

THE ROLE OF DEFECT AND SPLITTING IN FINITE GENERATION OF EXTENSIONS OF ASSOCIATED GRADED RINGS ALONG A VALUATION

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ABSTRACT. Suppose that R is a 2 dimensional excellent local domain with quotient field K , K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S . Let ν be the restriction of ν^* to K . The associated graded ring $\text{gr}_\nu(R)$ was introduced by Bernard Teissier. It plays an important role in local uniformization. We show in Theorem 0.1 that the extension $(K, \nu) \rightarrow (K^*, \nu^*)$ of valued fields is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blow up of R , S_1 is a local ring of a blowup of S , ν^* dominates S_1 , S_1 dominates R_1 and the associated graded ring $\text{gr}_{\nu^*}(S_1)$ is a finitely generated $\text{gr}_\nu(R_1)$ -algebra.

We also investigate the role of splitting of the valuation ν in K^* in finite generation of the extensions of associated graded rings along the valuation. We will say that ν does not split in S if ν^* is the unique extension of ν to K^* which dominates S . We show in Theorem 0.5 that if R and S are regular local rings, ν^* has rational rank 1 and is not discrete and $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_\nu(R)$ -algebra, then ν does not split in S . We give examples showing that such a strong statement is not true when ν does not satisfy these assumptions. As a consequence of Theorem 0.5, we deduce in Corollary 0.6 that if ν has rational rank 1 and is not discrete and if $R \rightarrow R'$ is a nontrivial sequence of quadratic transforms along ν , then $\text{gr}_\nu(R')$ is not a finitely generated $\text{gr}_\nu(R)$ -algebra.

Suppose that K is a field. Associated to a valuation ν of K is a value group Φ_ν and a valuation ring V_ν with maximal ideal m_ν . Let R be a local domain with quotient field K . We say that ν dominates R if $R \subset V_\nu$ and $m_\nu \cap R = m_R$ where m_R is the maximal ideal of R . We have an associated semigroup $S^R(\nu) = \{\nu(f) \mid f \in R\}$, as well as the associated graded ring along the valuation

$$(1) \quad \text{gr}_\nu(R) = \bigoplus_{\gamma \in \Phi_\nu} \mathcal{P}_\gamma(R) / \mathcal{P}_\gamma^+(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R) / \mathcal{P}_\gamma^+(R)$$

which is defined by Teissier in [44]. Here

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\} \text{ and } \mathcal{P}_\gamma^+(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

This ring plays an important role in local uniformization of singularities ([44] and [45]). The ring $\text{gr}_\nu(R)$ is a domain, but it is often not Noetherian, even when R is.

Suppose that $K \rightarrow K^*$ is a finite extension of fields and ν^* is a valuation which is an extension of ν to K^* . We have the classical indices

$$e(\nu^*/\nu) = [\Phi_{\nu^*} : \Phi_\nu] \text{ and } f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$$

as well as the defect $\delta(\nu^*/\nu)$ of the extension. Ramification of valuations and the defect are discussed in Chapter VI of [49], [21] and Kuhlmann's papers [33] and [35]. A survey

partially supported by NSF.

is given in Section 7.1 of [16]. By Ostrowski's lemma, if ν^* is the unique extension of ν to K^* , we have that

$$(2) \quad [K^* : K] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

where p is the characteristic of the residue field V_ν/m_ν . From this formula, the defect can be computed using Galois theory in an arbitrary finite extension. If V_ν/m_ν has characteristic 0, then $\delta(\nu^*/\nu) = 0$ and $p^{\delta(\nu^*/\nu)} = 1$, so there is no defect. Further, if $\Phi_\nu = \mathbb{Z}$ and K^* is separable over K then there is no defect.

If K is an algebraic function field over a field k , then an algebraic local ring R of K is a local domain which is essentially of finite type over k and has K as its field of fractions. In [10], it is shown that if $K \rightarrow K^*$ is a finite extension of algebraic function fields over a field k of characteristic zero, ν^* is a valuation of K^* (which is trivial on k) with restriction ν to K and if $R \rightarrow S$ is an inclusion of algebraic regular local rings of K and K^* such that ν^* dominates S and S dominates R then there exists a commutative diagram

$$(3) \quad \begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

where the vertical arrows are products of blowups of nonsingular subschemes along the valuation ν^* (monoidal transforms) and $R_1 \rightarrow S_1$ is dominated by ν^* and is a monomial mapping; that is, there exist regular parameters x_1, \dots, x_n in R_1 , regular parameters y_1, \dots, y_n in S_1 , units $\delta_i \in S_1$, and a matrix $A = (a_{ij})$ of natural numbers with $\text{Det}(A) \neq 0$ such that

$$(4) \quad x_i = \delta_i \prod_{j=1}^n y^{a_{ij}} \text{ for } 1 \leq i \leq n.$$

In [16], it is shown that this theorem is true, giving a monomial form of the mapping (4) after appropriate blowing up (3) along the valuation, if $K \rightarrow K^*$ is a separable extension of two dimension algebraic function fields over an algebraically closed field, which has no defect. This result is generalized to the situation of this paper, that is when R is a two dimensional excellent local ring, in [14]. However, it may be that such monomial forms do not exist, even after blowing up, if the extension has defect, as is shown by examples in [12].

In the case when k has characteristic zero and for separable defectless extensions of two dimensional algebraic function fields in positive characteristic, it is further shown in [16] that the expressions (3) and (4) are stable under further simple sequences of blow ups along ν^* and the form of the matrix A stably reflects invariants of the valuation.

We always have an inclusion of graded domains $\text{gr}_\nu(R) \rightarrow \text{gr}_{\nu^*}(S)$ and the index of their quotient fields is

$$(5) \quad [\text{QF}(\text{gr}_{\nu^*}(S)) : \text{QF}(\text{gr}_\nu(R))] = e(\nu^*/\nu)f(\nu^*/\nu)$$

as shown in Proposition 3.3 [13]. Comparing with Ostrowski's lemma (2), we see that the defect has disappeared in equation (5).

Even though $\text{QF}(\text{gr}_{\nu^*}(S))$ is finite over $\text{QF}(\text{gr}_\nu(R))$, it is possible for $\text{gr}_{\nu^*}(S)$ to not be a finitely generated $\text{gr}_\nu(R)$ -algebra. Examples showing this for extensions $R \rightarrow S$ of two dimensional algebraic local rings over arbitrary algebraically closed fields are given in Example 9.4 of [17].

It was shown by Ghezzi, Hà and Kashcheyeva in [23] for extensions of two dimensional algebraic function fields over an algebraically closed field k of characteristic zero and later

by Ghezzi and Kashcheyeva in [24] for defectless separable extensions of two dimensional algebraic functions fields over an algebraically closed field k of positive characteristic that there exists a commutative diagram (3) such that $\text{gr}_{\nu^*}(S_1)$ is a finitely generated $\text{gr}_{\nu}(R_1)$ -algebra. Further, this property is stable under further suitable sequences of blow ups.

In Theorem 1.6 [13], it is shown that for algebraic regular local rings of arbitrary dimension, if the ground field k is algebraically closed of characteristic zero, and the valuation has rank 1 and is zero dimensional ($V_{\nu}/m_{\nu} = k$) then we can also construct a commutative diagram (3) such that $\text{gr}_{\nu^*}(S_1)$ is a finitely generated $\text{gr}_{\nu}(R_1)$ -algebra and this property is stable under further suitable sequences of blow ups.

An example is given in [8] of an inclusion $R \rightarrow S$ in a separable defect extension of two dimensional algebraic function fields such that $\text{gr}_{\nu^*}(S_1)$ is stably not a finitely generated $\text{gr}_{\nu}(R_1)$ -algebra in diagram (3) under sequences of blow ups. This raises the question of whether the existence of a finitely generated extension of associated graded rings along the valuation implies that K^* is a defectless extension of K .

We find that we must impose the condition that K^* is a separable extension of K to obtain a positive answer to this question, as there are simple examples of inseparable defect extensions such that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra, such as in the following example, which is Example 8.6 [33]. Let k be a field of characteristic $p > 0$ and $k((x))$ be the field of formal power series over k , with the x -adic valuation ν_x . Let $y \in k((x))$ be transcendental over $k(x)$ with $\nu_x(y) > 0$. Let $\tilde{y} = y^p$, and $K = k(x, \tilde{y}) \subset K^* = k(x, y)$. Let $\nu^* = \nu_x|_{K^*}$ and $\nu = \nu_x|_K$. Then we have equality of value groups $\Phi_{\nu} = \Phi_{\nu^*} = \nu(x)\mathbb{Z}$ and equality of residue fields of valuation rings $V_{\nu}/m_{\nu} = V_{\nu^*}/m_{\nu^*} = k$, so $e(\nu^*/\nu) = 1$ and $f(\nu^*/\nu) = 1$. We have that ν^* is the unique extension of ν to K^* since K^* is purely inseparable over K . By Ostrowski's lemma (2), the extension $(K, \nu) \rightarrow (K^*, \nu^*)$ is a defect extension with defect $\delta(\nu^*/\nu) = 1$. Let $R = k[x, \tilde{y}]_{(x, \tilde{y})} \rightarrow S = k[x, y]_{(x, y)}$. Then we have equality

$$\text{gr}_{\nu}(R) = k[t] = \text{gr}_{\nu^*}(S)$$

where t is the class of x .

In this paper we show that the question does have a positive answer for separable extensions in the following theorem.

Theorem 0.1. *Suppose that R is a 2 dimensional excellent local domain with quotient field K . Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S . Let ν be the restriction of ν^* to K . Then the extension $(K, \nu) \rightarrow (K^*, \nu^*)$ is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blow up of R , S_1 is a local ring of a blowup of S , ν^* dominates S_1 , S_1 dominates R_1 and $\text{gr}_{\nu^*}(S_1)$ is a finitely generated $\text{gr}_{\nu}(R_1)$ -algebra.*

We immediately obtain the following corollary for two dimensional algebraic function fields.

Corollary 0.2. *Suppose that $K \rightarrow K^*$ is a finite separable extension of two dimensional algebraic function fields over a field k and ν^* is a valuation of K^* with restriction ν to K . Then the extension $(K, \nu) \rightarrow (K^*, \nu^*)$ is without defect if and only if there exist algebraic regular local rings R of K and S of K^* such that ν^* dominates S , S dominates R and $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra.*

We see from Theorem 0.1 that the defect, which is completely lost in the extension of quotient fields of the associated graded rings along the valuation (5), can be recovered

from knowledge of all extensions of associated graded rings along the valuation of regular local rings $R_1 \rightarrow S_1$ within the field extension which dominate $R \rightarrow S$ and are dominated by the valuation.

