ON THE BRAUER p-DIMENSIONS OF HENSELIAN DISCRETE VALUED FIELDS OF RESIDUAL CHARACTERISTIC $p > 0$

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Abstract. Let (K, v) be a Henselian discrete valued field with residue field $\widehat{K}$ of characteristic p, and $\mathrm{Brd}_p(K)$, $\mathrm{abrd}_p(K)$ be the Brauer p-dimension and absolute Brauer p-dimension of K, respectively. This paper shows that $\mathrm{abrd}_p(K) \geq n$ and $\mathrm{Brd}_p(K) \geq n - [n/3] - 1$, if $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$; the second inequality is strict when $n \neq 5$. We show that $\mathrm{Brd}_p(K) = \infty$, if $[\widehat{K}\colon \widehat{K}^p] = \infty$, and we find $\mathrm{Brd}_p(K)$ and $\mathrm{abrd}_p(K)$ in case $[\widehat{K}\colon \widehat{K}^p] \leq p$.

1. Introduction

Let E be a field, $\mathrm{Br}(E)$ its Brauer group, $s(E)$ the class of associative finite-dimensional central simple algebras over E, $d(E)$ the subclass of division algebras $D \in s(E)$, and for each $A \in s(E)$, let $\deg(A)$, $\mathrm{ind}(A)$ and $\exp(A)$ be the degree, the Schur index and the exponent of A, respectively. It is well-known (cf. [28], Sect. 14.4) that $\exp(A)$ divides $\mathrm{ind}(A)$ and shares with it the same set of prime divisors; also, $\mathrm{ind}(A) \mid \deg(A)$, and $\deg(A) = \mathrm{ind}(A)$ if and only if $A \in d(E)$. Note that $\mathrm{ind}(B_1 \otimes_E B_2) = \mathrm{ind}(B_1)\mathrm{ind}(B_2)$ whenever $B_1, B_2 \in s(E)$ and g.c.d.$\{\mathrm{ind}(B_1), \mathrm{ind}(B_2)\} = 1$; equivalently, $B_1' \otimes_E B_2' \in d(E)$, if $B_j' \in d(E)$, $j = 1, 2$, and g.c.d.$\{\deg(B_1'), \deg(B_2')\} = 1$ (see [28], Sect. 13.4). Since $\mathrm{Br}(E)$ is an abelian torsion group, and $\mathrm{ind}(A)$, $\exp(A)$ are invariants both of A and its equivalence class $[A] \in \mathrm{Br}(E)$, these results indicate that the study of the restrictions on the pairs $\mathrm{ind}(A)$, $\exp(A)$, $A \in s(E)$, reduces to the special case of p-primary pairs, for an arbitrary fixed prime p. The Brauer p-dimensions $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers, contain essential (sometimes, complete) information on these restrictions. We say that $\mathrm{Brd}_p(E) = n < \infty$, for a given $p \in \mathbb{P}$, if n is the least integer ≥ 0, for which $\mathrm{ind}(A_p) \mid \exp(A_p)^n$ whenever $A_p \in s(E)$ and $[A_p]$ lies in the p-component $\mathrm{Br}(E)_p$ of $\mathrm{Br}(E)$; if no such n exists, we put $\mathrm{Brd}_p(E) = \infty$. For instance, $\mathrm{Brd}_p(E) \leq 1$, for all $p \in \mathbb{P}$, if and only if E is a stable field, i.e. $\deg(D) = \exp(D)$, for each $D \in d(E)$; $\mathrm{Brd}_{p'}(E) = 0$, for some $p' \in \mathbb{P}$, if and only if $\mathrm{Br}(E)_{p'}$ is trivial. The absolute Brauer p-dimension of E is defined as the supremum $\mathrm{abrd}_p(E)$ of $\mathrm{Brd}_p(R)\colon R \in \mathrm{Fe}(E)$, where $\mathrm{Fe}(E)$ is the set of finite extensions of E in a separable closure E_{sep}. We have $\mathrm{abrd}_p(E) \leq 1$, $p \in \mathbb{P}$, if E is an absolutely stable field, i.e. its finite extensions are stable fields. Important examples of this type are provided

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by class field theory, which shows that $\mathrm{Brd}_p(\Phi) = \mathrm{abrd}_p(\Phi) = 1$, $p \in \mathbb{P}$, if Φ is a global or local field (see, e.g., [30], (31.4) and (32.19)).

The sequence $\mathrm{Brd}_p(E), \mathrm{abrd}_p(E) \colon p \in \mathbb{P}$, contains useful information about the behaviour of index-exponent relations over finitely-generated transcendental extensions of E [8]. As it turns out, an essential part of it can be derived of the description in [9] of the set of sequences $\mathrm{Brd}_p(K_q)$, $\mathrm{abrd}_p(K_q)$, $p \in \mathbb{P}$, where K_q runs across the class of fields with Henselian valuations v_q whose residue fields $\widehat{K}_q$ are perfect of characteristic $q \geq 0$, and such that their absolute Galois groups are projective profinite groups, in the sense of [33]. The description is complete, if $q = 0$ as well as in the subclass of maximally complete fields (K_q, v_q) with $\mathrm{char}(K_q) = q > 0$, which contain finitely many roots of unity (cf. [10], Sect. 3). By definition, a maximally complete field means a valued field (K, v) not admitting immediate proper extensions, i.e. valued extensions $(K', v') \neq (K, v)$ with $\widehat{K}' = \widehat{K}$ and value groups $v'(K') = v(K)$. These fields are singled out in valuation theory by Krull's theorem (see [37], Theorem 31.24 and page 483), stated as follows:

(1.1) Every nontrivially valued field (L_0, λ_0) possesses an immediate extension (L_1, λ_1) that is a maximally complete field.

The description of the set of sequences $\mathrm{Brd}_p(K_q), \mathrm{abrd}_p(K_q)$, $p \neq q$, is based on formulae for $\mathrm{Brd}_p(K_q)$ and $\mathrm{abrd}_p(K_q)$, $p \neq q$, deduced from lower and upper bounds on $\mathrm{Brd}_p(K)$ and $\mathrm{abrd}_p(K)$ (including infinity criterions). The bounds in question have been found under the hypothesis that (K, v) is a Henselian (valued) field with $\mathrm{abrd}_p(\widehat{K}) < \infty$ and $p \neq \mathrm{char}(\widehat{K})$. The formulae for $\mathrm{Brd}_p(K_q)$ depend only on whether or not $\widehat{K}_q$ contains a primitive p-th root of unity, and on invariants of $\widehat{K}_q$ and the value group $v_q(K_q)$. In fact, on the dimension $\tau(p)$ of the quotient group $v_q(K_q)/pv_q(K_q)$ as a vector space over the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and on the rank $r_p(\widehat{K}_q)$ of the Galois group $\mathcal{G}(\widehat{K}_q(p)/\widehat{K}_q)$, where $\widehat{K}_q(p)$ is the maximal p-extension of $\widehat{K}_q$ in $\widehat{K}_{q,\mathrm{sep}}$ (we put $\tau(p) = \infty$ if $v_q(K_q)/pv_q(K_q)$ is infinite, and $r_p(\widehat{K}_q) = 0$ if $\widehat{K}_q(p) = \widehat{K}_q$).
When $q > 0$, the noted restrictions on (K_q, v_q) allow one to find a formula for $\mathrm{Brd}_q(K_q)$ as well (see [10], Proposition 3.5). At the same time, they make it easy to show that $\mathrm{Brd}_p(K)$ does not depend only on $\widehat{K}$ and $v(K)$, when (K, v) runs across the class of Henselian fields of characteristic p. Specifically, it turns out (see [10], Exercise 3.7) that, for any integer $t \geq 2$, the iterated formal Laurent power series field $Y_t = \mathbb{F}_p((T_1)) \dots ((T_t))$ in t variables over $\mathbb{F}_p$ possesses subfields K_∞ and K_n, $n \in \mathbb{N}$, such that:

(1.2) (a) $\mathrm{Brd}_p(K_\infty) = \infty$; $n + t - 1 \leq \mathrm{Brd}_p(K_n) \leq n + t$, for each $n \in \mathbb{N}$;
(b) The valuations v_m of K_m, $m \leq \infty$, induced by the standard $\mathbb{Z}^t$-valued valuation of Y_t are Henselian with $\widehat{K}_m = \mathbb{F}_p$ and $v_m(K_m) = \mathbb{Z}^t$; here $\mathbb{Z}^t$ is viewed as an abelian group endowed with the inverse-lexicographic ordering.

Statement (1.2) motivates the study of Brauer p-dimensions of Henselian fields of residual characteristic $p > 0$, which lie in suitably chosen special classes. As a step in this direction, the present paper considers $\mathrm{Brd}_p(K)$ and $\mathrm{abrd}_p(K)$, for a Henselian discrete valued field (abbr, an HDV-field) (K, v) with $\mathrm{char}(\widehat{K}) = p$. This topic is related to the problem of describing index-exponent relations over finitely-generated field extensions (see, e.g., [8], Sect. 6 and Theorem 2.1, with its proof). When $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$, our main results give a lower bound for $\mathrm{Brd}_p(K)$ and improve the lower bound for $\mathrm{abrd}_p(K)$, provided by [27], Theorem 2. Combined with [27], Theorem 2, and [38], Proposition 2.1, they yield $\mathrm{Brd}_p(K) = \infty$ if and only if $[\widehat{K}\colon \widehat{K}^p] = \infty$, and also determine $\mathrm{Brd}_p(K)$ in case $[\widehat{K}\colon \widehat{K}^p] \leq p$.

2. **Statements of the main results**

Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$. As shown by Parimala and Suresh [27], $\mathrm{abrd}_p(K)$ satisfies the following: $[n/2] \leq \mathrm{abrd}_p(K) \leq 2n$, if $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$; $\mathrm{abrd}_p(K) = \infty$, if $[\widehat{K}\colon \widehat{K}^p] = \infty$. Bhaskhar and Haase have recently proved [4] that, in the former case, when n is odd, we have $\mathrm{abrd}_p(K) \geq 1 + [n/2]$. The proofs of these results show that they hold for $\mathrm{Brd}_p(K)$, if K contains a primitive p-th root of unity. The main purpose of the present paper is to improve the lower bounds in these results as well as to extend their scope, and also, to determine $\mathrm{Brd}_p(K)$ in the case where $n = 1$. Our first main result can be stated as follows:

Theorem 2.1. *Let (K, v) be an* HDV*-field with a residue field $\widehat{K}$ of characteristic $p > 0$. Then:*

(a) $\mathrm{Brd}_p(K)$ *is infinite if and only if $\widehat{K}/\widehat{K}^p$ is an infinite extension;*

(b) $\mathrm{abrd}_p(K) \geq n$, *provided that $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$; in this case, we have* $\mathrm{Brd}_p(K) \geq n$, *if char*$(K) = p$ *or* $\mathrm{char}(K) = 0$ *and there exists $\lambda \in K^*$ algebraic over $\mathbb{Q}$ of value $v(\lambda) \notin pv(K)$;*

(c) $\mathrm{Brd}_p(K) \geq n - [n/3] - 1$, $\mathrm{char}(K) = 0$ *and* $[\widehat{K}\colon \widehat{K}] = p^n < \infty$*; the inequality is strict except, possibly, in the case where $n = 5$, K does not contain a primitive p-th root of unity and $v(p) \in p^2v(K) \setminus p^3v(K)$.*

The lower bound for $\mathrm{Brd}_p(K)$ given by Theorem 2.1 (c) is better than those provided by [27], Theorem 2, and [4], Proposition 4.15, for $n \neq 1, 2, 3$ and 5. It is unlikely, however, that this bound is optimal. For example, by the proof of [8], Proposition 6.3 (see also [4], Theorem 5.2), we have $\mathrm{Brd}_p(K) \geq n+1$, if $\widehat{K}$ is a finitely-generated extension of $\mathbb{F}_p$ of transcendency degree n. Theorem 2.1 (b), this result and [4], Theorem 4.16, agree with the following conjecture (stated in [4] for complete discrete valued fields):

(2.1) If (K, v) is an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$, then $n \leq \mathrm{Brd}_p(K) \leq n + 1$.

When $\mathrm{char}(K) = p$, the conclusions of (2.1) and Theorem 2.1 (a) and (b) follow from our next result:

(2.2) If (K', v') is a Henselian field, $\mathrm{char}(K') = p > 0$, $v'(K') \neq pv'(K')$, and $\tau(p)$ is defined in accordance with Section 1, then:

(a) $\mathrm{Brd}_p(K') = \infty$, if $[\widehat{K}'\colon \widehat{K}'^p] = \infty$ or $\tau(p) = \infty$; when (K', v') is maximally complete, this holds if and only if $[K'\colon K'^p] = \infty$;

(b) $n + \tau(p) - 1 \leq \mathrm{Brd}_p(K')$, provided $n, \tau(p) < \infty$ and $[\widehat{K}'\colon \widehat{K}'^p] = p^n$; in addition, if (K', v') is maximally complete, then $[K'\colon K'^p] = p^{n'}$ and $\mathrm{Brd}_p(K') \leq n'$, where $n' = n + \tau(p)$;

(c) When (K', v') is maximally complete with $\widehat{K}'$ perfect, $\mathrm{Brd}_p(K') = \tau(p)$ if $r_p(\widehat{K}') \geq \tau(p)$; $\mathrm{Brd}_p(K') = \tau(p) - 1$ if $r_p(\widehat{K}') < \tau(p)$.

The former part of (2.2) (a) and the lower bound on $\mathrm{Brd}_p(K)$ in (2.2) (b) follow from [8], Lemma 4.2. Formula (2.2) (c) is contained in [10], Proposition 3.5, and can be deduced from this bound, [3], Theorem 3.3, and [7], Lemmas 4.1, 4.3. The other assertions of (2.2) are easily proved, using the version of Ostrowski's theorem for maximally complete fields and Albert's theory of p-algebras (see (3.2) (b) below and [2], Ch. VII, Theorem 28).

