SPACES OF $\mathbb{R}$-PLACES OF FUNCTION FIELDS OVER REAL CLOSED FIELDS

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Abstract. In this paper an answer to the problem "When do different orderings of R(X) (where R is a real closed field) lead to the same $\mathbb{R}$-place?" is given. We use this result to show that if R is a dense real closed subfield of a real closed field $\tilde{R}$, then the spaces of $\mathbb{R}$-places of function fields over R and $\tilde{R}$ are homeomorphic. We also discuss the problem of metrizibility of the space $M(R(X))$.

1. Introduction

Studies of real places of formally real fields were initiated by Dubois [6] and Brown [3], and since then have been continued in several papers by Brown and Marshall [4], Harman [10], Schülting [15], Becker and Gondard [2] and Gondard and Marshall [9]. We shall briefly outline some basic notions of this theory. We will use the notation and terminology introduced by Lam [12], where also most of the results that we recall in this section can be found. We assume that the reader is somewhat familiar with valuation theory and theory of formally real (ordered) fields.

Let K be an ordered field. The set $\mathcal{X}(K)$ of all orderings of K can be made into a topological space by introducing a subbasis for the topology on $\mathcal{X}(K)$ consisting of Harrison sets, i.e., sets of the form

$$H_K(a) := \{P \in \mathcal{X}(K) : a \in P\}, \;\; a \in \dot{K} = K \setminus \{0\}.$$

The space $\mathcal{X}(K)$ is known to be Boolean (i.e., compact, Hausdorff and totally disconnected - see [12, p.2]).

For a fixed ordering P of K the set

$$A(P) := \{a \in K : \exists q \in \mathbb{Q}^+ (-q <_P a <_P q)\}$$

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is a valuation ring in K with the maximal ideal

$$I(P) := \{a \in K : \forall q \in \mathbb{Q}^+ (-q <_P a <_P q)\}.$$

There is a natural ordering on the residue field $K(P) := A(P)/I(P)$, which is Archimedean, namely $\bar{P} = (P \cap U(P)) + I(P)$, where $U(P) = A(P) \setminus I(P)$ is the set of units of $A(P)$. Thus $K(P)$ is naturally embedded in the field $\mathbb{R}$; this embedding composed with the place $K \to K(P) \cup \{\infty\}$ associated to the ordering P, gives a real-valued place, or an $\mathbb{R}$-place, for short. Conversely, every place of K with values in $\mathbb{R}$ is determined by some ordering of K in the way described above (see [12, Prop. 9.1]). The set of all $\mathbb{R}$-places of the field K will be denoted by $M(K)$.

The above described correspondence between orderings and $\mathbb{R}$-places defines a surjective map

$$\lambda_K : \mathcal{X}(K) \longrightarrow M(K),$$

which, in turn, allows us to equip $M(K)$ with the quotient topology inherited from $\mathcal{X}(K)$. $M(K)$ is a Hausdorff space (see [12, Cor. 9.9]). It is also compact as a continuous image of a compact space. Unlike $\mathcal{X}(K)$, the space $M(K)$ need not be Boolean. However, every Boolean space is realized as a space of $\mathbb{R}$-places of some formally real field ([13]). On the other hand, there are many examples of fields for which the space of $\mathbb{R}$-places has a finite number of connected components, or even is connected. In particular, if K is a real closed field, then the space $M(K)$ has only one point, and the space $M(\mathbb{R}(X))$ is homeomorphic to a circle (see [2], [15]). A slightly more general result states that the space of $\mathbb{R}$-places of a rational function field $K(X)$ is connected if and only if $M(K)$ is connected (see [10],[15]).

The main objective of this paper is to describe the space of $\mathbb{R}$-places of the field $R(X)$, where R is a real closed field. The main theorem of Section 2 explains how the map $\lambda_{R(X)} : \mathcal{X}(R(X)) \longrightarrow M(R(X))$ "glues" points. We then apply this result, in Section 3, to show that if a field R is a dense real closed subfield of a real closed field $\tilde{R}$, then the spaces $M(R(X))$ and $M(\tilde{R}(X))$ are naturally homeomorphic. In the last section we find conditions of metrizibility of the space $M(R(X))$.

Throughout this paper we shall denote by $\dot{S}$ the set $S \setminus \{0\}$, for any subset S of a field. We shall also use the familiar notion of intervals: if $(S, <)$ is a linearly ordered set, then

$$(a, b) = \{c \in S : a < c \wedge c < b\}.$$

Similarly we define $[a, b]$, $[a, b)$, (a, ∞) etc. If A, B are subsets of an ordered set S, then by $A < B$ we mean that $a < b$ for every $a \in A$ and every $b \in B$.

2. The $\mathbb{R}$-places of $R(X)$

Let R be a real closed field with its unique ordering $\dot{R}^2$. Denote by v the natural valuation of R, i.e., associated to $A(\dot{R}^2)$, by Γ the value group of v and by k the residue field of v. Since R is real closed, Γ is a divisible group and k is a real closed field (see [7, Th. 4.3.7]). Moreover, using Hensel's Lemma one can show that k can be considered as a subfield of R.

There is a one-to-one correspondence between orderings of $R(X)$ and cuts of R (see [8], [16]). The cut (A_P, B_P) corresponding to P is given by $A_P = \{a \in R : a <_P X\}$ and $B_P = \{b \in R : b >_P X\}$. Conversely, if (A, B) is a cut in R, then the set

$$Q = \{f \in R(X) : \exists a \in A\ \exists b \in B\ \forall c \in (a, b)\ \ (f(c) \in \dot{R}^2)\}$$

is an ordering of $R(X)$, and $(A_Q, B_Q) = (A, B)$.

