the quotient field of the free Tate algebra $T_n(K)$ is a defectless field, provided that K is. This in turn is used to deduce the **Grauert–Remmert Finiteness Theorem**, in a generalized version due to Gruson; see [9, pp. 214–220] for "a simplified version of Gruson's approach".

In contrast to the approaches that use methods of non-archimedean analysis, we give in [37], [38] and [45] a new proof which replaces the analytic methods by valuation theoretical arguments. Such arguments seem to be more adequate for a theorem that is of (Krull) valuation theoretical nature.

Our approach has much in common with Abhyankar's method of using ramification theory in order to reduce the question of resolution of singularities to the study of purely inseparable extensions and of Galois extensions of degree p and the search for suitable normal forms of Artin-Schreier-like minimal polynomials (cf. [2]). Given a finite separable extension $(L'|L, v)$ of henselian fields of positive characteristic, we can study its properties by lifting it up to the absolute ramification fields. From Lemma 4.17 we know that the defect of $(L'|L, v)$ is equal to the defect of $(L'.L^r|L^r, v)$. From Lemma 4.9 we know that the extension $L'.L^r|L^r$ is a tower of Artin-Schreier extensions.

Abhyankar's ramification thoretical reduction to Artin-Schreier extensions and purely inseparable extensions is also used by Vincent Cossart and Olivier Piltant in [13] to reduce resolution of singularities of threefolds in positive characteristic to local uniformization on Artin-Schreier and purely inseparable coverings. The Artin-Schreier extensions appearing through this reduction are not necessarily defect extensions. According to Piltant, those that are, are harder to treat than the defectless ones.

In the situation of Theorem 5.1, we have to prove for $L = F^h$ that $(L'|L, v)$ is defectless, or equivalently, that each Artin-Schreier extension in the tower is defectless. Looking at the first one in the tower, assume that it is generated by a root ϑ of a polynomial $X^p - X - a$ with $a \in L^r$. Using the additivity of the Frobenius in characteristic p, we see that the element $\vartheta - c$, which generates the same extension, has minimal polynomial $X^p - X - (a - c^p + c)$. Hence if a contains some p-th power c^p, we can replace it by c without changing the extension. Using this fact and the special structure of (F^h, v) given by the assumptions of Theorem 5.1 on (F, v), we deduce normal forms for a which allow us to read off that the extension is defectless. This fact in turn implies that $(F^h(\vartheta), v)$ is again of the same special form as (F^h, v), which enables us to proceed by induction over the extensions in the tower.

Note that when algebraic geometers work with Artin-Schreier extensions they usually work with polynomials of the form $X^p - dX - a$. The reason is that they work over rings and not over fields. A polynomial like $X^p - b^{p-1}X - a$ over a ring R can be transformed into the polynomial $X^p - X - a/b^p$, as we have seen in Example 3.17, but a/b^p does in general not lie in the ring anymore. Working with a polynomial of the form $X^p - X - a$ is somewhat easier than with a polynomial of the form $X^p - dX - a$, and it suffices to derive normal forms as needed for the proof of Theorem 5.1, and of Theorem 5.10 which we will discuss below.

In the case of mixed characteristic, where the valued fields have characteristic 0 and their residue fields have positive characteristic, Artin-Schreier extensions are replaced by Kummer extensions (although re-written with corresponding Artin-Schreier polynomials), and additivity is replaced by quasi-additivity (cf. Lemma 3.19).

Related normal form results can be found in the work of Helmut Hasse, George Whaples, and in Matignon's proof of his version of Theorem 5.1. See also Helmut Epp's paper [20], in particular the proof of Theorem 1.3. This proof contains a gap which was filled in [43].