The fact that there exists $R_1 \rightarrow S_1$ as in the conclusions of the theorem if the assumptions of the theorem hold and the extension is without defect is proven within 2-dimensional algebraic function fields over an algebraically closed field in [23] and [24], and in the generality of the assumptions of Theorem 0.1 in Theorems 4.3 and 4.4 of [14]. Further, if the assumptions of the theorem hold and the defect $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} is not finitely generated by Theorem 7.3 [16] in the case of algebraic function fields over an algebraically closed field. With the full generality of the hypothesis of Theorem 0.1, the defect is zero by Corollary 18.7 [21] in the case of discrete, rank 1 valuations and the defect is zero by Theorem 3.7 [14] in the case of rational rank 2 valuations, so by Abhyankar's inequality, Proposition 2 [1] or Appendix 2 [49], if the defect $\delta(\nu^*/\nu) \neq 0$, then the value group Φ_{ν^*} has rational rank 1 and is not discrete and V_{ν^*}/m_{ν^*} is algebraic over S/m_S . Thus to prove Theorem 0.1, we have reduced to proving the following proposition, which we establish in this paper.

Proposition 0.3. *Suppose that R is a 2 dimensional excellent local domain with quotient field K . Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S . Let ν be the restriction of ν^* to K .*

Suppose that ν^ has rational rank 1 and ν^* is not discrete. Further suppose that there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blow up of R , S_1 is a local ring of a blowup of S , ν^* dominates S_1 , S_1 dominates R_1 and $\text{gr}_{\nu^*}(S_1)$ is a finitely generated $\text{gr}_{\nu}(R_1)$ -algebra. Then the defect $\delta(\nu^*/\nu) = 0$.*

Another factor in the question of finite generation of extensions of associated graded rings along a valuation is the splitting of ν in K^* . We will say that ν does not split in S if ν^* is the unique extension of ν to K^* such that ν^* dominates S . After a little blowing up, we can always obtain non splitting, as the following lemma shows.

Lemma 0.4. *Given an extension $R \rightarrow S$ as in the hypotheses of Theorem 0.1, there exists a normal local ring R' which is a local ring of a blow up of R such that ν dominates R' and if*

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

is a commutative diagram of normal local rings, where R_1 is a local ring of a blow up of R and S_1 is a local ring of a blow up of S , ν^ dominate S_1 and R_1 dominates R' , then ν does not split in S_1 .*

Lemma 0.4 will be proven in Section 1.

We have the following theorem.

Theorem 0.5. *Suppose that R is a 2 dimensional excellent regular local ring with quotient field K . Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional regular local ring with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S . Let ν be the restriction of ν^* to K . Further suppose that ν^* has rational rank 1 and ν^* is not discrete. Suppose that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra. Then S is a localization of the integral closure of R in K^* , the defect $\delta(\nu^*/\nu) = 0$ and ν^* does not split in S .*

We give examples showing that the condition rational rank 1 and discrete on ν^* in Theorem 0.5 are necessary.

As an immediate consequence of Theorem 0.5, we obtain the following corollary.

Corollary 0.6. *Suppose that R is a 2 dimensional excellent regular local ring with quotient field K . Suppose that ν is a valuation of K such that ν dominates R . Further suppose that ν has rational rank 1 and ν is not discrete. Suppose that $R \rightarrow R'$ is a nontrivial sequence of quadratic transforms along ν . Then $gr_\nu(R')$ is not a finitely generated $gr_\nu(R)$ -algebra.*

In [47], Michel Vaquié extends MacLane's theory of key polynomials [37] to show that if $(K, \nu) \rightarrow (K^*, \nu^*)$ is a finite extension of valued fields with $\delta(\nu^*/\nu) = 0$ and ν^* is the unique extension of ν to K^* , then ν^* can be constructed from ν by a finite sequence of augmented valuations. This suggests that a converse of Theorem 0.5 may be true.

We thank Bernard Teissier for discussions on the topics of this paper.

1. LOCAL DEGREE AND DEFECT

We will use the following criterion to measure defect, which is Proposition 3.4 [14]. This result is implicit in [16] with the assumptions of Proposition 0.3.

Proposition 1.1. *Suppose that R is a 2 dimensional excellent local domain with quotient field K . Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S , the residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S and the value group Φ_{ν^*} of ν^* has rational rank 1. Let ν be the restriction of ν^* to K . There exists a local ring R' of K which is essentially of finite type over R , is dominated by ν and dominates R such that if we have a commutative diagram*

$$(6) \quad \begin{array}{ccc} V_\nu & \rightarrow & V_{\nu^*} \\ \uparrow & & \uparrow \\ R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R' & & \uparrow \\ \uparrow & & \\ R & \rightarrow & S \end{array}$$

where R_1 is a regular local ring of K which is essentially of finite type over R and dominates R , S_1 is a regular local ring of K^* which is essentially of finite type over S and dominates S , R_1 has a regular system of parameters u, v and S_1 has a regular system of parameters x, y such that there is an expression

$$u = \gamma x^a, v = x^b f$$

where $a > 0$, $b \geq 0$, γ is a unit in S , $x \nmid f$ in S_1 and f is not a unit in S_1 , then

$$(7) \quad ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu) f(\nu^*/\nu) p^{\delta(\nu^*/\nu)}$$

where $d = \bar{\nu}(f \bmod x)$ with $\bar{\nu}$ being the natural valuation of the DVR S/xS .

We now prove Lemma 0.4 from the introduction. Let $\nu_1 = \nu^*, \nu_2, \dots, \nu_r$ be the extensions of ν to K^* . Let T be the integral closure of V_ν in K^* . Then $T = V_{\nu_1} \cap \dots \cap V_{\nu_r}$ is the integral closure of V_{ν^*} in K^* (by Propositions 2.36 and 2.38 [3]). Let $m_i = m_{\nu_i} \cap T$

be the maximal ideals of T . By the Chinese remainder theorem, there exists $u \in T$ such that $u \in m_1$ and $u \notin m_i$ for $2 \leq i \leq r$. Let

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0$$

be an equation of integral dependence of u over V_ν . Let A be the integral closure of $R[a_1, \dots, a_n]$ in K and let $R' = A_{A \cap m_\nu}$. Let T' be the integral closure of R' in K^* . We have that $u \in T' \cap m_i$ if and only if $i = 1$. Let $S' = T'_{T' \cap m_1}$. Then ν does not split in S' and R' has the property of the conclusions of the lemma.

2. GENERATING SEQUENCES

Given an additive group G with $\lambda_0, \dots, \lambda_r \in G$, $G(\lambda_0, \dots, \lambda_r)$ will denote the subgroup generated by $\lambda_0, \dots, \lambda_r$. The semigroup generated by $\lambda_0, \dots, \lambda_r$ will be denoted by $S(\lambda_0, \dots, \lambda_r)$.

In this section, we will suppose that R is a regular local ring of dimension two, with maximal ideal m_R and residue field R/m_R . For $f \in R$, let \bar{f} or $[f]$ denote the residue of f in R/m_R .

The following theorem is Theorem 4.2 of [17], as interpreted by Remark 4.3 [17].

Theorem 2.1. *Suppose that ν is a valuation of the quotient field of R dominating R . Let $L = V_\nu/m_\nu$ be the residue field of the valuation ring V_ν of ν . For $f \in V_\nu$, let $[f]$ denote the class of f in L . Suppose that x, y are regular parameters in R . Then there exist $\Omega \in \mathbb{Z}_+ \cup \{\infty\}$ and $P_i(\nu, R) \in m_R$ for $i \in \mathbb{Z}_+$ with $i < \min\{\Omega + 1, \infty\}$ such that $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and for $1 \leq i < \Omega$, there is an expression*

$$(8) \quad P_{i+1}(\nu, R) = P_i(\nu, R)^{n_i(\nu, R)} + \sum_{k=1}^{\lambda_i} c_k P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)}$$

with $n_i(\nu, R) \geq 1$, $\lambda_i \geq 1$,

$$(9) \quad 0 \neq c_k \text{ units in } R$$

for $1 \leq k \leq \lambda_i$, $\sigma_{i,s}(k) \in \mathbb{N}$ for all s, k , $0 \leq \sigma_{i,s}(k) < n_s(\nu, R)$ for $s \geq 1$. Further,

$$n_i(\nu, R)\nu(P_i(\nu, R)) = \nu(P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \cdots P_i(\nu, R)^{\sigma_{i,i}(k)})$$

for all k .

For all $i \in \mathbb{Z}_+$ with $i < \Omega$, the following are true:

- 1) $\nu(P_{i+1}(\nu, R)) > n_i(\nu, R)\nu(P_i(\nu, R))$.
- 2) Suppose that $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $j_k(l) \in \mathbb{N}$ for $1 \leq l \leq m$ and $0 \leq j_k(l) < n_k(\nu, R)$ for $1 \leq k \leq r$ are such that $(j_0(l), j_1(l), \dots, j_r(l))$ are distinct for $1 \leq l \leq m$, and $\nu(P_0(\nu, R)^{j_0(l)} P_1(\nu, R)^{j_1(l)} \cdots P_r(\nu, R)^{j_r(l)}) = \nu(P_0(\nu, R)^{j_0(1)} \cdots P_r(\nu, R)^{j_r(1)})$ for $1 \leq l \leq m$. Then

$$1, \left[\frac{P_0(\nu, R)^{j_0(2)} P_1(\nu, R)^{j_1(2)} \cdots P_r(\nu, R)^{j_r(2)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}} \right], \dots, \left[\frac{P_0(\nu, R)^{j_0(m)} P_1(\nu, R)^{j_1(m)} \cdots P_r(\nu, R)^{j_r(m)}}{P_0(\nu, R)^{j_0(1)} P_1(\nu, R)^{j_1(1)} \cdots P_r(\nu, R)^{j_r(1)}} \right]$$

are linearly independent over R/m_R .

- 3) Let

$$\bar{n}_i(\nu, R) = [G(\nu(P_0(\nu, R)), \dots, \nu(P_i(\nu, R))) : G(\nu(P_0(\nu, R)), \dots, \nu(P_{i-1}(\nu, R)))]$$

Then $\bar{n}_i(\nu, R)$ divides $\sigma_{i,i}(k)$ for all k in (8). In particular, $n_i(\nu, R) = \bar{n}_i(\nu, R)d_i(\nu, R)$ with $d_i(\nu, R) \in \mathbb{Z}_+$

- 4) There exists $U_i(\nu, R) = P_0(\nu, R)^{w_0(i)} P_1(\nu, R)^{w_1(i)} \dots P_{i-1}(\nu, R)^{w_{i-1}(i)}$ for $i \geq 1$ with $w_0(i), \dots, w_{i-1}(i) \in \mathbb{N}$ and $0 \leq w_j(i) < n_j(\nu, R)$ for $1 \leq j \leq i-1$ such that $\nu(P_i(\nu, R)^{\bar{n}_i}) = \nu(U_i(\nu, R))$ and setting

$$\alpha_i(\nu, R) = \left[\frac{P_i(\nu, R)^{\bar{n}_i(\nu, R)}}{U_i(\nu, R)} \right]$$

then

$$b_{i,t} = \left[\sum_{\sigma_{i,i(k)} = t\bar{n}_i(\nu, R)} c_k \frac{P_0(\nu, R)^{\sigma_{i,0}(k)} P_1(\nu, R)^{\sigma_{i,1}(k)} \dots P_{i-1}(\nu, R)^{\sigma_{i,i-1}(k)}}{U_i(\nu, R)^{(d_i(\nu, R) - t)}} \right] \in R/m_R(\alpha_1(\nu, R), \dots, \alpha_{i-1}(\nu, R))$$

for $0 \leq t \leq d_i(\nu, R) - 1$ and

$$f_i(u) = u^{d_i(\nu, R)} + b_{i,d_i(\nu, R)-1} u^{d_i(\nu, R)-1} + \dots + b_{i,0}$$

is the minimal polynomial of $\alpha_i(\nu, R)$ over $R/m_R(\alpha_1(\nu, R), \dots, \alpha_{i-1}(\nu, R))$.