The cuts $(\emptyset, R)$ and $(R, \emptyset)$ are called the *improper cuts*. The orderings determined by these cuts are

$$P_\infty^- = \{f \in R(X) : \exists b \in R\ \forall c < b\ \ (f(c) \in \dot{R}^2)\}$$

and

$$P_\infty^+, = \{f \in R(X) : \exists a \in R\ \forall c > b\ \ (f(c) \in \dot{R}^2)\},$$

respectively. A cut (A, B) of R is called *normal* if it satisfies the following condition:

$$\forall c \in \dot{R}^2\ \exists a \in A\ \exists b \in B\ \ (b - a < c).$$

If A has a maximal element or B has a minimal element, then (A, B) is called *a principal cut*. Principal cuts are normal. Every $a \in R$ defines two principal cuts: $((-\infty, a), [a, \infty))$, with the corresponding ordering denoted by P_a^-, and

$((-\infty, a], (a, \infty))$, with the corresponding ordering denoted by P_a^+. Note that if R is a real closed subfield of $\mathbb{R}$, then all proper cuts of R are normal. Moreover, $R = \mathbb{R}$ if and only if all proper cuts are principal.

If A does not have a maximal element and B does not have a minimal element, then we say that (A, B) is a *free cut* or a *gap*. If R is not contained in $\mathbb{R}$, then R has *abnormal* gaps, i.e., gaps which are not normal. For example, if

$$A = (-\infty, 0] \cup I(\dot{R}^2)$$

and

$$B = (0, \infty) \setminus I(\dot{R}^2),$$

then (A, B) is an abnormal gap in R.

In fact, we have three kinds of proper cuts:

(1) principal cuts,

(2) normal (but not principal) gaps,

(3) abnormal gaps.

Note that the correspondence between cuts in R and orderings of $R(X)$ makes the set $\mathcal{X}(R(X))$ linearly ordered: if Q is another ordering of $R(X)$, then let

$$P \prec Q \Longleftrightarrow A_P \subset A_Q.$$

The set $\mathcal{X}(R(X))$ has a minimal element P_∞^- and a maximal element P_∞^+. Consider the two orderings corresponding to the principal cuts determined by $a \in R$, P_a^- and P_a^+. Then $P_a^- \prec P_a^+$. Observe that the interval (P_a^-, P_a^+) is empty – we thus say that $\prec$ has *a step* in a.

Proposition 2.1. *The Harrison topology on the space $\mathcal{X}(R(X))$ coincides with the topology induced by the ordering defined above.*

Proof. Take the Harisson set $H_{R(X)}(\frac{f}{g}) = H(fg) \subset \mathcal{X}(R(X))$. Note that $H_{R(X)}(fg)$ is a finite union of intervals (P_a^-, P_b^+) such that:

1. $a, b \in R \cup \{\infty\}$ and if $a, b \in R$, then they are roots of fg;

2. fg has positive values on (a, b).

So, $H_{R(X)}(\frac{f}{g})$ is open in the order topology of $\mathcal{X}(R(X))$.

On the other hand, an interval $(P,Q) \subset \mathcal{X}(R(X))$ can be replaced by a union of Harrison sets $H_{R(X)}(f)$, where f runs through all quadratic polynomials with roots $a < b \in B_P \cap A_Q$ such that f is positive on (a,b). □

Consider the map

$$\lambda_{R(X)} : \mathcal{X}(R(X)) \longrightarrow M(R(X)).$$

Note that $\lambda_{R(X)}$ annihilates every step by "gluing" orderings, that is by mapping both P_a^- and P_a^+ onto the same real place. Our goal is to answer the following question: *Which points of $\mathcal{X}(R(X))$ are glued by $\lambda_{R(X)}$, that is, for which orderings P_1 and P_2 of $R(X)$, $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$?*

We shall make use of the following Separation Criterion [12, Prop. 9.13], which we recall now in the version useful for this paper:

Theorem 2.2. [Separation Criterion] *Let P_1 and P_2 be two different orderings of $R(X)$. Then $\lambda_{R(X)}(P_1) \neq \lambda_{R(X)}(P_2)$ if and only if there exists $f \in R(X)$ such that $f \in U(P_1) \cap P_1$ and $-f \in P_2$.*

We shall refer to f as to a "separating element". In view of the above described duality between orderings of $R(X)$ and cuts of R, we can also speak of a "separating element" of two cuts of R.

Claim 2.3. *Let P be an ordering of $R(X)$ with corresponding (proper) cut (A,B) in R. Then*

(1) $A(P) = \{f \in R(X) : \exists a \in A\ \exists b \in B\ \forall c \in [a,b]\ \ (f(c) \in A(\dot{R}^2))\}$
(2) $U(P) = \{f \in R(X) : \exists a \in A\ \exists b \in B\ \forall c \in [a,b]\ \ (f(c) \in U(\dot{R}^2))\}$

Proof. (1) Suppose that $f \in A(P)$. Then there exists $q \in \mathbb{Q}^+$ such that $q \pm f \in P$. It means that there exist $a \in A$, $b \in B$, such that for every $c \in (a,b)$,

$$-q < f(c) < q.$$

Then for every $c \in [a,b]$,

$$-q \leq f(c) \leq q.$$

Thus $f(c) \in A(\dot{R}^2)$.