Let us reconsider Examples 2.2 and 2.7 in the light of Theorem 5.1. In Example 2.2 we have an extension without transcendence defect if and only the transcendence degree is 1. In this case, $(K(t)^h, v)$ is a defectless field, and we have that $F \subset K(t)^h$. In the case of higher transcendence degree, this may not be the case, as Example 3.1 shows. At least we know that every separable extension of $K(T)^h$ is defectless since it is discretely valued. The situation is different in Example 2.7. If the extension is without transcendence defect, then again, $(K(t)^h, v)$ is a defectless field, and moreover, vF/vK and $Fv|Kv$ are finitely generated ([42, Corollary 2.7]). But if $\operatorname{char} K > 0$, then there are valuations v on $K(x, y)$, trivial on K, such that $K(x, y)v = K$, $vK(x, y)$ not finitely generated, and such that $(K(x, y), v)$ admits an infinite tower of Artin-Schreier defect extensions ([42, Theorem 1.2]).

Applications of Theorem 5.1 are:

- **Elimination of ramification.** In [34] we use Theorem 5.1 to prove:

Theorem 5.6. *Take a defectless field* (K,v) *and a valued function field* $(F|K,v)$ *without transcendence defect. Assume that* $Fv|Kv$ *is a separable extension and* vF/vK *is torsion free. Then* $(F|K,v)$ *admits elimination of ramification in the following sense: there is a transcendence basis* $T=\{x_1,\ldots,x_r,y_1,\ldots,y_s\}$ *of* $(F|K,v)$ *such that*

a) $vF = vK \oplus \mathbb{Z}vx_1 \oplus \ldots \oplus \mathbb{Z}vx_r$,

b) $y_1v,\ldots,y_sv$ *form a separating transcendence basis of* $Fv|Kv$.

For each such transcendence basis T *and every extension of* v *to the algebraic closure of* F, $(F^h|K(T)^h,v)$ *is unramified.*

Corollary 5.7. *Let* $(F|K,v)$ *be a valued function field without transcendence defect. Fix an extension of* v *to* $\tilde{F}$. *Then there is a finite extension* $L_0|K$ *and a transcendence basis* T *of* $(L_0.F|L_0,v)$ *such that for every algebraic extension* L *of* K *containing* L_0, *the extension* $((L.F)^h|L(T)^h,v)$ *is unramified.*

• **Local uniformization in positive and in mixed characteristic.** We consider places P and their associated valuations $v=v_P$ of a function field $F|K$, by which we mean that $P|_K$ is the identity and hence $v|_K$ is trivial. We write $aP=av$ and denote by $\mathcal{O}$ the valuation ring of v on F. Rewriting our earlier definition, we say that P admits **smooth local uniformization** if there is a model for F on which P is centered at a smooth point, that is, if there are $x_1,\ldots,x_n\in\mathcal{O}$ such that $F=K(x_1,\ldots,x_n)$ and the point $x_1P,\ldots,x_nP$ is smooth. (Note that in [34] and [35] we add a further condition, which we drop here for simplicity.) The place P is called an **Abhyankar place** if equality holds in the Abhyankar inequality, which in the present case means that $\operatorname{trdeg} F|K = \operatorname{rr} vF + \operatorname{trdeg} FP|K$.

Theorem 5.6 is a crucial ingredient for the following result (cf. [34, Theorem 1.1], [39]):

Theorem 5.8. *Assume that* P *is an Abhyankar place of the function field* $F|K$ *such that* $FP|K$ *is separable. Then* P *admits smooth local uniformization.*

The analogous arithmetic case ([34, Theorem 1.2]) uses Theorem 5.1 in mixed characteristic. Note that the condition "$FP|K$ is separable" is necessary since it is implied by elimination of ramification.

• **Model theory of valued fields.** In [47] we use Theorem 5.6 to prove the following Ax–Kochen–Ershov Principle:

Theorem 5.9. *Take a henselian defectless valued field* (K,v) *and an extension* $(L|K,v)$ *of finite transcendence degree without transcendence defect. If* vK *is existentially closed in* vL *and* Kv *is existentially closed in* Lv, *then* (K,v) *is existentially closed in* (L,v).