Before stating our second main result, let us recall that a field E is said to be p-quasilocal, for a given $p \in \mathbb{P}$, if one of the following conditions is fulfilled: (i) $\mathrm{Brd}_p(E) = 0$ or $r_p(E) = 0$; (ii) $\mathrm{Brd}_p(E) \neq 0$, $r_p(E) > 0$ and every degree p extension of E in $E(p)$ embeds as an E-subalgebra in each $D \in d(E)$ of degree p. We say that E is a quasilocal field, if its finite extensions are p'-quasilocal fields, for each $p' \in \mathbb{P}$. Both types of fields have been studied in [6]. Specifically, it has been proved there that if E is a p-quasilocal field, then so are the extensions of E in $E(p)$, and in case $\mathrm{char}(E) = p$, $\mathrm{Brd}_p(E) \leq 1$ and purely inseparable extensions of E are p-quasilocal as well. Note also that one can find in [10], Sect. 4, a formula for $\mathrm{Brd}_p(L)$, where (L, λ) is a Henselian field, such that $\widehat{L}$ is p-quasilocal, $\mathrm{char}(\widehat{L}) \neq p$, $r_p(\widehat{L}) \neq 0$ and $\lambda(L) \neq p\lambda(L)$. By an almost perfect field, we mean a field Φ whose finite extensions are simple; it is well-known that Φ is almost perfect if and only if either it is perfect or $\mathrm{char}(\Phi) = p > 0$ and $[\Phi\colon \Phi^p] = p$. The introduced notions are used in the following characterization of those HDV-fields with $\mathrm{char}(\widehat{K}) = p$, which satisfy the inequality $\mathrm{Brd}_p(K) \leq 1$.

Theorem 2.2. *Let (K, v) be an* HDV*-field with* $\mathrm{char}(\widehat{K}) = p > 0$. *Then* $\mathrm{Brd}_p(K) \leq 1$ *if and only if $\widehat{K}$ is p-quasilocal and almost perfect; in order that* $\mathrm{Brd}_p(K) = 0$ *it is necessary and sufficient that $\widehat{K}$ be perfect and $r_p(\widehat{K}) = 0$.*

The validity of Theorem 2.2 in the case where $\widehat{K}_{\mathrm{sep}} = \widehat{K}$ has been proved in [4]; this result is also contained in [38], Proposition 2.1. When K contains a primitive p-th root of unity and $\mathrm{char}(\widehat{K}) = p$, it has been shown in [4], Sect. 4, and in [5], Sect. 2, that $\mathrm{Brd}_p(K) \leq 1$ implies $[\widehat{K}\colon \widehat{K}^p] \leq p$.

Corollary 2.3. *For an* HDV*-field (K, v) with* $\mathrm{char}(\widehat{K}) = p > 0$*, we have* $\mathrm{Brd}_p(K) = 2$*, provided that $[\widehat{K}\colon \widehat{K}^p] = p$ and $\widehat{K}$ is not p-quasilocal.*

Corollary 2.3 follows from Theorem 2.2 and [27], Theorem 2. Theorem 2.2 and this corollary fully determine $\mathrm{Brd}_p(K)$ in the case where (K, v) is an HDV-field with $\mathrm{char}(\widehat{K}) = p$ and $[\widehat{K}\colon \widehat{K}^p] \leq p$. Theorem 2.2 also allows us to supplement [6], Proposition 2.3, and the main results of [5] as follows:

Corollary 2.4. *Let (K, v) be an* HDV*-field. Then K is stable if and only if $\widehat{K}$ is stable, almost perfect and p-quasilocal, for each $p \in \mathbb{P}$; K is absolutely stable if and only if $\widehat{K}$ is quasilocal and almost perfect.*

The proofs of the main results of this paper essentially rely on the following properties of HDV-fields (K, v), which specify and supplement (2.2):

(2.3) (a) The scalar extension map $\mathrm{Br}(K) \to \mathrm{Br}(K_v)$, where K_v is a completion of K with respect to the topology of v, is an injective homomorphism which preserves Schur indices and exponents (cf. [11], Theorem 1, and [31], Ch. 2, Theorem 9); hence, $\mathrm{Brd}_{p'}(K) \leq \mathrm{Brd}_{p'}(K_v)$, for every $p' \in \mathbb{P}$;

(b) The valued field $(K_v, \bar{v})$, where $\bar{v}$ is the valuation of K_v continuously extending v, is maximally complete (see [31], Ch. 2, Theorem 8; in addition, $(K_v, \bar{v})$ is an immediate extension of (K, v);

(c) $\mathrm{Brd}_p(K) = \infty \leftrightarrow [\widehat{K} \colon \widehat{K}^p] = \infty$; $n \leq \mathrm{Brd}_p(K) \leq n+1$ if $[\widehat{K} \colon \widehat{K}^p] = p^n$.

At the same time, it follows from (2.2) (b), (1.1) and (1.2) that the conclusion of the latter part of (2.3) (a) need not be true, for $p' = p$, if (K, v) is merely Henselian of prime characteristic p, and $(K_v, \bar{v})$ is replaced by an immediate extension of (K, v) that is a maximally complete field.

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [8]. Throughout, Brauer and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. For any field E, E^* stands for its multiplicative group, $E^{*n} = \{a^n \colon\ a \in E^*\}$, for each $n \in \mathbb{N}$, $\mathcal{G}_E = \mathcal{G}(E_{\mathrm{sep}}/E)$ is the absolute Galois group of E, and for each $p \in \mathbb{P}$, ${}_p\mathrm{Br}(E) = \{b_p \in \mathrm{Br}(E) \colon\ pb_p = 0\}$. As usual, $\mathrm{Br}(E'/E)$ denotes the relative Brauer group of any field extension E'/E. We write $\pi_{E'/E}$ for the scalar extension map of $\mathrm{Br}(E)$ into $\mathrm{Br}(E')$, and $\mathrm{I}(E'/E)$ for the set of intermediate fields of E'/E; when E'/E is separable of finite degree $[E' \colon E]$, $N(E'/E)$ stands for the norm group of E'/E. By a p-basis of a field extension Y'/Y, such that $\mathrm{char}(Y) = p$, $Y'^p \subseteq Y$ and $[Y' \colon Y] = p^n < \infty$, we mean a generating set of Y'/Y of n elements.

The paper is organized as follows: Section 3 includes preliminaries used in the sequel, and also, a proof of Theorem 2.1 (a). Theorems 2.1 (b), (c) and 2.2 are proved in Sections 4 and 5, respectively. Section 6 presents applications to m-dimensional local fields (i.e. m-discretely valued fields with finite m-th residue fields, see [35], [14], [40]). It is shown that a field K_m of this type is absolutely stable, if $m \leq 2$, and K_m is not stable, otherwise (when $\mathrm{char}(K_m) > 0$, this is contained in [5], Corollaries 4.4, 4.5). We also prove that $\mathrm{abrd}_p(K_m) \geq m - 1$ and usually $\mathrm{Brd}_p(K_m) \geq m - 1$, for $m \geq 3$ and $p = \mathrm{char}(K_0)$, where K_0 is the m-th residue field of K_m. This result and the upper bound $\mathrm{abrd}_p(K_m) \leq m$, in fact found in [19], agree with (2.1).

3. **Preliminaries and proof of Theorem 2.1 (a)**

Let K be a field with a nontrivial valuation v, $O_v(K) = \{a \in K \colon\ v(a) \geq 0\}$ the valuation ring of (K, v), $M_v(K) = \{\mu \in K \colon\ v(\mu) > 0\}$ the maximal ideal of $O_v(K)$, $O_v(K)^* = \{u \in K \colon\ v(u) = 0\}$ the multiplicative group of $O_v(K)$, $v(K)$ and $\widehat{K} = O_v(K)/M_v(K)$ the value group and the residue field of (K, v), respectively. For each $\gamma \in v(K)$, $\gamma \geq 0$, we denote by $\nabla_\gamma(K)$ the set $\{\lambda \in K \colon\ v(\lambda - 1) > \gamma\}$. We say that the valuation v is Henselian, if it extends uniquely, up-to an equivalence, to a valuation v_L on each algebraic extension L of K. For example, maximally complete fields are Henselian, since Henselizations of any valued field are its immediate extensions (see [13], Theorem 15.3.5). In order that v be Henselian, it is necessary and sufficient that any of the following three equivalent conditions holds (cf. [13], Sect. 18.1, and [21], Ch. XII, Sect. 4):

(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the formal derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c-a) = v(f(a)/f'(a))$;

(b) K is separably closed in the completion K_v, and the valuation $\bar{v}$ of K_v continuously extending v is Henselian;

(c) For each normal extension Ω/K, $v'(\tau(\mu)) = v'(\mu)$ whenever $\mu \in \Omega$, v' is a valuation of Ω extending v, and τ is a K-automorphism of Ω.

When v is Henselian, so is v_L, for every algebraic field extension L/K. In this case, we denote by $\widehat{L}$ the residue field of (L, v_L), and put $O_v(L) = O_{v_L}(L)$, $M_v(L) = M_{v_L}(L)$ and $v(L) = v_L(L)$. Clearly, $\widehat{L}$ is an algebraic extension of $\widehat{K}$, and $v(K)$ is an ordered subgroup of $v(L)$; the index $e(L/K)$ of $v(K)$ in $v(L)$ is called a ramification index of L/K. Suppose further that $[L\colon K]$ is finite. Then, by Ostrowski's theorem, $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ has no divisor $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$. We say that L/K is defectless, if $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$. It is clear from Ostrowski's theorem that L/K is defectless, provided that $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$. The same holds in the following two cases:

(3.2) (a) If (K, v) is HDV and L/K is separable (see [35], Proposition 2.2);

(b) When (K, v) is maximally complete (cf. [37], Theorem 31.22).

Assume that (K, v) is a nontrivially valued field. We say that a finite extension R of K is inertial with respect to v, if R has a unique (up-to an equivalence) valuation v_R extending v, the residue field $\widehat{R}$ of (R, v_R) is separable over $\widehat{K}$, and $[R\colon K] = [\widehat{R}\colon \widehat{K}]$; R/K is called totally ramified with respect to v, if v has a unique prolongation v_R on R, and the index of $v(K)$ in $v_R(R)$ equals $[R\colon K]$. When v is Henselian, this amounts to saying that $e(R/K) = [R\colon K]$. In this case, inertial extensions have the following frequently used properties (see [16], Theorems 2.8, 2.9, and [34], Theorem A.24, or the remarks between Proposition 3.1 and Theorem 3.2 of [36]):

(3.3) (a) An inertial extension R'/K in is Galois if and only if $\widehat{R}'/\widehat{K}$ is Galois. When this holds, $\mathcal{G}(R'/K)$ and $\mathcal{G}(\widehat{R}'/\widehat{K})$ are canonically isomorphic.

(b) The compositum K_{ur} of inertial extensions of K in K_{sep} is a Galois extension of K with $\mathcal{G}(K_{\mathrm{ur}}/K)$ isomorphic to the absolute Galois group $\mathcal{G}_{\widehat{K}}$.

(c) Finite extensions of K in K_{ur} are inertial, and the natural mapping of $I(K_{\mathrm{ur}}/K)$ into $I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$ is bijective.

The Henselity of (K, v) guarantees that v extends to a unique, up-to an equivalence, valuation v_D, on each $D \in d(K)$ (cf. [31], Ch. 2, Sect. 7). Put $v(D) = v_D(D)$ and denote by $\widehat{D}$ the residue division ring of (D, v_D). It is known that $\widehat{D}$ is a division $\widehat{K}$-algebra, $v(D)$ is an ordered abelian group and

$v(K)$ is an ordered subgroup of $v(D)$ of finite index $e(D/K)$ (called a ramification index of D/K). Note further that $[\widehat{D}\colon \widehat{K}] < \infty$, and by the Ostrowski-Draxl theorem [12], $[\widehat{D}\colon \widehat{K}]e(D/K) \mid [D\colon K]$ and $[D\colon K][\widehat{D}\colon \widehat{K}]^{-1}e(D/K)^{-1}$ has no prime divisor $p \neq \mathrm{char}(\widehat{K})$. When (K, v) is an HDV-field, the following condition holds (cf. [35], Theorem 3.1):

(3.4) D/K is defectless, i.e. $[D\colon K] = [\widehat{D}\colon \widehat{K}]e(D/K)$.

Next we give examples of central division K-algebras of exponent p, which are specific for HDV-fields (K, v) with $\mathrm{char}(\widehat{K}) = p$ and $\widehat{K} \neq \widehat{K}^p$. Suppose first that there exists a totally ramified Galois extension M/K, such that $\mathcal{G}(M/K)$ is an elementary abelian p-group of order p^n, for some $n \in \mathbb{N}$. Then, by Galois theory, M equals the compositum $L_1 \dots L_n$ of degree p (cyclic) extensions L_j of K, $j = 1, \dots, n$. Fix a generator σ_j of $\mathcal{G}(L_j/K)$ and an element $a_j \in K^*$, and denote by Δ_j the cyclic K-algebra $(L_j/K, \sigma_j, a_j)$, for each j. Arguing by the method of proving [8], Lemma 4.2, one obtains that

(3.5) The tensor product $D_n = \otimes_{j=1}^n \Delta_j$, where $\otimes = \otimes_K$, lies in $d(K)$, provided that $a_j \in O_v(K)^*$, $j = 1, \dots, n$, and $\hat{a}_1, \dots, \hat{a}_n$ are p-independent over $\widehat{K}^p$, i.e. $\widehat{K}^p(\hat{a}_1, \dots, \hat{a}_n)/\widehat{K}^p$ is a field extension of degree p^n; $\widehat{D}_n$ is a root field over $\widehat{K}$ of the binomials $X^p - \hat{a}_j$, $j = 1, \dots, n$, so $[\widehat{D}_n\colon \widehat{K}] = p^n$.

It is likely that, for every HDV-field (K, v) with $\widehat{K}$ infinite and $\mathrm{char}(\widehat{K}) = p > 0$, there are totally ramified Galois extensions M_n/K, $n \in \mathbb{N}$, such that $\mathcal{G}(M_n/K)$ is an elementary abelian p-group of order p^n, for each index n. By [8], Lemma 4.2, this holds if $\mathrm{char}(K) = p$, and the following two lemmas prove the existence of such extensions, when $\mathrm{char}(K) = 0$ and $v(p) \notin pv(K)$.