Now suppose that there exists $a \in A$, $b \in B$, such that for every $c \in [a,b]$, $f(c) \in A(\dot{R}^2)$. Then f has no poles in $[a,b]$, and as it is a semialgebraic function, it is continuous on $[a,b]$. Therefore f has a minimum and a maximum on $[a,b]$, i.e., there exists $c_{min}, c_{max} \in [a,b]$ such that for every $c \in [a,b]$, $f(c_{min}) \le f(c) \le f(c_{max})$. Since $f(c_{min}), f(c_{max}) \in A(\dot{R}^2)$, there exists $q \in \mathbb{Q}^+$ such that

$$-q < f(c_{min}) \le f(c) \le f(c_{max}) < q,$$

for every $c \in [a,b]$, so $f \in A(P)$.
(2) By definition of the set of units of a valuation ring, $U(P)$ contains the functions $f \in R(X)$ such that $f \in A(P)$ and $1/f \in A(P)$. Suppose that $f \in U(P)$. Then there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that for every $c \in [a_1, b_1]$, $f(c) \in A(\dot{R}^2)$ and for every $c \in [a_2, b_2]$, $1/f(c) \in A(\dot{R}^2)$. Let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$. Then for every $c \in [a,b]$, $f(c)$ and $1/f(c)$ belong to $A(\dot{R}^2)$, i.e., $f(c) \in U(\dot{R}^2)$.

Supose that $f(c) \in U(\dot{R}^2)$ for every $c \in [a,b]$, where $a \in A$ and $b \in B$. Then $f, 1/f \in A(P)$, so $f \in U(P)$. $\square$

Remark 2.4. In a similar way one can show that

$$A(P_\infty^+) = \{f \in R(X) : \exists a \in R\, \forall c > a \;\; (f(c) \in A(\dot{R}^2))\},$$

$$U(P_\infty^+) = \{f \in R(X) : \exists a \in R\, \forall c > a \;\; (f(c) \in U(\dot{R}^2))\},$$

$$A(P_\infty^-) = \{f \in R(X) : \exists a \in R\, \forall c < a \;\; (f(c) \in A(\dot{R}^2))\},$$

$$U(P_\infty^+) = \{f \in R(X) : \exists a \in R\, \forall c < a \;\; (f(c) \in U(\dot{R}^2))\}.$$

Remark 2.5. By *a closed neighborhood* of a proper cut (A,B) in R we mean an interval $[a,b] \subset R$ such that $[a,b] \cap A \neq \emptyset$ and $[a,b] \cap B \neq \emptyset$. Note that $A(P)$ is the set of those functions which, on some closed neighborhood of (A_P, B_P), have values in $A(\dot{R}^2)$. Functions that belong to $U(P)$ are the ones that, on some closed neighborhood of (A_P, B_P), have values in $U(\dot{R}^2)$.

By [16, Lem. 2.2.1], every (proper) cut of R determines a lower cut set

$$S = \{v(b-a) :\; a \in A,\, b \in B\},$$

in the value group Γ. Note that if (A,B) is a normal cut, then $S = \Gamma$. For improper cuts, take $S = \emptyset$. The sets S allow us to compare gaps as follows: we can say that a gap (A_1, B_1) is "coarser" than (A_2, B_2) if $S_1 \subsetneq S_2$.

From now on let (A_1, B_1) and (A_2, B_2) be the cuts in R corresponding to two fixed orderings P_1 and P_2 of $R(X)$, respectively. Relabeling suitably, if necessary, we may assume that $A_1 \subset A_2$. Consider the set

$$U = \{v(a' - a) : \ a, a' \in B_1 \cap A_2, \ a < a'\}.$$

Denote by S_1 and S_2 the lower cuts in Γ determined by (A_1, B_1) and (A_2, B_2).

Lemma 2.6. *The set U is an upper cut set in Γ. Moreover, $\Gamma \setminus (S_1 \cap S_2) \subset U$.*

Proof. We shall show that if $\gamma \in U$ and $\gamma' \in \Gamma$ with $\gamma < \gamma'$, then $\gamma' \in U$. We have that $\gamma = v(a' - a)$, where a and a' are as in the definition of U and $\gamma' = v(c)$, where c is a positive element of R. Since $v(a' - a) < v(c)$, $v(\frac{c}{a'-a}) \in I(\dot{R}^2)$. Then $\frac{c}{a'-a} < 1$. So $a + c < a'$. Thus $a + c \in B_1 \cap A_2$. We have $v(a + c - a) = v(c) = \gamma' \in U$.

Now suppose that $\gamma \in \Gamma \setminus (S_1 \cap S_2)$. Let c be a positive element of R with $v(c) = \gamma$. Fix an element $a \in B_1 \cap A_2$. Assume that $\gamma \notin S_1$. Then $a - c \in B_1 \cap A_2$. Thus $\gamma = v(c) = v(a - (a - c)) \in U$. If $\gamma \notin S_2$, then $a + c \in B_1 \cap A_2$ and $\gamma = v(c) = v(a + c - a) \in U$. □

Theorem 2.7. *Let P_1, P_2 be the orderings as above. Then $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$ if and only if $S_1 = S_2 =: S$ and $S \cap U = \emptyset$.*

Proof. We consider three cases:

CASE 1. Suppose that $S_1 \subset S_2$. Then there exist $a \in B_1 \cap A_2$ and $b \in B_2$ such that $v(b - a) \notin S_1$. Consider a linear polynomial $f(X) = \frac{X-a}{b-a} + 1$. This polynomial has a root $x_0 = a - (b - a)$. If $x_0 \in A_1$, then $v(b - a) = v(a - x_0) \in S_1$, a contradiction. Therefore $x_0 \in B_1$. Moreover, $f(a) = 1$ and $f(b) = 2$. Thus f has positive values in some closed neighbourhood of (A_2, B_2) which are units in $A(\dot{R}^2)$ and negative values in some closed neighbourhood of (A_1, B_1). By Remark 2.5 and by the Separation Criterion, $\lambda_{R(X)}(P_1) \neq \lambda_{R(X)}(P_2)$. If $S_2 \subset S_1$, we proceed in a similar manner.

