ON FIELDS OF TOTALLY S-ADIC NUMBERS

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Abstract. Given a finite set S of places of a number field, we prove that the field of totally S-adic algebraic numbers is not Hilbertian.

The field of totally real algebraic numbers $\mathbb{Q}_{\mathrm{tr}}$, the field of totally p-adic algebraic numbers $\mathbb{Q}_{\mathrm{tot},p}$, and, more generally, fields of totally S-adic algebraic numbers $\mathbb{Q}_{\mathrm{tot},S}$, where S is a finite set of places of $\mathbb{Q}$, play an important role in number theory and Galois theory, see for example [5, 8, 9, 7]. The objective of this note is to show that none of these fields is Hilbertian (see [3, Chapter 12] for the definition of a Hilbertian field).

Although it is immediate that $\mathbb{Q}_{\mathrm{tr}}$ is not Hilbertian, it is less clear whether the same holds for $\mathbb{Q}_{\mathrm{tot},p}$. For example, every finite group that occurs as a Galois group over $\mathbb{Q}_{\mathrm{tr}}$ is generated by involutions (in fact, the converse also holds, see [4]) although over a Hilbertian field all finite abelian groups (for example) occur. In contrast, over $\mathbb{Q}_{\mathrm{tot},p}$ every finite group occurs, see [2]. In fact, although (except in the case of $\mathbb{Q}_{\mathrm{tr}}$) it was not clear whether these fields are actually Hilbertian, certain weak forms of Hilbertianity were proven and used, both explicitly and implicitly, for example in [4, 6]. Also, any proper finite extension of any of these fields is actually Hilbertian, see [3, Theorem 13.9.1].

The non-Hilbertianity of $\mathbb{Q}_{\mathrm{tot},p}$ was actually implicitly stated and proven in [1, Examples 5.2] but this result seems to have escaped the notice of the community and was forgotten. We give a short elementary proof (which is closely related to the proof in [1]) of the following more general result.

Theorem 1. *For any finite set S of real archimedean or ultrametric discrete absolute values on a field K, the maximal extension $K_{\mathrm{tot},S}$ of K in which every element of S totally splits is not Hilbertian.*

Note that $K_{\mathrm{tot},S}$ is the intersection of all Henselizations and real closures of K with respect to elements of S. We would like to stress that S does not necessarily consist of *local* primes in the sense of [7].

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Let

$$\gamma(Y,T) \quad = \quad (Y^{-1}+T^{-1}Y)^{-1} \quad = \quad \frac{YT}{Y^2+T}$$

and

$$f(X,Z) \quad = \quad X^2+X-Z^2.$$

Lemma 2. *If (F,v) is a discrete valued field with uniformizer $t \in F$, then $v(\gamma(y,t)) > 0$ for each $y \in F$.*

Proof. If $v(y)=0$, then $v(t^{-1}y) < 0 = v(y^{-1})$, so $v(y^{-1}+t^{-1}y) < 0$. If $v(y) < 0$, then $v(y^{-1}) > 0$ and $v(t^{-1}y) < 0$, so $v(y^{-1}+t^{-1}y) < 0$. If $v(y) > 0$, then $v(y^{-1}) < 0$ and $v(t^{-1}y) \geq 0$ since t is a uniformizer, so again $v(y^{-1}+t^{-1}y) < 0$. Thus, in each case, $v(\gamma(y,t)) = -v(y^{-1}+t^{-1}y) > 0$. □

Lemma 3. *Let F be a field and $t \in F \smallsetminus \{0,-1\}$. If $\mathrm{char}(F)=2$, assume in addition that t is not a square in F. Then $f(X,\gamma(Y,t))$ is irreducible over $F(Y)$.*

Proof. If $\mathrm{char}(F) \neq 2$, then $f(X,\gamma(Y,t))$ is reducible if and only if the discriminant $1+4\gamma(Y,t)^2$ is a square in $F(Y)$. This is the case if and only if $(Y^2+t)^2+4(tY)^2$ is a square. Writing

$$(Y^2+t)^2+4(tY)^2 = (Y^2+aY+b)^2$$

and comparing coefficients we get that $a=0$, $b^2=t^2$, and $a^2+2b = 2t(1+2t)$. Hence, $t=0$ or $t=-1$.

If $\mathrm{char}(F)=2$, then $f(X,\gamma(Y,t))$ is irreducible if and only if

$$g(X) := f(X+\gamma(Y,t),\gamma(Y,t)) = X^2+X+\gamma(Y,t)$$

is irreducible. If v denotes the normalized valuation on $F(Y)$ corresponding to the irreducible polynomial $Y^2+t \in F[Y]$, then $v(\gamma(Y,t)) = -1$. This implies that a zero x of $g(X)$ in $F(Y)$ would satisfy $v(x) = -\frac{1}{2}$, so $g(X)$ has no zero in $F(Y)$ and is therefore irreducible. □

Proof of Theorem 1. Without loss of generality assume that $S \neq \emptyset$ and that the absolute values in S are pairwise inequivalent. Let $F = K_{\mathrm{tot},S}$.

The weak approximation theorem gives an element $t \in K \smallsetminus \{0,-1\}$ that is a uniformizer for each of the ultrametric absolute values in S. Clearly, if S contains an ultrametric discrete absolute value (in particular if $\mathrm{char}(K)=2$), then t is not a square in F. Hence, by Lemma 3, $f(X,\gamma(Y,t))$ is irreducible over $F(Y)$.

Assume, for the purpose of contradiction, that F is Hilbertian. Then there exists $y \in F$ such that $f(X,\gamma(y,t))$ is defined and irreducible over F.

Let $|\cdot| \in S$. If $|\cdot|$ is archimedean (this means we are in the case $\mathrm{char}(K) \neq 2$), let $\leq$ be an ordering corresponding to an extension of

$|\cdot|$ to F, and let E be a real closure of $(F, \leq)$. Since $\gamma(y,t)^2 \geq 0$, there exists $x \in E$ such that $f(x, \gamma(y,t)) = 0$ (note that the map $E_{\geq 0} \to E_{\geq 0}$, $\xi \mapsto \xi^2 + \xi$ is surjective). If $|\cdot|$ is ultrametric and v is a discrete valuation corresponding to an extension of $|\cdot|$ to F, let E be a Henselization of (F, v). Since $v(\gamma(y,t)) > 0$ by Lemma 2, $f(X, \gamma(y,t)) \in \mathcal{O}_v[X]$ and

$$\overline{f(X, \gamma(y,t))} = X(X+1)$$

has a simple root, so by Hensel's lemma there exists $x \in E$ with $f(x, \gamma(y,t)) = 0$.

Thus in each case, $f(X, \gamma(y,t))$ has a root in E, so since it is of degree 2 all of its roots are in E. Since F is the intersection over all such E, all roots of $f(X, \gamma(y,t))$ lie in F, contradicting the irreducibility of $f(X, \gamma(y,t))$. □

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