Lemma 3.1. *Assume that (K, v) is an* HDV*-field with* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(\widehat{K}) = p > 0$*, and also, that (Φ, ω) is a valued subfield of (K, v), such that p does not divide the index of $\omega(\Phi)$ in $v(K)$. Let Ψ be a finite extension of Φ in K_{sep} of p-primary degree, and suppose that Ψ is totally ramified over Φ relative to ω. Then $\Psi K/K$ is totally ramified and $[\Psi K\colon K] = [\Psi\colon \Phi]$.*

Proof. Our assumptions guarantee that (K, v) contains as a valued subfield a Henselization (Φ', ω') of (Φ, ω) (cf. [13], Theorem 15.3.5). Also, the condition that Ψ is totally ramified over Φ relative to ω means that Ψ/Φ possesses a primitive element θ whose minimal polynomial $f_\theta(X)$ over Φ is Eisensteinian relative to $O_\omega(\Phi)$ (see [15], Ch. 2, (3.6), and [21], Ch. XII, Sects. 2, 3 and 6). Since $(\Phi', \omega')/(\Phi, \omega)$ is immediate, $f_\theta(X)$ remains Eisensteinian relative to $O_{\omega'}(\Phi')$, whence, irreducible over Φ'. In other words, the field $\Psi' = \Phi'(\theta) = \Psi\Phi'$ is a totally ramified extension of Φ' and $[\Psi'\colon \Phi'] = [\Psi\colon \Phi]$. Put $m = [\Psi\colon \Phi]$ and $\theta_1 = \theta$, denote by $\theta_1, \dots, \theta_m$ the roots of $f_\theta(X)$ in K_{sep}, and let $M' = \Phi'(\theta_1, \dots, \theta_m)$. Applying (3.1) (c) to the extension M'/Φ', one obtains that $\omega'_{M'}(\theta_j) = \omega'_{M'}(\theta)$, $j = 1, \dots, m$. At the same time, by the Eisensteinian property of $f_\theta(X)$ relative to $O_{\omega'}(\Phi')$, the free term of $f_\theta(X)$ is a uniform element of (Φ', ω'). As $m = [\Psi\colon \Phi]$ is a p-primary number, v is discrete and p does not divide the index $|v(K)\colon \omega(\Phi)|$, the presented observations indicate that $f_\theta(X)$ is irreducible over K, and the field $\Psi' K = \Psi K$ is a totally ramified extension of K of degree m, as claimed by Lemma 3.1. □

Let (K, v) be an HDV-field with $\mathrm{char}(K) = 0$ and $\mathrm{char}(\widehat{K}) = p > 0$, and let ε be a primitive p-th root of unity in $K_{\rm sep}$. It is known (cf. [21], Ch. VIII, Sect. 3) that then $K(\varepsilon)/K$ is a cyclic extension and $[K(\varepsilon)\colon K] \mid p-1$; also, it is easy to see that $v_{K(\varepsilon)}(1-\varepsilon) = v(p)/(p-1)$. These facts enable one to deduce the following assertions from (3.1) (a):

(3.6) $K^{\ast p} = K(\varepsilon)^{\ast p} \cap K^{\ast}$, and for each $\beta \in \nabla_{\gamma'}(K(\varepsilon))$, where $\gamma' = pv(p)/(p-1)$, the polynomial $g_{\beta}(X) = (1-\varepsilon)^{-p}((1-\varepsilon)X+1)^p - \beta)$ lies in $O_v(K(\varepsilon))[X]$ and has a root in $K(\varepsilon)$ (see also [35], Lemma 2.1). Hence, $\nabla_{\gamma'}(K(\varepsilon)) \subset K(\varepsilon)^{\ast p}$ and $\nabla_{\gamma}(K) \subset K^{\ast p}$, in case $\gamma \in v(K)$ and $\gamma \geq \gamma'$.

An element $\lambda \in \nabla_0(K)$ is said to be normal over K, if $\lambda \notin K^{\ast p}$ and $v(\lambda - 1) \geq v(\lambda' - 1)$ whenever λ' lies in the coset $\lambda K^{\ast p}$. Let $\pi = \lambda - 1$ and K' be an extension of K generated by a p-th root of λ. It is easy to see that λ is normal over K if and only if one of the following conditions is fulfilled:

(3.7) (a) $v(\pi) \notin pv(K)$ and $(p-1)v(\pi) < pv(p)$; when this holds, K' is totally ramified over K;

(b) $(p-1)v(\pi) < pv(p)$ and $\pi = \pi_1^p a$, for a pair $\pi_1 \in K$, $a \in O_v(K)^{\ast}$, such that $\hat{a} \notin \widehat{K}^{\ast p}$; in this case, $\widehat{K}'/\widehat{K}$ is purely inseparable of degree p;

(c) $\pi = \pi_1^p a$, for some $\pi_1 \in K$, $a \in O_v(K)^{\ast}$, such that $(p-1)v(\pi_1) = v(p)$ and the polynomial $X^p - X - \hat{a}$ is irreducible over $\widehat{K}$; when this occurs, K'/K is inertial and $v(\pi) = pv(p)/(p-1)$.

Statements (3.6) and (3.7) show that, for each $\alpha \in \nabla_0(K) \setminus K^{\ast p}$, $\alpha K^{\ast p}$ contains a normal element over K. In addition, it follows from (3.7) and (3.2) (a) that if α is normal over K, then it is normal over any finite extension K_1 of K of degree not divisible by p. This, applied to the field $K_1 = K(\varepsilon)$, facilitates the construction of the abelian p-extensions of K needed to prove Theorem 2.1 (b) and (c). We use repeatedly this technique, beginning with the proof of the former assertion of the following lemma.

Lemma 3.2. *Let (K, v) be an* HDV*-field with* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(\widehat{K}) = p > 0$. *Suppose that one of the following two conditions holds:*

(a) *$\widehat{K}$ is an infinite perfect field;*

(b) *$\widehat{K}$ is imperfect and there exists $\theta \in K$ algebraic over the field $\mathbb{Q}$ of rational numbers (the prime subfield of K), and of value $v(\theta) \notin pv(K)$.*

Then there exist totally ramified Galois extensions M_n/K, $n \in \mathbb{N}$, such that $[M_n\colon K] = p^n$ and $\mathcal{G}(M_n/K)$ is an elementary abelian p-group, for each n.

Proof. Let ε be a primitive p-th root of unity in $K_{\rm sep}$, $m = [K(\varepsilon)\colon K]$, and $\mathbb{F}$ the prime subfield of $\widehat{K}$. As noted above, then $K(\varepsilon)/K$ is a cyclic extension and $m \mid p-1$. Fix a generator φ of $\mathcal{G}(K(\varepsilon)/K)$, and an element $\pi \in K$ with $0 < v(\pi) \leq v(p)$ and $v(\pi) \notin pv(K)$, and let s, l be integers satisfying $\varphi(\varepsilon) = \varepsilon^s$ and $sl \equiv 1 (\mathrm{mod}\ p)$. For any $\alpha \in O_v(K)^{\ast}$, denote by L'_{α} the extension of $K(\varepsilon)$ in $K_{\rm sep}$ obtained by adjunction of a p-th root η_{α} of the element $\alpha' = \prod_{j=0}^{m-1}[1 + (1-\varphi^j(\varepsilon))^p \pi^{-1}\alpha]^{l(j)}$, where $l(j) = l^j$, for each j. It is verified by direct calculations that $\varphi(\alpha')\alpha'^{-s} \in K(\varepsilon)^{\ast p}$. Observing that $s^p \equiv s (\mathrm{mod}\ p)$ and $v_{K(\varepsilon)}(1-\varphi^j(\varepsilon)) = v(p)/(p-1)$, $0 \leq j \leq m-1$, one obtains similarly that $v_{K(\varepsilon)}(\alpha' - 1 - m(1-\varepsilon)^p\pi^{-1}\alpha) > v_{K(\varepsilon)}(m(1-\varepsilon)^p\pi^{-1}\alpha)$ and $v(m\alpha) = 0$. These calculations show that α' and $1 + m(1-\varepsilon)^p\pi^{-1}\alpha$ are normal elements over $K(\varepsilon)$. Therefore, $\alpha' \notin K(\varepsilon)^{*p}$, and since $\varphi(\alpha')\alpha'^{-s} \in K(\varepsilon)^{*p}$, this enables one to deduce the following statement from Albert's theorem (characterizing cyclic extensions of degree p, see [1], Ch. IX, Theorem 6):

(3.8) L'_α/K is a cyclic field extension of degree pm; in particular, there exists a unique cyclic extension L_α of K in L'_α of degree p.

Suppose now that $\widehat{K}$ is infinite and perfect. The infinity of $\widehat{K}$ ensures the existence of a sequence $b = b_n \in O_v(K)^*$, $n \in \mathbb{N}$, such that the system $\bar{b} = \hat{b}_n \in \widehat{K}$, $n \in \mathbb{N}$, is linearly independent over the field $\mathbb{F}$. Denote by K' the compositum of the fields L_{b_n}, $n \in \mathbb{N}$. Clearly, $K' = \cup_{n=1}^{\infty} M_n$, where $M_n = L_{b_1} \dots L_{b_n}$, for each n, and it follows that K'/K is a Galois extension and $\mathcal{G}(K'/K)$ is an infinite abelian pro-p-group of period p. In addition, it is easily obtained from (3.8) and the choice of b that every degree p extension of K in K' is totally ramified. As $\widehat{K}$ is perfect, whence its finite extensions in $\widehat{K}'$ have inertial lifts over K, that embed in K' as K-subalgebras (see (3.3)), this result, combined with Galois theory and (3.2) (a), implies finite extensions of K in K' are totally ramified. Therefore, the preceding argument shows that the fields M_n, $n \in \mathbb{N}$, have the properties claimed by Lemma 3.2.

The idea of the proof of Lemma 3.2 (b) is borrowed from [25], 2.2.1. Identifying $\mathbb{Q}$ with the prime subfield of K, put $\Phi = \mathbb{Q}(\theta)$, and $E_0 = \Phi(t_0)$, where $t_0 \in O_v(K)^*$ is chosen so that $\hat{t}_0 \notin \widehat{K}^p$ (whence, $\hat{t}_0$ is transcendental over $\mathbb{F}$). Denote by ω and v_0 the valuations induced by v upon Φ and E_0, respectively, and fix a system $t_n \in K_{\mathrm{sep}}$, $n \in \mathbb{N}$, such that $t_n^p = t_{n-1}$, for each $n > 0$. It is easy to see that the fields $E_n = \Phi(t_n)$, $n \in \mathbb{N}$, are purely transcendental extensions of Φ. Let v_n be the restricted Gaussian valuation of E_n extending ω, for each $n \in \mathbb{N}$. Clearly, for any pair of indices ν, n with $0 < \nu \le n$, $E_{\nu-1}$ is a subfield of E_n and v_n is the unique prolongation of $v_{\nu-1}$ on E_n. Hence, the union $E_\infty = \cup_{n=0}^{\infty} E_n$ is a field with a unique valuation v_∞ extending v_n, for every $n < \infty$. Denote by $\widehat{E}_n$ the residue field of (E_n, v_n), for each $n \in \mathbb{N} \cup \{0, \infty\}$. The Gaussian property of v_n, $n < \infty$, guarantees that $v_n(E_n) = \omega(\Phi)$, $\hat{t}_n$ is a transcendental element over $\widehat{\Phi}$ and $\widehat{E}_n = \widehat{\Phi}(\hat{t}_n)$ (cf. [13], Example 4.3.2). Observing also that $\hat{t}_n^p = \hat{t}_{n-1}$, $n \in \mathbb{N}$, $\widehat{E}_\infty = \cup_{n=1}^{\infty} \widehat{E}_n$ and $\widehat{\Phi}^p = \widehat{\Phi}$, one concludes that $\widehat{E}_\infty$ is an infinite perfect field. It is therefore clear from Lemma 3.2 (a) and the Grunwald-Wang theorem [22], that if (E'_∞, v'_∞) is a Henselization of (E_∞, v_∞) with $E'_\infty \subset K_{\mathrm{sep}}$, then there exist totally ramified Galois extensions T'_n/E'_∞ and T_n/E_∞, $n \in \mathbb{N}$, such that $[T_n \colon E_\infty] = [T'_n \colon E'_\infty] = p^n$, $T'_n = T_n E'_\infty$ and $\mathcal{G}(T_n/E_\infty)$ is an elementary abelian p-group isomorphic to $\mathcal{G}(T'_n/E'_\infty)$, for every n. This means that T_n/E_∞ possesses a primitive element θ_n whose minimal polynomial $f_n(X)$ over E_∞ is Eisensteinian relative to $O_{v_n}(E_\infty)$. Since $E_\infty = \cup_{n=1}^{\infty} E_n$ and $E_n \subset E_{n+1}$, $n \in \mathbb{N}$, it is easy to see (e.g., from [6], (1.3)) that, for each n, there exists $k_n \in \mathbb{N}$, such that $f_n(X) \in E_{k_n}[X]$ and $E_{k_n}(\theta_n)/E_{k_n}$ is a Galois extension. This shows that $[E_{k_n}(\theta_n) \colon E_{k_n}] = p^n$, which implies $\mathcal{G}(E_{k_n}(\theta_n)/E_{k_n}) \cong \mathcal{G}(T_n/E_\infty)$. As v_∞ extends v_{k_n} and $v_\infty(E_\infty) = v_{k_n}(E_{k_n})$, it is also clear that $f_n(X) \in O_{v_{k_n}}(E_{k_n})[X]$ and $f_n(X)$ is Eisensteinian relative to $O_{v_{k_n}}(E_{k_n})$. Let now $\psi_n \colon E_{k_n} \to E_0$ be the Φ-isomorphism mapping

t_{k_n} into t_0, and let $\bar{\psi}_n$ be the isomorphism of $E_{k_n}[X]$ upon $E_0[X]$, which extends ψ_n so that $\bar{\psi}_n(X) = X$. Then the polynomial $g_n(X) = \bar{\psi}_n(f_n(X))$ lies in $O_{v_0}(E_0)[X]$, it is Eisensteinian relative to $O_{v_0}(E_0)$, and $p^n = [L_n\colon E_0]$, where L_n is a root field of $g_n(X)$ over E_0. The polynomials $g_n(X)$, $n \in \mathbb{N}$, preserve the noted properties also when (E_0, v_0) is replaced by its Henselization (E_0', v_0'). As $v_0(E_0) = \omega(\Phi)$ is a subgroup of $v(K)$ of index not divisible by p, these results, combined with Lemma 3.1, prove Lemma 3.2 (b). □

Assume now that (K, v) is an HDV-field with $\mathrm{char}(\widehat{K}) = p$, and for some $m \in \mathbb{N}$, $O_v(K)^*$ contains elements $c_1, b_1, \dots, c_m, b_m$, such that the system $\hat{c}_1, \hat{b}_1, \dots, \hat{c}_m, \hat{b}_m$ is p-independent over $\widehat{K}^p$, and also, there is an extension C_j of K in $K(p)$ with $[C_j\colon K] = p$ and $\widehat{C}_j = \widehat{K}(\sqrt[p]{\hat{c}_j})$, for $j = 1, \dots, m$. Fix a generator τ_j of $\mathcal{G}(C_j/K)$, and put $V_j = (C_j/K, \tau_j, b_j)$, for each index j. It follows from [24], Theorem 1, that:

(3.9) The K-algebra $W_m = \otimes_{j=1}^m V_j$ lies in $d(K)$ ($\otimes = \otimes_K$), and $\widehat{W}_m/\widehat{K}$ is a field extension obtained by adjunction of p-th roots of $\hat{c}_j, \hat{b}_j$, $j = 1, \dots, m$.