CASE 2. Suppose that $S_1 = S_2 =: S$ but $S \cap U \neq \emptyset$. Let $\gamma \in U \cap S$. Then there exist $a \in A_1$, $b \in B_1$ and $c, d \in B_1 \cap A_2$ such that $\gamma = v(b - a) = v(d - c)$.

We shall show that one can fix γ in such a way that $a < b \leqslant c < d$. If $c < b$, then we take $\gamma' = v(c - a)$. We have $\gamma' \geqslant \gamma$, so $\gamma' \in U$. If $\gamma' > \gamma$, then $v(c-a) > v(d-c)$. Thus $c-a < d-c$ and $c+(c-a) < d$. Therefore $c+(c-a) \in B_1 \cap A_2$. So we can take γ' as γ, c as b, and $c+(c-a)$ as d.

Since $v(b-a) = v(d-c)$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{b-a}{d-c} < n$. Then $\frac{b-a}{n} < d-c$. Consider a linear polynomial $f(X)$ such that $f(a) = n+1$ and $f(b) = 1$, that is $f(X) = \frac{n(b-X)}{b-a} + 1$. This polynomial has a root $x_0 = b + \frac{b-a}{n} < b+d-c < d$. Thus f has positive values in a closed neighborhood of (A_1, B_1) which are units in $A(\dot{R}^2)$ and negative values in a closed neighborhood of (A_2, B_2). Using the Separation Criterion we get $\lambda_{R(X)}(P_1) \neq \lambda_{R(X)}(P_2)$.

CASE 3. Suppose that $S_1 = S_2 =: S$ and $S \cap U = \emptyset$. By Lemma 2.6, $U = \Gamma \setminus S$. We can assume that $0 \in B_1 \cap A_2$. Indeed, if $a \in B_1 \cap A_2$, then consider the cuts: $(A_1 - a, B_1 - a)$ and $(A_2 - a, B_2 - a)$. Then $f(X)$ is a "separating element" for (A_1, B_1) and (A_2, B_2) if and only if $f(X+a)$ is a "separating element" for $(A_1 - a, B_1 - a)$ and $(A_2 - a, B_2 - a)$, and consequently, the $\mathbb{R}$-places determined by orderings associated to $(A_1 - a, B_1 - a)$ and $(A_2 - a, B_2 - a)$ are equal if and only if $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$. Note that U and S remain unchanged under translation of cuts.

The cuts (A_1, B_1) and (A_2, B_2) are symmetric about 0, i.e., $a \in B_1 \cap A_2 \Rightarrow -a \in B_1 \cap A_2$, and consequently, $a \in B_2 \Rightarrow -a \in A_1$. Indeed, if $a \in B_1 \cap A_2$ and $-a \in A_1$, then $S \ni v(a) = v(-a) \in U$, a contradiction to $S \cap U = \emptyset$.

Let $A := B_1 \cap A_2$ and $B = A_1 \cup B_2$. Then $A \cup B = R$, $A \cap B = \emptyset$, $A = -A$, $B = -B$. Further, $v(A) = U \cup \{\infty\}$. Since R is the disjoint union of A and B and Γ is the disjoint union of U and S, it follows that $v(B) = S$. We assume $B \neq \emptyset$; the proof can easily be adapted to the case of $B = \emptyset$, the case of improper cuts.

Let v_1 be the valuation determined by ordering P_1 and let v_2 be the valuation determined by P_2. Both v_1 and v_2 are extensions of v. We show that

$$v(a) \geqslant v_i(X) \geqslant v(b), \text{ for } a \in A, b \in B, i = 1, 2.$$

Indeed, since $v(a) = v(-a)$, $v(b) = v(-b)$, $A = -A$, $B = -B$, we may assume that $b \in B_2$, $-b \in A_1$, $a \geq 0$, $-a \leq 0$. It follows that $-b <_{P_1} X <_{P_1} -a \leq 0$ and $0 \leq a <_{P_2} X <_{P_2} b$, whence our claim.

Suppose that $v_i(X) = v(a)$ for some $a \in A$, then $v_i(\frac{X}{a}) = 0$. Thus the function $\frac{X}{a}$ has Archimedean values in some closed neighbourhood of (A_i, B_i). Therefore there exists $b \in B$ such that $v(\frac{b}{a}) = 0$. So $S \ni v(b) = v(a) \in U \cup \{\infty\}$, a contradiction to $S \cap U = \emptyset$. Similarly, $v_i(X) \neq v(b)$ for $b \in B, i = 1, 2$. Therefore

$$v(a) > v_i(X) > v(b), \text{ for } a \in A, b \in B, i = 1, 2.$$

So, $v_i(X) \notin \Gamma$ and since Γ is divisible, $n \cdot v_i(X) \notin \Gamma$ for $n \in \mathbb{N}$. By [7, Cor. 2.2.3], $v_1 = v_2$ and k is a the residue field of v_i.

By [12, Cor. 2.13], two orderings determine the same $\mathbb{R}$-place if they determine the same valuation and the same ordering on the residue field. Since the residue field of v_i is real closed, $\lambda_{R(X)}(P_1) = \lambda_{R(X)}(P_2)$ $\square$

Remark 2.8. The set U allows us to compare gaps by the "distance" between them. Theorem 2.7 shows that the map $\lambda_{R(X)}$ glues the abnormal gaps which are "close" to each other in the sense that the distance between them is smaller than their "size". For example, the orderings determined by gaps (A_1, B_1) and (A_2, B_2) where
$A_1 = -R^2 \setminus I(\dot{R}^2)$, $B_1 = I(\dot{R}^2) \cup R^2$,
$A_2 = -R^2 \cup I(\dot{R}^2)$, $B_2 = R^2 \setminus I(\dot{R}^2)$,
are always glued. More generally, we can replace $I(\dot{R}^2)$ by any convex subgroup $\{c \in R : v(c) \in U \text{ or } c = 0\}$ where U is a final segment of Γ. Then vX will satisfy the gap $(\Gamma \setminus U, U)$ in Γ. The proof of Theorem 2.7 shows that these are all possible cases, up to translation of cuts by adding an element of R.