Let us continue our discussion from the end of Section 2.2. The conditions "$vK \prec_\exists vL$" and "$Kv \prec_\exists Lv$" imply that vF/vK is torsion free and $Fv|Kv$ is a separable extension. So we can apply Theorem 5.6 to obtain the transcendence basis $T = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ of $(F|K, v)$ with the properties as specified in that theorem. Because of these properties, the embeddings of vL in v^*K^* and of Lv in K^*v^* lift to an embedding ι_0 of $(K(T), v)$ in (K^*, v^*) over K. Using Hensel's Lemma and the fact that $Fv|K(T)v$ is separable, one finds in $F^h|K(T)$ a subextension $F_1|K(T)$ with $F_1v = Fv$ and $[F_1 : K(T)] = [F_1v : K(T)v]$ ($Fv|K(T)v$ is finite by the remark preceding Theorem 5.1). Using Hensel's Lemma and the embedding of Lv in K^*v^* again, one extends ι_0 to an embedding ι_1 of (F_1, v) in (K^*, v^*). The extension $(F^h|F_1^h, v)$ is immediate, and as it is an extension inside the unramified extension $(F^h|K(T)^h, v)$, it must be defectless and hence trivial. As (K^*, v^*) is henselian, being an elementary extension of the henselian field (K, v), one can now use the universal property of henselizations to extend ι_1 to an embedding ι_2 of (F^h, v) in (K^*, v^*). The restriction of ι_2 to F is the desired embedding which transfers every existential sentence valid in (F, v) into (K^*, v^*).

5.2. **Henselian Rationality of Immediate Function Fields.** Let us return to Example 2.8. If (F, v) does not lie in the henselization $K(x)^h$, we are lost. This happens if and only if $(F^h|K(x)^h, v)$ has non-trivial defect (the equivalence holds because $(F|K(x), v)$ is finite and immediate, $F^h = F.K(x)^h$ and henselizations are immediate extensions).

So the question arises: how can we avoid the defect in the case of immediate extensions? The answer is a theorem proved in [37] (cf. [38] and [48]):

Theorem 5.10. (Henselian Rationality)
Let (K, v) be a tame field and $(F|K, v)$ an immediate function field of transcendence degree 1. Then

$$\text{there is } x \in F \text{ such that } (F^h, v) = (K(x)^h, v)\ , \tag{5.4}$$

that is, $(F|K, v)$ is henselian generated. The same holds over a separably tame field (K, v) if in addition $F|K$ is separable.

Since the assertion says that F^h is equal to the henselization of a rational function field, we also call F **henselian rational** in this case. For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless. Indeed, take any $x \in F \setminus K$. Then $K(x)|K$ cannot be algebraic since otherwise, $(K(x)|K, v)$ would be a proper finite immediate (and hence defect) extension of the tame field (K, v), a contradiction to the definition of "tame". Hence, $F|K(x)$ is algebraic and immediate. Therefore, $(F^h|K(x)^h, v)$ is algebraic and immediate too. But since it cannot have a non-trivial defect, it must be trivial. This proves that $(F, v) \subset (K(x)^h, v)$. In contrast to this, in the case of positive residue characteristic only a very carefully chosen $x \in F \setminus K$ will do the job. As for the Generalized Stability Theorem, the proof of Theorem 5.10 in positive characteristic uses ramification theory and the deduction of normal forms for Artin-Schreier extensions. This time however, all Artin-Schreier extensions are immediate and hence defect extensions. The normal forms serve a different purpose, namely, to find a suitable generator x. The proof also uses significantly a theory of immediate extensions which builds on Kaplansky's paper [29, Sections 2 and 3].