Our next lemma allows us to use (3.9) for proving Theorem 2.1 (a).

Lemma 3.3. *Let* (K, v) *be an* HDV*-field with* $\mathrm{char}(K) = 0$*,* $v(p) \in pv(K)$*,* $\mathrm{char}(\widehat{K}) = p > 0$*, and* $\widehat{K} \neq \widehat{K}^p$*, and let* $\widetilde{\Lambda}/\widehat{K}$ *be an inseparable field extension of degree p. Then there exists an extension* Λ *of* K *in* $K(p)$*, such that* $[\Lambda\colon K] = p$ *and* $\widehat{\Lambda}$ *is* $\widehat{K}$*-isomorphic to* $\widetilde{\Lambda}$*.*

Proof. Let ε be a primitive p-th root of unity in K_{sep}, φ a generator of $\mathcal{G}(K(\varepsilon)/K)$, and s and l be integers satisfying $\varphi(\varepsilon) = \varepsilon^s$ and $sl \equiv 1 (\mathrm{mod}\ p)$. Suppose that $[K(\varepsilon)\colon K] = m$, and fix elements $\lambda \in O_v(K)^*$ and $\pi \in K$ so that the extension of $\widehat{K}$ obtained by adjunction of a p-th root of $\hat{\lambda}$ be $\widehat{K}$-isomorphic to $\widetilde{\Lambda}$. If $\varepsilon \in K$, then one may take as Λ the root field in K_{sep} of the binomial $X^p - \lambda \in K[X]$, so we assume that $\varepsilon \notin K$. Putting $\lambda_1 = \prod_{j=0}^{m-1} [1 + (\varphi^j(1-\varepsilon)^p \pi^{-p} \lambda)]^{l(j)}$, where $l(j) = l^j$, for $j = 0, 1, \dots, m-1$, we show that the root field $\Lambda_1 \in \mathrm{Fe}(K)$ of $f_\lambda(X) = X^p - \lambda_1$ (over $K(\varepsilon)$) is a cyclic extension of K of degree pm. It is easily verified that $\varphi(\lambda_1)\lambda_1^{-s} \in K(\varepsilon)^p$, so it follows from Albert's theorem that it suffices to prove that $\lambda_1 \notin K(\varepsilon)^p$. Our argument relies on the fact that $v_{K(\varepsilon)}(\lambda_1 - \lambda_1') > v_{K(\varepsilon)}(\lambda_1)$, where $\lambda_1' = 1 + m(\varepsilon - 1)^p \pi^{-p} \lambda$. This implies that if $\eta_1 \in K_{\mathrm{sep}}$ is a root of $f_\lambda(X)$, then the minimal polynomial over $K(\varepsilon)$, say $h_\lambda(X)$, of the element $\eta = \pi(\eta_1 - 1)(1 - \varepsilon)^{-1}$ has the presentation $h_\lambda(X) = X^p - m\lambda + \tilde{h}(X)$, for some $\tilde{h}(X) \in M_{v_{K(\varepsilon)}}[X]$ of degree $\leq m - 1$. The obtained result indicates that $\hat{\lambda} \in \widehat{\Lambda}_1^{*p}$, and since $\hat{\lambda} \notin \widehat{K}^{*p}$, it proves that $\eta \notin K(\varepsilon)$ and $\eta_1 \notin K(\varepsilon)$. Therefore, $[\Lambda_1\colon K] = pm$, and it follows from Galois theory and the cyclicity of Λ_1/K, that there exists a cyclic extension Λ of K in Λ_1 of degree p. As $m \mid p-1$, one finally concludes that $\hat{\lambda} \in \widehat{\Lambda}^{*p}$, which completes our proof. □

It is now easy to show that an HDV-field (K, v) with $\mathrm{char}(\widehat{K}) = p > 0$ satisfies $\mathrm{Brd}_p(K) = \infty$ if and only if $[\widehat{K}\colon \widehat{K}^p] = \infty$. In view of (2.3) (c), one may consider only the case of $\mathrm{char}(K) = 0$. Then the implication

$\mathrm{Brd}_p(K) = \infty \to [\widehat{K}\colon \widehat{K}^p] = \infty$ follows from [27], Theorem 2, so it remains to be seen that $\mathrm{Brd}_p(K) = \infty$, if $[\widehat{K}\colon \widehat{K}^p] = \infty$. For the purpose, one uses (3.5) and (3.9) together with Lemmas 3.1, 3.2 and 3.3, and thereby concludes that there exist $D_n \in d(K)$, $n \in \mathbb{N}$, with $\exp(D_n) = p$ and $\deg(D_n) = p^n$, for each n. This yields $\mathrm{Brd}_p(K) = \infty$, so Theorem 2.1 (a) is proved.

4. **Proof of Theorem 2.1 (b) and (c)**

The aim of this Section is to complete the proof of Theorem 2.1. Our argument relies implicitly on the fact that each generating set of a finite extension Y'/Y with $\mathrm{char}(Y) = p > 0$ and $Y'^p \subseteq Y$ contains as a subset a p-basis of Y'/Y (see the proof of (4.2)). We also use the following lemma.

Lemma 4.1. *Let (K, v) be an* HDV*-field, such that* $\mathrm{char}(\widehat{K}) = p > 0$, $\widehat{K}^p \neq \widehat{K}$, $[\widehat{K}\colon \widehat{K}^p] = p^n < \infty$, *and $v(p) \notin p^n v(K)$. Then K possesses a finite Galois extension M in K_{sep} satisfying the following conditions:*

(a) *$\mathcal{G}(M/K)$ is an elementary abelian p-group;*

(b) *There is $M_0 \in I(M/K)$ with $[M\colon M_0] = p^n$, M/M_0 totally ramified and $[\widehat{M_0}\colon \widehat{K}] \mid p^\nu$, where ν is the greatest integer for which $v(p) \in p^\nu v(K)$.*

Proof. Lemma 3.2 allows us to consider only the case where $v(p) \in pv(K)$. Denote by $\mathbb{F}$ the prime subfield of K, fix an element $\alpha \in O_v(K)^*$ so that $\hat{\alpha} \notin \widehat{K}^p$, put $\Phi = \mathbb{F}(\alpha)$, and let ω_0 be the valuation of $\mathbb{F}$ induced by v. Identifying $\mathbb{F}$ with the field $\mathbb{Q}$, one obtains that ω_0 is equivalent to the p-adic valuation of $\mathbb{F}$. Consider now the valuation ω of Φ induced by v. It follows from the choice of α and the definition of Φ that α is transcendental over $\mathbb{F}$ and ω is a restricted Gaussian valuation extending ω_0. Note also that $v(p)$ is a generator of $\omega(\Phi)$, $\widehat{\Phi} = \widehat{\mathbb{F}}(\hat{\alpha})$, and $\hat{\alpha}$ is transcendental over $\widehat{\mathbb{F}}$, where $\widehat{\mathbb{F}}$ and $\widehat{\Phi}$ are the residue fields of $(\mathbb{F}, \omega_0)$ and (Φ, ω), respectively. Now choose a Henselization (Φ', ω') among the valued subfields of (K, v). The valued extension $(\Phi', \omega')/(\Phi, \omega)$ is immediate, so the preceding observations indicate that $v(p)$ is a generator of $\omega'(\Phi')$ and $\widehat{\Phi}' \neq \widehat{\Phi}'^p$. Hence, by Lemma 3.2, there exist totally ramified Galois extensions Ψ'_m, $m \in \mathbb{N}$, of Φ' in K_{sep} with $[\Psi'_m\colon \Phi'] = p^m$ and $\mathcal{G}(\Psi'_m/\Phi')$ an elementary abelian p-group, for each m. Observe that $[(K \cap \Psi'_m\colon \Phi'] \mid p^\nu$, where ν is defined in the statement of Lemma 4.1 (b); since ω' is Henselian and $(K, v)/(\Phi', \omega')$ is a valued extension, this can be deduced from (3.2) (a). Therefore, it is easily obtained from Galois theory and the assumptions on $\mathcal{G}(\Psi'_m/\Phi')$, $m \in \mathbb{N}$, that Ψ'_m can be chosen so that $\Psi'_m \cap K = \Phi'$, for every m. This amounts to saying that the fields $\Psi_m = \Psi'_m K$ are Galois extensions of K with $\mathcal{G}(\Psi_m/K) \cong \mathcal{G}(\Psi'_m/\Phi')$. As Ψ'_m/Φ' are totally ramified, one also sees that $[\widehat{\Psi}_m\colon \widehat{K}] \mid p^\nu$, for all $m \in \mathbb{N}$. In the rest of the proof, we suppose that m is fixed so that $m \geq (\nu + 1)n$ and take into account that Ψ_m equals the compositum $L_{m1} \dots L_{mm}$, for some degree p extensions L_{mj} of K in M, $j = 1, \dots, m$. Put $W_0 = K$, $W_j = L_{m1} \dots L_{mj}$, for $j = 1, \dots, m$, and denote by Σ the set of those indices $j > 0$, for which $e(W_j/W_{j-1}) = 1$. Clearly, $[W_{j''}\colon W_{j'}] = p^{j''-j'}$ in case $0 \leq j' < j'' \leq m$; in particular, $W_m = \Psi_m$, so it follows from (3.2) (a) and the divisibility of p^ν by $[\widehat{W}_m\colon \widehat{K}]$ that Σ consists of at most ν elements. As $m \geq (\nu+1)n$, one may also assume, for the proof of Lemma 4.1, that $\Sigma \neq \phi$. Thus it turns out that some of the following two conditions holds:

(4.1) (i) $n < \mu$ or $\mu \leq m - n$, for each $\mu \in \Sigma$;
(ii) Σ contains indices-neighbours μ' and μ'', such that $\mu'' - \mu' > n$.

The conclusions of our lemma are immediate consequences of (4.1). □

In the setting of Lemma 4.1, we have $[\widehat{L}\colon \widehat{L}^p] = p^n$, for every finite extension L/K. Therefore, (3.5) and Lemma 4.2 (b) ensure that $\mathrm{Brd}_p(M_0) \geq n$, which yields $\mathrm{abrd}_p(K) \geq n$ and so completes the proof of Theorem 2.1 (b). Also, it is easily obtained from Lemma 4.1 (a) and Galois theory that if $\nu < n$, then there exists a field $R \in I(M/K)$, such that $[R\colon K] = p^{n-\nu}$ and $R \cap M_0 = K$. This allows us to supplement Lemma 4.1 as follows:

(4.2) $\mathrm{Brd}_p(K) \geq n - \nu$. Specifically, one can find an algebra $\Delta \in d(K)$ so that $\exp(\Delta) = p$, $\deg(\Delta) = p^{n-\nu}$, $[\Delta] \in \mathrm{Br}(R/K)$, $\Delta \otimes_K M_0 \in d(M_0)$, $v(\Delta \otimes_K M_0) = v(M_0R)$, and the residue division ring of $\Delta \otimes_K M_0$ is a field that is a purely inseparable extension of $\widehat{M_0}$ of degree $p^{n-\nu}$.

The proof of (4.2) relies on the existence of elements $a_1, \dots, a_{n-\nu}$ of $O_v(K)^*$, such that $\hat{a}_1, \dots, \hat{a}_{n-\nu}$ are p-independent over $\widehat{M}_0^p$. Take cyclic extensions $R_1, \dots, R_{n-\nu}$ of K of degree p so that $R_1 \dots R_{n-\nu} = R$, fix a generator ρ_u of $\mathcal{G}(R_u/K)$, $u = 1, \dots, n-\nu$, and put $\Delta = \otimes_{u=1}^{n-\nu}\Delta_u$, where $\Delta_u = (R_u/K, \rho_u, a_u)$, for each u. It follows from Lemma 4.1 and the conditions on $a_1, \dots, a_{n-\nu}$ that the M_0-algebra $\Delta' = \Delta \otimes_K M_0$ can be defined in accordance with (3.5). This ensures that $\Delta' \in d(M_0)$ and $\Delta \in d(K)$, which implies in conjunction with (3.2) that Δ' has the properties required by (4.2).

To prove our next lemma, we need the following known characterization of finite extensions of K_v in $K_{v,\mathrm{sep}}$, for v Henselian and discrete (cf. [21], Ch. XII, Sects. 2,3 and 6, and the lemma on page 380 of [20]):

(4.3) (a) Every $L \in \mathrm{Fe}(K_v)$ is K_v-isomorphic to $\widetilde{L} \otimes_K K_v$ and $\widetilde{L}_v$, where $\widetilde{L}$ is the separable closure of K in L. The extension L/K_v is Galois if and only if so is $\widetilde{L}/K$; when this holds, $\mathcal{G}(L/K_v)$ and $\mathcal{G}(\widetilde{L}/K)$ are isomorphic.
(b) $K_{\mathrm{sep}} \otimes_K K_v$ is a field and there exist canonical isomorphisms $K_{\mathrm{sep}} \otimes_K K_v \cong K_{v,\mathrm{sep}}$ and $\mathcal{G}_K \cong \mathcal{G}_{K_v}$.