Remark 2.9. The orderings P_a^+ and P_a^- determine the same $\mathbb{R}$-place, since $S_1 = S_2 = \Gamma$ and $U = \emptyset$. Also P_∞^+ and P_∞^- determine the same $\mathbb{R}$-place, since $S_1 = S_2 = \emptyset$ and $U = \Gamma$.

Remark 2.10. If R is a real closed subfield of $\mathbb{R}$, then every cut of R is normal. Then it is easy to deduce from Theorem 2.7 that the space $M(R(X))$ is homeomorphic to $M(\mathbb{R}(X))$. We will prove a more general result in Theorem 3.3 below.

Remark 2.11. At most two orderings determine the same $\mathbb{R}$-place. Let $P_1 \prec P_2 \prec P_3$ be orderings of $R(X)$ with corresponding cuts $(A_1, B_1), (A_2, B_2)$ and (A_3, B_3)

in R and corresponding lower cut sets S_1, S_2 and S_3 in Γ, respectively. Suppose that $S_1 = S_2 = S_3$. Let U_{13} be the upper cut set determined by the orderings P_1 and P_3. Take $a \in B_1 \cap A_2$ and $b \in B_2 \cap A_3$. Then $v(b-a) \in U_{13} \cap S_2$, so P_1 and P_3 do not determine the same $\mathbb{R}$-place.

Another way to see this is as follows. If an $\mathbb{R}$-place of $R(X)$ has the same value group as R, which is divisible and has no nontrivial 2-character, then there is only one ordering of $R(X)$ compatible with it. If it does not have the same value group as R, then it is of the form $\Gamma \oplus \mathbb{Z}$, having two 2-characters, and hence there are two distinct orderings compatible with it. Indeed, if two distinct orderings are glued, i.e., if $S_1 = S_2 =: S$ and $S \cap U = \emptyset$, then Case 3 of the proof of Theorem 2.7 shows that $v(X-a) \notin \Gamma$ for some $a \in R$.

3. Extension theory of $M(R(X))$

Let L/K be an extension of ordered fields. Then we have restriction maps

$$\rho : \mathcal{X}(L) \to \mathcal{X}(K), \qquad \rho(P) = P \cap K,$$

and

$$\rho : M(L) \to M(K), \qquad \rho(\xi) = \xi\,|_K\ .$$

The restriction maps are continuous and the diagram

$$\begin{array}{ccc} \mathcal{X}(L) & \xrightarrow{\lambda_F} & M(L) \\ \rho\big\downarrow & & \rho\big\downarrow \\ \mathcal{X}(K) & \xrightarrow{\lambda_K} & M(K) \end{array}$$

commutes (see [6, 7.2.]).

Note that the surjectivity of the map $\rho : \mathcal{X}(L) \to \mathcal{X}(K)$ implies the surjectivity of the map $\rho : M(L) \to M(K)$.

Lemma 3.1. *Let $R \subset \tilde{R}$ be an extension of real closed fields and let $\tilde{P}$ be an ordering of $\tilde{R}(X)$ with corresponding cut $(\tilde{A}, \tilde{B})$ in $\tilde{R}$. Then $(\tilde{A} \cap R, \tilde{B} \cap R)$ is a cut in R whose corresponding ordering $P \in \mathcal{X}(R(X))$ is a restriction of $\tilde{P}$. The map $\rho : \mathcal{X}(\tilde{R}(X)) \to \mathcal{X}(R(X))$ is surjective.*

Proof. It is easy to see that $(\tilde{A}\cap R, \tilde{B}\cap R)$ is a cut in R. If $(\tilde{A}, \tilde{B})$ is an improper cut in $\tilde{R}$, then $(\tilde{A}\cap R, \tilde{B}\cap R)$ is an improper cut in R, as well.

Recall that if $(\tilde{A}, \tilde{B})$ is a proper cut in $\tilde{R}$, then $f \in \tilde{P}$ iff there exists a closed neighbourhood $[a,b]$ of $(\tilde{A}, \tilde{B})$ such that for every $c \in [a,b]$, $f(c) > 0$. Since R is a real closed field, all real roots of a polynomial $f \in R[X]$ are in R. This implies that $\tilde{P} \cap R(X) = P$.

To show the last assertion, take $P \in \mathcal{X}(R(X))$ with corresponding cut (A, B) in R. Set $\tilde{A} = \{\tilde{a} \in \tilde{R} \mid \tilde{a} < B\}$ and $\tilde{B} = R \setminus \tilde{A}$. Then $(\tilde{A}, \tilde{B})$ is a cut in $\tilde{R}$ and $(\tilde{A}\cap R, \tilde{B}\cap R) = (A, B)$. Let $\tilde{P} \in \mathcal{X}(\tilde{R}(X))$ be the ordering corresponding to this cut. By what we have already proved, $\tilde{P} \cap R(X) = P$. □

Corollary 3.2. *Let $R \subset \tilde{R}$ be an extension of real closed fields. Then the map $\rho : M(\tilde{R}(X)) \to M(R(X))$ is surjective.*

Theorem 3.3. *Let $R \subset \tilde{R}$ be an extension of real closed fields. If R is dense in $\tilde{R}$, then $M(\tilde{R}(X))$ and $M(R(X))$ are homeomorphic.*

Proof. The restriction map $\rho : M(\tilde{R}(X)) \longrightarrow M(R(X))$ is surjective and continuous. Since both spaces are compact and Hausdorff, we need only to show that it is injective.

