

PERRON TRANSFORMS AND HIRONAKA'S GAME

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ABSTRACT. In this paper we present a matricial result that generalizes Hironaka's game and Perron transforms simultaneously. We also show how one can deduce the various forms in which the algorithm of Perron appears in proofs of local uniformization from our main result.

1. INTRODUCTION

The algorithm of Perron appears as an important tool in various proofs of local uniformization for valuations centered on algebraic varieties. For instance, in [9], Zariski applies this algorithm in the proof of the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. Then, in [10], he uses this theorem to prove resolution of singularities for algebraic surfaces (in characteristic 0). The local uniformization problem over base fields of any characteristic is still open. In [7], Knaf and Kuhlmann use a similar algorithm in the proof that Abhyankar places admit local uniformization in any characteristic. Also, in [2], Cutkosky and Mourtada use a version of Perron transforms in the proof that reduction of the multiplicity of a characteristic $p > 0$ hypersurface singularity along a valuation is possible if there is a finite linear projection which is defectless.

The Hironaka's game was proposed by Hironaka in [5] and [6]. This game encodes the combinatorial part of the resolution of a given singularity. Different winning strategies for this game allow different resolutions for that singularity. The existence of a winning strategy for Hironaka's game was first proved by Spivakovsky in [8]. An alternative solution was presented in [11]. In [4], Hauser presents a detailed relation between Hironaka's game and its applications on resolution of singularities.

The main goal of this paper is to explicitly relate Perron transforms and Hironaka's game. Our main result (Theorem 2.1), which is given in terms of matrices with non-negative integer entries, implies the existence of a winning strategy for the Hironaka's game and also the existence of Perron transforms with some required properties.

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This paper is divided as follows. In Section 2, we present and prove our main theorem. In Section 3, we present and prove Lemma 4.2 of [7] (Theorem 3.3 below). Knaf and Kuhlmann use this result as an important step to prove that every Abhyankar valuation admits local uniformization. A proof of Theorem 3.3 can be found in [3], but we show here that it follows easily from Theorem 2.1. In Section 4, we present and prove Lemma 4.1 of [2] (Theorem 4.1 below). In [2], the authors refer to a proof of it in [1]. That proof is based on the original algorithm of Zariski to prove local uniformization. Again, we show that Theorem 4.1 follows from Theorem 2.1. In section 5, we present the Hironaka's game (also known as Hironaka's polyhedra game), and deduce from Theorem 2.1 that it admits a winning strategy (Theorem 5.2).

2. MAIN THEOREM

Let J be a subset of $\{1, \dots, n\}$ and $j \in J$. We define the $n \times n$ matrix

$$A_{J,j} = (a_{rs})_{1 \leq r,s \leq n}$$

by

$$a_{rs} = \begin{cases} 1 & \text{if } r = s \text{ or if } r = j \text{ and } s \in J \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $\det A_{J,j} = 1$ and if we think of $A_{J,j}$ as a mapping from \mathbb{N}^n to \mathbb{N}^n we have

$$A_{J,j}(\alpha_1, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_{j-1}, \sum_{i \in J} \alpha_i, \alpha_{j+1}, \dots, \alpha_n).$$

The main result of this paper is the following:

Theorem 2.1. *Let $\alpha, \beta \in \mathbb{N}^n$. Then there exist $l \in \mathbb{N}$, subsets $J_1, \dots, J_l \subseteq \{1, \dots, n\}$ and j_1, \dots, j_l such that for every k , $1 \leq k \leq l$, J_k is chosen in function of the set*

$$\{\alpha, \beta, J_1, \dots, J_{k-1}, j_1, \dots, j_{k-1}\},$$

and j_k is randomly assigned in J_k such that

$$A\alpha \leq A\beta \text{ or } A\beta \leq A\alpha$$

componentwise, where

$$A = A_{J_l, j_l} \dots A_{J_1, j_1}.$$

Given $\alpha = (a_1, \dots, a_n), \beta = (b_1, \dots, b_n) \in \mathbb{N}^n$, we define $\tau(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ in the following way: set

$$\gamma_{\alpha\beta} = (c_1, \dots, c_n) \text{ where } c_i := \min\{a_i, b_i\},$$