Statement (3.5) and our next lemma prove that $\mathrm{Brd}_p(K) \geq n - [n/3]$, if (K,v) is an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K}\colon \widehat{K}^p] = p^n \leq p^3$. They also imply in conjunction with (3.9), Lemma 3.3 and [24], Theorem 1, that if $[\widehat{K}\colon \widehat{K}^p] = p^n$, for an integer $n \geq 4$, then there exist $D_j \in d(K)$, $j = 1, 2$, such that $D_1 \otimes_K D_2 := D \in d(K)$, $\deg(D_1) = e(D_1/K) = p^2$, $e(D_2/K) = 1$, $\deg(D_2) = \exp(D_1) = p$, and $\widehat{D}$ is a field with $[\widehat{D}\colon \widehat{K}] = p^4$ and $\widehat{D}^p \subseteq \widehat{K}$. This shows that $\mathrm{Brd}_p(K) \geq 3$, which proves Theorem 2.1 (c) in case $n = 4$.

Lemma 4.2. *Let (K,v) be an* HDV*-field with* $\mathrm{char}(\widehat{K}) = p > 0$. *Then there exists a totally ramified abelian noncyclic extension M/K of degree p^2 unless* $p > 2$, $\mathrm{char}(K) = 0$, $\widehat{K} = \mathbb{F}_p$ *and $v(p)$ is a generator of $v(K)$.*

Proof. In view of (4.3) and [8], Lemma 4.2, it is sufficient to consider only the special case where $(K, v) = (K_v, \bar{v})$ and $\mathrm{char}(K) = 0$. As in the proofs of Lemmas 3.2 and 3.3, ε denotes a primitive p-th root of unity in K_{sep}. Suppose first that $\widehat{K}$ is finite. It is well-known that then K can be viewed as a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers (cf. [15], Ch. IV, (1.1)); in addition, if $[K\colon \mathbb{Q}_p] = \mu$, then $\mu+1 \leq r_p(K) \leq \mu+2$ and $r_p(K) = \mu+2$ if and only if $\varepsilon \in K$ (cf. [33], Ch. II, Theorems 3 and 4). Since $r_p(\widehat{K}) = 1$, and by (3.3), $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$, this implies the existence of a Galois extension M' of K in K_{sep}, such that $M' \cap K_{\mathrm{ur}} = K$, $[M'\colon K] = p^{\mu}$ and $\mathcal{G}(M'/K)$ is an elementary abelian p-group. In particular, degree p extensions of K in M' are totally ramified. As $\widehat{K} = \widehat{K}^p$, this means that M'/K is totally ramified (see the end of the proof of Lemma 3.2 (a)). Thus it becomes clear that if $K \neq \mathbb{Q}_p$, then $\mu \geq 2$ and every $M \in I(M'/K)$ of degree p^2 has the properties required by Lemma 4.2. When $K = \mathbb{Q}_p$, the assertion of the lemma is deduced in the same way from the following facts: $r_2(\mathbb{Q}_2) = 3$; if $p > 2$, then $\varepsilon \notin \mathbb{Q}_p$, $r_p(\mathbb{Q}_p) = 2$ and $K_{\mathrm{ur}} \cap K(p)$ is a $\mathbb{Z}_p$-extension of K.

Our objective now is to prove the existence of an extension M/K admissible by Lemma 4.2, assuming that $\widehat{K}$ is infinite. We consider only the case where $v(p) \in pv(K)$ and $\widehat{K} \neq \widehat{K}^p$ (this is allowed by Lemma 3.2). The condition on $v(p)$ and the cyclicity of $v(K)$ ensure that there exists $\pi \in K$ satisfying $0 < v(\pi) \leq v(p)/p$ and $v(\pi) \notin pv(K)$. Suppose first that $\varepsilon \in K$, and put $M_1 = K(\xi)$, $M_2 = K(\eta)$ and $M = M_1M_2$, where ξ and η are p-th roots in K_{sep} of π and $1+\pi$, respectively. It is clear from Kummer theory and the noted properties of π that M/K is a noncyclic Galois extension and $[M\colon K] = p^2$. Moreover, it is easily verified that degree p extensions of K in M are totally ramified. Observe now that $v_{M_1}(\xi) \notin v(K)$ and the norm $N_{M_1}^{M}(1+\xi-\eta)$ equals $(1+\xi)^p - \eta^p$. Taking also into account that $v(p) \in pv(K)$ and $N_{M_1}^{M}(1+\xi-\eta) \in pv(M)$, and applying Newton's binomial formula to the element $(1+\xi)^p$, one concludes that $v_{M_1}(\xi) \in pv(M)$. This, combined with (3.2) (a), shows that M/K is totally ramified, which completes the proof of Lemma 4.2 in the case where $\varepsilon \in K$.

Suppose finally that $\varepsilon \notin K$, i.e. $[K(\varepsilon)\colon K] = m \geq 2$, and as in the proofs of Lemmas 3.2 and 3.3, let φ be a generator of $\mathcal{G}(K(\varepsilon)/K)$, and s, l be integers with $\varphi(\varepsilon) = \varepsilon^s$ and $sl \equiv 1 (\mathrm{mod}\ p)$. For each $\alpha \in O_v(K)^*$, put $\pi_\alpha = \pi\alpha^p$ and $\alpha' = \prod_{j=0}^{m-1}[1 + (1-\varphi^j(\varepsilon))^p\pi_\alpha^{-1}]^{l(j)}$, where $l(j) = l^j$, for $j = 0, 1, \dots, m-1$, and denote by L'_α the extension of $K(\varepsilon)$ in K_{sep} obtained by adjunction of a p-th root η_α of α'. Then, by (3.8), L'_α/K is a cyclic degree pm extension, so L'_α contains as a subfield a degree p (cyclic) extension L_α of K. Now fix α so that $\hat{\alpha}^p \neq \hat{\alpha}$ and put $M_\alpha = L_1L_\alpha$ and $M'_\alpha = M_\alpha(\varepsilon)$. Using Kummer theory and arguing by the method of proving (3.8), one obtains that $M'_\alpha/K(\varepsilon)$ and M_α/K are noncyclic Galois extensions of degree p^2, so it remains for the proof of Lemma 4.2 to show that M_α is totally ramified over K. Since $m \mid p-1$, this is the same as to prove that $M'_\alpha/K(\varepsilon)$ is totally ramified. Put $\beta = \alpha^{p-1}$, $\eta'_1 = \pi_1(\eta_1 - 1)/(1-\varepsilon)$, and $\eta'_\alpha = \pi_\alpha(\eta_\alpha - 1)/(1-\varepsilon)$, and denote by $f_1(X)$ and $f_\alpha(X)$ the minimal polynomials over $K(\varepsilon)$ of η'_1 and η'_α, respectively. It is easily verified that $N_{L'_1}^{M'_\alpha}(\beta\eta'_1 - \eta'_\alpha) = f_\alpha(\beta\eta'_1) = f_\alpha(\beta\eta'_1) - \beta^p f_1(\eta'_1)$. Note also that the equalities

$v_{K(\varepsilon)}(1-\varepsilon^i) = v(p)/(p-1)$, $i = 1, \dots, p-1$,
and the inequalities $0 < v(\pi) \le v(p)/p$
and $v_{K(\varepsilon)}((1-\varphi^j(\varepsilon))/(1-\varepsilon) - s^j) \ge v_{K(\varepsilon)}(1-\varepsilon)$, $j = 1, \dots, m$, imply that $v_{K(\varepsilon)}(t_1) = v_{K(\varepsilon)}(t_\alpha) = v(\pi^{p-1})$ and $v_{K(\varepsilon)}(\beta^p t_1 - t_\alpha) \ge v(p) + v(\pi^{p-1})$, where t_1 and t_α are the free terms of $f_1(X)$ and $f_\alpha(X)$, respectively. At the same time, we have $f_1(X) = (\pi/(1-\varepsilon))^p((1-\varepsilon)\pi^{-1}X + 1) - 1'$ and $f_\alpha(X) = (\pi_\alpha/(1-\varepsilon))^p((1-\varepsilon)\pi_\alpha^{-1}X + 1)^p - \alpha'$, so it follows from the observations on $v(\pi)$ and $v_{K(\varepsilon)}(1-\varepsilon^i)$, $i = 1, \dots, p-1$, that the coefficients of the polynomials
$f_1(X) - X^p - t_1 = \sum_{j=1}^{p-1} r_{1,j}X^j$ and $f_\alpha(X) - X^p - t_\alpha = \sum_{j=1}^{p-1} r_{\alpha,j}X^j$
(defined by the rule $r_{1,j} = \binom{p}{j}(\pi/(1-\varepsilon))^{p-j}$ and $r_{\alpha,j} = \binom{p}{j}(\pi_\alpha/(1-\varepsilon))^{p-j}$, for each index j) lie in the ideal $M_v(K(\varepsilon))$ and form the same value sequence $v(r_{1,j}) = v_{K(\varepsilon)}(r_{\alpha,j}) = v(p) + (p-j)(v(\pi) - v_{K(\varepsilon)}(1-\varepsilon))$, $j = 1, \dots, p-1$, which strictly increases. It is now easy to see that
$v_{L_1'}(\eta_1') = v_{L_\alpha'}(\eta_\alpha') = v_{K(\varepsilon)}(t_1)/p = v(\pi^{p-1})/p \le (p-1)v(p)/p^2$,
$v(\pi^{p-1}) + v_{L_1'}(\eta_1') = (1 + (1/p))v(\pi^{p-1}) \le (1 - (1/p^2))v(p) < v(p)$ and
$v_{L_1'}(N_{L_1'}^{M_\alpha'}(\beta\eta_1' - \eta_\alpha')) = v_{L_1'}(f_\alpha(\beta\eta_1') - \beta^p f_1(\eta_1')) = v_{L_1'}(\pi_\alpha^{p-1}\beta\eta_1' - \beta^p\pi^{p-1}\eta_1') =$
$v_{L_1'}(\pi^{p-1}(\beta^p.\beta\eta_1' - \beta^p\eta_1')) = v(\pi^{p-1}(\beta^{p+1} - \beta^p)) + v_{L_1'}(\eta_1') = v(\pi^{p-1}) + v_{L_1'}(\eta_1')$.
As $v(\pi) \notin pv(K)$, the obtained result indicates that $v_{L_1'}(N_{L_1'}^{M_\alpha'}(\beta\eta_1' - \eta_\alpha'))$ lies in the complement $pv(M_\alpha') \setminus pv(L_1')$, which enables one to deduce from (3.2) (a) that M_α'/K is totally ramified. Lemma 4.2 is proved. □

In order to complete the proof of Theorem 2.1 it remains to be seen that if (K, v) is an HDV-field with $\mathrm{char}(K) = 0$, $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K} \colon \widehat{K}^p] = p^n$, for some integer $n \ge 5$, then $\mathrm{Brd}_p(K) \ge n - [n/3]$ except, possibly, when $n = 5$, K does not contain a primitive p-th root of unity, and $v(p) \in p^2v(K) \setminus p^3v(K)$. Statements (3.5), (4.2) and Lemmas 4.1 and 4.2 imply the desired inequality in the case of $v(p) \notin p^2v(K)$. For the rest of the proof of Theorem 2.1 (c), we need the following lemma.

Lemma 4.3. *Let (K, v) be an* HDV*-field with* $\mathrm{char}(\widehat{K}) = p > 0$, $\widehat{K} \ne \widehat{K}^p$, $\mathrm{char}(K) = 0$ *and* $v(p) \in pv(K)$. *Suppose that $\varepsilon \in K_{\mathrm{sep}}$ is a primitive p-th root of unity, $v(p) \in p^3v(K)$ or $\varepsilon \in K$, and $\lambda \in O_v(K)^*$ is chosen so that $\hat{\lambda} \notin \widehat{K}^p$. Then there is a noncyclic Galois extension M/K, such that $[M \colon K] = [\widehat{M} \colon \widehat{K}] = p^2$, $\hat{\lambda} \in \widehat{M}^p$ and $\widehat{M}/\widehat{K}$ is simple and purely inseparable.*