Take two distinct places $\xi_1, \xi_2 \in M(\tilde{R}(X))$ and let P_1, P_2 corresponding orderings of $\tilde{R}(X)$ and $(\tilde{A}_1, \tilde{B}_1)$ and $(\tilde{A}_2, \tilde{B}_2)$ be the cuts in $\tilde{R}$ associated with them. Let $\tilde{v}$ be the valuation corresponding to the unique ordering of $\tilde{R}$. Consider $\tilde{U} = \{v(\tilde{a}' - \tilde{a}) :\ \tilde{a}, \tilde{a}' \in \tilde{B}_1 \cap \tilde{A}_2,\ \tilde{a} < \tilde{a}'\}$. Set $A_i = \tilde{A}_i \cap R$ and $B_i = \tilde{B}_i \cap R$ for $i = 1, 2$. If $\tilde{U} = \emptyset$, then also $U = \{v(a' - a) :\ a, a' \in B_1 \cap A_2,\ a < a'\} = \emptyset$. If $\tilde{U} = \{0\}$, then $\tilde{R}$ and R are archimedean and $\tilde{B}_1 \cap \tilde{A}_2 \neq \emptyset$, so by density of R in $\tilde{R}$, $B_1 \cap A_2 \neq \emptyset$, which implies $U = \{0\} = \tilde{U}$. Now assume that U has at least two elements and hence has no last element. Then by the density of R in $\tilde{R}$, for all $\tilde{a}, \tilde{a}' \in \tilde{B}_1 \cap \tilde{A}_2$ with $\tilde{a} < \tilde{a}'$ there are a so close to $\tilde{a}$ and a' so close to $\tilde{a}$ with $\tilde{a} < a < a' < \tilde{a}'$ such that $v(\tilde{a} - a) > v(\tilde{a}' - \tilde{a})$ and $v(\tilde{a}' - a') > v(\tilde{a}' - \tilde{a})$. It follows that $a, a' \in B_1 \cap A_2$ with $v(a' - a) = v(\tilde{a}' - \tilde{a})$. Hence $U = \tilde{U}$.

In the same way, one shows that $\tilde{S}_i = \{v(\tilde{b} - \tilde{a}) :\ \tilde{a} \in \tilde{A}_i,\, \tilde{b} \in \tilde{B}_i\} = \{v(b - a) :\ a \in A_i,\, b \in B_i\} = S_i$ for $i = 1, 2$. Now it follows from Theorem 2.7 that the restrictions of ξ_1 and ξ_2 to $R(x)$ remain distinct. □

Recall that an ordered field K is called *continuously closed* if every normal cut in K is principal. We say that an ordered field $\tilde{K}$ is *a continuous closure of* K if $\tilde{K}$ is continuously closed and K is dense in $\tilde{K}$. The continuous closure $\tilde{K}$ is uniquely determined for every ordered field K. Moreover, if K is real closed, then $\tilde{K}$ is also real closed (see [1]). In fact, the continuous closure $\tilde{K}$ of K is a completion of K with respect to:
1) order topology if K is Archimedean;
2) valuation topology if K is not Archimedean
(see [14]).

So we have:

Corollary 3.4. *If $\tilde{R}$ is the continuous closure of R, then $M(\tilde{R}(X))$ and $M(R(X))$ are homeomorphic.*

4. Metrizibility of the space $M(R(X))$

First we shall recall some basic topological facts. By Urysohn's metrization theorem (see [11, p. 125]) a compact Hausdorff space is metrizable if and only if it is second-countable. Every second-countable space is separable. Recall that the cellularity of a topological space M is

$$\sup\{|\mathcal{F}|: \mathcal{F} \text{ is a family of pairwise disjoint open subsets of } M\}.$$

The cellurality is not smaller than the density of M.

Recall that *the real holomorphy ring* $\mathcal{H}_K$ of a formally real field K is the intersection of all real valuation rings of K, i.e.,

$$\mathcal{H}_K = \bigcap\{A(P), P \in \mathcal{X}(K)\}.$$

By [12, Th. 9.11], a subbasis for the space $M(K)$ is given by the family of the sets $U(a) = \{\xi \in M(K) \mid \xi(a) > 0\}$, where $a \in \mathcal{H}_K$. If K is countable, then this subbasis (and consequently, also a basis) of $M(K)$ is countable, so $M(K)$ is second-countable. So we have:

Corollary 4.1. *If K is a countable field, then $M(K)$ is metrizable.*

As before we consider a real closed field R with natural valuation v, value group Γ, and residue field $k \subset R$.

Lemma 4.2. *Take $a \in R$. Then the set*

$$U_a = \bigcup_{\Gamma \ni \gamma > 0} \{\xi \in M(R(X)) \mid v_\xi(X-a) > \gamma\}$$

is open in $M(R(X))$.

Proof. If Γ is a trivial group, then $U_a = \emptyset$. So we assume that Γ is not trivial.

We shall show that

$$\lambda_{R(X)}^{-1}(U_a) = \bigcup_{c \in \dot{R}^2, v(c) > 0} (P_{a-c}^-, P_{a+c}^+),$$

where each (P_{a-c}^-, P_{a+c}^+) is an open interval in $\mathcal{X}(R(X))$.

Suppose that $P \in \lambda_{R(X)}^{-1}(U_a)$. Then there exists $\Gamma \ni \gamma > 0$ such that $v_P(X-a) > \gamma$. Let $c \in \dot{R}^2$ such that $v(c) = \gamma$. Then $-c <_P X - a <_P c$, and thus $a - c <_P X <_P a + c$, so $P \in (P_{a-c}^-, P_{a+c}^+)$.