The algorithm terminates with $\Omega < \infty$ if and only if either

$$(10) \quad \bar{n}_\Omega(\nu, R) = [G(\nu(P_0(\nu, R)), \dots, \nu(P_\Omega(\nu, R))) : G(\nu(P_0(\nu, R)), \dots, \nu(P_{\Omega-1}(\nu, R)))] = \infty$$

or

$$(11) \quad \bar{n}_\Omega(\nu, R) < \infty \text{ (so that } \alpha_\Omega(\nu, R) \text{ is defined as in 4)) and } d_\Omega(\nu, R) = [R/m_R(\alpha_1(\nu, R), \dots, \alpha_\Omega(\nu, R)) : R/m_R(\alpha_1(\nu, R), \dots, \alpha_{\Omega-1}(\nu, R))] = \infty.$$

If $\bar{n}_\Omega(\nu, R) = \infty$, set $\alpha_\Omega(\nu, R) = 1$.

Let notation be as in Theorem 2.1.

The following formula is formula $B(i)$ on page 10 of [17].

Suppose that M is a Laurent monomial in $P_0(\nu, R), P_1(\nu, R), \dots, P_i(\nu, R)$ and $\nu(M) = 0$. Then there exist $s_i \in \mathbb{Z}$ such that

$$(12) \quad M = \prod_{j=1}^i \left[\frac{P_j(\nu, R)^{\bar{n}_j}}{U_j(\nu, R)} \right]^{s_j},$$

so that

$$[M] \in R/m_R[\alpha_1(\nu, R), \dots, \alpha_i(\nu, R)].$$

Define $\beta_i(\nu, R) = \nu(P_i(\nu, R))$ for $0 \leq i$.

Since ν is a valuation of the quotient field of R , we have that

$$(13) \quad \Phi_\nu = \cup_{i=1}^\infty G(\beta_0(\nu, R), \beta_1, \dots, \beta_i(\nu, R))$$

and

$$(14) \quad V_\nu/m_\nu = \cup_{i=1}^\infty R/m_R[\alpha_1(\nu, R), \dots, \alpha_i(\nu, R)]$$

The following is Theorem 4.10 [17].

Theorem 2.2. Suppose that ν is a valuation dominating R . Let

$$P_0(\nu, R) = x, P_1(\nu, R) = y, P_2(\nu, R), \dots$$

be the sequence of elements of R constructed by Theorem 2.1. Suppose that $f \in R$ and there exists $n \in \mathbb{Z}_+$ such that $\nu(f) < n\nu(m_R)$. Then there exists an expansion

$$f = \sum_I a_I P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r} + \sum_J \varphi_J P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r} + h$$

where $r \in \mathbb{N}$, a_I are units in R , $I, J \in \mathbb{N}^{r+1}$, $\nu(P_0(\nu, R)^{i_0} P_1(\nu, R)^{i_1} \cdots P_r(\nu, R)^{i_r}) = \nu(f)$ for all I in the first sum, $0 \leq i_k < n_k(\nu, R)$ for $1 \leq k \leq r$, $\nu(P_0(\nu, R)^{j_0} \cdots P_r(\nu, R)^{j_r}) > \nu(f)$ for all terms in the second sum, $\varphi_J \in R$ and $h \in m_R^n$. The terms in the first sum are uniquely determined, up to the choice of units a_i , whose residues in R/m_R are uniquely determined.

Let $\sigma_0(\nu, R) = 0$ and inductively define

$$(15) \quad \sigma_{i+1}(\nu, R) = \min\{j > \sigma_i(\nu, R) \mid n_j(\nu, R) > 1\}.$$

In Theorem 2.2, we see that all of the monomials in the expansion of f are in terms of the P_{σ_i} .

We have that

$$S(\beta_0(\nu, R), \beta_1(\nu, R), \dots, \beta_{\sigma_j(\nu, R)}) = S(\beta_{\sigma_0}(\nu, R), \beta_{\sigma_1(\nu, R)}, \dots, \beta_{\sigma_j(\nu, R)})$$

for all $j \geq 0$ and

$$\begin{aligned} & R/m_R[\alpha_1(\nu, R), \alpha_2(\nu, R), \dots, \alpha_{\sigma_j(\nu, R)}(\nu, R)] \\ &= R/m_R[\alpha_{\sigma_1(\nu, R)}(\nu, R), \alpha_{\sigma_2(\nu, R)}(\nu, R), \dots, \alpha_{\sigma_j(\nu, R)}(\nu, R)] \end{aligned}$$

for all $j \geq 1$.

Suppose that R is a regular local ring of dimension two which is dominated by a valuation ν . The quadratic transform T_1 of R along ν is defined as follows. Let u, v be a system of regular parameters in R . Then $R[\frac{v}{u}] \subset V_\nu$ if $\nu(u) \leq \nu(v)$ and $R[\frac{u}{v}] \subset V_\nu$ if $\nu(u) \geq \nu(v)$. Let

$$T_1 = R \left[\frac{v}{u} \right]_{R[\frac{v}{u}] \cap m_\nu} \quad \text{or} \quad T_1 = R \left[\frac{u}{v} \right]_{R[\frac{u}{v}] \cap m_\nu},$$

depending on if $\nu(u) \leq \nu(v)$ or $\nu(u) > \nu(v)$. T_1 is a two dimensional regular local ring which is dominated by ν . Let

$$(16) \quad R \rightarrow T_1 \rightarrow T_2 \rightarrow \cdots$$

be the infinite sequence of quadratic transforms along ν , so that $V_\nu = \cup_{i \geq 1} T_i$ (Lemma 4.5 [3]) and $L = V_\nu/m_\nu = \cup_{i \geq 1} T_i/m_{T_i}$.

For $f \in R$ and $R \rightarrow R^*$ a sequence of quadratic transforms along ν , we define a strict transform of f in R^* to be f_1 if $f_1 \in R^*$ is a local equation of the strict transform in R^* of the subscheme $f = 0$ of R . In this way, a strict transform is only defined up to multiplication by a unit in R^* . This ambiguity will not be a difficulty in our proof. We will denote a strict transform of f in R^* by $\text{st}_{R^*}(f)$.

We use the notation of Theorem 2.1 and its proof for R and the $\{P_i(\nu, R)\}$. Recall that $U_1 = U^{w_0(1)}$. Let $w = w_0(1)$. Since $\bar{n}_1(\nu, R)$ and w are relatively prime, there exist $a, b \in \mathbb{N}$ such that

$$\varepsilon := \bar{n}_1(\nu, R)b - wa = \pm 1.$$

Define elements of the quotient field of R by

$$(17) \quad x_1 = (x^b y^{-a})^\varepsilon, y_1 = (x^{-w} y^{\bar{n}_1(\nu, R)})^\varepsilon.$$

We have that

$$(18) \quad x = x_1^{\bar{n}_1(\nu, R)} y_1^a, y = x_1^w y_1^b.$$

Since $\bar{n}_1(\nu, R)\nu(y) = w\nu(x)$, it follows that

$$\bar{n}_1(\nu, R)\nu(x_1) = \nu(x) > 0 \text{ and } \nu(y_1) = 0.$$

We further have that

$$(19) \quad \alpha_1(\nu, R) = [y_1]^\varepsilon \in V_\nu/m_\nu.$$

Let $A = R[x_1, y_1] \subset V_\nu$ and $m_A = m_\nu \cap A$.

Let $R_1 = A_{m_A}$. We have that R_1 is a regular local ring and the divisor of xy in R_1 has only one component ($x_1 = 0$). In particular, $R \rightarrow R_1$ is “free” (Definition 7.5 [16]). $R \rightarrow R_1$ factors (uniquely) as a product of quadratic transforms and the divisor of xy in R_1 has two distinct irreducible factors in all intermediate rings.

The following is Theorem 7.1 [17].

Theorem 2.3. *Let R be a two dimensional regular local ring with regular parameters x, y . Suppose that R is dominated by a valuation ν . Let $P_0(\nu, R) = x$, $P_1(\nu, R) = y$ and $\{P_i(\nu, R)\}$ be the sequence of elements of R constructed in Theorem 2.1. Suppose that $\Omega \geq 2$. Then there exists some smallest value i in the sequence (16) such that the divisor of xy in $\text{Spec}(T_i)$ has only one component. Let $R_1 = T_i$. Then $R_1/m_{R_1} \cong R/m_R(\alpha_1(\nu, R))$, and there exists $x_1 \in R_1$ and $w \in \mathbb{Z}_+$ such that $x_1 = 0$ is a local equation of the exceptional divisor of $\text{Spec}(R_1) \rightarrow \text{Spec}(R)$, and $Q_0 = x_1$, $Q_1 = \frac{P_2}{x_1^{w n_1}}$ are regular parameters in R_1 . We have that*

$$P_i(\nu, R_1) = \frac{P_{i+1}(\nu, R)}{P_0(\nu, R_1)^{w n_1(\nu, R) \cdots n_i(\nu, R)}}$$

for $1 \leq i < \max\{\Omega, \infty\}$ satisfy the conclusions of Theorem 2.1 for the ring R_1 .