Open problem (HR): Improve Theorem 5.10 by finding versions that work with weaker assumptions. For instance, can the assumption "tame" be replaced by "henselian and perfect" or just "perfect", or can it even be dropped altogether? Then, even with a weaker assumption on (K,v), can the assumption "immediate" be replaced by "vF/vK is a torsion group and $Fv = Kv$"?

Note that in order to allow $Fv|Kv$ to be any algebraic extension, a possible generalization of Theorem 5.10 would have to replace (5.4) by

$$\text{there is } x \in F \text{ such that } \; (F^i,v) \;=\; (K(x)^i,v)\;. \tag{5.5}$$

Applications of Theorem 5.10 in conjunction with Theorem 5.1 are:

• **Local uniformization in positive and in mixed characteristic.** Theorem 5.10 together with Theorem 5.6 is a crucial ingredient for the proof of "local uniformization by alteration" (cf. [35, Theorem 1.2], [39]):

Theorem 5.11. *Assume that P is a place of the function field $F|K$. Then there is a finite extension $F'|F$ and an extension P' of P from F to F' such that P' admits smooth local uniformization. The extension $F'|F$ can be chosen to be Galois. Alternatively, it can be chosen such that $(F',P')|(F,P)$ is purely wild, hence $v_{P'}F'/v_PF$ is a p-group and $F'P'|FP$ is purely inseparable.*

The analogous arithmetic case ([35, Theorem 1.4]) uses Theorems 5.10 and 5.1 in mixed characteristic. While local uniformization by alteration follows from de Jong's resolution of singularities by alteration (see [3]), the additional information on the extension $F'|F$ does not follow. Moreover, the proofs of Theorems 5.8 and 5.11 use only valuation theory.

Recently, Michael Temkin ([61, Corollary 1.3]) proved "Inseparable Local Uniformization":

Theorem 5.12. *In the setting of Theorem 5.11, the extension $F'|F$ can also be chosen to be purely inseparable.*

It is interesting that local uniformization has now been proved up to separable alteration on the one hand, and up to purely inseparable alteration on the other. These two results are somewhat "orthogonal" to each other. Can they be put together to get rid of alteration? While this appears to be an attractive thought at first sight, one should keep in mind Example 3.17 which shows that every purely inseparable defect extension of degree p of (L,v) which does not lie in the completion of (L,v) can be transformed into an Artin-Schreier defect extension. Thus, the "same" defect may appear in a separable extension and in a purely inseparable extension (see the next section for details), which leaves us the choice to kill it either with separable or with inseparable alteration. So this fact does not in itself indicate whether we need or do not need alteration for local uniformization.

Open problem (LU): Prove (or disprove) local uniformization without extension of the function field.

In fact, one reason for the extension of the function field in our approach is the fact that we apply Theorem 5.10 to fields of lower transcendence degree than the function field itself. However, subfunction fields are too small to be tame fields, so we enlarge our intermediate fields so that they become (separably) tame, and once we have found local uniformization in this larger configuration, we collect the only finitely many new elements that are needed for it and adjoin them to the original function field. So we see that if we can weaken the assumptions of Theorem 5.10, then possibly we will need smaller extensions of our function field. Temkin's work contains several developments in this direction, one of which we will discuss in more detail in the next section.

• **Model theory of valued fields.** In [47] we use Theorem 5.10 together with Theorem 5.6 to prove the following:

Theorem 5.13. *a) If (K,v) is a tame field, then the Ax–Kochen–Ershov Principle (2.3) holds.*

b) The Classification Problem for valued fields has a positive solution for tame fields: If (K,v) and (L,v) are tame fields such that vK and vL are elementarily equivalent as fields and Kv and Lv are elementarily equivalent as ordered groups, then (K,v) and (L,v) are elementarily equivalent as valued fields.

This theorem comprises several classes of valued fields for which the classification had already been known to hold, such as the already mentioned henselian fields with residue fields of characteristic 0.

