The Immersion of Ultrametric Spaces into Hahn Spaces

by

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Abstract

In this paper we prove that there is an immersion of every ultrametric space X into a Hahn space associated to X. It is not assumed that the set of distances of X is totally ordered.

Introduction

In Ultrametric Dynamics one of the most central theorems is the Fixed Point Theorem. In its original formulation it states that if X is an ultrametric space, if φ is a self-map of X, which is contracting and strictly contracting on orbits, then φ has a fixed point in X, provided X is spherically complete.

The dynamic situation when X is not spherically complete calls for an immersion of X into a spherically complete ultrametric space X' which therefore contains the fixed points of φ. The elements of X', not in X, should be approched arbitrarily close by the elements of X and X' ought to be described explicitly.

A very special case is offered by the ultrametric space $\mathbb{Q}$ of rational numbers, with the ultrametric associated to the p-adic valuation; $\mathbb{Q}$ is embedded into the spherically complete ultrametric space $\mathbb{Q}_p$ of p-adic numbers.

The immersion of an ultrametric space with totally ordered set of distances into a spheri-

Subject classification: 13F25

cally complete ultrametric space was proved independently by Priess & Ribenboim [4] and by Schörner [6]. In this paper we prove the theorem for ultrametric spaces having set of distances which is not assumed to be totally ordered. We also prove the similar theorem for ultrametric torsion-free abelian additive groups.

This paper consists of several parts:

(A) Preliminaries

§1. Ultrametric Spaces

(1º) Definitions and Relevant Results.

We give the definitions and results which are required in the sequel. For more details, the reader may consult the papers listed in the references.

(1.1) Let $(\Gamma, \leq)$ be an ordered set with smallest element 0. Let X be a non-empty set. A mapping $d\colon X \times X \to \Gamma$ is called an *ultramectric distance function* when the following properties are satisfied for all $x, y, z \in X$:

d1) $d(x, y) = 0$ if and only if $x = y$.

d2) $d(x, y) = d(y, x)$.

d3) If $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$ then $d(x, z) \leq \gamma$, for all $\gamma \in \Gamma$.

(X, d, Γ) is called an *ultrametric space* and $d(x, y)$ is the *ultrametric distance* between x and y.

The ultrametric space is *trivial*, if there exists $\gamma \in \Gamma$ such that for all $x, y \in X$, $x \neq y$, $d(x, y) = \gamma$.

X is said to be *solid*, if for every $\gamma \in \Gamma$ and $x \in X$ there exists $y \in X$ such that $d(x, y) = \gamma$. If X is solid then $d(X \times X) = \Gamma$.

If $(\Gamma, \leq)$ is totally ordered, (d3) becomes:

d3') $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

Let (X, d, Γ) and (X', d', Γ') be ultrametric spaces such that $X \subseteq X'$ and $\Gamma \subseteq \Gamma'$. Assume that Γ has the induced order, the same 0 as Γ' and that moreover, for all $x, y \in X$, $d(x, y) = d'(x, y)$. Then we say that (X, d, Γ) is a *subspace* of (X', d', Γ'), or also that (X', d', Γ') is an *extension* of (X, d, Γ).

(1.2) Let $\Gamma^\bullet = \Gamma \backslash \{0\}$, $\gamma \in \Gamma^\bullet$ and let $B_\gamma(x) = \{y \in X \mid d(y, x) \leq \gamma\}$.

A set $B \subseteq X$ is called a *ball* if there exists $\gamma \in \Gamma^\bullet$ and $x \in X$ such that $B = B_\gamma(x)$. In this situation x is a *center* of B and γ is a *radius* of B.

(1.3) An ultrametric space X is said to be *spherically complete* when every chain of balls of X (that is, every set of balls which is totally ordered by inclusion) has a non-empty intersection.

(1.4) An ultrametric space X is spherically complete if and only if the following property is satisfied: for every limit ordinal λ, every strictly decreasing family $(B_\iota)_{\iota<\lambda}$ of balls has a non-empty intersection.

(1.5) Let (X, d, Γ) and (X', d', Γ') be ultrametric spaces, let $\theta\colon X \to X'$ and $\underline{\theta}\colon \Gamma \to \Gamma'$. The pair $(\theta, \underline{\theta})$ is called an *expanding mapping* from X to X' when the following conditions are satisfied:

1) $\underline{\theta}$ is order-preserving and $\underline{\theta}(0) = 0'$ (the smallest element of Γ').