and denote $\bar{\alpha} := \alpha - \gamma_{\alpha\beta}$ and $\bar{\beta} := \beta - \gamma_{\alpha\beta}$. Then

$$\tau(\alpha, \beta) = (\min\{|\bar{\alpha}|, |\bar{\beta}|\}, \max\{|\bar{\alpha}|, |\bar{\beta}|\}) \in \mathbb{N} \times \mathbb{N},$$

where $|\cdot|$ denotes the sum norm. Observe that $\alpha \leq \beta$ or $\beta \leq \alpha$ componentwise if, and only if, the first coordinate of $\tau(\alpha, \beta)$ is 0.

Proposition 2.2. *Let $\alpha, \beta \in \mathbb{N}^n$ such that $0 < |\bar{\alpha}|$ and $0 < |\bar{\beta}|$, where $\bar{\alpha} = \alpha - \gamma_{\alpha\beta}$ and $\bar{\beta} = \beta - \gamma_{\alpha\beta}$. Then there exists $J \subseteq \{1, \dots, n\}$ such that, for every $j \in J$, we have*

$$\tau(A_{J,j}\alpha, A_{J,j}\beta) <_{lex} \tau(\alpha, \beta),$$

where $<_{lex}$ denotes the lexicographic order.

We proof now Theorem 2.1 using Proposition 2.2, and we will prove Proposition 2.2 in the sequence.

Proof of Theorem 2.1 assuming Proposition 2.2. We set

$$\alpha^{(0)} := \alpha, \beta^{(0)} := \beta$$

and for $k \geq 1$, if $\alpha^{(k-1)}, \beta^{(k-1)}, J_k$ and j_k have been defined, we set

$$\alpha^{(k)} := A_{J_k, j_k} \alpha^{(k-1)}, \beta^{(k)} := A_{J_k, j_k} \beta^{(k-1)}.$$

We have to show that for some $l \in \mathbb{N}$, the first coordinate of $\tau(\alpha^{(l)}, \beta^{(l)})$ is 0, where we choose J_k , and j_k is randomly assigned in J_k for all $k \leq l$.

If the first coordinate of $\tau(\alpha, \beta)$ is 0, then nothing needs to be done. Suppose that the first coordinate of $\tau(\alpha, \beta)$ is different than 0. By Proposition 2.2, there exists $J_1 \subseteq \{1, \dots, n\}$ such that for any $j_1 \in J_1$ we have

$$\tau(\alpha^{(1)}, \beta^{(1)}) <_{lex} \tau(\alpha, \beta).$$

If $\tau(\alpha^{(1)}, \beta^{(1)})$ has first coordinate 0, it is done. If not, we apply proposition 1.2 again. Iterating this process, we produce a strictly descending sequence

$$\tau(\alpha, \beta) >_{lex} \tau(\alpha^{(1)}, \beta^{(1)}) >_{lex} \tau(\alpha^{(2)}, \beta^{(2)}) >_{lex} \dots$$

Since $\mathbb{N} \times \mathbb{N}$ is well ordered with respect to the lexicographic order, a strictly descending sequence must be finite. Then there is $l \in \mathbb{N}$ such that $\tau(\alpha^{(l)}, \beta^{(l)})$ have the first coordinate equals to 0. \square

Now we will prove Proposition 2.2.

Proof of Proposition 2.2. Let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$, and assume, without loss of generality, that $|\bar{\alpha}| \leq |\bar{\beta}|$. Since $|\bar{\alpha}| > 0$ and $|\bar{\beta}| > 0$, there exists $i, i' \in \{1, \dots, n\}$ such that $a_i > b_i$ and $a_{i'} < b_{i'}$. Hence, renumbering the indexes, there exists $r \in \{1, \dots, n-1\}$ such that $a_i > b_i$ for $i \leq r$ and $a_i \leq b_i$ for $i > r$. Then we have

$$\bar{\alpha} = (\bar{a}_1, \dots, \bar{a}_r, 0, \dots, 0) \text{ and } \bar{\beta} = (0, \dots, 0, \bar{b}_{r+1}, \dots, \bar{b}_n),$$

with $\bar{a}_i = a_i - b_i$, $1 \leq i \leq r$ and $\bar{b}_j = b_j - a_j$, $r < j \leq n$. We also assume that $\bar{b}_{r+1}, \dots, \bar{b}_n$ are in descending order. The set J and $j \in J$ will be chosen after this permutation, and then we can return to the original configuration with the inverse permutation.