We have that

$$G(\beta_0(\nu, R_1), \dots, \beta_i(\nu, R_1)) = G(\beta_0(\nu, R), \dots, \beta_{i+1}(\nu, R))$$

for $i \geq 1$ so that

$$\bar{n}_i(\nu, R_1) = \bar{n}_{i+1}(\nu, R) \text{ for } i \geq 1$$

and

$$R_1/m_{R_1}[\alpha_1(\nu, R_1), \dots, \alpha_i(\nu, R_1)] = R/m_R[\alpha_1(\nu, R), \dots, \alpha_{i+1}(\nu, R)] \text{ for } i \geq 1$$

so that

$$d_i(\nu, R_1) = d_{i+1}(\nu, R) \text{ and } n_i(\nu, R_1) = n_{i+1}(\nu, R) \text{ for } i \geq 1.$$

Let $\sigma_0(\nu, R_1) = 0$ and inductively define

$$\sigma_{i+1}(\nu, R_1) = \min\{j > \sigma_i(1) \mid n_j(\nu, R_1) > 1\}.$$

We then have that $\sigma_0(\nu, R_1) = 0$ and for $i \geq 1$, $\sigma_i(\nu, R_1) = \sigma_{i+1}(\nu, R) - 1$ if $n_1(\nu, R) > 1$ and $\sigma_i(\nu, R_1) = \sigma_i(\nu, R) - 1$ if $n_1(\nu, R) = 1$, and for all $j \geq 0$,

$$S(\beta_0(\nu, R_1), \beta_1(\nu, R_1), \dots, \beta_{\sigma_{j+1}(\nu, R_1)}(\nu, R_1)) = S(\beta_{\sigma_0(1)}(\nu, R), \beta_{\sigma_1(\nu, R)}, \dots, \beta_{\sigma_j(\nu, R)}(\nu, R))$$

Iterating this construction, we produce a sequence of sequences of quadratic transforms along ν ,

$$R \rightarrow R_1 \rightarrow \cdots \rightarrow R_{\sigma_1(\nu, R)}.$$

Now $x, \bar{y} = P_{\sigma_1(\nu, R)}$ are regular parameters in R . By (17) (with y replaced with \bar{y}) we have that $R_{\sigma_1(\nu, R)}$ has regular parameters

$$(20) \quad x_1 = (x^b \bar{y}^{-a})^\varepsilon, \quad y_1 = (x^{-\omega} \bar{y}^{\bar{n}_{\sigma_1(\nu, R)}(\nu, R)})^\varepsilon$$

where $\omega, a, b \in \mathbb{N}$ satisfy $\varepsilon = \bar{n}_{\sigma_1(\nu, R)}(\nu, R)b - \omega a = \pm 1$.

Further, $R_{\sigma_1(\nu, R_1)}$ has regular parameters $x_{\sigma_1(\nu, R)}, y_{\sigma_1(\nu, R)}$ where $x = \delta x_{\sigma_1(\nu, R_1)}^{\bar{n}_{\sigma_1(\nu, R)}(\nu, R_1)}$ and $y_{\sigma_1(\nu, R_1)} = \text{st}_{R_{\sigma_1(\nu, R_1)}} P_{\sigma_1(\nu, R)}(\nu, R)$ with $\delta \in R_{\sigma_1(\nu, R)}$ a unit.

For the remainder of this section, we will suppose that R is a two dimensional regular local ring and ν is a non discrete rational rank 1 valuation of the quotient field of R with valuation ring V_ν , so that V_ν/m_ν is algebraic over R/m_R . Suppose that $f \in R$ and $\nu(f) = \gamma$. We will denote the class of f in $\mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) \subset \text{gr}_\nu(R)$ by $\text{in}_\nu(f)$. By Theorem 2.2, we have that $\text{gr}_\nu(R)$ is generated by the initial forms of the $P_i(\nu, R)$ as an R/m_R -algebra. That is,

$$\begin{aligned} \text{gr}_\nu(R) &= R/m_R[\text{in}_\nu(P_0(\nu, R)), \text{in}_\nu(P_1(\nu, R)), \dots] \\ &= R/m_R[\text{in}_\nu(P_{\sigma_0(\nu, R)}(\nu, R)), \text{in}_\nu(P_{\sigma_1(\nu, R)}(\nu, R)), \dots]. \end{aligned}$$

Thus the semigroup $S^R(\nu) = \{\nu(f) \mid f \in R\}$ is equal to

$$S^R(\nu) = S(\beta_0(\nu, R), \beta_1(\nu, R), \dots) = S(\beta_{\sigma_0(\nu, R)}(\nu, R), \beta_{\sigma_1(\nu, R)}(\nu, R), \dots)$$

and the value group

$$\Phi_\nu = G(\beta_0(\nu, R), \beta_1(\nu, R) \dots)$$

and the residue field of the valuation ring

$$V_\nu/m_\nu = R/m_R[\alpha_1(\nu, R), \alpha_2(\nu, R), \dots] = R/m_R[\alpha_{\sigma_1}(\nu, R), \alpha_{\sigma_2}(\nu, R), \dots]$$

By 1) of Theorem 2.1, every element $\beta \in S^R(\nu)$ has a unique expression

$$\beta = \sum_{i=0}^r a_i \beta_i(\nu, R)$$

for some r with $a_i \in \mathbb{N}$ for all i and $0 \leq a_i < n_i(\nu, R)$ for $1 \leq i$. In particular, if $a_i \neq 0$ in the expansion then $\beta_i(\nu, R) = \beta_{\sigma_j(\nu, R)}(\nu, R)$ for some j .

Lemma 2.4. *Let*

$$\sigma_i = \sigma_i(\nu, R), \beta_i = \beta_i(\nu, R), P_i = P_i(\nu, R), n_i = n_i(\nu, R), \bar{n}_i = \bar{n}_i(\nu, R),$$

$$\sigma_i(1) = \sigma_i(\nu, R_{\sigma_1}), \beta_i = \beta_i(\nu, R_{\sigma_1}), P_i(1) = P_i(\nu, R_{\sigma_1}), n_i(1) = n_i(\nu, R_{\sigma_1}), \bar{n}_i(1) = \bar{n}_i(\nu, R_{\sigma_1}).$$

Suppose $i \in \mathbb{N}$, $r \in \mathbb{N}$ and $a_j \in \mathbb{N}$ for $j = 0, \dots, r$ with $0 \leq a_j < n_{\sigma_j}$ for $j \geq 1$ are such that

$$\nu(P_{\sigma_0}^{a_0} \dots P_{\sigma_r}^{a_r}) > \nu(P_{\sigma_i})$$

or $r < i$ and

$$\nu(P_{\sigma_0}^{a_0} \dots P_{\sigma_r}^{a_r}) = \nu(P_{\sigma_i}).$$

By (18) and Theorem 2.3, we have expressions in

$$R_{\sigma_1} = R[x_1, y_1]_{m_\nu \cap R[x_1, y_1]}$$

where x_1, y_1 are defined by (20)

$$P_{\sigma_0}^{a_0} \dots P_{\sigma_r}^{a_r} = y_1^{a_0 + b a_1} P_{\sigma_1(1)}(1)^{a_2} \dots P_{\sigma_{r-1}(1)}(1)^{a_r} P_{\sigma_0(1)}(1)^t$$

where $t = \bar{n}_{\sigma_1} a_0 + \omega a_1 + \omega n_{\sigma_1} a_2 + \dots + \omega n_{\sigma_1} \dots n_{\sigma_{r-1}} a_r$ and

$$P_{\sigma_i} = \begin{cases} y_1^a P_{\sigma_0(1)}(1)^{\bar{n}_{\sigma_1}} & \text{if } i = 0 \\ y_1^b P_{\sigma_0(1)}(1)^\omega & \text{if } i = 1 \\ P_{\sigma_{i-1}(1)}(1) P_{\sigma_0(1)}(1)^{\omega n_{\sigma_1} \dots n_{\sigma_{i-1}}} & \text{if } i \geq 2. \end{cases}$$

Let

$$\lambda = \begin{cases} \bar{n}_{\sigma_1} & \text{if } i = 0 \\ \omega & \text{if } i = 1 \\ \omega n_{\sigma_1} \cdots n_{\sigma_{i-1}} & \text{if } i \geq 2. \end{cases}$$

Then

$$t > \lambda,$$

except in the case where $i = 1$, $P_{\sigma_0}^{a_0} \cdots P_{\sigma_r}^{a_r} = P_{\sigma_0}$, and $\bar{n}_{\sigma_1} = \omega = 1$. In this case $\lambda = t$.

Proof. First suppose that $i \geq 2$ and $r \geq i$. Then

$$t - \lambda = (\bar{n}_{\sigma_1} a_0 + \omega a_1 + \omega n_{\sigma_1} a_2 + \cdots + \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} a_r) - \omega n_{\sigma_1} \cdots n_{\sigma_{i-1}} > 0.$$

Now suppose that $i \geq 2$ and $r < i$. We have that

$$\begin{aligned} & (\bar{n}_1 a_0 + \omega a_1 + \cdots + \omega n_1 \cdots n_{r-1} a_r - \omega n_1(\nu, R) \cdots n_{i-1}) \beta_{\sigma_0(1)}(1) \\ & \geq \beta_{\sigma_{i-1}(1)}(1) - a_2 \beta_{\sigma_1(1)}(1) - \cdots - a_r \beta_{\sigma_{r-1}(1)}(1) > 0 \end{aligned}$$

since $n_{\sigma_j(1)}(1) = n_{\sigma_{j+1}}$ for all j , and so $n_{\sigma_{j+1}} \beta_{\sigma_j(1)}(1) < \beta_{\sigma_{j+1}(1)}(1)$ for all j .

Now suppose that $i = 1$. As in the proof for the case $i \geq 2$ we have that $t - \lambda > 0$ if $r \geq 1$, so suppose that $i = 1$ and $r = 0$. Then $\bar{n}_{\sigma_1} \beta_{\sigma_1} = \omega \beta_{\sigma_0}$. From our assumption $a_0 \nu(P_0) \geq \nu(P_1)$ we obtain $t - \lambda = \bar{n}_{\sigma_1} a_0 - \omega \geq 0$ with equality if and only if $a_0 = \omega = \bar{n}_{\sigma_1} = 1$ since $\gcd(\omega, \bar{n}_{\sigma_1}) = 1$.

Now suppose $i = 0$. As in the previous cases, we have $t - \lambda > 0$ if $r > 1$ and $t - \lambda > 0$ if $r = 1$ except possibly if $P_0^{a_0} \cdots P_r^{a_r} = P_1^{a_1}$. We then have that $\nu(P_{\sigma_1}^{a_1}) > \nu(P_{\sigma_0})$, and so

$$a_1 \frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} > 1.$$

Since

$$\frac{\beta_{\sigma_1}}{\beta_{\sigma_0}} = \frac{\omega}{\bar{n}_{\sigma_1}},$$

we have that $t - \lambda = \omega a_1 - \bar{n}_{\sigma_1} > 0$. □

Lemma 2.5. *Let notation be as in Lemma 2.4. Suppose that $f \in R$, with $\nu(f) = \nu(P_{\sigma_i})$ for some $i \geq 0$, and that f has an expression of the form of Theorem 2.2,*

$$f = c P_{\sigma_i} + \sum_{j=1}^s c_j P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)} + h$$

where $s, r \in \mathbb{N}$, c, c_j are units in R , with $0 \leq a_k(j) < n_k$ for $1 \leq k \leq r$ for $1 \leq j \leq s$,

$$\nu(f) = \nu(P_{\sigma_i}) \leq \nu(P_{\sigma_0}^{a_0(j)} P_{\sigma_1}^{a_1(j)} \cdots P_{\sigma_r}^{a_r(j)})$$

for $1 \leq j \leq s$, $a_k(j) = 0$ for $k \geq i$ if $\nu(f) = \nu(P_{\sigma_0}^{a_0(j)} \cdots P_{\sigma_r}^{a_r(j)})$ and $h \in m_R^n$ with $n > \nu(f)$. Then $\text{st}_{R_{\sigma_1}}(f)$ is a unit in R_{σ_1} if $i = 0$ or 1 and if $i > 1$, there exists a unit \bar{c} in R_{σ_1} and $\Omega \in R_{\sigma_1}$ such that

$$\text{st}_{R_{\sigma_1}}(f) = \bar{c} P_{\sigma_{i-1}(1)}(1) + x_1 \Omega$$

with $\nu(\text{st}_{R_{\sigma_1}}(f)) = \nu(P_{\sigma_{i-1}(1)}(1))$ and $\nu(P_{\sigma_{i-1}(1)}(1)) \leq \nu(x_1 \Omega)$.