2) $\underline{\theta}(d(x, y)) \leq d'(\theta x, \theta y)$ for all $x, y \in X$.

The pair $(\theta, \underline{\theta})$ is called a *contracting mapping* when condition (1) above is satisfied as well as:

3) $\underline{\theta}(d(x, y)) \geq d'(\theta x, \theta y)$ for all $x, y \in X$.

If $\underline{\theta}(d(x,y)) = d'(\theta x, \theta y)$ the pair $(\theta, \underline{\theta})$ is called a *morphism* from X to X'.

If $(\theta, \underline{\theta})$ is a morphism and θ and $\underline{\theta}$ are injective, we say that $(\theta, \underline{\theta})$ is an *immersion* from X to X'.

A morphism $(\theta, \underline{\theta})$ such that both θ and $\underline{\theta}$ are bijections is called an *isometry.*

An isometry $(\theta, \underline{\theta})$ such that $(\theta^{-1}, \underline{\theta}^{-1})$ is also an isometry is called an *isomorphism.*

(1.6) Lemma. Let $(\theta, \underline{\theta})$ be an immersion from (X, d, Γ) into (X', d', Γ'). Then there exists an extension $(\widehat{X}, \hat{d}, \widehat{\Gamma})$ of (X, d, Γ) which is isomorphic to (X', d', Γ').

Proof. Let $\widehat{X}$ be the disjoint union of X and $X' \backslash \theta(X)$ and let $\widehat{\Gamma}$ be the disjoint union of Γ and $\Gamma' \backslash \underline{\theta}(\Gamma)$. Let $\widehat{\theta} \colon \widehat{X} \to X'$ be defined as follows:

$$\begin{cases} \widehat{\theta}(x) = \theta(x) \text{ if } x \in X, \\ \widehat{\theta}(\hat{x}) = \hat{x} \text{ if } \hat{x} \in \widehat{X} \backslash X = X' \backslash \theta(X). \end{cases}$$

Let $\widehat{\underline{\theta}} \colon \widehat{\Gamma} \to \Gamma'$ be defined as follows:

$$\begin{cases} \widehat{\underline{\theta}}(\gamma) = \underline{\theta}(\gamma) \quad \text{for all} \quad \gamma \in \Gamma, \\ \widehat{\underline{\theta}}(\hat{\gamma}) = \hat{\gamma} \quad \text{for all} \quad \hat{\gamma} \in \widehat{\Gamma} \backslash \Gamma = \Gamma' \backslash \underline{\theta}(\Gamma). \end{cases}$$

The mappings $\widehat{\theta}$ and $\widehat{\underline{\theta}}$ are bijections.

We define the relation $\leq$ on $\widehat{\Gamma}$ as follows:

$$\hat{\gamma}_1 \leq \hat{\gamma}_2 \quad \text{when} \quad \widehat{\underline{\theta}}(\hat{\gamma}_1) \leq \widehat{\underline{\theta}}(\hat{\gamma}_2)\,.$$

Then $(\widehat{\Gamma}, \leq)$ is an ordered set with smallest element 0.

We define $\hat{d} \colon \widehat{X} \times \widehat{X} \to \widehat{\Gamma}$ as follows: $\hat{d}(\hat{x}, \hat{y}) = \widehat{\underline{\theta}}^{-1}(d'(\widehat{\theta}(\hat{x}), \widehat{\theta}(\hat{y})))$. Then $(\widehat{X}, \hat{d}, \widehat{\Gamma})$ is an ultrametric space and it is straightforward to verify that $(\widehat{\theta}, \widehat{\underline{\theta}})$ is an isomorpism between $(\widehat{X}, \hat{d}, \widehat{\Gamma})$ and (X', d', Γ'). $\square$

(1.7) Let $(\Gamma, \leq)$ be an ordered set. A subset Δ of Γ is said to be *noetherian* when either one of the following two equivalent conditions is satisfied:

a) every strictly increasing sequence of elements of Δ is finite,

b) every non-empty subset of Δ has a maximal element.

If $\Delta_1, \Delta_2, \dots, \Delta_n$ are noetherian subsets of Γ, then $\Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_n$ is noetherian.

(1.8) Let $(\Gamma, \leq)$ be an ordered set. A subset Δ of Γ consisting of pairwise order incomparable elements is called an *antichain* of Γ. In particular, the empty set is an antichain of Γ.