Since $0 < |\bar{\alpha}| \leq |\bar{\beta}|$, there exists $k \in \{r+1, \dots, n\}$ such that

$$(1) \quad |\bar{\alpha}| \leq \sum_{i=1}^k \bar{b}_i \text{ and } |\bar{\alpha}| > \sum_{i=1}^{k-1} \bar{b}_i.$$

Take $J = \{1, \dots, k\}$. For any fixed $j \in J$ we set

$$\alpha' = A_{J,j}\alpha \text{ and } \beta' = A_{J,j}\beta.$$

Then

$$\alpha' = (a_1, \dots, a'_j = \sum_{i=1}^k a_i, \dots, a_n) \text{ and } \beta' = (b_1, \dots, b'_j = \sum_{i=1}^k b_i, \dots, b_n).$$

To calculate $\tau(\alpha', \beta')$, we denote

$$\bar{\alpha}' = \alpha' - \gamma_{\alpha'\beta'} \text{ and } \bar{\beta}' = \beta' - \gamma_{\alpha'\beta'}.$$

One can show that

$$|\bar{\alpha}'| = |\bar{\alpha}| - \bar{a}_j + \bar{a}'_j \text{ and } |\bar{\beta}'| = |\bar{\beta}| - \bar{b}_j + \bar{b}'_j,$$

where \bar{a}'_j is the j th coordinate of $\bar{\alpha}'$, and \bar{b}'_j is the j th coordinate of $\bar{\beta}'$.

We claim that $\bar{a}'_j = 0$. Indeed, we know that $\bar{a}'_j = a'_j - \min\{a'_j, b'_j\} = \max\{a'_j - b'_j, 0\}$, but

$$\begin{aligned} a'_j - b'_j &= \sum_{i=1}^k a_i - \sum_{i=1}^k b_i = \sum_{i=1}^r (a_i - b_i) - \sum_{i=r+1}^k (b_i - a_i) \\ &= \sum_{i=1}^k \bar{a}_i - \sum_{i=1}^k \bar{b}_i = |\bar{\alpha}| - \sum_{i=1}^k \bar{b}_i \leq 0, \end{aligned}$$

because $|\bar{\alpha}| \leq \sum_{i=1}^k \bar{b}_i$.

We will analyze the cases $j \leq r$ and $j > r$ separately.

If $j \leq r$, then $\bar{a}_j > 0$, and we have

$$|\bar{\alpha}'| = |\bar{\alpha}| - \bar{a}_j < |\bar{\alpha}|.$$

Hence

$$\tau(\alpha', \beta') = (\min\{|\bar{\alpha}'|, |\bar{\beta}'|\}, \max\{|\bar{\alpha}'|, |\bar{\beta}'|\}) \leq_{lex} (|\alpha'|, |\beta'|) <_{lex} (|\alpha|, |\beta|) = \tau(\alpha, \beta)$$

and the result follows.

If $j > r$, we have $|\bar{\alpha}'| = |\bar{\alpha}|$, since $\bar{a}'_j = \bar{a}_j = 0$. However, in this case, we claim that $\bar{b}'_j = \max\{b'_j - a'_j, 0\} < \bar{b}_j$. Indeed, since $k \geq j > r$ and the \bar{b}'_i 's are decreasing for $i > r$, the inequalities (1) guarantee that $\bar{b}_j > 0$. For the inequality $b'_j - a'_j < \bar{b}_j$, we will use that $\bar{b}_j \geq \bar{b}_k$, since the \bar{b}'_i 's are decreasing for $i > r$ and $r < j \leq k$. We have

$$b'_j - a'_j = \sum_{i=1}^k \bar{b}_i - |\bar{\alpha}| \leq \bar{b}_j + \sum_{i=1}^{k-1} \bar{b}_i - |\bar{\alpha}|.$$