Proof. Let

$$\lambda = \begin{cases} \bar{n}_1 & \text{if } i = 0 \\ \omega & \text{if } i = 1 \\ \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} & \text{if } i \geq 2 \end{cases}$$

Then

$$f = cH_i + \sum_{j=1}^s c_j (y_1)^{aa_0(j)+ba_1(j)} P_{\sigma_0(1)}(1)^{t_j} P_{\sigma_1(1)}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^t h'$$

with

$$H_i = \begin{cases} (y_1)^a P_{\sigma_0(1)}(1)^{\bar{n}_1} & \text{if } i = 0 \\ (y_1)^b P_{\sigma_0(1)}(1)^\omega & \text{if } i = 1 \\ P_{\sigma_0(1)}(1)^{\omega n_1 \cdots n_{i-1}} P_{\sigma_{i-1}(1)}(1) & \text{if } i \geq 2 \end{cases}$$

and

$$t_j = \bar{n}_1 a_0(j) + \omega a_1(j) + \omega n_{\sigma_1} a_2 + \cdots + \omega n_{\sigma_1} \cdots n_{\sigma_{r-1}} a_r(j)$$

for $1 \leq j \leq s$, $t > \lambda$ and $h' \in R_{\sigma_1}$. By Lemma 2.4, if $i \geq 2$ or $i = 0$, we have that $t_j > \lambda$ for all j . Thus $f = P_{\sigma_0(1)}(1)^\lambda \bar{f}$ where

$$\bar{f} = cG_i + \sum_{j=1}^s c_j P_{\sigma_0(1)}(1)^{t_j-\lambda} P_{\sigma_1(1)}(1)^{a_2(j)} \cdots P_{\sigma_{r-1}(1)}(1)^{a_r(j)} + P_{\sigma_0(1)}(1)^{t-\lambda} h'$$

with

$$G_i = \begin{cases} (y_1)^a & \text{if } i = 0 \\ (y_1)^b & \text{if } i = 1 \\ P_{\sigma_{i-1}(1)}(1) & \text{if } i \geq 2 \end{cases}$$

is a strict transform $\bar{f} = \text{st}_{R_1}(f)$ of f in R_1 .

If $i = 1$, then by Lemma 2.4, $t_j > \lambda$ for all j , except possibly for a single term (that we can assume is t_1) which is P_{σ_0} , and we have that $\omega = \bar{n}_{\sigma_1} = 1$. In this case $t_1 = \lambda$. Then

$$\left[\frac{P_{\sigma_1}}{P_{\sigma_0}} \right] = \alpha_{\sigma_1}(\nu, R) \in V_\nu/m_\nu$$

which has degree $d_{\sigma_1}(\nu, R) = n_{\sigma_1} > 1$ over R/m_R . By (18), $x = x_1$, $y = x_1 y_1$ and

$$f = x_1 [c + c_1 y_1 + x_1 \Omega]$$

with $\Omega \in R_{\sigma_1}$. We have that $c + c_1 y_1$ is a unit in R_{σ_1} since

$$[y_1] = \left[\frac{P_{\sigma_0}}{P_{\sigma_1}} \right] \notin R/m_R.$$

□

3. FINITE GENERATION IMPLIES NO DEFECT

Suppose that R is a two dimensional regular local ring of K and S is a two dimensional regular local ring such that S dominates R . Let K be the quotient field of R and K^* be the quotient field of S . Suppose that $K \rightarrow K^*$ is a finite separable field extension. Suppose that ν^* is a non discrete rational rank 1 valuation of K^* such that V_{ν^*}/m_{ν^*} is algebraic over S/m_S and that ν^* dominates S . Then we have a natural graded inclusion $\text{gr}_\nu(R) \rightarrow \text{gr}_{\nu^*}(S)$, so that for $f \in R$, we have that $\text{in}_\nu(f) = \text{in}_{\nu^*}(f)$. Let $\nu = \nu^*|_K$. Let $L = V_{\nu^*}/m_{\nu^*}$. Suppose that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_\nu(R)$ -algebra.

Let x, y be regular parameters in R , with associated generating sequence to ν , $P_0 = P_0(\nu, R) = x, P_1 = P_1(\nu, R) = y, P_2 = P_2(\nu, R), \dots$ in R as constructed in Theorem 2.1,

with $U_i = U_i(\nu, R)$, $\beta_i = \beta_i(\nu, R) = \nu(P_i)$, $\gamma_i = \alpha_i(\nu, R)$, $m_i = m_i(\nu, R)$, $\bar{m}_i = \bar{m}_i(\nu, R)$, $d_i = d_i(\nu, R)$ and $\sigma_i = \sigma_i(\nu, R)$ defined as in Section 2.

Let u, v be regular parameters in S , with associated generating sequence to ν^* , $Q_0 = P_0(\nu^*, S) = u$, $Q_1 = P_1(\nu^*, S) = v$, $Q_2 = P_2(\nu^*, S), \dots$ in S as constructed in Theorem 2.1, with $V_i = U_i(\nu^*, S)$, $\gamma_i = \beta_i(\nu^*, S) = \nu^*(Q_i)$, $\delta_i = \alpha_i(\nu^*, S)$, $n_i = n_i(\nu^*, S)$, $\bar{n}_i = \bar{n}_i(\nu^*, S)$, $e_i = \alpha_i(\nu^*, S)$ and $\tau_i = \sigma_i(\nu^*, S)$ defined as in Section 2.

With our assumption that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra, we have that for all sufficiently large l ,

$$(21) \quad \text{gr}_{\nu^*}(S) = \text{gr}_{\nu}(R)[\text{in}_{\nu^*}Q_{\tau_0}, \dots, \text{in}_{\nu^*}Q_{\tau_l}].$$

Proposition 3.1. *With our assumption that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra, there exist integers $s > 1$ and $r > 1$ such that for all $j \geq 0$,*

$$\begin{aligned} \beta_{\sigma_{r+j}} &= \gamma_{\tau_{s+j}}, \bar{m}_{\sigma_{r+j}} = \bar{n}_{\tau_{s+j}}, d_{\sigma_{r+j}} = e_{\tau_{s+j}}, m_{\sigma_{r+j}} = n_{\tau_{s+j}}, \\ G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}}) &\subset G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}), \\ [G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}})] &= e(\nu^*/\nu), \\ R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}] &\subset S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}] \end{aligned}$$

and

$$[S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}] : R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}]] = f(\nu^*/\nu).$$

Proof. Let l be as in (21). For $s \geq l$, define the sub algebra A_{τ_s} of $\text{gr}_{\nu^*}(S)$ by

$$A_{\tau_s} = S/m_S[\text{in}_{\nu^*}Q_{\tau_0}, \dots, \text{in}_{\nu^*}Q_{\tau_s}].$$

For $s \geq l$, let

$$\begin{aligned} r_s &= \max\{j \mid \text{in}_{\nu^*}P_{\sigma_j} \in A_{\tau_s}\}, \\ \lambda_s &= [G(\gamma_{\tau_0}, \dots, \gamma_{\tau_s}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r_s}})], \end{aligned}$$

and

$$\chi_s = [S/m_S[\varepsilon_{\tau_0}, \dots, \varepsilon_{\tau_s}] : R/m_R[\delta_{\sigma_0}, \dots, \delta_{\sigma_{r_s}}]].$$

To simplify notation, we will write $r = r_s$.

We will now show that $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$. Suppose that $\beta_{\sigma_{r+1}} > \gamma_{\tau_{s+1}}$. We have that

$$\text{in}_{\nu^*}Q_{\tau_{s+1}} \in \text{gr}_{\nu}(R)[\text{in}_{\nu^*}Q_{\tau_0}, \dots, \text{in}_{\nu^*}Q_{\tau_s}].$$

Since

$$\beta_{\sigma_{r+1}} < \beta_{\sigma_{r+2}} < \dots$$

we then have that $\text{in}_{\nu^*}Q_{\tau_{s+1}} \in A_{\tau_s}$ which is impossible. Thus $\beta_{\sigma_{r+1}} \leq \gamma_{\tau_{s+1}}$. If $\beta_{\sigma_{r+1}} < \gamma_{\tau_{s+1}}$, then since

$$\gamma_{\tau_{s+1}} < \gamma_{\tau_{s+2}} < \dots$$

and $\text{in}_{\nu^*}P_{\sigma_{r+1}} \in \text{gr}_{\nu^*}(S)$, we have that $\text{in}_{\nu^*}P_{\sigma_{r+1}} \in A_{\tau_s}$, which is impossible. Thus $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$.

We will now establish that either we have a reduction $\lambda_{s+1} < \lambda_s$ or

$$(22) \quad \lambda_{s+1} = \lambda_s, \beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}} \text{ and } \bar{m}_{\sigma_{r+1}} = \bar{n}_{\tau_{s+1}}.$$

Let ω be a generator of the group $G(\gamma_{\tau_1}, \dots, \gamma_{\tau_s})$, so that $G(\gamma_{\tau_1}, \dots, \gamma_{\tau_s}) = \mathbb{Z}\omega$. We have that

$$G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+1}}) = \frac{1}{\bar{n}_{\tau_{s+1}}} \mathbb{Z}\omega$$

and

$$G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+1}}) = \frac{1}{\bar{m}_{\sigma_{r+1}}} \mathbb{Z}(\lambda_s \omega).$$

There exists a positive integer f with $\gcd(f, \bar{n}_{\tau_{s+1}}) = 1$ such that

$$\gamma_{\tau_{s+1}} = \frac{f}{\bar{n}_{\tau_{s+1}}} \omega$$

There exists a positive integer g with $\gcd(g, \bar{m}_{\sigma_{r+1}}) = 1$ such that

$$\beta_{\sigma_{r+1}} = \frac{g}{\bar{m}_{\sigma_{r+1}}} \lambda_s \omega.$$

Since $\beta_{\sigma_{r+1}} = \gamma_{\tau_{s+1}}$, we have

$$g \lambda_s \bar{n}_{\tau_{s+1}} = f \bar{m}_{\sigma_{r+1}}.$$

Thus $\bar{n}_{\tau_{s+1}}$ divides $\bar{m}_{\sigma_{r+1}}$ and $\bar{m}_{\sigma_{r+1}}$ divides $\lambda_s \bar{n}_{\tau_{s+1}}$, so that

$$a = \frac{\bar{m}_{\sigma_{r+1}}}{\bar{n}_{\tau_{s+1}}}$$

is a positive integer and defining

$$\bar{\lambda} = \frac{\lambda_s}{a},$$

we have that $\bar{\lambda}$ is a positive integer with

$$\frac{\lambda_s}{\bar{m}_{\sigma_{r+1}}} = \frac{\bar{\lambda}}{\bar{n}_{\tau_{s+1}}}$$

and

$$\bar{\lambda} = [G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+1}}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+1}})].$$

Since $\lambda_{s+1} \leq \bar{\lambda}$, either $\lambda_{s+1} < \lambda_s$ or $\lambda_{s+1} = \lambda_s$ and $\bar{m}_{\sigma_{r+1}} = \bar{n}_{\tau_{s+1}}$.