(2º) Examples of Ultrametric Spaces

(1.9) Example when $(\Gamma, \leq)$ is totally ordered. Let Δ be a totally ordered abelian additive group, let ∞ be a symbol such that $\infty \notin \Delta$, and $\delta + \infty = \infty + \delta = \infty$, $\infty + \infty = \infty$, $\delta < \infty$ for all $\delta \in \Delta$. We denote by 0 the neutral element of Δ, that is $0 + \delta = \delta$ for every $\delta \in \Delta$. Let K be a commutative field, let $v \colon K \to \Delta \cup \{\infty\}$ be a valuation of K, so we have:

v1) $v(x) = \infty$ if and only if $x = 0$.

v2) $v(xy) = v(x) + v(y)$ for all $x, y \in K$.

v3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

Let $\Gamma^\bullet$ be a totally ordered abelian multiplicative group with neutral element 1, let 0 be a symbol such that $0 \notin \Gamma^\bullet$, $0\gamma = \gamma 0 = 0$, $0 \cdot 0 = 0$, $0 < \gamma$ for every $\gamma \in \Gamma^\bullet$. Let $\theta \colon \Delta \cup \{\infty\} \to \Gamma = \Gamma^\bullet \cup \{0\}$ be an order reversing bijection such that $\theta(\infty) = 0$, $\theta(\delta + \delta') = \theta(\delta) \cdot \theta(\delta')$, so $\theta(0) = 1$.

Let $d \colon K \times K \to \Gamma$ be defined by $d(x, y) = \theta(v(x - y))$.

Then (K, d, Γ) is an ultrametric space which is said to be *associated to the valued field* $(K, v, \Delta \cup \{\infty\})$.

(1.10) Another example where Γ is totally ordered. Let Γ be a totally ordered set with smallest element 0, let $\Gamma^\bullet = \Gamma \backslash \{0\}$. Let R be a non-empty set with a distinguished element 0. For each $f \colon \Gamma^\bullet \to R$ let $\operatorname{supp}(f) = \{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq 0\}$ be the *support* of f. Let $R[[\Gamma]]$ be the set of all $f \colon \Gamma^\bullet \to R$ with support which is empty or anti-well ordered. Let $d \colon R[[\Gamma]] \times R[[\Gamma]] \to \Gamma$ be defined by $d(f, f) = 0$ and if $f \neq g$, $d(f, g)$ is the largest element of the set $\{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq g(\gamma)\}$. Then $(R[[\Gamma]], d, \Gamma)$ is an ultrametric space which is solid and spherically complete.

(1.11) Examples when Γ is not totally ordered. Let I be a set with at least two elements, let $(X_i)_{i \in I}$ be a family of sets X_i, each one having at least two elements. Let $X = \prod_{i \in I} X_i$. Let

$\mathcal{P}(I)$ be the set of all subsets of I, ordered by inclusion. And let $d\colon X \times X \to \mathcal{P}(I)$ be defined by $d(f,g) = \{i \in I \mid f_i \neq g_i\}$, where $f = (f_i)_{i\in I}$ and $g = (g_i)_{i\in I}$. Then $(X, d, \mathcal{P}(I))$ is a solid and spherically complete ultrametric space. If each $X_i = \{0,1\}$ we obtain the ultrametric space $(\mathcal{P}(I), d, \mathcal{P}(I))$ with $d(A,B) = (A \cup B)\backslash(A \cap B)$ for all $A, B \subseteq I$.

(1.12) Other examples. Let X be a topological space, let Y be a discrete topological space, let $C(X,Y)$ denote the set of continuous functions from X to Y and let $\mathcal{C}\ell(X)$ denote the set of clopen (i.e. closed and open) subsets of X. The mapping $d\colon C(X,Y) \times C(X,Y) \to \mathcal{C}\ell(X)$ is defined by $d(f,g) = \{x \in X \mid f(x) \neq g(x)\}$. Then $(C(X,Y), d, \mathcal{C}\ell(X))$ is a solid ultrametric space. If $\mathcal{C}\ell(X)$ is a complete sub-Boolean algebra of $\mathcal{P}(X)$ then $(C(X,Y), d, \mathcal{C}\ell(X))$ is spherically complete (see [4]).

(B) The Skeleton of an Ultrametric Space

Let (X, d, Γ) be a non-trivial ultrametric space with surjective distance, that is $d(X \times X) = \Gamma$. We shall define the skeleton of X. It is the family $\mathcal{E} = (\mathcal{E}_v)_{v\in V}$ where V and each set $\mathcal{E}_v$ will be introduced in §3.