Open problem (AKE): Prove Ax–Kochen–Ershov Principles for classes of non-perfect valued fields of positive characteristic. This problem is connected with the open problem whether the elementary theory of $\mathbb{F}_p((t))$ is decidable (cf. [38], [41], [47]).

6. Two types of Artin-Schreier defect extensions

In this section, we assume all fields to have characteristic $p>0$. In Section 3 we have given several examples of Artin-Schreier defect extensions, i.e., Artin-Schreier defect extensions with non-trivial defect. Note that every such extension is immediate. Some of our examples were derived from immediate purely inseparable extensions of degree p (Examples 3.17 and 3.18). If an Artin-Schreier defect extension is derived from a purely inseparable defect extension of degree p as in Example 3.17, then we call it a **dependent Artin-Schreier defect extension**. If it cannot be derived in this way, then we call it an **independent Artin-Schreier defect extension**. More precisely, an Artin-Schreier defect extension $(L(\vartheta)|L,v)$ with $\vartheta^p-\vartheta\in L$ is defined to be dependent if there is a purely inseparable extension $(L(\eta)|L,v)$ of degree p such that

$$\text{for all } c\in L,\;\; v(\vartheta-c)\;=\;v(\eta-c)\;.$$

The extension $(L(\vartheta)|L,v)$ constructed in Example 3.12 is an independent Artin-Schreier defect extension. This is obvious if we choose $K=\mathbb{F}_p(t)$ or $K=\mathbb{F}_p((t))$ because then L is the perfect hull of K and does not admit any purely inseparable defect extensions at all. But if for instance, $K=\mathbb{F}_p(t,s)$ with $s\in\mathbb{F}_p((t))$ transcendental over $\mathbb{F}_p(t)$, then L is not perfect. How do we know then that $(L(\vartheta)|L,v)$ is an independent Artin-Schreier defect extension? The answer is given by the following characterization proved in [37] (see [38] and [46]):

Theorem 6.1. *Take an Artin-Schreier defect extension $(L(\vartheta)|L,v)$ with $\vartheta^p-\vartheta\in L$. Then this extension is independent if and only if*

$$v(\vartheta-L)\;+\;v(\vartheta-L)\;=\;v(\vartheta-L)\;. \tag{6.1}$$

Note that $v(\vartheta-L)+v(\vartheta-L):=\{\alpha+\beta\mid\alpha,\beta\in v(\vartheta-L)\}$ and that the sum of two initial segments of a value group is again an initial segment. Equation (6.1) means that $v(\vartheta-L)$ defines a cut in vL which is idempotent under addition of cuts (defined through addition of the left cut sets). If vL is archimedean, then there are only four possible idempotent cuts, corresponding to $v(\vartheta-L)=\emptyset$ (which is impossible), $v(\vartheta-L)=(vL)^{<0}$, $v(\vartheta-L)=(vL)^{\leq 0}$, and $v(\vartheta-L)=vL$ (which means that ϑ lies in the completion of (L,v)).

It is important to note that $v(\vartheta-K)\subseteq(vL)^{<0}$. Indeed, if there were some $c\in K$ such that $v(\vartheta-c)\geq 0$, then

$$0\;\leq\;v((\vartheta-c)^p-(\vartheta-c))\;\leq\;v(\vartheta^p-\vartheta-(c^p-c))\;.$$

But a polynomial X^p-X-a with $va\geq 0$ splits completely in the absolute inertia field of (L,v) and thus cannot induce a defect extension. Therefore, if vL is archimedean, then (6.1) holds if and only if $v(\vartheta-L)=(vL)^{<0}$. This shows that the extension $(L(\vartheta)|L,v)$ of Example 3.12 is an independent Artin-Schreier defect extension even if L is not perfect. On the other hand, the extension $(L(\vartheta_0)|L,v)$ of Example 3.17, where ϑ_0/b is a root of the polynomial X^p-X-1/b^pt, is a dependent Artin-Schreier defect extension as it was obtained from the purely inseparable defect extension $(L(\eta)|L,v)$. And indeed,