We will now suppose that s is sufficiently large that (22) holds. Since

$$\text{in}_{\nu^*} Q_{\tau_{s+1}} \in \text{gr}_{\nu^*}(S) = \text{gr}_{\nu}(R)[\text{in}_{\nu^*} Q_{\tau_0}, \dots, \text{in}_{\nu^*} Q_{\tau_s}],$$

if $\bar{n}_{\tau_{s+1}} > 1$ we have an expression

$$(23) \quad \text{in}_{\nu^*} P_{\sigma_{r+1}} = \text{in}_{\nu^*}(\alpha) \text{in}_{\nu^*} Q_{\tau_{s+1}}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}_{\gamma_{\tau_{s+1}}}^+(S)$ with α a unit in S and if $\bar{n}_{\tau_{s+1}} = 1$, since $\text{in}_{\nu^*} P_{\sigma_{r+1}} \notin A_{\tau_s}$, we have an expression

$$(24) \quad \text{in}_{\nu^*} P_{\sigma_{r+1}} = \text{in}_{\nu^*}(\alpha) \text{in}_{\nu^*} Q_{\tau_{s+1}} + \sum \text{in}_{\nu^*}(\alpha_J) (\text{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\text{in}_{\nu^*} Q_{\tau_s})^{j_s}$$

in $\mathcal{P}_{\gamma_{\tau_{s+1}}}(S)/\mathcal{P}_{\gamma_{\tau_{s+1}}}^+(S)$ with α a unit in S and the sum is over certain $J = (j_0, \dots, j_s) \in \mathbb{N}^s$ such that the α_J are units in S , and the terms $\text{in}_{\nu^*} Q_{\tau_{s+1}}$ and the $(\text{in}_{\nu^*} Q_{\tau_0})^{j_0} \cdots (\text{in}_{\nu^*} Q_{\tau_s})^{j_s}$ are linearly independent over S/m_S .

The monomial $U_{\sigma_{r+1}}$ in $P_{\sigma_0}, \dots, P_{\sigma_r}$ and the monomial $V_{\tau_{s+1}}$ in $Q_{\tau_0}, \dots, Q_{\tau_s}$ both have the value $\bar{n}_{\tau_{s+1}} \gamma_{\tau_{s+1}} = \bar{m}_{\sigma_{r+1}} \beta_{\sigma_{r+1}}$, and satisfy

$$\varepsilon_{\tau_{s+1}} = \left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}} \right]$$

and

$$\delta_{\sigma_{r+1}} = \left[\frac{P_{\sigma_{r+1}}^{\bar{m}_{\sigma_{r+1}}}}{U_{\sigma_{r+1}}} \right].$$

Since $U_{\sigma_{r+1}}, V_{\tau_{s+1}} \in A_{\tau_s}$ and by (12) and 2) of Theorem 2.1, we have that

$$\left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \in S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}].$$

If $\bar{n}_{\tau_{s+1}} > 1$, then by (23), we have

$$\left[\frac{P_{\sigma_{r+1}}^{\bar{n}_{\tau_{s+1}}}}{U_{\sigma_{r+1}}} \right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \left([\alpha]^{\bar{n}_{\tau_{s+1}}} \left[\frac{Q_{\tau_{s+1}}^{\bar{n}_{\tau_{s+1}}}}{V_{\tau_{s+1}}} \right] \right)$$

in $L = V_{\nu^*}/m_{\nu^*}$, and if $\bar{n}_{\tau_{s+1}} = 1$, then by (24), we have

$$\left[\frac{P_{\sigma_{r+1}}}{U_{\sigma_{r+1}}} \right] = \left[\frac{V_{\tau_{s+1}}}{U_{\sigma_{r+1}}} \right] \left([\alpha] \left[\frac{Q_{\tau_{s+1}}}{V_{\tau_{s+1}}} \right] + \sum [\alpha_J] \left[\frac{Q_{\tau_0}^{j_0} \cdots Q_{\tau_s}^{j_s}}{V_{\tau_{s+1}}} \right] \right).$$

Thus by equation (12),

$$(25) \quad S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}][\varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}].$$

We have a commutative diagram

$$\begin{array}{ccc} S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}] & \rightarrow & S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}] \\ \uparrow & & \uparrow \\ R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_r}] & \rightarrow & R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_r}][\delta_{\sigma_{r+1}}]. \end{array}$$

Let

$$\bar{\chi} = [S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] : R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_r}, \delta_{\sigma_{r+1}}]].$$

Since

$$S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}, \varepsilon_{\tau_{s+1}}] = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}][\delta_{\sigma_{r+1}}],$$

we have that $e_{\tau_{s+1}} | d_{\sigma_{r+1}}$. Further,

$$\frac{d_{\sigma_{r+1}}}{e_{\tau_{s+1}}} \bar{\chi} = \chi_s,$$

whence $\bar{\chi} \leq \chi_s$. Thus $\chi_{s+1} \leq \chi_s$ and if $\chi_{s+1} = \chi_s$, then $d_{\sigma_{r+1}} = e_{\tau_{s+1}}$ and $r_{s+1} = r_s + 1$ since $P_{\sigma_{r+2}} \in A_{\tau_{s+1}}$ implies $\lambda_{s+1} < \lambda_s$ or $\chi_{s+1} < \chi_s$.

We may thus choose s sufficiently large that there exists an integer $r > 1$ such that for all $j \geq 0$,

$$\beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}, \bar{m}_{\sigma_{r+j}} = \bar{n}_{\tau_{s+j}}, d_{\sigma_{r+j}} = e_{\tau_{s+j}}, m_{\sigma_{r+j}} = n_{\tau_{s+j}},$$

$$G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}}) \subset G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}),$$

there is a constant λ (which does not depend on j) such that

$$[G(\gamma_{\tau_0}, \dots, \gamma_{\tau_{s+j}}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_{r+j}})] = \lambda$$

$$R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}] \subset S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}]$$

and there is a constant χ (which does not depend on j) such that

$$[S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+j}}] : R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+j}}]] = \chi.$$

Then

$$\Phi_{\nu^*} = \cup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \mathbb{Z}\omega$$

where $G(\gamma_{\tau_0}, \dots, \gamma_{\tau_s}) = \mathbb{Z}\omega$, and

$$\Phi_{\nu} = \cup_{j \geq 1} \frac{1}{\bar{m}_{\sigma_{r+1}} \cdots \bar{m}_{\sigma_{r+j}}} \lambda \mathbb{Z}\omega = \cup_{j \geq 1} \frac{1}{\bar{n}_{\tau_{s+1}} \cdots \bar{n}_{\tau_{s+j}}} \lambda \mathbb{Z}\omega$$

so that

$$\lambda = [\Phi_{\nu^*} : \Phi_{\nu}] = e(\nu^*/\nu).$$

For $i \geq 0$, let $K_i = R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_{r+i}}]$ and $M_i = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_{s+i}}]$. We have that $M_{i+1} = M_i[\delta_{\sigma_{r+i+1}}]$ for $i \geq 0$ and $\chi = [M_i : K_i]$ for all i . Further,

$$\cup_{i=0}^{\infty} M_i = V_{\nu^*}/m_{\nu^*} \text{ and } \cup_{i=0}^{\infty} K_i = V_{\nu}/m_{\nu}.$$

Thus if $g_1, \dots, g_{\lambda} \in M_0$ form a basis of M_0 as a K_0 -vector space, then g_1, \dots, g_{λ} form a basis of M_i as a K_i -vector space for all $i \geq 0$. Thus

$$\chi = [V_{\nu^*}/m_{\nu^*} : V_{\nu}/m_{\nu}] = f(\nu^*/\nu).$$

□

Let r and s be as in the conclusions of Proposition 3.1. There exists τ_t with $t \geq s$ such that we have a commutative diagram of inclusions of regular local rings (with the notation introduced in Section 2)

$$\begin{array}{ccc} R_{\sigma_r} & \rightarrow & S_{\tau_t} \\ \uparrow & & \uparrow \\ R & \rightarrow & S. \end{array}$$

After possibly increasing s and r , we may assume that $R' \subset R_{\sigma_r}$, where R' is the local ring of the conclusions of Proposition 1.1. Recall that R has regular parameters $x = P_0$, $y = P_1$ and S has regular parameters $u = Q_0$, $v = Q_1$, R_{σ_r} has regular parameters x_{σ_r} , y_{σ_r} such that

$$x = \delta x_{\sigma_r}^{\bar{m}_{\sigma_1} \cdots \bar{m}_{\sigma_r}}, \quad y_{\sigma_r} = \text{st}_{R_{\sigma_r}} P_{\sigma_{r+1}}$$

where δ is a unit in R_{σ_r} and S_{τ_t} has regular parameters u_{τ_t} , v_{τ_t} such that

$$u = \varepsilon u_{\tau_t}^{\bar{n}_{\tau_1} \cdots \bar{n}_{\tau_t}}, \quad v_{\tau_t} = \text{st}_{S_{\tau_t}} Q_{\tau_{t+1}}$$

where ε is a unit in S_{τ_t} . We may choose $t \gg 0$ so that we we have an expression

$$(26) \quad x_{\sigma_r} = \varphi u_{\tau_t}^{\lambda}$$

for some positive integer λ where φ is a unit in S_{τ_t} , since $\cup_{i=0}^{\infty} S_{\tau_t} = V_{\nu^*}$.

We have expressions $P_i = \psi_i x_{\sigma_r}^{c_i}$ in R_{σ_r} where ψ_i are units in R_{σ_r} for $i \leq \sigma_r$ so that $P_i = \psi_i^* u_{\tau_t}^{c_i \lambda}$ in S_{τ_t} where ψ_i^* are units in S_{τ_t} for $i \leq \sigma_r$ by (26).

Lemma 3.2. *For $j \geq 1$ we have*

$$\text{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) = u_{\tau_t}^{\lambda_j} \text{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$$

for some $\lambda_j \in \mathbb{N}$, where we regard $P_{\sigma_{r+j}}$ as an element of R on the left hand side of the equation and regard $P_{\sigma_{r+j}}$ as an element of S on the right hand side.