Since $|\bar{\alpha}| > \sum_{i=1}^{k-1} \bar{b}_i$, we have $\sum_{i=1}^{k-1} \bar{b}_i - |\bar{\alpha}| < 0$, and then

$$b'_j - a'_j \leq \bar{b}_j + \sum_{i=1}^{k-1} \bar{b}_i - |\bar{\alpha}| < \bar{b}_j.$$

Finally,

$$|\bar{\beta}'| = |\bar{\beta}| - \bar{b}_j + \bar{b}'_j < |\bar{\beta}|,$$

and therefore

$$\tau(\alpha', \beta') = (\min\{|\bar{\alpha}'|, |\bar{\beta}'|\}, \max\{|\bar{\alpha}'|, |\bar{\beta}'|\}) \leq_{lex} (|\alpha'|, |\beta'|) <_{lex} (|\alpha|, |\beta|) = \tau(\alpha, \beta).$$

□

3. KUHLMANN AND KNAF'S PERRON TRANSFORM

Let Γ be a finitely generated ordered abelian group and $\mathcal{B} = \{\gamma_1, \dots, \gamma_n\}$ a basis of Γ (i.e., $\Gamma = \gamma_1\mathbb{Z} \oplus \dots \oplus \gamma_n\mathbb{Z}$) formed by positive elements. Such basis exists because every ordered abelian group is free; see [3].

Definition 3.1. A *simple Perron transform* on \mathcal{B} is a new basis $\mathcal{B}_1 = \{\gamma_1^{(1)}, \dots, \gamma_n^{(1)}\}$ of Γ , obtained in the following way: let $J \subseteq \{1, \dots, n\}$ and $j \in J$ such that $\gamma_j \leq \gamma_i$ for all $i \in J$. Then

$$\gamma_i^{(1)} = \begin{cases} \gamma_i - \gamma_j & \text{if } i \in J \setminus \{j\} \\ \gamma_i & \text{otherwise} \end{cases}.$$

Observe that \mathcal{B}_1 is indeed a basis of Γ and is formed by positive elements, since $\gamma_i > \gamma_j$ for all $i \in J \setminus \{j\}$. We define a *Perron transform* on \mathcal{B} as a basis \mathcal{B}' , obtained by perform finitely many successive simple Perron transforms starting from \mathcal{B} . We denote, whenever necessary,

$$(2) \quad \mathcal{B}' = (\mathcal{B}_0 = \mathcal{B}, \mathcal{B}_1, \dots, \mathcal{B}_{l-1}, \mathcal{B}_l = \mathcal{B}'),$$

where \mathcal{B}_k is a simple Perron transform of \mathcal{B}_{k-1} , for $k = 1, \dots, l$.

Let $\alpha \in \Gamma$. If α is written on the basis \mathcal{B} by

$$[\alpha]_{\mathcal{B}} = (a_1, \dots, a_n) = a_1\gamma_1 + \dots + a_n\gamma_n,$$

then α is written on the basis \mathcal{B}_1 by

$$(3) \quad [\alpha]_{\mathcal{B}_1} = (a_1, \dots, a_{j-1}, \sum_{i \in J} a_i, a_{j+1}, \dots, a_n) = [A_{J,j}\alpha]_{\mathcal{B}}.$$

The matrix $A_{J,j}$ is the matrix of change of basis, from \mathcal{B} to \mathcal{B}_1 , which we also denoted by $A_{\mathcal{B}, \mathcal{B}_1}$. If \mathcal{B}' is the Perron transform (2), then

$$A_{\mathcal{B}, \mathcal{B}'} = A_{\mathcal{B}_{l-1}, \mathcal{B}'} \dots A_{\mathcal{B}, \mathcal{B}_1}.$$

For a subset \mathcal{D} of Γ , we denote

$$\langle \mathcal{D} \rangle_+ := \left\{ \sum_{i=1}^n m_i d_i \mid m_i \in \mathbb{N}, d_i \in \mathcal{D} \right\}.$$

We see by (3) that

$$\langle \mathcal{B} \rangle_+ \subseteq \langle \mathcal{B}_1 \rangle_+ \subseteq \dots \subseteq \langle \mathcal{B}' \rangle_+.$$

Lemma 3.2. *Let Γ be a finitely generated ordered abelian group, \mathcal{B} a basis of Γ formed by positive elements and $\alpha \in \Gamma$ a positive element. Then there exists a Perron transform \mathcal{B}' of \mathcal{B} such that $\alpha \in \langle \mathcal{B}' \rangle_+$.*

Proof. Note that $\alpha \in \langle \mathcal{B}' \rangle_+$ if, and only if, α has non-negative coordinates on the basis \mathcal{B}' .