Proof. The idea of our proof is the same as the one of Lemma 4.2 in the case where $\widehat{K}$ is infinite, so we point out only the basic steps and omit details. As above, $m := [K(\varepsilon) \colon K]$, φ is a generator of $\mathcal{G}(K(\varepsilon)/K)$, s and l are integers chosen so that $\varphi(\varepsilon) = \varepsilon^s$ and $sl \equiv 1 (\mathrm{mod}\ p)$. When $\varepsilon \in K$ (e.g., if $p = 2$), we put $M = K(\xi, \eta)$, where ξ and η are p-th roots in K_{sep} of λ and $\lambda + 1$, respectively; also, we put $\theta = 1 + \xi - \eta$ and take an element $\pi_0 \in K$ so

that $v(\pi_0^p) = v(p)$. Clearly, M/K is a noncyclic abelian extension of degree p^2, $\widehat{M}$ is purely inseparable over $\widehat{K}$ (degree p extensions of K in M are not inertial), $v_{K(\xi)}(N_{K(\xi)}^M(\theta)/\pi_0^p) = 0$ and the residue class of $N_{K(\xi)}^M(\theta)/\pi_0^p$ lies in $\widehat{M}^p \setminus \widehat{K}^p$. In view of (3.2) (a), this proves that M/K has the properties claimed by Lemma 4.3, so we assume further that $\varepsilon \notin K$ and $v(p) \in p^3v(K)$. Take some $\pi \in K^{*p}$ so that $0 < v(\pi) \leq v(p)/p^2$, put $\pi_\alpha = \pi\alpha^{-p}$ and $\alpha' = \prod_{j=0}^{m-1}[1 + (1 - \varphi^j(\varepsilon))^p\pi_\alpha^{-p}\lambda]^{l(j)}$, for each $\alpha \in O_v(K)^*$, where $l(j) = l^j$, for $j = 0, \dots, m-1$, and denote by L'_α some extension of $K(\varepsilon)$ in K_{sep} obtained by adjunction of a p-th root η_α of α'. It is easily verified that $v_{K(\varepsilon)}(\alpha' - 1 - m(1-\varepsilon)^p\pi_\alpha^{-p}\lambda) > v(p) + v_{K(\varepsilon)}((1-\varepsilon)^p\pi_\alpha^{-p}\lambda)$. Since $m \mid p-1$, this implies the binomial $X^p - m\lambda$ and the minimal polynomial $f_\alpha(X)$ of the element $\eta'_\alpha = \pi_\alpha(\eta_\alpha - 1)/(1-\varepsilon)$ over $K(\varepsilon)$ have the same reduction modulo $M_v(K(\varepsilon))$. Therefore, $\eta'_\alpha \in O_v(K(\varepsilon))^*$ and $\hat{\eta}'^p_\alpha = m\hat{\lambda} = m^p\hat{\lambda}$; in particular, $\hat{\lambda} \in \widehat{L}'^p_\alpha$. Observing also that $\varphi(\alpha')\alpha'^{-s} \in K(\varepsilon)^{*p}$, one deduces from Albert's theorem that L'_α/K is cyclic of degree pm. Thus L'_α contains as a subfield a cyclic extension L_α of K of degree p, and $\hat{\lambda} \in \widehat{L}^p_\alpha$. Now fix α so that $\hat{\alpha}^p \neq \hat{\alpha}$, put $M_\alpha = L_1L_\alpha$ and $M'_\alpha = L'_1L'_\alpha$, and denote by θ_1 and θ_α the free terms of $f_1(X)$ and $f_\alpha(X)$, respectively. It follows from Kummer theory, the choice of α and the preceding observations that M_α/K is a noncyclic Galois extension, $[M_\alpha \colon K] = p^2$, and $\widehat{\Lambda} \cong \widehat{L}_1$ over $\widehat{K}$, for each $\Lambda \in I(M_\alpha/K)$ with $[\Lambda \colon K] = p$; hence, $\widehat{M_\alpha}/\widehat{K}$ is purely inseparable. Observe that $0 < v(\pi^p) \leq v(p)/p$, so it is verified by direct calculations that
$v_{K(\varepsilon)}(\theta_1 - \theta_\alpha) \geq \min\{v_{K(\varepsilon)}(\theta_1 + m\lambda), v_{K(\varepsilon)}(\theta_\alpha + m\lambda)\} \geq v(p)$ whereas
$v(\pi^{p-1}) = v(\pi_\alpha^{p-1}) = v(\pi^{p-1} - \pi_\alpha^{p-1}) \leq (p-1)v(p)/p^2 < v(p)$.
Note also that if $\delta'_\alpha = (\eta'_1 - \eta'_\alpha)/\tau^{p-1}$, where $\tau \in K$ is a p-th root of π, then the norm $\xi_\alpha = N_{L'_1}^{M'_\alpha}(\delta'_\alpha)$ is equal to $f_\alpha(\eta'_1)/\pi^{p-1} = (f_\alpha(\eta'_1) - f_1(\eta'_1))/\pi^{p-1}$.
These calculations make it easy to show that $v_{M'_\alpha}(\delta'_\alpha) = 0$, $v_{L'_1}(\xi_\alpha) = 0$ and $\hat{\xi}_\alpha$ a p-th root of $(\hat{\alpha}^{p(p-1)} - 1)^p.m^p\hat{\lambda}$ lying in $\widehat{M}'^p_\alpha$. Since $\hat{\lambda} \notin \widehat{K}^p$, m is equal to $[K(\varepsilon) \colon K]$, $[L'_1 \colon L_1]$ and $[M'_\alpha \colon M_\alpha]$, and $p \dagger m$, the obtained result leads to the conclusion that $[\widehat{M}_\alpha \colon \widehat{K}] = [M_\alpha \colon K] = p^2$ and the field $\widehat{L}_1 \cap \widehat{M}^p_\alpha$ contains a p-th root of $\hat{\lambda}$. At the same time, it follows that $\widehat{M}_\alpha/\widehat{K}$ is a simple purely inseparable extension, so Lemma 4.3 is proved. □

Remark 4.4. Note that if (K, v) is an HDV-field with $\mathrm{char}(K) = 0$, $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K} \colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$, then there exists an integer $c(K) \geq 0$, such that $e(L_m/K) > 1$, for each finite Galois extension L_m/K satisfying the following: $\mathcal{G}(L_m/K)$ is an elementary abelian p-group of order $p^m > p^{c(K)}$; $\widehat{L}_m/\widehat{K}$ is purely inseparable. Moreover, it can be deduced from (3.6), (3.7) and Albert's theorem that $c(K) \leq n\mu$, μ being the index of $\langle pv(p)/(p-1)\rangle$ as a subgroup of $v(K(\varepsilon))$, where $\varepsilon \in K_{\mathrm{sep}}$ and $\varepsilon \neq 1 = \varepsilon^p$.

We are now in a position to complete the proof of Theorem 2.1. Let (K, v) be an HDV-field with $\mathrm{char}(K) = 0$, $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K} \colon \widehat{K}^p] = p^n$, for some integer $n \geq 5$. Suppose that we are not in the case where $n = 5$, $v(p) \in$

$p^2v(K) \setminus p^3v(K)$ and K does not contain a primitive p-th root of unity (this restriction is allowed by (4.2)). Fix $k \in \mathbb{N}$ and elements $a_j, b_j, c_j \in O_v(K)^*$, $j = 1, \dots, k$, so that $p^{3k} \leq [\widehat{K}\colon \widehat{K}^p]$ and $\hat{a}_j, \hat{b}_j, \hat{c}_j \in \widehat{K}$, $j = 1, \dots, k$, are p-independent over $\widehat{K}^p$. Then, by Lemma 4.3, one can find, for each index j, Galois extensions L_j, L'_j and M_j of K in $K(p)$, such that $M_j = L_jL'_j$, $[L_j\colon K] = [L'_j\colon K] = p$, $[M_j\colon K] = p^2$, and $\widehat{M}_j/\widehat{K}$ be purely inseparable, $[\widehat{M}_j\colon \widehat{K}] = p^2$ and $\hat{a}_j \in \widehat{M}_j^p$. Fix generators $\tau_1, \tau'_1, \dots, \tau_k, \tau'_k$ of $\mathcal{G}(L_1/K), \mathcal{G}(L'_1/K), \dots, \mathcal{G}(L_k/K), \mathcal{G}(L'_k/K)$, respectively, and put $\Delta_k = \otimes_{j=1}^k A_j$, where $\otimes = \otimes_K$ and $A_j = (L_j/K, \tau_j, b_j) \otimes_K (L'_j/K, \tau'_j, c_j)$, for $j = 1, \dots, k$. It follows from [24], Theorem 1, that $\Delta_k \in d(K)$, $\exp(\Delta_k) = p$ and $\deg(\Delta_k) = p^{2k}$. The obtained result proves Theorem 2.1 (c) in the special case of $n = 3k$. Assume now that $n = 3k + u$, where $u \in \{1, 2\}$, fix a totally ramified Galois extension T_u/K with $[T_u\colon K] = p^u$ and $\mathcal{G}(T_u/K)$ of period p, and take elements $a_i \in O_v(K)^*$, $i = 1, \dots, u$, so that $\widehat{K}^p(\hat{a}_i, i \leq u; \hat{b}_j, \hat{c}_j, j = 1, \dots k) = \widehat{K}$. Then (3.5) and [24], Theorem 1, enable one to attach to T_u/K and a_i, $i \leq u$, an algebra $\Theta_u \in d(K)$ with $\exp(\Theta_u) = p$, $\deg(\Theta_u) = p^u$, $[\Theta_u] \in \mathrm{Br}(T_u/K)$ and $\Theta_u \otimes_K \Delta_k \in d(K)$. This ensures that $\exp(\Theta_u \otimes_K \Delta_k) = p$ and $\deg(\Theta_u \otimes_K \Delta_k) = p^{n'}$, where $n' = 2k + u = n - [n/3]$, which completes the proof of Theorem 2.1.

Remark 4.5. An HDV-field (K, v) with $\mathrm{char}(K) = 0$, $\mathrm{char}(\widehat{K}) = p > 0$, and $[\widehat{K}\colon \widehat{K}^p] = p^n < \infty$ satisfies $\mathrm{Brd}_p(K) \geq n - [n/3]$ also in the case of $e(K(\varepsilon)/K) = 1$, ε being a primitive p-th root of unity in K_{sep}. In view of Theorem 2.1, one may assume, for the proof, that $v(p) \in p^2v(K)$ and $n \geq 5$. Then there is $\pi \in K^{*p}$, such that $0 < v(\pi) \leq v(p)/(p^2 - p)$, which implies K has an extension claimed by Lemma 4.3, and so proves the stated inequality.

Theorem 2.1, [4], Proposition 4.5, and (2.1) raise interest in the question of whether $\mathrm{Brd}_p(K) = n$, if (K, v) is an HDV-field, $\mathrm{char}(\widehat{K}) = p > 0$, $\widehat{K}_{\mathrm{sep}} = \widehat{K}$ and $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$. An affirmative answer would agree with the well-known conjecture that $\mathrm{abrd}_p(F) < \nu$ whenever F is a field of type (C_ν), for some $\nu \in \mathbb{N}$, i.e. each homogeneous polynomial $f(X_1, \dots, X_m) \in F[X_1, \dots, X_m]$ of degree $d > 0$, $d^\nu < m$, has a nontrivial zero over F. This is particularly clear in the special case where F/E is a finitely-generated field extension of transcendency degree n, and E has a Henselian discrete valuation ω, such that $\widehat{E}$ is algebraically closed, $\mathrm{char}(\widehat{E}) = p$, and in case $\mathrm{char}(E) = p$, E is complete relative to the topology of ω. Indeed, then E is of type (C_1), by Lang's theorem [20], so it follows from the Lang-Nagata-Tsen theorem [26], that F is of type (C_{n+1}) (for more information on the (C_ν) property, see [33], Ch. II, 3.2 and 4.5). The assumptions on F and E also imply the existence of a discrete valuation ω' of F extending ω with $\widehat{F}/\widehat{E}$ a finitely-generated extension of transcendency degree n; in particular, $[\widehat{F}'\colon \widehat{F}'^p] = p^n$, for every finite extension F'/F. This enables one to deduce (e.g., from [8], Lemmas 3.1 and 4.3) that if (L, w) is a Henselization of (F, ω'), then $\mathrm{abrd}_p(L) \leq \mathrm{Brd}_p(F)$. Therefore, the stated conjecture requires that $\mathrm{abrd}_p(L) \leq n$. On the other hand, $(L, w)/(F, \omega')$ is immediate, so $[\widehat{L}\colon \widehat{L}^p] = p^n$, and by Theorem 2.1 (b), $\mathrm{abrd}_p(L) \geq n$. Moreover, one obtains by the method of proving [8], Proposition 6.3, that $\mathrm{Brd}_p(L) \geq n$. Thus the assertion that $\mathrm{Brd}_p(L) = n$ can be viewed as a special case of the conjecture.

5. **HDV-fields with almost perfect residue fields**

This Section is devoted to the proof of Theorem 2.2. Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$. Theorem 2.1 and [6], Proposition 2.1, show that $\mathrm{Brd}_p(K) \geq 2$, if $[\widehat{K}\colon \widehat{K}^p] \geq p^2$ or $\widehat{K}$ is not p-quasilocal. Therefore, we assume that $\widehat{K}$ is p-quasilocal with $\mathrm{char}(\widehat{K}) = p > 0$ and $[\widehat{K}\colon \widehat{K}^p] \leq p$, and we prove that $\mathrm{Brd}_p(K)$ is determined by Theorem 2.2. Our argument is facilitated by (2.3) (a), (b) and (4.3), which indicate that it is sufficient to settle the special case where $(K, v) = (K_v, \bar{v})$. Suppose first that $\widehat{K}$ is perfect and $r_p(\widehat{K}) = 0$. Then $\mathrm{Br}(\widehat{K})_p = \{0\}$, by [2], Ch. VII, Theorem 22, so it follows from Witt's theorem that $\mathrm{Br}(K)_p = \{0\}$, i.e. $\mathrm{Brd}_p(K) = 0$, as claimed. Next we consider the case of $\widehat{K}$ perfect and $r_p(\widehat{K}) > 0$. Then Witt's theorem ensures the existence of a nicely semiramified (abbr, NSR) algebra $\Delta_p \in d(K)$, in the sense of [16], of degree p and thereby proves that $\mathrm{Brd}_p(K) \geq 1$. On the other hand, (3.4) and [2], Ch. VII, Theorem 22, imply $e(D_p/K) = \deg(D_p/K)$, for each $D_p \in d(K)$ with $[D_p] \in \mathrm{Br}(K)_p$. Hence, by (3.2) (a), (3.4), [29], (3.19), and the cyclicity of the group $v(D_p)/v(K)$, $\deg(D_p) \mid \exp(D_p)$, which yields $\mathrm{Brd}_p(K) \leq 1$. It remains to be seen that $\mathrm{Brd}_p(K) \leq 1$, if $[\widehat{K}\colon \widehat{K}^p] = p$. Our proof relies on the following lemma.