§2. Compatible Equivalence Relations on (X, d, Γ)

An equivalence relation α on X is identified with the set $\{(x,y) \in X \times X \mid x\,\alpha\,y\}$; this set shall also be denoted by α. The set of all equivalence relations on X is ordered by inclusion, thus $\alpha \subseteq \beta$ means that if $x, y \in X$ and $x\,\alpha\,y$ then $x\,\beta\,y$. The equality relation is the smallest equivalence relation, the largest equivalence relation is the trivial relation, that is $x(\mathrm{tr})y$ for all $x, y \in X$. The ordered set of equivalence relations is a complete lattice, the intersection of an arbitrary family of equivalence relations is an equivalence relation.

If $(\alpha_i)_{i\in I}$ is any family of equivalence relations, the supremum of the family contains, but may not be equal to $\bigcup_{i\in I} \alpha_i$. However, if $\{\alpha_i \mid i \in I\}$ is totally ordered, the $\bigcup_{i\in I} \alpha_i$ is an equivalence relation, which is therefore the supremum of $(\alpha_i)_{i\in I}$.

We denote by $[x]_\alpha$ the α-equivalence class which contains $x \in X$ and by X/α the set of all α-equivalence classes in X.

The equivalence relation α is *compatible* with the distance d when the following property is satisfied: if $x\,\alpha\,y$, and $d(x', y') \le d(x,y)$ then $x'\alpha y'$. We denote by $\equiv(X, d, \Gamma)$, or simply by $\equiv(X)$, the set of compatible equivalence relations. The equality relation and the trivial relation are compatible with the distance and $\equiv(X)$ is a complete lattice. The intersection of any family of compatible equivalence relations is again a compatible equivalence relation. If $(\alpha_i)_{i\in I}$ is a family of compatible equivalence relations such that $\{\alpha_i \mid i \in I\}$ is totally ordered then $\bigcup_{i\in I} \alpha_i$ is a compatible equivalence relation.

For every $\gamma \in \Gamma$ let $(\equiv_\gamma)$ be the relation defined by $x(\equiv_\gamma)y$ when $d(x, y) \le \gamma$. Then $(\equiv_\gamma)$ is a compatible equivalence relation. For simplicity we shall write $[x]_\gamma$ instead of $[x]_{\equiv_\gamma}$ and X/γ instead of $X/(\equiv_\gamma)$. We observe that if $\gamma \in \Gamma^\bullet$ then $[x]_\gamma = B_\gamma(x)$ (the ball with center x and radius γ).

We shall repeatedly use the following remark:

(2.1) Remark. If $\alpha \in \equiv(X)$, $\alpha \subset (\equiv_\gamma)$ and $d(x, y) = \gamma$ then x (not α)y.

Proof. We assume that $x\,\alpha\,y$. If $d(x', y') \le \gamma = d(x, y)$ then $x'\,\alpha\,y'$, hence $(\equiv_\gamma) \subseteq \alpha$. This is absurd. □

§3 Virtual Points and the Skeleton.

Let $\alpha, \beta \in \equiv(X)$. We say that β *covers* α, or α is *covered* by β when $\alpha \subset \beta$ and there does not exist $\alpha' \in \equiv(X)$ such that $\alpha \subset \alpha' \subset \beta$.

Let $V = V(X)$ be the set of pairs (α, β) such that $\alpha, \beta \in \equiv(X)$ and β covers α. The elements of V are called the *virtual points* of X, or simply the *virtuals* of* X.

(3.1) Lemma. Let $\alpha, \beta \in \equiv(X)$ be such that $\alpha \subset \beta$, let $x, y \in X$ be such that $x\,\beta\,y$, x (not α)y. Then there exists $(\alpha_0, \beta_0) \in V$ such that $\alpha \subseteq \alpha_0 \subset \beta_0 \subseteq \beta$, $x\,\beta_0\,y$ and x (not α_0)y. In particular $V \ne \varnothing$.

Proof. We consider the set S of all $\bar\alpha \in \equiv(X)$ such that $\alpha \subseteq \bar\alpha \subset \beta$, and x (not $\bar\alpha$)y. Thus $S \ne \varnothing$ because $\alpha \in S$. If T is a chain of elements of S then $\alpha' = \bigcup_{\bar\alpha\in T} \bar\alpha \in \equiv(X)$, $\alpha \subseteq \alpha' \subseteq \beta$, x (not α')y, so $\alpha' \subset \beta$, hence $\alpha' \in S$. Thus S is inductive. By Zorn's Lemma there exists a maximal element $\alpha_0 \in S$, so $\alpha \subseteq \alpha_0 \subset \beta$. Let β_0 be the intersection of all $\bar\beta \in \equiv(X)$ such that $\alpha_0 \subset \bar\beta \subseteq \beta$ and $x\,\bar\beta\,y$. So $\alpha_0 \subseteq \beta_0$, but x (not α_0)y, and $x\,\beta_0\,y$, so $\alpha_0 \subset \beta_0$.