Write

$$\alpha = \alpha_+ - \alpha_-,$$

where $[\alpha_+]_{\mathcal{B}}$ and $[\alpha_-]_{\mathcal{B}}$ have non-negative coordinates. By Theorem 2.1, there is a matrix

$$A = A_{J_l, j_l} \dots A_{J_1, j_1}$$

such that j_1, \dots, j_l is given so that A is the change-of-basis matrix of a Perron transform \mathcal{B}' on \mathcal{B} , and

$$A[\alpha_+]_{\mathcal{B}} \geq A[\alpha_-]_{\mathcal{B}} \text{ or } A[\alpha_+]_{\mathcal{B}} \leq A[\alpha_-]_{\mathcal{B}}$$

componentwise. Since \mathcal{B}' is formed by positive elements and α is positive, the equation

$$[\alpha]_{\mathcal{B}'} = [\alpha_+]_{\mathcal{B}'} - [\alpha_-]_{\mathcal{B}'}$$

ensures that

$$[\alpha_+]_{\mathcal{B}'} = A[\alpha_+]_{\mathcal{B}} \geq A[\alpha_-]_{\mathcal{B}} = [\alpha_-]_{\mathcal{B}'}$$

componentwise. Then

$$[\alpha]_{\mathcal{B}'} = [\alpha_+]_{\mathcal{B}'} - [\alpha_-]_{\mathcal{B}'}$$

has non-negative coordinates, and therefore

$$\alpha \in \langle \mathcal{B}' \rangle_+.$$

□

In [7] Knaf and Kuhlmann use the following result as an important step to prove that every Abhyankar valuation admits local uniformization.

Theorem 3.3 (Lemma 4.2 of [7]). *Let Γ be a finitely generated ordered abelian group and $\alpha_1, \dots, \alpha_l \in \Gamma$ positive elements. Then there exists a basis \mathcal{B} of Γ , formed by positive elements, such that*

$$\alpha_1, \dots, \alpha_l \in \langle \mathcal{B} \rangle_+.$$

Proof. We start with a basis \mathcal{B}_0 of Γ formed by positive elements. By Lemma 3.1, there is a Perron transform \mathcal{B}_1 of \mathcal{B}_0 such that

$$\alpha_1 \in \langle \mathcal{B}_1 \rangle_+.$$

By Lemma 3.2, there is a Perron transform \mathcal{B}_2 of \mathcal{B}_1 such that

$$\alpha_2 \in \langle \mathcal{B}_2 \rangle_+.$$

Since $\langle \mathcal{B}_1 \rangle_+ \subseteq \langle \mathcal{B}_2 \rangle_+$, we have

$$\alpha_1, \alpha_2 \in \langle \mathcal{B}_2 \rangle_+.$$

Repeating this process to include all elements $\alpha_1, \dots, \alpha_l$, take $\mathcal{B} = \mathcal{B}_l$, and we have

$$\alpha_1, \dots, \alpha_l \in \langle \mathcal{B} \rangle_+.$$

□

4. CUTKOSKY AND MOURTADA'S PERRON TRANSFORM

Let $k[x_1, \dots, x_m]$ be a polynomial ring over a field k . Let ν be a valuation in $k[x_1, \dots, x_m]$ with center (x_1, \dots, x_m) , that is, $\nu(k^\times) = 0$ and $\nu(f) > 0$ for all $f \in (x_1, \dots, x_m)$. Suppose that $\mathcal{B} = \{\nu(x_1), \dots, \nu(x_n)\}$ is a rational basis of $\Gamma_\nu \otimes \mathbb{Q}$, where Γ_ν is the value group of ν . Let x'_1, \dots, x'_m be such that

$$x_i = \begin{cases} \prod_{j=1}^n (x'_j)^{a_{ij}} & \text{if } 1 \leq i \leq n \\ x'_i & \text{if } n < i \leq m \end{cases},$$

where $a_{ij} \in \mathbb{N}$, $\det(a_{ij}) = 1$ and $0 < \nu(x'_i)$ for every $1 \leq i \leq m$. In [2], the authors call the inclusion map $k[x_1, \dots, x_m] \rightarrow k[x'_1, \dots, x'_m]$ a Perron transform of type (6).