Lemma 5.1. *Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p$ and $[\widehat{K}\colon \widehat{K}^p] = p$, and let Y/K be a field extension, such that $[Y\colon K] = [\widehat{Y}\colon \widehat{K}] = p$. Suppose that $\widehat{K}$ is p-quasilocal and $\widehat{Y}$ is normal over $\widehat{K}$. Then $\mathrm{Br}(Y/K)$ includes the group ${}_p\mathrm{Br}(K) \cap \mathrm{IBr}(K)$, and the homomorphism $\pi_{Y/K}\colon \mathrm{Br}(K) \to \mathrm{Br}(Y)$, maps $\mathrm{Br}(K)_p \cap \mathrm{IBr}(K)$ surjectively upon $\mathrm{Br}(Y)_p \cap \mathrm{IBr}(Y)$.*

Proof. It follows from [6], Theorem 4.1, and Albert-Hochschild's theorem (cf. [33], Ch. II, 2.2) that $\pi_{\widehat{K}/\widehat{Y}}$ maps $\mathrm{Br}(\widehat{K})_p$ surjectively upon $\mathrm{Br}(\widehat{Y})_p$. At the same time, we have $\mathrm{Br}(\widehat{Y}/\widehat{K}) = {}_p\mathrm{Br}(\widehat{K})$, by [6], Theorem 4.1, if $\widehat{Y}/\widehat{K}$ is separable, and by [2], Ch. VII, Theorem 28, when $\widehat{Y}/\widehat{K}$ is inseparable. Note further that $\mathrm{IBr}(Y)$ includes the image of $\mathrm{IBr}(K)$ under $\pi_{K/Y}$, and the natural mappings $r_{K/\widehat{K}}\colon \mathrm{IBr}(K) \to \mathrm{Br}(\widehat{K})$, and $r_{Y/\widehat{Y}}\colon \mathrm{IBr}(Y) \to \mathrm{IBr}(\widehat{Y})$, are index-preserving group isomorphisms (see [16], Theorems 5.6 and 2.8). Observing also that $(\pi_{\widehat{K}/\widehat{Y}} \circ r_{K/\widehat{K}})([D]) = (r_{Y/\widehat{Y}} \circ \pi_{K/Y})([D])$ (in $\mathrm{Br}(\widehat{Y})$) whenever $D \in d(K)$ is inertial over K, one proves the latter part of the assertion of Lemma 5.1, as well as the fact that $\mathrm{ind}(D_p \otimes_K Y) = \deg(D_p)/p$, for each $D_p \in d(K)$ with $[D_p] \neq 0$ and $[D_p] \in (\mathrm{Br}(K)_p \cap \mathrm{IBr}(K))$. In view of the Corollary in [28], Sect. 13.4, this completes our proof. □

Next we show that Theorem 2.2 will be proved, if we deduce the equality $\deg(\Delta) = p$, assuming that $\Delta \in d(K)$ and $\exp(\Delta) = p$. It follows from (3.4) and [16], Proposition 1.7, that each $D \in d(K)$ with $\deg(D) = p$ possesses a maximal subfield Y satisfying the conditions of Lemma 5.1. Hence, $\widehat{Y}$ is p-quasilocal (cf. [6], Theorem 4.1 and Proposition 4.4), which enables one to obtain from the claimed property of Δ, by the method of proving [7], Lemma 4.1, that if $\Delta_n \in d(K)$ and $\exp(\Delta_n) = p^n$, then Δ_n has a splitting field Y_n with $[Y_n \colon K] = p^n$, $v(Y_n) = v(K)$ and $\widehat{Y}_n \in I(\widehat{Y}'/\widehat{K})$, where $\widehat{Y}'$ is a perfect closure of $\widehat{K}(p)$. This result gives the desired reduction. Since, by Merkur'ev's theorem [23], Sect. 4, Theorem 2, each $\Delta \in d(K)$ with $\exp(\Delta) = p$ is Brauer equivalent to a tensor product of degree p algebras from $d(K)$, we need only prove that if $D_j \in d(K)$ and $\deg(D_j) = p$, $j = 1, 2$, then $D_1 \otimes_K D_2 \notin d(K)$. This can be deduced from the following lemma.

Lemma 5.2. *Let (K, v) be an* HDV*-field with* $\mathrm{char}(\widehat{K}) = p$*, $\widehat{K}$ p-quasilocal and $[\widehat{K} \colon \widehat{K}^p] = p$. Then* $\exp(\Delta) = p^2$*, for any $\Delta \in d(K)$ of degree p^2.*

Proof. Let Δ be a K-algebra satisfying the conditions of the lemma. As $\widehat{K}$ is almost perfect, this implies p^2 is divisible by the dimension of any commutative $\widehat{K}$-subalgebra of $\widehat{\Delta}$. At the same time, it follows from (3.4) and the cyclicity of $v(\Delta)$ that $e(\Delta/K) \mid p^2$. Suppose first that $e(\Delta/K) = 1$. Then Δ/K is inertial, by (3.4), which makes it easy to deduce from [16], Theorem 2.8, [2], Ch. VII, Theorem 28, and [6], Theorem 3.1, that $\deg(\Delta) = \exp(\Delta)$, as claimed by Lemma 5.2. Henceforth, we assume that $e(\Delta/K) \neq 1$. Our first objective is to prove the following:

(5.1) (a) If U is a central K-subalgebra of Δ of degree p, then U is neither an inertial nor an NSR-algebra over K;

(b) If $e(\Delta/K) = p$, then totally ramified extensions of K of degree p are not embeddable in Δ as K-subalgebras.

The proof of (5.1) (a) relies on the Double Centralizer Theorem (see [28], Sect. 12.7), which implies that Δ is K-isomorphic to $U \otimes_K U'$, for some $U' \in d(K)$ with $\deg(U') = p$. Suppose for a moment that U/K is inertial. Applying (3.2) (a), (3.4) and [16], Theorem 2.8 and Proposition 1.7, one concludes that $e(U'/K) = p$, $\widehat{U}'/\widehat{K}$ is a normal field extension of degree p, and U' contains as a K-subalgebra an extension Y of K with $\widehat{Y} = \widehat{U}'$. Therefore, by Lemma 5.1, Y is embeddable in U as a K-subalgebra, which means that $U \otimes_K Y \notin d(Y)$. Since $\Delta \in d(K)$ and $U \otimes_K Y$ is a K-subalgebra of Δ, this is a contradiction ruling out the possibility that U/K be inertial.

We turn to the proof of (5.1) (b), so we assume that $e(\Delta/K) = p$. Suppose that our assertion is false, i.e. Δ contains as a K-subalgebra a totally ramified extension T of K of degree p, and let W' be the centralizer of T in Δ. It is clear from the Double Centralizer Theorem that $W' \in d(T)$ and $\deg(W') = p$, and it follows from (3.4) and the assumptions on Δ/K and T/K that $[\widehat{W'} \colon \widehat{T}] = p^2$. As $\widehat{K}$ is almost perfect, these facts show that $\widehat{W'} \in d(\widehat{T})$. Taking into account that $\widehat{T} = \widehat{K}$, and applying [16], Theorem 2.8, one concludes that $W' \cong W \otimes_K T$ as a T-algebra, where $W \in d(K)$ is an inertial lift of $\widehat{W'}$ over K. Our conclusion, however, contradicts (5.1) (a), since it requires that W embed in Δ as a K-subalgebra, so (5.1) (b) is proved.

We continue with the proof of Lemma 5.2 in the case of $e(\Delta/K) = p$. Clearly, (3.4) yields $[\widehat{\Delta}\colon \widehat{K}] = p^3$, so the assumption that $[\widehat{K}\colon \widehat{K}^p] = p$ implies that $\widehat{\Delta}$ is noncommutative. This means that $[\widehat{\Delta}\colon Z(\widehat{\Delta})] = p^2$ and $[Z(\widehat{\Delta})\colon \widehat{K}] = p$, where $Z(\widehat{\Delta})$ is the centre of $\widehat{\Delta}$. First we prove that $\exp(\Delta) = p^2$, under the extra hypothesis that Δ possesses a K-subalgebra Δ_0, such that $[\Delta_0\colon K] = p^3$ and $\widehat{\Delta}_0$ is $\widehat{K}$-isomorphic to $\widehat{\Delta}$; by [16], Theorem 2.9, this holds in the special case where $Z(\widehat{K})$ is a separable extension of $\widehat{K}$. It follows from [16], Proposition 1.7, our extra hypothesis and the cyclicity of $v(K)$ that $Z(\widehat{\Delta})/\widehat{K}$ is a normal extension of degree p. Hence, by Lemma 5.1, we have $[\Delta_0] = [D \otimes_K Z(\Delta_0)]$ (in $\mathrm{Br}(Z(\Delta_0))$), for some $D \in d(K)$ inertial over K. The obtained result indicates that $[\Delta \otimes_K D^{\mathrm{op}}] \in \mathrm{Br}(Z(\Delta_0)/K)$, which requires that $\exp(\Delta \otimes_K D^{\mathrm{op}}) \mid p$. Taking finally into account that $\deg(D) = \exp(D) = p^2$, one concludes that $\exp(\Delta) = p^2$, as claimed.

We are now prepared to consider the case of $e(\Delta/K) = p$ in general. The preceding part of our proof allows us to assume that $Z(\widehat{\Delta})$ is a purely inseparable extension of $\widehat{K}$. Note also that $[Z(\widehat{\Delta})\colon \widehat{K}] = p$, and it follows from [6], Theorem 3.1, and [2], Ch. VII, Theorem 28, that $\widehat{\Delta}$ is a cyclic $Z(\widehat{\Delta})$-algebra of degree p. Therefore, there exists $\eta \in \Delta$, which generates an inertial cyclic extension of K of degree p. Hence, by the Skolem-Noether theorem (cf. [28], Sect. 12.6), there is $\xi \in \Delta^*$, such that $\xi\eta'\xi^{-1} = \varphi(\eta')$, for every $\eta' \in K(\eta)$, where φ is a generator of $\mathcal{G}(K(\eta)/K)$. Denote by B the K-subalgebra of Δ generated by η and ξ. It is easy to see that $K(\xi^p) = Z(B)$, $\deg(B) = p$ and B is either an inertial or an NSR-algebra over $K(\xi^p)$. In view of (5.1) (a), this means that $\xi^p \notin K$ which gives $[K(\xi^p)\colon K] = p$, and combined with (5.1) (b), proves that $v(K(\xi^p)) = v(K)$. In other words, $K(\xi^p)^* = O_v(K(\xi^p))^* K^*$. As $e(\Delta/K) = p$, the obtained properties of B and $K(\xi^p)$ indicate that if $B/K(\xi^p)$ is inertial (equivalently, if $v_\Delta(\xi) \in v(K)$, see [16], Theorem 5.6 (a)), then $\widehat{B} \cong \widehat{\Delta}$ over $\widehat{K}$. This means that Δ/K is subject to the extra hypothesis, which yields $\exp(\Delta) = p^2$. When $B/K(\xi^p)$ is NSR, these properties imply with (5.1) (b) and [28], Sect. 15.1, Proposition b, the existence of an algebra $\Theta \in d(K)$ satisfying the following conditions:

(5.2) (a) Θ is isomorphic to the cyclic K-algebra $(K(\eta)/K, \varphi, \pi')$, for some $\pi' \in K^*$; Θ/K is NSR, whence Θ does not embed in Δ as a K-subalgebra;
(b) $\mathrm{ind}(\Delta \otimes_K \Theta) = p^2$ (see also [28], Sect. 13.4, and [8], (1.1)(b)), the underlying division K-algebra Δ' of $\Delta \otimes_K \Theta$ has a K-subalgebra Z' isomorphic to $Z(B)$, and the centralizer $C_{\Delta'}(Z') := C$ is an inertial Z'-algebra.

Note here that $[\Delta'] \in \mathrm{Br}(K(\xi^p, \eta)/K)$. Using (3.2) (a), (3.4) and (5.2), one concludes that $[C\colon K] = p^3$ (see also [28], Sect. 12.7) and either Δ'/K is inertial or $e(\Delta'/K) = p$ and $\widehat{C} \cong \widehat{\Delta}'$ as a $\widehat{K}$-algebra. As shown above, this alternative on Δ' requires that $\exp(\Delta') = p^2$. In view of (5.2) (b) and the equality $\deg(\Theta) = \exp(\Theta) = p$, it thereby proves that $\exp(\Delta) = p^2$ as well.

It remains to consider the case where $e(\Delta/K) = p^2$. We first show that one may assume without loss of generality that $\mathrm{Brd}_p(\widehat{K}) = 0$. It follows from (5.1) (a), (3.4) and the equality $e(\Delta/K) = p^2$ that $\widehat{\Delta}/\widehat{K}$ is a field extension of degree p^2. Using [16], Theorem 3.1, one obtains that $\Delta \otimes_K U \in d(U)$ and $e((\Delta \otimes_K U)/U) = p^2$ whenever U is an extension of K in $K(p) \cap K_{\mathrm{ur}}$, such that no proper extension of $\widehat{K}$ in $\widehat{U}$ is embeddable in $\widehat{\Delta}$ as a $\widehat{K}$-subalgebra. Note also that $\widehat{\Delta} \otimes_{\widehat{K}} \widehat{U}$ is $\widehat{U}$-isomorphic to the residue field of $\Delta \otimes_K U$, which enables one to prove (by applying Galois theory and Zorn's lemma) that U can be chosen so as to satisfy the condition $r_p(\widehat{U}) \leq 1$. Then, by [17], Proposition 4.4.8, $\mathrm{Br}(\widehat{U})_p = \{0\}$, which leads to the desired reduction.

We suppose further that $\mathrm{Brd}_p(\widehat{K}) = 0$ and prove the following assertion:

(5.3) If Δ possesses a K-subalgebra Z, such that $[Z\colon K] = [\widehat{Z}\colon \widehat{K}] = p$ and $\widehat{Z}$ is purely inseparable over $\widehat{K}$, then $\widehat{\Delta}/\widehat{K}$ is purely inseparable.