* One often says "primes" instead of "prime numbers" even though "prime" is not a noun. By a similar grammatical abuse, we shall say "virtuals" instead of "virtual points".

We show that β_0 covers α_0. If $\beta' \in \equiv (X)$ and $\alpha_0 \subseteq \beta' \subset \beta_0$ then x (not β')y, so by the maximality of α_0, $\beta' = \alpha_0$. Thus $(\alpha_0, \beta_0) \in V$, $x\,\beta_0\,y$ and x (not α_0)y.

Taking $\alpha = (=)$, $\beta = (\mathrm{tr})$ and $x \neq y$, we deduce that $V \neq \varnothing$. □

(3.2) Lemma. Let $\gamma \in \Gamma^\bullet$ and $\alpha \in \equiv (X)$ be such that $\alpha \subset (\equiv_\gamma)$. Then there exists $\alpha_0 \in \equiv (X)$ such that $\alpha \subseteq \alpha_0$ and α_0 is covered by $(\equiv_\gamma)$.

Proof. Let $x, y \in X$ be such that $d(x, y) = \gamma$, so $x(\equiv_\gamma)y$. By (2.1), x (not α)y. By (3.1) there exists a virtual $(\alpha_0, \beta_0) \in V$ such that $\alpha \subseteq \alpha_0 \subset \beta_0 \subseteq (\equiv_\gamma)$, with $x\,\beta_0\,y$, x (not α_0)y. Then $(\equiv_\gamma) \subseteq \beta_0$, so $\beta_0 = (\equiv_\gamma)$ and α_0 is covered by $(\equiv_\gamma)$. □

If $\gamma \in \Gamma^\bullet$ let $PrV(\gamma)$ be the set of all virtuals $(\alpha, \equiv_\gamma)$. By (3.2) $PrV(\gamma) \neq \varnothing$. We shall write (α, γ) instead of $(\alpha, \equiv_\gamma)$.

If $\gamma \neq \gamma'$ then $PrV(\gamma) \cap PrV(\gamma') = \varnothing$. The elements of $PrV = \bigcup_{\gamma \in \Gamma^\bullet} PrV(\gamma)$ are called the *principal virtual points of* X, or simply the *principal virtuals* of X.

In general $PrV \subset V$.

For each $\gamma \in \Gamma^\bullet$ let $V(\gamma)$ be the set of virtuals $(\alpha, \beta) \in V$ such that there exist $x, y \in X$ satisfying $d(x, y) = \gamma$, $x\,\beta\,y$, x (not α)y. Since the distance is surjective, it is equivalent to state that if $d(x, y) = \gamma$ then $x\,\beta\,y$ and x (not α)y.

(3.3) Lemma. 1) For every $\gamma \in \Gamma^\bullet$, $PrV(\gamma) \subseteq V(\gamma)$.

2) The mapping $\gamma \mapsto V(\gamma)$ is injective.

Proof. 1) Let $(\alpha, \gamma) \in PrV(\gamma)$, let $x, y \in X$ be such that $d(x, y) = \gamma$. By (2.1) x (not α)y, so $(\alpha, \gamma) \in V(\gamma)$.

2) Let $V(\gamma) = V(\gamma')$. Since $PrV(\gamma) \neq \varnothing$, let $(\alpha, \gamma) \in PrV(\gamma) \subseteq V(\gamma) = V(\gamma')$. So there exist $x', y' \in X$ such that $d(x', y') = \gamma'$, $d(x', y') \leq \gamma$, $x'(\text{not}\alpha)y'$. So $\gamma' \leq \gamma$ and similarly $\gamma \leq \gamma'$, hence $\gamma = \gamma'$. □

If (α, β) and (α', β') are virtuals, we define $(\alpha, \beta) < (\alpha', \beta')$ when $\beta \subseteq \alpha'$. It is easy to verify that the relation $\leq$ is an order on the set V of virtuals.