Theorem 4.1 (Lemma 4.1 of [2]). *Let $M_1 = x_1^{d_1} \dots x_n^{d_n}$ and $M_2 = x_1^{e_1} \dots x_n^{e_n}$ two monomials, with $d_1, \dots, d_n, e_1, \dots, e_n \geq 0$ and $\nu(M_1) < \nu(M_2)$. Then there exists a Perron transform of type (6) such that M_1 divides M_2 in $k[x'_1, \dots, x'_m]$.*

Proof. Since \mathcal{B} is a rational basis of $\Gamma_\nu \otimes \mathbb{Q}$, we have that \mathcal{B} is a basis, formed by positive elements, of the ordered subgroup $\Gamma := \nu(x_1)\mathbb{Z} \oplus \dots \oplus \nu(x_n)\mathbb{Z}$ of Γ_ν . By Lemma 3.2, since $\nu(M_2) - \nu(M_1) > 0$, there is a Perron transform $\mathcal{B}' = \{\gamma'_1, \dots, \gamma'_n\}$ such that $\nu(M_2) - \nu(M_1) \in \langle \mathcal{B}' \rangle_+$.

Since $\langle \mathcal{B} \rangle_+ \subseteq \langle \mathcal{B}' \rangle_+$, we have, for all $i = 1, \dots, n$,

$$\nu(x_i) = \sum_{j=1}^n a_{ij} \gamma'_j,$$

where $a_{ij} \in \mathbb{N}$ for all $i, j = 1, \dots, n$. We have that (a_{ij}) is a matrix with non-negative entries and $\det(a_{ij}) = 1$, since it represents a change of basis of a Perron transform.

Define x'_1, \dots, x'_m by the equations

$$x_i = \begin{cases} \prod_{j=1}^n (x'_j)^{a_{ij}} & \text{if } 1 \leq i \leq n \\ x'_i & \text{if } n < i \leq m \end{cases}.$$

We have $\nu(x'_j) = \gamma'_j > 0$ for every j , $1 \leq j \leq n$. Furthermore,

$$\nu(M_2 M_1^{-1}) = \nu(M_2) - \nu(M_1) = b_1 \gamma'_1 + \dots + b_n \gamma'_n,$$

where $b_1, \dots, b_n \in \mathbb{N}$, since $\nu(M_2) - \nu(M_1) \in \langle \mathcal{B}' \rangle_+$. Then

$$M_2 M_1^{-1} = x_1^{b_1} \dots x_n^{b_n} \in k[x'_1, \dots, x'_n],$$

and therefore M_1 divides M_2 in $k[x'_1, \dots, x'_n]$. □

If \mathcal{B}' is a simple Perron transform, then the first $n \times n$ quadrant of the matrix (a_{ij}) above is the matrix $A_{\mathcal{B}, \mathcal{B}'}$, which is of the form $A_{J,j}$, for some $J \subseteq \{1, \dots, n\}$ and $j \in J$. Then the x'_i 's are defined by

$$x'_i = \begin{cases} \frac{x_i}{x_j} & \text{if } i \in J \setminus \{j\} \\ x_i & \text{otherwise} \end{cases}.$$

If \mathcal{B}' is a Perron transform, then the x_i 's are defined by iteration of the above definition to simple Perron transforms.

As a corollary of Theorem 4.1 we have the following:

Theorem 4.2. *Let $f \in k[x_1, \dots, x_m]$ be a polynomial. Then there is a Perron transform of type (6) such that f is written by*

$$f = x'_1{}^{b_1} \dots x'_n{}^{b_n} g,$$

where $b_1, \dots, b_n \in \mathbb{N}$ and $g \in k[x'_1, \dots, x'_m] \setminus (x'_1, \dots, x'_n)$.