Now suppose that $P \in (P_{a-c}^-, P_{a+c}^+)$ for some $c \in \dot{R}^2, v(c) > 0$, i.e., $a - c \leq_P X \leq_P a+c$. Then $-c \leq_P X-a \leq_P c$ and thus $v_P(X-a) \geq v(c) > \frac{1}{2}v(c) > 0$. □

Proposition 4.3. *Let R be a non-Archimedean real closed field such that k is an uncountable field or Γ is an uncountable group. Then is not metrizable.*

Proof. Suppose that $k \subset R$ is an uncountable field. For every $a \in k$ take an open set U_a as in the previous lemma.

Note that U_a is nonempty, because the place determined by the principal cuts in a belongs to U_a.

Suppose that $U_a \cap U_b \neq \emptyset$ for $a \neq b$. Let $\xi \in U_a \cap U_b$, i.e., $v_\xi(X-a) > \gamma_1 > 0$ and $v_\xi(X-b) > \gamma_2 > 0$, for some $\gamma_1, \gamma_2 \in \Gamma$. Then $v_\xi(a-b) = v_\xi((X-a)-(X-b)) > 0$, a contradiction, because k is Archimedean.

Now suppose that Γ is uncountable. For every $\Gamma \ni \gamma < 0$ choose an element $a \in R$ with $v(a) = \gamma$ and consider sets U_a, which are like previously open and nonempty.

Suppose that $\xi \in U_a \cap U_b$ for $a \neq b$. Then $v_\xi(X-a) > 0$ and $v_\xi(X-b) > 0$, thus $v_\xi(a-b) = v_\xi((X-a)-(X-b)) > 0$. But $v_\xi(a-b) = \min\{v_\xi(a), v_\xi(b)\} < 0$, a contradiction.

In both cases the family of the sets U_a is an uncountable family of pairwise disjoint open sets in $M(R(X))$, so the cellularity of $M(R(X))$ is uncountable, hence $M(R(X))$ cannot be metrizable. $\square$

Remark 4.4. The proof shows that, without the assumptions on k or Γ, the cellularity of $M(R(X))$ is bigger or equal to $\max\{|k|,|\Gamma|\}$.

Lemma 4.5. *Let N be a dense subset in $M(R(X))$. Then $\lambda_{R(X)}^{-1}(N)$ is a dense subset of $\mathcal{X}(R(X))$.*

Proof. Take a basic open set in $\mathcal{X}(R(X))$, i.e., the set of all cuts in an interval $(a,b) \subset R$. Consider a polynomial $f(X) \in R[X]$, $f(X) = \frac{-4(X-a)(X-b)}{(b-a)^2}$ and let $g = \frac{f}{1+f^2}$. Note that g is positive only on interval (a,b) and $g(\frac{a+b}{2}) = \frac{1}{2}$. Therefore the subbasic set $U(g)$ is nonempty (the $\mathbb{R}$-place determined by the principal cuts in $\frac{a+b}{2}$ belongs to $U(g)$), and by density of N in $M(R(X))$, there exists $\xi \in N \cap U(g)$. Let $P \in \lambda_{R(X)}^{-1}(\xi)$ and let (A,B) be a cut corresponding to P. Since $\xi(g) > 0$, $g \in P$. So there exists $a' \in A, b' \in B$ such that for every $c \in (a',b')$, $g(c) > 0$. So, $(a',b') \subseteq (a,b)$ and P corresponds to a cut in (a,b). $\square$

Theorem 4.6. *Let R be a real closed field. Then $M(R(X))$ is metrizable if and only if R contains a countable dense subfield.*

Proof. Suppose that $M(R(X))$ is metrizable. Therefore both, the residue field k and the value group Γ of the natural valuation v of R are countable. Since $M(R(X))$ is compact, it is separable. Let N be a countable, dense subset of $M(R(X))$. Then, by the previous lemma, the set $\lambda_{R(X)}^{-1}(N)$ is dense in $\mathcal{X}(R(X))$. Using this set we shall describe a construction of a countable, dense subset of R.

For every $\gamma \in \Gamma$ choose an element $c_\gamma \in \dot{R}^2$ such that $v(c_\gamma) = \gamma$.

Let (A,B) be a cut in R with corresponding ordering $P \in \lambda_{R(X)}^{-1}(N)$ and let S be the corresponding lower cut set in Γ. For every γ which is not the maximal element in S choose a pair of elements $a_\gamma^P \in A$ and $b_\gamma^P \in B$ such that $v(b_\gamma^P - a_\gamma^P) = \gamma$. If γ_0 is a maximal element in S then choose $a \in A$, $b \in B$ such that $v(b-a) = \gamma_0$. As pointed out earlier in the paper, we may assume that the residue field k is a subfield of R. Then

$$(\{\bar{d} \in k \,:\, a + \bar{d}c_{\gamma_0} \in A\}, \{\bar{e} \in k \,:\, a + \bar{e}c_{\gamma_0} \in B\})$$

is a cut in k. We note that $\bar{e}$ is in the left cut set, and that $a+\bar{f}c_{\gamma_0} \in B$ for every $\bar{f} \in k$ such that $\bar{f} > \frac{b-a}{c_{\gamma_0}}$. Hence, this cut is proper. For every $\bar{c} \in \dot{k}^2$ we can thus choose $\bar{d}$ in the left and $\bar{e}$ in the right cut set such that $\bar{e} - \bar{d} = \bar{c}$. Setting $a^P_{\bar{c}} = a + \bar{d}c_{\gamma_0} \in A$ and $b^P_{\bar{c}} = a + \bar{e}c_{\gamma_0} \in B$ we obtain that $v(b^P_{\bar{c}} - a^P_{\bar{c}}) = \gamma_0$ and $\xi_{\dot{R}^2}(\frac{b^P_{\bar{c}} - a^P_{\bar{c}}}{c_{\gamma_0}}) = \bar{c}$.