$$\begin{aligned}v\left(\frac{\vartheta_0}{b}-L\right)+v\left(\frac{\vartheta_0}{b}-L\right)\;&=\;\{\alpha\in vL\mid\alpha<vb\}\;+\;\{\alpha\in vL\mid\alpha<vb\}\\&\neq\;\{\alpha\in vL\mid\alpha<vb\}\end{aligned}$$

since $vb\neq 0$. Note that since vL is p-divisible, we in fact have that

$$\{\alpha\in vL\mid\alpha<vb\}+\{\alpha\in vL\mid\alpha<vb\}\;=\;\{\alpha\in vL\mid\alpha<2vb\}\;.$$

This example also shows that the criterion of Theorem 6.1 only works for roots of Artin-Schreier polynomials. Indeed, $v(\vartheta_0-L)=v(\eta-L)=(vL)^{<0}$, which does not contradict the theorem since the minimal polynomial of ϑ_0 is X^p-dX-1/t with $vd\neq 0$.

Each of the perfect fields (L,v) from Example 3.12 provides an example of a valued field without dependent Artin-Schreier defect extensions, but admitting an independent Artin-Schreier defect extension. Valued fields without independent Artin-Schreier defect extensions but admitting dependent Artin-Schreier defect extensions are harder to find; one example is given in [46].

The classification of Artin-Schreier defect extensions and Theorem 6.1 are the main tool in the proof of the following criterion ([37]; see [38] and [46]):

Theorem 6.2. *A valued field (L,v) of positive characteristic is henselian and defectless if and only if it is algebraically maximal and inseparably defectless.*

This criterion is very useful when one tries to construct examples of defectless fields with certain properties, as was done in [41, Section 4]. How can one construct defectless fields? One possibility is to take any valued field and pass to its maximal immediate extension. Every maximal field is defectless. But it is in general an extension of very large transcendence degree. If we want something smaller, then we could use Theorem 5.1. But that talks only about function fields (or their henselizations). If we want to construct a field with a certain value group (as in [41]), we may have to pass to an infinite algebraic extension. If we replace that by any of its maximal immediate algebraic extensions, we obtain an algebraically maximal field (M,v). But Example 3.25 shows that such a field may not be defectless. If, however, we can make sure that (M,v) is also inseparably defectless, then Theorem 6.2 tells us that (M,v) is defectless.

How do we know that a valued field (L,v) is inseparably defectless? In the case of finite **p-degree** $[K:K^p]$ (also called **Ershov invariant of** K), Delon ([16]) gave a handy characterization of inseparably defectless valued fields:

Theorem 6.3. *Let L be a field of characteristic $p>0$ and finite p-degree $[L:L^p]$. Then for the valued field (L,v), the property of being inseparably defectless is equivalent to each of the following properties:*

a) $[L:L^p]=(vL:pvL)[Lv:Lv^p]$, *i.e.,* $(L|L^p,v)$ *is a defectless extension*

b) $(L^{1/p}|L,v)$ *is a defectless extension*

c) *every immediate extension of* (L,v) *is separable*

d) *there is a separable maximal immediate extension of* (L,v).

The very useful upward direction of the following lemma was also stated by Delon ([16], Proposition 1.44):

Lemma 6.4. *Let $(L'|L,v)$ be a finite extension of valued fields of characteristic $p>0$. Then (L,v) is inseparably defectless and of finite p-degree if and only if (L',v) is.*

The condition of finite p-degree is necessary, as Example 3.23 shows. In that example, $(k((t))|K,v)$ is a purely inseparable defect extension of degree p. Hence (K,v) is not inseparably defectless. But $(k((t)),v)$ is, since it is a maximal and hence defectless field.