Proof. Let $R = k[x_{n+1}, \dots, x_m]$. Write f as

$$f = g_1 M_1 + \dots + g_k M_k,$$

where $g_i \in R \setminus \{0\}$ and M_i are monomials of $R[x_1, \dots, x_n]$ with coefficients equal to 1, for all $i = 1, \dots, k$. We may assume that $\nu(M_1) < \nu(M_i)$ for all $i = 2, \dots, k$. Applying Theorem 4.1 successively, there exists a type (6) Perron transform such that M_1 divides M_i in $k[x'_1, \dots, x'_m]$ for all $i = 1, \dots, k$. Then

$$f = M_1 \left(g_1 + g_2 \frac{M_2}{M_1} + \dots + g_k \frac{M_k}{M_1} \right),$$

where

$$g_1 + g_2 \frac{M_2}{M_1} + \dots + g_k \frac{M_k}{M_1} \in k[x'_1, \dots, x'_m] \setminus (x'_1, \dots, x'_n).$$

□

5. HIRONAKA'S GAME

Let V be a finite number of points in \mathbb{N}^n with positive convex hull $N := \text{conv}(V + \mathbb{R}_{\geq 0}^n)$. Consider two “players”, \mathcal{P}_1 and \mathcal{P}_2 , competing in the following game (known as Hironaka’s polyhedra game): player \mathcal{P}_1 chooses a subset J of $\{1, \dots, n\}$ and, afterwards, player \mathcal{P}_2 chooses an element $j \in J$. After this “round”, the set V is replaced by the set V_1 obtained as follows: for each element $\alpha = (a_1, \dots, a_n) \in V$ the corresponding element $\alpha^1 = (b_1, \dots, b_n) \in V_1$ will be

$$b_j := \sum_{i \in J} a_i \text{ e } b_k := a_k \text{ se } k \neq j.$$

We define then $N_1 := \text{conv}(V_1 + \mathbb{R}_{\geq 0}^n)$. Player \mathcal{P}_1 wins the game if, after finitely many rounds, the set N_l becomes an “ortant”, i.e., a set of the form $\alpha + \mathbb{R}_{\geq 0}^n$ for some $\alpha \in \mathbb{N}^n$. The main result of [8] is the following:

Theorem 5.1. *There exists a winning strategy for player \mathcal{P}_1 .*

Observe that α^1 above is equal to $A_{J,j}\alpha$, and player \mathcal{P}_1 wins the game after the l th round if, and only if, the set V_l has minimum element which respect to the componentwise order. Then we can rephrase Theorem 5.1 in the following equivalent theorem:

Theorem 5.2. *Let $V \subset \mathbb{N}^n$ be a finite non-empty set. Then there exist $l \in \mathbb{N}$ and finite sequences $J_1, \dots, J_l \subseteq \{1, \dots, n\}$ and j_1, \dots, j_l such that J_k is chosen in function of the set*

$$V \cup \{J_1, \dots, J_{k-1}, j_1, \dots, j_{k-1}\},$$

and

$$j_k \text{ is randomly assigned in } J_k,$$

such that there exists $\alpha \in V$ for which

$$A\alpha \leq A\beta \text{ componentwise, for every } \beta \in V,$$

where

$$A = A_{J_l, j_l} \dots A_{J_1, j_1}.$$

Proof. We prove by induction on $|V|$. If $|V| = 1$ there is nothing to prove. The case $|V| = 2$ is just Theorem 2.1 and it was already proved in Section 2. Suppose, by induction, that that the Theorem is true for set V' with $|V'| = m$. Let $V = \{\alpha_1, \dots, \alpha_m, \beta\}$. Applying the induction hypothesis to $V' := \{\alpha_1, \dots, \alpha_m\}$, we may assume that $\alpha_1 \leq \alpha_k$ componentwise for all $k = 1, \dots, m$. Since a matrix of the type $A_{J,j}$ preserves componentwise inequalities, we apply the case $|V| = 2$ to $\{\alpha_1, \beta\}$ and the theorem is proved. \square

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