Assuming the opposite and using (3.2) (a) and (3.4), one obtains that Z has an inertial extension M which is a maximal subfield of Δ. As v is Henselian, the assumptions on Z and M ensure that $M = LZ$, for some inertial extension L of K in M of degree p. Note further that $[M\colon K] = [\widehat{M}\colon \widehat{K}] = [\widehat{\Delta}\colon \widehat{K}] = p^2$, which means that $\widehat{M} = \widehat{\Delta}$. The obtained result enables one to deduce from [16], Proposition 1.7, and the Henselity of v that L/K is a cyclic extension. At the same time, the equality $\mathrm{Brd}_p(\widehat{K}) = 0$ and the Albert-Hochschild theorem, applied to the extension $\widehat{Z}/\widehat{K}$, indicate that $\mathrm{Brd}_p(\widehat{Z}) = 0$. Therefore, the norm group $N(M/Z)$ includes $O_v(Z)^*$ (cf. [28], Sect. 15.1, Proposition b), which enables one to deduce from the Skolem-Noether theorem and the Double Centralizer Theorem that there is a Z-isomorphism $C_\Delta(Z) \cong (M/Z, \psi', \gamma)$, for some $\gamma \in K^*$ and some generator ψ' of $\mathcal{G}(M/Z)$. This in turn implies $\Delta \cong D_1 \otimes_K D_2$ as a K-algebra, where $D_1 = (L/K, \psi, \gamma)$, ψ being the K-automorphism of L induced by ψ', $D_2 \in d(K)$ and $[D_2] \in \mathrm{Br}(Z/K)$. As $\mathrm{Brd}_p(K) = 0$ and $\deg(D_2) = p$, one obtains further that D_2 contains as a subfield a totally ramified extension T of K of degree p. It is now easy to see that $(L \otimes_K T)/T$ is an inertial and cyclic extension of degree p, and to deduce consecutively from here that $N((L \otimes_K T)/T)$ includes $O_v(T)^*$ and K^*. Observing also that $D_1 \otimes_K T$ is T-isomorphic to $((L \otimes_K T)/T, \psi_T, \gamma)$, where ψ_T is the T-isomorphism of $L \otimes_K T$ extending ψ, one obtains from [28], Sect. 15.1, Proposition b, that $D_1 \otimes_K T \notin d(T)$. Since $D_1 \otimes_K T$ is a K-subalgebra of $D_1 \otimes_K D_2 \cong \Delta \in d(K)$, this is a contradiction proving (5.3).

It is now easy to prove Lemma 5.2. If $\widehat{\Delta}/\widehat{K}$ is a purely inseparable field extension, then it follows from [38], Proposition 2.1, that $\exp(\Delta) = p^2$. Suppose finally that $\widehat{\Delta}$ is a field and $\widehat{\Delta}/\widehat{K}$ is not purely inseparable. In view of [16], Proposition 1.7 and Theorem 2.9, this ensures the existence of an inertial cyclic extension Λ of K of degree p, which embeds in Δ as a K-subalgebra. Our goal is to show that there is an infinite extension W of K in an algebraic closure $\overline{K}$, satisfying the following conditions:

(5.4) $v(W) = v(K)$, $\widehat{W}$ is purely inseparable over $\widehat{K}$ and $\Delta \otimes_K W \in d(W)$.

Note that (5.4) implies $\exp(\Delta) = p^2$. Indeed, it follows from (3.2) (a), (5.4) and the equality $[\widehat{K}\colon \widehat{K}^p] = p$ that $\widehat{W}$ is perfect and $(\Delta \otimes_K W)/W$ is NSR. Hence, $\exp(\Delta \otimes_K W) = \deg(\Delta \otimes_K W) = p^2$, and since $\exp(\Delta \otimes_K W) \mid \exp(\Delta)$ and $\exp(\Delta) \mid \deg(\Delta) = p^2$, this gives $\exp(\Delta) = p^2$, as required.

Finally, we prove (5.4). Fix an element $a_0 \in O_v(K)^*$ so that $\hat{a}_0 \notin \widehat{K}^p$, take a system $a_n \in \overline{K}$, $n \in \mathbb{N}$, satisfying $a_n^p = a_{n-1}$, for each n, and let

W be the union of the fields $W_n = K(a_n)$, $n \in \mathbb{N}$. It is easily verified that $[W_n : K] = [\widehat{W_n} : \widehat{K}] = p^n$ and $\widehat{W_n}/\widehat{K}$ is purely inseparable, for every $n \in \mathbb{N}$, so it follows from (3.2) (a), the equality $[\widehat{K} : \widehat{K}^p] = p$ and the inclusions $W_n \subset W_{n+1}$, $n \in \mathbb{N}$, that W is a field, $v(W) = v(K)$ and $\widehat{W}$ a perfect closure of $\widehat{K}$. Arguing by induction on n, taking into account that $\Delta \otimes_K W_{n+1} \cong (\Delta \otimes_K W_n) \otimes_{W_n} W_{n+1}$ as W_n-algebras, and using (5.3), the noted properties of W_n, and the behaviour of Schur indices under scalar extensions of finite degrees (cf. [28], Sect. 13.4), one obtains that, for each $n \in \mathbb{N}$, $\Delta \otimes_K W_n \in d(W_n)$, and $\Lambda \otimes_K W_n$ is an inertial cyclic extension of W_n of degree p, embeddable in $\Delta \otimes_K W_n$ as a W_n-subalgebra. Therefore, $\Delta \otimes_K W \in d(W)$, so (5.4), Lemma 5.2 and Theorem 2.2 are proved. □

Corollary 5.3. *Assume that (K, v) is an* HDV*-field, such that $\widehat{K}$ is of type (C_1). Then K is absolutely stable.*

Proof. The field $\widehat{K}$ is almost perfect with $\mathrm{abrd}_p(\widehat{K}) = 0\colon p \in \mathbb{P}$ (cf. [33], Ch. II, 3.2), so $\widehat{K}$ is quasilocal, and by Corollary 2.4, K is absolutely stable. □

When $\mathrm{char}(K) = \mathrm{char}(\widehat{K})$, the assertion of Corollary 5.3 is contained in [39], Theorem 2; it is a special case of [5], Corollary 4.6, if $\widehat{K}$ is perfect.

6. An application to m-dimensional local fields

The first result of this Section contains information on the sequence $\mathrm{Brd}_{p'}(K_m)$, $p' \in \mathbb{P}$, for an m-dimensional local field K_m, which is complete in case $\mathrm{char}(K_m) > 0$. Specifically, it shows that K_2 is absolutely stable. As noted in Section 2, this property of K_2 is known in characteristic $p > 0$; the crucial inequality $\mathrm{abrd}_p(K_2) \leq 1$ can be deduced from [2], Ch. XI, Theorem 3, and results of Aravire, Jacob, Merkurjev and Tignol (see [3], Theorem 3.3 and Corollary 3.4, as well as the Appendix to [3]).

Proposition 6.1. *Let K_m be an m-dimensional local field with an m-th residue field K_0. Then:*

(a) $\mathrm{Brd}_{p'}(K_m) = 1$, *if $p' \in \mathbb{P}$, $p' \neq \mathrm{char}(K_0)$ and K_0 does not contain a primitive p'-th root of unity;* $\mathrm{Brd}_{p'}(K_m) = [(1 + m)/2]$, *when $p' \in \mathbb{P}$ and K_0 contains a primitive p'-th root of unity;*

(b) $\mathrm{Brd}_p(K_m) = m - 1$, *if* $\mathrm{char}(K_m) = p > 0$ *and $m \geq 2$;*

(c) *K_m is stable iff $m \leq 2$; when this holds, it is absolutely stable.*

Proof. Our assumptions imply the existence of a valuation v_m of K_m, such that (K_m, v_m) is maximally complete with $\widehat{K}_m \cong K_0$ and $v_m(K_m)$ is isomorphic to the inversely-lexicographically ordered abelian group $\mathbb{Z}^m$. Thus Proposition 6.1 (b) and (a) reduces to a consequence of (2.2) (c) and [10], Theorem 4.1 (see also [18], for a refinement of the latter part of Proposition 6.1 (a)). It remains to prove Proposition 6.1 (c). In view of Proposition 6.1 (a) and (b), it suffices to consider the special case where $\mathrm{char}(\widehat{K}_m) = p > 0$ and $\mathrm{char}(K_m) = 0$. Moreover, one need only prove that $\mathrm{Brd}_p(K_m) \leq 1$ if and only if $m \leq 2$. If $m = 1$, then K_m is a local field, whence, it is absolutely stable (e.g., by Corollary 2.4); in particular, $\mathrm{Brd}_p(K_m) = 1$. We assume further that $m \geq 2$. In this case, K_m is complete with respect to some discrete valuation w_m whose residue field K_{m-1} is an $(m-1)$-dimensional local field with last residue field isomorphic to K_0. Therefore, (K_m, w_m) is an HDV-field, and it follows from [16], Theorem 2.8, that $\mathrm{Brd}_{p'}(K_{m-1}) \leq \mathrm{Brd}_p(K_m)$, for each $p' \in \mathbb{P}$. Suppose now that $m = 2$. Then, by local class field theory (cf. [32], Ch. XIII, Sect. 3), K_1 is a quasilocal field with $\mathrm{Br}(K_1) \cong \mathbb{Q}/\mathbb{Z}$; hence, by Corollary 2.4, K_2 is absolutely stable, as claimed. More precisely, it is easy to see that $\mathrm{Brd}_{p'}(K_u) = 1$, $u = 1, 2$, $p' \in \mathbb{P}$. Note also that $r_p(K_1) \geq 2$. Indeed, [8], Lemma 4.2, shows that $r_p(K_1) = \infty$ if $\mathrm{char}(K_1) = p$, and when $\mathrm{char}(K_1) = 0$, our assertion follows from (4.3) and [33], Ch. II, Theorems 3 and 4. The inequality $r_p(K_1) \geq 2$ implies together with (3.3) and [16], Exercise 4.3 and Theorem 5.15 (a), the existence of $\Delta_p \in d(K_2)$ and a cyclic extension L_p/K, such that Δ_p/K is NSR and L_p/K is inertial relative to w_2, $\Delta_p \otimes_K L_p \in d(L_p)$ and $\deg(\Delta_p) = [L_p \colon K] = p$. This means that K_2 is not p-quasilocal. Assuming finally that $m \geq 3$, summing-up the obtained results, and using [6], Proposition 2.1 and Theorem 3.1, one concludes that $\mathrm{Brd}_p(K_j) \geq 2$, $j = 3, \dots, m$, which completes our proof. □

Proposition 6.1 describes the sequence $\mathrm{Brd}_{p'}(K_m)$, $p' \in \mathbb{P}$, $p' \neq \mathrm{char}(K_0)$. In addition, Proposition 6.1 (b), statements (2.3) (a) and the concluding result of this paper prove (2.1) in the special case where (K, v) is an HDV-field, such that $\widehat{K}$ is an n-dimensional local field of characteristic $p > 0$.

Proposition 6.2. *In the setting of Proposition 6.1, suppose that $m \geq 3$, $\mathrm{char}(K_m) = 0$ and $\mathrm{char}(K_0) = p$. Then $m-1 \leq \mathrm{abrd}_p(K_m) \leq m$. Moreover, $\mathrm{Brd}_p(K_m) \geq m-1$ unless $m \geq 4$, $\mathrm{char}(K_1) = 0$ and $r_p(K_1) < m-1$, where K_1 is the last but one residue field of K_m.*

Proof. It is well-known that finite extensions of K_m are m-dimensional local fields, so the equality $\mathrm{abrd}_p(K_m) \leq m$ reduces to a consequence of [7], Lemma 4.1, and the Corollary to [19], Theorem 2. To prove the other inequalities stated in Proposition 6.2, we consider the i-th residue field K_{m-i} of K_m, where $i \geq 0$ is the maximal integer for which $\mathrm{char}(K_{m-i}) = 0$. Clearly, if $i > 0$, then K_m has a $\mathbb{Z}^i$-valued Henselian valuation v_i with a residue field K_{m-i}. When $i = m-1$, Theorem 4.1 of [10], applied to (K_m, v_i), gives a formula for $\mathrm{Brd}_p(K_m)$, which indicates that $\mathrm{Brd}_p(K_m) \leq m-1$ and equality holds if and only if $r_p(K_1) \geq m-1$. This, combined with (4.3) and [33], Ch. II, Theorems 3 and 4, proves that $\mathrm{abrd}_p(K_m) = m-1$. It remains to be seen that $\mathrm{Brd}_p(K_m) \geq m-1$, provided that $i < m-1$. Then $K_{m-i'}$, $i' = i, i+1$, is an $(m-i')$-dimensional local field with last residue field K_0; in particular, $K_{m-i'}$ is complete with respect to a discrete valuation $\omega_{m-i'}$ whose residue field is $K_{m-i'-1}$. In view of [8], Lemma 4.2, and formula (2.2)(c), this means that $r_p(K_{m-i-1}) = \infty$, and in the case where $i < m-2$, $\mathrm{Brd}_p(K_{m-i-1}) = m-i-2$. More precisely, there exist $D_0 \in d(K_{m-i-1})$, defined as in (3.5) when $i < m-2$, and totally ramified Galois extensions M'_n/K_{m-i-1}, $n \in \mathbb{N}$, relative to ω_{m-i-1}, such that $\deg(D_0) = p^{m-i-2}$,

$[D_0] \in {}_p\mathrm{Br}(K_{m-i-1})$, $e(D_0/K) = p^{m-i-2}$, $\widehat{D}_0$ is a field with $\widehat{D}_0^p \subseteq \widehat{K}$, and for each index n, $D_0 \otimes_{K_{m-i-1}} M'_n \in d(M'_n)$ and $\mathcal{G}(M'_n/K_{m-i-1})$ is elementary abelian of order p^n. Let D and M_n be inertial lifts over K_{m-i} (relative to ω_{m-i}) of D_0 and M'_n, respectively. Then M_n/K_{m-i} are inertial Galois extensions, $\mathcal{G}(M_n/K_{m-i}) \cong \mathcal{G}(M'_n/K_{m-i-1})$ and $D \otimes_{K_{m-i}} M_n$ lies in $d(M_n)$, for every $n \in \mathbb{N}$. This enables one to deduce (in the spirit of the proof of [8], Proposition 6.3) from [16], Exercise 4.3 (or [7], (3.6) (a)), and [24], Theorem 1, that there exists $T \in d(K_{m-i})$ with $\deg(T) = p$, T/K_{m-i} NSR relative to ω_{m-i}, and $\Sigma \in d(K_{m-i})$, where $\Sigma = D \otimes_{K_{m-i}} T$. Clearly, $\exp(\Sigma) = p$ and $\deg(\Sigma) = p^{m-i-1}$, so $\mathrm{Brd}_p(K_{m-i}) \geq m - i - 1$, proving Proposition 6.2 in case $i = 0$. Let finally $i > 0$. Considering inertial lifts over K_m relative to v_i of Σ and any $L_i \in I(M_{i+1}/K_{m-i})$ with $\Sigma \otimes_{K_{m-i}} L_i \in d(L_i)$ and $[L_i \colon K_{m-i}] = p^i$, one obtains similarly that $\mathrm{Brd}_p(K_m) \geq m - 1$. □

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