Let $\mathcal{A}_P$ be a set of all $a^P_\gamma, b^P_\gamma, a^P_{\bar{c}}, b^P_{\bar{c}}$ with $\gamma \in S$, $\bar{c} \in \dot{k}^2$. Note that $\mathcal{A}_P$ is a countable set because S and $\dot{k}^2$ are countable. Let $\mathcal{A} = \bigcup\{\mathcal{A}_P : P \in \lambda^{-1}_{R(X)}(N)\}$. Then $\mathcal{A}$ is countable. We will show that it is dense in R.

Suppose that $a < b \in R$. By density of $\lambda^{-1}_{R(X)}(N)$ in $\mathcal{X}(R(X))$, there exists $P \in \lambda^{-1}_{R(X)}(N)$ such that $P^+_a \prec P \prec P^-_b$. Let (A, B) be a cut in R corresponding to P and let S be the corresponding lower cut set in Γ. Then $v(b-a) \in S$. If $v(b-a)$ is not the maximal element in S, then $v(b-a) < \gamma$ for some $\gamma \in S$. In this case, consider $a^P_\gamma, b^P_\gamma \in \mathcal{A}_P$. Since $v(b-a) < v(b^P_\gamma - a^P_\gamma)$, we have $a^P_\gamma \in (a,b)$ or $b^P_\gamma \in (a,b)$. If $v(b-a) = \gamma_0$ is the maximal element in S, then $\xi_{\dot{R}^2}(\frac{b-a}{c_{\gamma_0}}) = \bar{d} \in \dot{k}^2$. Take $\bar{c} \in \dot{k}^2$, $\bar{c} < \bar{d}$. Then for $a^P_{\bar{c}}, b^P_{\bar{c}} \in \mathcal{A}_P$,

$$\xi_{\dot{R}^2}(\frac{b-a}{c_{\gamma_0}} - \frac{b^P_{\bar{c}} - a^P_{\bar{c}}}{c_{\gamma_0}}) > 0\,.$$

Thus, $(b-a) - (b^P_{\bar{c}} - a^P_{\bar{c}}) > 0$.

If $a^P_{\bar{c}} < a$, then $0 < (b-a) - (b^P_{\bar{c}} - a^P_{\bar{c}}) < b - b^P_{\bar{c}}$, and thus $b^P_{\bar{c}} < b$. Similarly, if $b < b^P_{\bar{c}}$, then $a < a^P_{\bar{c}}$. So the interval (a,b) contains an element from $\mathcal{A}$. Since $\mathcal{A}$ is dense in R the field $k(\mathcal{A})$ is dense in R and countable, because $\mathcal{A}$ is countable.

Now suppose that K is a countable, dense subfield of R. Let R' be the real closure of K inside of R. Then $R' \subset R$ and R' is countable and dense in R. By Theorem 3.3, $M(R'(X)) \cong M(R(X))$, and by Corollary 4.1, $M(R'(X))$ is metrizable. $\square$

The following example shows that the converse of Proposition 4.3 does not hold:

Example 4.7. Let be a countable, Archimedean field k and a countable, non-trivial, ordered, divisible group Γ. The field $k((\Gamma))$ is real closed, with its natural valuation v being its t-adic valuation with value group Γ and residue field k. Take R to be the real closure of $k(\Gamma)$ in $k((\Gamma))$.

Consider the function field $R(X)$. Since R is countable, $M(R(X))$ is metrizable. We shall show that $M(k((\Gamma))(X))$ is not metrizable.

Since Γ is divisible, $\mathbb{Q} \subseteq \Gamma$. Fix an increasing sequence of rational numbers (γ_n) converging to 0. Consider a Cantor set given as a family of functions

$$\sigma \in \{0,1\}^{\{\gamma_n : n \in \mathbb{N}\}} .$$

Now define a family of sets U_σ of cardinality $2^{\aleph_0}$ as follows: U_σ contains all $\mathbb{R}$-places determined by cuts of the interval (a^σ, b^σ), where

$$a^\sigma_\delta = \begin{cases} \sigma(\delta) & \delta = \gamma_n \\ -1 & \delta = 0 \\ 0 & otherwise \end{cases}$$

$$b^\sigma_\delta = \begin{cases} \sigma(\delta) & \delta = \gamma_n \\ +1 & \delta = 0 \\ 0 & otherwise. \end{cases}$$

Take $\sigma, \tau \in \{0,1\}^{\{\gamma_n : n \in \mathbb{N}\}}$, a cut (A_1, B_1) in (a^σ, b^σ) with corresponding lower cut set S_1 in Γ, and a cut (A_2, B_2) in (a^τ, b^τ) with corresponding lower cut set S_2. Then both S_1 and S_2 contain $(-\infty, 0)$. Let U be the upper cut set in Γ corresponding to (A_1, B_1) and (A_2, B_2). If $\sigma \neq \tau$, then U contains an element $\gamma < 0$. Thus $U \cap S_1 \neq \emptyset$ and by Theorem 2.7, the $\mathbb{R}$-places determined by orderings of $k((\Gamma))(X)$ associated to the cuts (A_1, B_1) and (A_2, B_2) are distinct. Therefore $U_\sigma \cap U_\tau = \emptyset$ for $\sigma \neq \tau$, and thus cellularity of $M(k((\Gamma))(X))$ is uncountable.

More generally, take any real closed subfield R' of $k((\Gamma))$. If it is included in a subfield of $k((\Gamma))$ that is of countable transcendence degree over the completion of R, then by Theorem 4.6 $M(R'(X))$ is metrizable. It can be shown that also the converse is true: if the compositum of R' with the completion is of uncountable transcendence degree over the completion, then there are again uncountably many $\sigma \in R'$ that one can use for the above definition of the intervals U_σ.

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