Work in progress.

a) An analogue of the classification of Artin-Schreier defect extensions and of Theorem 6.2 has to be developed for the mixed characteristic case (valued fields of characteristic 0 with residue fields of characteristic p). Temkin and other authors have already done part of the work.

b) The classification of Artin-Schreier defect extensions is also reflected in their higher ramification groups. This will be worked out in [50].

We have seen in Example 3.17 that every purely inseparable defect extension of degree p of (L,v) which does not lie in the completion of (L,v) can be transformed into a (dependent) Artin-Schreier defect extension. This can be used to prove the following result (cf. [46]):

Proposition 6.5. *Assume that (L,v) does not admit any dependent Artin-Schreier extension. Then every immediate purely inseparable extension lies in the completion of (L,v).*

Corollary 6.6. *Every non-trivially valued Artin-Schreier closed field lies dense in its perfect hull. In particular, the algebraic closure of a non-trivially valued separable-algebraically closed field (L,v) lies in the completion of (L,v).*

Which of the Artin-Schreier defect extensions are the more harmful, the dependent or the independent ones? There are some indications that the dependent ones are more harmful. Temkin's work (especially [61, Theorem 3.2.3]) seems to indicate that there is a generalization of Theorem 5.10 which already works over henselian perfect instead of tame valued fields (K,v). When K is perfect, then (K,v) does not have dependent Artin-Schreier defect extensions. The independent ones do not seem to matter here, at least when (K,v) has rank 1. This improvement is one of the keys to Theorem 5.12. Let us give an example which illustrates what is going on here.

Example 6.7. Assume that $(F|K,v)$ is an immediate function field of transcendence degree 1, rank 1 and characteristic $p>0$, and that we have chosen $x\in F$ such that $(F^h|K(x)^h,v)$ is of degree p. We want to improve our choice of x, that is, find $y\in F$ such that $F^h=K(y)^h$. The procedure given in [37, 48] uses the fact that because $(F|K,v)$ is immediate, x is a **pseudo limit** of a **pseudo Cauchy sequence** $(a_\nu)_{\nu<\lambda}$ in K which has no pseudo limit in K ([29, Section 2]). The hypothesis that (K,v) be tame (or separably tame and $F|K$ separable) guarantees that $(a_\nu)_{\nu<\lambda}$ is of **transcendental type**, that is, if f is any polynomial in one variable over K, then the value $vf(a_\nu)$ is fixed for all large enough $\nu<\lambda$ ([29, p. 306]). This is essential in our procedure. If we drop the tameness hypothesis, then $(a_\nu)_{\nu<\lambda}$ may not be of transcendental type and may in fact have some element in some immediate algebraic extension of (K,v) as a pseudo limit. Now suppose that this element is the root ϑ of an Artin-Schreier polynomial over K. The fact that both x and ϑ are limits of $(a_\nu)_{\nu<\lambda}$ implies that $v(x-\vartheta)>v(\vartheta-K)$. If the Artin-Schreier defect extension $(K(\vartheta)|K,v)$ is independent, then because of our rank 1 assumption, it follows that $v(x-\vartheta)\geq 0$. If we assume in addition that Kv is algebraically closed, then there is some $c\in K$ such that $v(x-\vartheta-c)>0$. But then, by a special version of Krasner's Lemma (cf. [42, Lemma 2.21]), the polynomial $X^p-X-(\vartheta^p-\vartheta)$ splits in $(K(x)^h,v)$, so that $\vartheta\in K(x)^h$. This shows that K is not relatively algebraically closed in $K(x)^h$. Replacing K by its relative algebraic closure in $K(x)^h$, we will avoid this special case of pseudo Cauchy sequences that are not of transcendental type.