Proof. Using (26), we have

$$P_{\sigma_{r+j}} = \text{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) x_{\sigma_r}^{f_j} = \text{st}_{R_{\sigma_r}}(P_{\sigma_{r+j}}) u_{\tau_t}^{\lambda f_j} \varphi^{f_j}$$

where $f_j \in \mathbb{N}$. Viewing $P_{\sigma_{r+j}}$ as an element of S , we have that

$$P_{\sigma_{r+j}} = \text{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) u_{\tau_t}^{g_j}$$

for some $g_j \in \mathbb{N}$. Since $u_{\tau_t} \notin \text{st}_{S_{\tau_t}}(P_{\sigma_{r+j}})$, we have that $f_j \lambda \leq g_j$ and so $\lambda_j = g_j - f_j \lambda \geq 0$. □

By induction in the sequence of quadratic transforms above R and S in Lemma 2.5, and since $\nu^*(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}}$ by Proposition 3.1, we have by (23) and (24) an expression

$$(27) \quad \text{st}_{S_{\tau_t}}(P_{\sigma_{r+j}}) = c \text{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}) + u_{\tau_t} \Omega$$

with $c \in S_{\tau_t}$ a unit, $\Omega \in S_{\tau_t}$ and $\nu^*(u_{\tau_t} \Omega) \geq \nu^*(\text{st}_{S_{\tau_t}}(Q_{\tau_{s+j}}))$ if $s + j > t$ and

$$(28) \quad S_{\tau_t}(P_{\sigma_{r+j}}) \text{ is a unit in } S_{\tau_t}$$

if $s + j \leq t$. Thus $P_{\sigma_{r+j}} = u_{\tau_t}^{d_j} \bar{\varphi}_j$ in S_{τ_t} where d_j is a positive integer and $\bar{\varphi}_j$ is a unit in S_{τ_t} if $s + j \leq t$.

Suppose $s < t$. Then

$$y_{\sigma_r} = \text{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = \tilde{\varphi} u_{\tau_t}^h$$

where $\tilde{\varphi}$ is a unit in S_{τ_t} and h is a positive integer. As shown in equation (20) of Section 2,

$$R_{\sigma_{r+1}} = R_{\sigma_r}[\bar{x}_1, \bar{y}_1]_{m_{\nu} \cap R_{\sigma_r}[\bar{x}_1, \bar{y}_1]}$$

where

$$\bar{x}_1 = (x_{\sigma_r}^b y_{\sigma_r}^{-a})^\varepsilon, \quad \bar{y}_1 = (x_{\sigma_r}^{-\omega} y_{\sigma_r}^{m_{\sigma_r}})^\varepsilon$$

with $\varepsilon = m_{\sigma_r} b - \omega a = \pm 1$, $\nu(\bar{x}_1) > 0$ and $\nu(\bar{y}_1) = 0$. Substituting

$$x_{\sigma_r} = \varphi u_{\tau_t}^\lambda \text{ and } y_{\sigma_r} = \tilde{\varphi} u_{\tau_t}^h$$

we see that $R_{\sigma_{r+1}}$ is dominated by S_{τ_t} . We thus have a factorization

$$R_{\sigma_r} \rightarrow R_{\sigma_{r+1}} \rightarrow S_{\tau_t}$$

with $x_{\sigma_{r+1}} = \bar{x}_1 = \hat{\varphi} u_{\tau_t}^{\lambda'}$ where $\hat{\varphi}$ is a unit in S_{τ_t} and λ' is a positive integer. We may thus replace s with $s + 1$, r with $r + 1$ and R_{σ_r} with $R_{\sigma_{r+1}}$.

Iterating this argument, we may assume that $s = t$ (with $r = r_s$) so that by Lemma 3.2, (28) and (27),

$$y_{\sigma_r} = \text{st}_{R_{\sigma_r}}(P_{\sigma_{r+1}}) = u_{\tau_s}^\mu \text{st}_{S_{\tau_s}}(P_{\sigma_{r+1}})$$

where

$$\text{st}_{S_{\tau_s}}(P_{\sigma_{r+1}}) = c \text{st}_{S_{\tau_s}}(Q_{\tau_{s+1}}) + u_{\tau_s} \Omega$$

with c a unit in S_{τ_s} and $\Omega \in S_{\tau_s}$. Thus by (26), we have an expression

$$x_{\sigma_r} = \varphi u_{\tau_s}^\lambda, \quad y_{\sigma_r} = \bar{\varepsilon} u_{\tau_s}^\alpha (v_{\tau_s} + u_{\tau_s} \Omega)$$

where λ is a positive integer, $\alpha \in \mathbb{N}$, φ and $\bar{\varepsilon}$ are units in S_{τ_s} and $\Omega \in S_{\tau_s}$.

We have that $\nu^*(x_{\sigma_r}) = \lambda \nu^*(u_{\tau_s})$,

$$\begin{aligned} \nu(x_{\sigma_r}) \mathbb{Z} &= G(\nu(x_{\sigma_r})) = G(\beta_{\sigma_0}, \dots, \beta_{\sigma_r}) \text{ and} \\ \nu^*(u_{\tau_s}) \mathbb{Z} &= G(\nu^*(u_{\tau_s})) = G(\gamma_{\tau_0}, \dots, \gamma_{\tau_s}). \end{aligned}$$

Thus

$$\lambda = [G(\gamma_{\tau_0}, \dots, \gamma_{\tau_s}) : G(\beta_{\sigma_0}, \dots, \beta_{\sigma_r})] = e(\nu^*/\nu)$$

by Proposition 3.1.

By Theorem 2.3, we have that

$$R_{\sigma_r}/m_{R_{\sigma_r}} = R/m_R[\delta_{\sigma_1}, \dots, \delta_{\sigma_r}] \text{ and } S_{\tau_s}/m_{S_{\tau_s}} = S/m_S[\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_s}].$$

Thus

$$[S_{\tau_s}/m_{S_{\tau_s}} : R_{\sigma_r}/m_{R_{\sigma_r}}] = f(\nu^*/\nu)$$

by Proposition 3.1.

Since the ring R' of Proposition 1.1 is contained in R_{σ_r} by our construction, we have by Proposition 1.1 that $(K, \nu) \rightarrow (K^*, \nu^*)$ is without defect, completing the proofs of Proposition 0.3 and Theorem 0.1.

4. NON SPLITTING AND FINITE GENERATION

In this section, we will have the following assumptions. Suppose that R is a 2 dimensional excellent local domain with quotient field K . Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R . Suppose that ν^* is a valuation of K^* such that ν^* dominates S . Let ν be the restriction of ν^* to K .

Suppose that ν^* has rational rank 1 and ν^* is not discrete. Then V_{ν^*}/m_{ν^*} is algebraic over S/m_S , by Abhyankar's inequality, Proposition 2 [1].

Lemma 4.1. *Let assumptions be as above. Then the associated graded ring $\text{gr}_{\nu^*}(S)$ is an integral extension of $\text{gr}_{\nu}(R)$.*

Proof. It suffices to show that $\text{in}_{\nu^*}(f)$ is integral over $\text{gr}_{\nu}(R)$ whenever $f \in S$. Suppose that $f \in S$. There exists $n_1 > 0$ such that $n_1\nu^*(f) \in \Phi_{\nu}$. Let $x \in m_R$ and $\omega = \nu(x)$. Then there exists a positive integer b and natural number a such that $bn_1\nu^*(f) = a\omega$, so

$$\nu^* \left(\frac{f^{bn_1}}{x^a} \right) = 0.$$

Let

$$\xi = \left[\frac{f^{bn_1}}{x^a} \right] \in V_{\nu^*}/m_{\nu^*},$$

and let $g(t) = t^r + \bar{a}_{r-1}t^{r-1} + \dots + \bar{a}_0$ with $\bar{a}_i \in R/m_R$ be the minimal polynomial of ξ over R/m_R . Let a_i be lifts of the \bar{a}_i to R . Then

$$\nu^*(f^{b_1 n_1 r} + a_{r-1} x^a f^{b_1 n_1 (r-1)} + \dots + a_0 x^{ar}) > \nu^*(f^{b_1 n_1 r}) = \nu^*(a_{r-1} x^a f^{b_1 n_1 (r-1)}) = \dots = \nu^*(a_0 x^{ar}).$$

Thus

$$\text{in}_{\nu^*}(f)^{b_1 n_1 r} + \text{in}_{\nu}(a_{r-1} x^a) \text{in}_{\nu^*}(f)^{b_1 n_1 (r-1)} + \dots + \text{in}_{\nu}(a_0 x^{ar}) = 0$$

in $\text{gr}_{\nu^*}(S)$. Thus $\text{in}_{\nu^*}(f)$ is integral over $\text{gr}_{\nu}(R)$. □

We now establish Theorem 0.5. Recall (as defined after Proposition 0.3) that ν^* does not split in S if ν^* is the unique extension of ν to K^* which dominates S .

Theorem 4.2. *Let assumptions be as above and suppose that R and S are regular local rings. Suppose that $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra. Then S is a localization of the integral closure of R in K^* , the defect $\delta(\nu^*/\nu) = 0$ and ν^* does not split in S .*

Proof. Let s and r be as in the conclusions of Proposition 3.1. We will first show that $P_{\sigma_{r+j}}$ is irreducible in \hat{S} for all $j > 0$. There exists a unique extension of ν^* to the quotient field of \hat{S} which dominates \hat{S} ([43], [17], [22]). The extension is immediate since ν^* is not discrete; that is, there is no increase in value group or residue field for the extended valuation. It has the property that if $f \in \hat{S}$ and $\{f_i\}$ is a Cauchy sequence in \hat{S} which converges to f , then $\nu^*(f) = \nu^*(f_i)$ for all $i \gg 0$.

Suppose that $P_{\sigma_{r+j}}$ is not irreducible in \hat{S} for some $j > 0$. We will derive a contradiction. With this assumption, $P_{\sigma_{r+j}} = fg$ with $f, g \in m_{\hat{S}}$. Let $\{f_i\}$ be a Cauchy sequence in S which converges to f and let $\{g_i\}$ be a Cauchy sequence in S which converges to g . For i sufficiently large, $f - f_i, g - g_i \in m_{\hat{S}}^n$ where n is so large that $n\nu^*(m_{\hat{S}}) =$

$n\nu^*(m_S) > \nu(P_{\sigma_{r+j}})$. Thus $P_{\sigma_{r+j}} = f_i g_i + h$ with $h \in m_{\hat{S}}^n \cap S = m_S^n$, and so $\text{in}_{\nu^*}(P_{\sigma_{r+j}}) = \text{in}_{\nu^*}(f_i)\text{in}_{\nu^*}(g_i)$. Now

$$\nu^*(f_i), \nu^*(g_i) < \nu(P_{\sigma_{r+j}}) = \beta_{\sigma_{r+j}} = \gamma_{\tau_{s+j}} = \nu^*(Q_{\tau_{s+j}})$$

so that

$$\text{in}_{\nu^*}(f_i), \text{in}_{\nu^*}(g_i) \in S/m_S[\text{in}_{\nu^*}(Q_{\tau_0}), \dots, \text{in}_{\nu^*}(Q_{\tau_{s+j-1}})]$$

which implies

$$\text{in}_{\nu^*}(P_{\sigma_{r+j}}) \in S/m_S[\text{in}_{\nu^*}(Q_{\tau_0}), \dots, \text{in}_{\nu^*}(Q_{\tau_{s+j-1}})].$$

But then (24) implies

$$\text{in}_{\nu^*}(Q_{\tau_{s+j}}) \in S/m_S[\text{in}_{\nu^*}(Q_{\tau_0}), \dots, \text{in}_{\nu^*}(Q_{\tau_{s+j-1}})]$$

which is impossible. Thus $P_{\sigma_{r+j}}$ is irreducible in \hat{S} for all $j > 0$.