If on the other hand, the extension $(K(\vartheta)|K,v)$ is dependent, then it does not follow that $v(x-\vartheta)\geq 0$. But if $v(x-\vartheta)<0$, Krasner's Lemma is of no use. However, by assuming that K is perfect we obtain that (K,v) does not admit dependent Artin-Schreier defect extensions. Assuming in addition that K is relatively algebraically closed in F^h, we obtain that $(a_\nu)_{\nu<\lambda}$ does not admit any Artin-Schreier root ϑ over K as a limit. This fact alone does not imply that under our additional assumptions, $(a_\nu)_{\nu<\lambda}$ must be of transcendental type. But with some more technical effort, building on results in [46], this can be shown to be true. $\diamondsuit$

Another indication may come from the paper [15] by Steven Dale Cutkosky and Olivier Piltant. They give an example of an extension $R\subset S$ of algebraic regular local rings of dimension two over a field k of positive characteristic and a valuation on the rational function field $\operatorname{Quot}R$, with $\operatorname{Quot}S|\operatorname{Quot}R$ being a tower of two Artin-Schreier defect extensions, such that strong monomialization in the sense of Theorem 4.8 of their paper does not hold for $R\subset S$ ([15, Theorem 7.38]).

Work in progress with Laura Ghezzi and Samar El-Hitti indicates that both extensions are dependent Artin-Schreier defect extensions. In fact, in Piltant's own words, he chose the valuation in the example such that it is "very close" to [the behaviour of] a valuation in a purely inseparable extension.

Open problem (CPE): Is there a version of the example of Cutkosky and Piltant involving independent Artin-Schreier defect extensions? Or are such extensions indeed less harmful than the dependent ones? Can strong monomialization always be proven when only independent Artin-Schreier defect extensions are involved?

7. Two languages

Algebraic geometers and valuation theorists often speak different languages. For example, while the defect was implicitly present already in Abhyankar's work, it has been explicitly studied rather by the early valuation theorists like Ostrowski, and by researchers interested in the model theory of valued fields in positive characteristic or in constant reduction, most of them using a field theoretic language and "living in the Kaplansky world of pseudo Cauchy sequences" (cf. [29, Section 2], [38]).

For instance, our joint investigation with Ghezzi and ElHitti of the example given by Cutkosky and Piltant is facing the difficulty that the valuation in the example is given by means of generating sequences, whereas our criterion for dependence/independence is by nature closer to the world of pseudo Cauchy sequences, which can also be used to describe valuations.

Open problem (CGS): Rewrite the criterion given in Theorem 6.1 in terms of generating sequences.

As to the problem of how to describe valuations, Michel Vaquié has done much work generalizing MacLane's approach using families of key polynomials. In this approach, he also showed how to read off defects as invariants of such families (see [63]). A closer look reveals that a set like $v(\vartheta - L)$ can be directly determined from Vaquié's families of key polynomials.

Open problem (CV): Is there an efficient algorithm to convert generating sequences into Vaquié's families of key polynomials? More generally, find algorithms that convert between generating sequences, key polynomials, pseudo Cauchy sequences and higher ramification groups.

A lot of work has been done by several authors on the description of valuations on rational function fields, working with key polynomials or pseudo Cauchy sequences. For references, see [42].

Open problem (RFF): Develop a thorough theory of valuations on rational function fields, bringing the different approaches listed in (CV) together, then generalize to algebraic function fields. Understand the defect extensions that can appear over rational function fields.

Problems (CGV), (CV) and (RFF) can be understood as parts of a larger program:

Open problem (DIC): Develop a "dictionary" between algebraic geometry and valuation theory. This would allow us to translate results known about the defect into results in algebraic geometry, and open questions in algebraic geometry into questions in valuation theory. It would help us to investigate critical examples from several points of view and to use them both in algebraic geometry and valuation theory.

Let us conclude with the following

Open problem (DAG): What exactly is the meaning of the defect in algebraic geometry? How can we locate and interpret it? What is the role of dependent and independent Artin-Schreier defect extensions, e.g., in the work of Abhyankar, of of Cutkosky and Piltant, or of Cossart and Piltant?

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