If S is not a localization of the integral closure of R in K^* , then by Zariski's Main Theorem (Theorem 1 of Chapter 4 [41]), $m_R S = fN$ where $f \in m_S$ and N is an m_S -primary ideal. Thus f divides P_i in S for all i , which is impossible since we have shown that $P_{\sigma_{r+j}}$ is analytically irreducible in S for all $j > 0$; we cannot have $P_{\sigma_{r+j}} = a_j f$ where a_j is a unit in S for $j > 0$ since $\nu(P_{\sigma_{r+j}}) = \nu^*(Q_{\tau_{s+j}})$ by Proposition 3.1.

Now suppose that ν^* is not the unique extension of ν to K^* which dominates S . Recall that V_ν is the union of all quadratic transforms above R along ν and V_{ν^*} is the union of all quadratic transforms above S along ν^* (Lemma 4.5 [3]).

Then for all $i \gg 0$, we have a commutative diagram

$$\begin{array}{ccc} R_{\sigma_i} & \rightarrow & T_i \\ \uparrow & & \uparrow \\ R & \rightarrow & T \end{array}$$

where T is the integral closure of R in K^* , T_i is the integral closure of R_{σ_i} in K^* , $S = T_{\mathfrak{p}}$ for some maximal ideal \mathfrak{p} in T which lies over m_R , and there exist $r \geq 2$ prime ideals $\mathfrak{p}_1(i), \dots, \mathfrak{p}_r(i)$ in T_i which lie over $m_{R_{\sigma_i}}$ and whose intersection with T is \mathfrak{p} . We may assume that $\mathfrak{p}_1(i)$ is the center of ν^* .

There exists an m_R -primary ideal I_i in R such that the blow up of I_i is $\gamma : X_{\sigma_i} \rightarrow \text{Spec}(R)$ where X_{σ_i} is regular and R_{σ_i} is a local ring of X_{σ_i} . Let Z_{σ_i} be the integral closure of X_{σ_i} in K^* . Let $Y_{\sigma_i} = Z_{\sigma_i} \times_{\text{Spec}(T)} \text{Spec}(S)$. We have a commutative diagram of morphisms

$$\begin{array}{ccc} Y_{\sigma_i} & \xrightarrow{\beta} & X_{\sigma_i} \\ \delta \downarrow & & \gamma \downarrow \\ \text{Spec}(S) & \xrightarrow{\alpha} & \text{Spec}(R) \end{array}$$

The morphism δ is projective (by Proposition II.5.5.5 [25] and Corollary II.6.1.11 [25]) and it is birational, so since Y_{σ_i} and $\text{Spec}(S)$ are integral, it is a blow up of an ideal J_i in S (Proposition III.2.3.5 [26]), which we can take to be m_S -primary since S is a regular local ring and hence factorial. Define curves $C = \text{Spec}(R/(P_{\sigma_i}))$ and $C' = \alpha^{-1}(C) = \text{Spec}(S/(P_{\sigma_i}))$. Denote the Zariski closure of a set W by \overline{W} . The strict transform C^* of C' in Y_{σ_i} is the Zariski closure

$$\begin{aligned} (29) \quad C^* &= \overline{\delta^{-1}(C' \setminus m_S)} = \overline{\delta^{-1}\alpha^{-1}(C \setminus m_R)} = \overline{\beta^{-1}\gamma^{-1}(C \setminus m_R)} \\ &= \beta^{-1}(\overline{\gamma^{-1}(C \setminus m_R)}) \text{ since } \beta \text{ is quasi finite} \\ &= \beta^{-1}(\tilde{C}) \end{aligned}$$

where \tilde{C} is the strict transform of C in X_{σ_i} . We have that $Z_{\sigma_i} \times_{X_{\sigma_i}} \text{Spec}(R_{\sigma_i}) \cong \text{Spec}(T_i)$, so

$$Y_{\sigma_i} \times_{X_{\sigma_i}} \text{Spec}(R_{\sigma_i}) \cong \text{Spec}(T_i \otimes_T S).$$

Let x_{σ_i} be a local equation in R_{σ_i} of the exceptional divisor of $\text{Spec}(R_{\sigma_i}) \rightarrow \text{Spec}(R)$ and let $y_{\sigma_i} = \text{st}_{R_{\sigma_i}}(P_{\sigma_i})$. Then $x_{\sigma_i}, y_{\sigma_i}$ are regular parameters in R_{σ_i} . We have that

$$\sqrt{m_{R_{\sigma_i}}(T_i \otimes_T S)} = \cap_{j=1}^r \mathfrak{p}_j(i)(T_i \otimes_T S).$$

The blow up of $J_i(S/(P_{\sigma_i}))$ in C' is $\bar{\delta} : C^* \rightarrow C'$, where $\bar{\delta}$ is the restriction of δ to C^* Corollary II.7.15 [28]). Since y_{σ_i} is a local equation of \tilde{C} in R_{σ_i} , we have by (29) that

$$\mathfrak{p}_1(i), \dots, \mathfrak{p}_r(i) \in \bar{\delta}^{-1}(m_S) \subset C^*.$$

Since $\bar{\delta}$ is proper and C' is a curve, $C^* = \text{Spec}(A)$ for some excellent one dimensional domain A such that the inclusion $S/(P_{\sigma_i}) \rightarrow A$ is finite (Corollary I.1.10 [39]). Let $B = A \otimes_{S/(P_{\sigma_i})} \hat{S}/(P_{\sigma_i})$. Then

$$C^* \times_{\text{Spec}(S/(P_{\sigma_i}))} \text{Spec}(\hat{S}/(P_{\sigma_i})) = \text{Spec}(B) \rightarrow \text{Spec}(\hat{S}/(P_{\sigma_i}))$$

is the blow up of $J_i(\hat{S}/(P_{\sigma_i}))$ in $\hat{S}/(P_{\sigma_i})$. The extension $\hat{S}/(P_{\sigma_i}) \rightarrow B$ is finite since $S/(P_{\sigma_i}) \rightarrow A$ is finite.

Now assume that $S/(P_{\sigma_i})$ is analytically irreducible. Then B has only one minimal prime since the blow up $\text{Spec}(B) \rightarrow \text{Spec}(\hat{S}/(P_{\sigma_i}))$ is birational.

Since a complete local ring is Henselian, B is a local ring (Theorem I.4.2 on page 32 of [39]), a contradiction to our assumption that $r > 1$. \square

As a consequence of the above theorem (Theorem 0.5), we now obtain Corollary 0.6.

Corollary 4.3. *Let assumptions be as above and suppose that R is a regular local ring. Suppose that $R \rightarrow R'$ is a nontrivial sequence of quadratic transforms along ν . Then $\text{gr}_{\nu}(R')$ is not a finitely generated $\text{gr}_{\nu}(R)$ -algebra.*

The conclusions of Theorem 0.5 do not hold if we remove the assumption that ν^* is not discrete, when V_{ν}/m_{ν} is finite over R/m_R . We give a simple example. Let k be an algebraically closed field of characteristic not equal to 2 and let $p(u)$ be a transcendental series in the power series ring $k[[u]]$ such that $p(0) = 1$. Then $f = v - up(u)$ is irreducible in the power series ring $k[[u, v]]$ and $k[[u, v]]/(f)$ is a discrete valuation ring with regular parameter u . Let ν be the natural valuation of this ring. Let $R = k[[u, v]]_{(u, v)}$ and $S = k[[x, y]]_{(x, y)}$. Define a k -algebra homomorphism $R \rightarrow S$ by $u \mapsto x^2$ and $v \mapsto y^2$. The series $f(x^2, y^2)$ factors as $f = (y - x\sqrt{p(x^2)})(y + x\sqrt{p(x^2)})$ in $k[[x, y]]$. Let $f_1 = y - x\sqrt{p(x^2)}$ and $f_2 = y + x\sqrt{p(x^2)}$. The rings $k[[x, y]]/(f_i)$ are discrete valuation rings with regular parameter x . Let ν_1 and ν_2 be the natural valuations of these ring.

Let ν be the valuation of the quotient field of R which dominates R defined by the natural inclusion $R \rightarrow k[[u, v]]/(f)$ and let ν_i for $i = 1, 2$ be the valuations of the quotient field of S which dominate S and are defined by the respective natural inclusions $S \rightarrow k[[x, y]]/(f_i)$. Then ν_1 and ν_2 are distinct extensions of ν to the quotient field of S which dominate S . However, we have that $\text{gr}_{\nu}(R) = k[\text{in}_{\nu}(u)]$ and $\text{gr}_{\nu_i}(S) = k[\text{in}_{\nu^*}(x)]$ with $\text{in}_{\nu^*}(x)^2 = \text{in}_{\nu}(u)$. Thus $\text{gr}_{\nu_i}(S)$ is a finite $\text{gr}_{\nu}(R)$ -algebra.

We now give an example where ν^* has rational rank 2 and ν splits in S but $\text{gr}_{\nu^*}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra. Suppose that k is an algebraically closed field of characteristic not equal to 2. Let $R = k[[x, y]]_{(x, y)}$ and $S = k[[u, v]]_{(u, v)}$. The substitutions

$u = x^2$ and $v = y^2$ make S into a finite separable extension of R . Define a valuation ν_1 of the quotient field K^* of S by $\nu_1(x) = 1$ and $\nu_1(y-x) = \pi+1$ and define a valuation ν_2 of the quotient field K^* by $\nu_2(x) = 1$ and $\nu_2(y+x) = \pi+1$. Since $u = x^2$ and $v-u = (y-x)(y+x)$, we have that $\nu_1(u) = \nu_2(u) = 2$ and $\nu_1(v-u) = \nu_2(v-u) = \pi+2$. Let ν be the common restriction of ν_1 and ν_2 to the quotient field K of R . Then ν splits in S . However, $\text{gr}_{\nu_1}(S)$ is a finitely generated $\text{gr}_{\nu}(R)$ -algebra since $\text{gr}_{\nu_1}(S) = k[\text{in}_{\nu_1}(x), \text{in}_{\nu_1}(y-x)]$ is a finitely generated k -algebra. Note that $\text{gr}_{\nu}(R) = k[\text{in}_{\nu}(u), \text{in}_{\nu}(v-u)]$ with $\text{in}_{\nu_1}(x)^2 = \text{in}_{\nu}(u)$ and $\text{in}_{\nu}(v-u) = 2\text{in}_{\nu_1}(y-x)\text{in}_{\nu_1}(x)$.

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