Let $\mathcal{A}(V)$ be the set of antichains A of V, that is, the subsets A of V which are trivially ordered. On the set $\mathcal{A}(V)$ we consider the relation $<$ defined as follows: if $A, A' \in \mathcal{A}(V)$ then $A < A'$ when for every $(\alpha, \beta) \in A$ there exists $(\alpha', \beta') \in A'$ such that $\beta \subseteq \alpha'$, that is

$(\alpha, \beta) < (\alpha', \beta')$ in V. As easily seen, the relation $\leq$ is an order relation on $\mathcal{A}(V)$. Since each set $\{(\alpha, \beta)\}$, where $(\alpha, \beta) \in V$, is an antichain, then $V \subseteq \mathcal{A}(V)$ (up to this identification) and the order on V is extended to the order on $\mathcal{A}(V)$.

We also note that if $A, A' \in \mathcal{A}(V)$ and $A \subseteq A'$ (as subsets of V) then $A \leq A'$.

(3.4) Lemma. 1) For every $\gamma \in \Gamma^\bullet$ the sets $PrV(\gamma)$ and $V(\gamma)$ are antichains of V.

2) If $0 < \gamma < \gamma'$ then $V(\gamma) < V(\gamma')$ and $PrV(\gamma) < PrV(\gamma')$.

3) If $V(\gamma) < V(\gamma')$ then $\gamma' \nleq \gamma$. If $PrV(\gamma) < PrV(\gamma')$ then $\gamma < \gamma'$.

Proof. 1) Let (α, β), $(\alpha', \beta') \in V(\gamma)$ and assume that $(\alpha, \beta) < (\alpha', \beta')$, so $\beta \subseteq \alpha'$. Let $x, y \in X$ be such that $d(x, y) = \gamma$, then $x\, \beta\, y$, $x(\text{not } \alpha)y$, $x\, \beta'\, y$ and $x(\text{not } \alpha')y$, which is a contradiction. So $V(\gamma)$ is an antichain.

It is obvious that $PrV(\gamma)$ is an antichain.

2) Let $(\alpha, \beta) \in V(\gamma)$; we show that there exists $(\alpha', \beta') \in V(\gamma')$ such that $(\alpha, \beta) \leq (\alpha', \beta')$. This implies that $V(\gamma) \leq V(\gamma')$ and since $\gamma \neq \gamma'$ then by (3.3) $V(\gamma) < V(\gamma')$.

If $(\alpha, \beta) \in V(\gamma')$ we take $(\alpha', \beta') = (\alpha, \beta)$. Let $(\alpha, \beta) \notin V(\gamma')$. By assumption, there exist $x, y \in X$ such that $d(x, y) = \gamma$, $x\, \beta\, y$, x (not α)y.

Let $x', y' \in X$ be such that $d(x', y') = \gamma'$. We have x' (not α)y', otherwise $(\equiv_\gamma) \subset (\equiv_{\gamma'}) \subseteq \alpha$, hence $x\, \alpha\, y$, which is absurd. Since $(\alpha, \beta) \notin V(\gamma')$ then x' (not β)y'. By (3.1), applied to $(\beta,\ \text{tr})$, there exists $(\alpha', \beta') \in V$ such that $(\alpha, \beta) < (\alpha', \beta')$, $x'\, \beta'\, y'$, x' (not α')y', so $(\alpha', \beta') \in V(\gamma')$.

Let $(\alpha, \gamma) \in PrV(\gamma)$. Since $\gamma < \gamma'$ by (3.2) there exists $\alpha' \in \equiv(X)$ such that $(\equiv_\gamma) \subseteq \alpha' \subset \equiv_{\gamma'}$ and $(\alpha', \gamma') \in PrV(\gamma')$, with $(\alpha, \gamma) < (\alpha', \gamma')$.

3) Let $V(\gamma) < V(\gamma')$. If $\gamma' \leq \gamma$ then $V(\gamma') \leq V(\gamma) < V(\gamma')$ which is impossible. So $\gamma' \nleq \gamma$. It is clear that if $PrV(\gamma) < PrV(\gamma')$ then $\gamma < \gamma'$. □

Now we define the skeleton of X.

For each $v = (\alpha, \beta) \in V$ let $\mathcal{E}_v = X/\alpha$.

Let $x, y \in X$ be such that $x\, \beta\, y$ but x (not α)y. Then $[x]_\alpha \neq [y]_\alpha$, so $\mathcal{E}_{(\alpha,\beta)}$ has at least two elements.

The family $\mathcal{E}_X = (\mathcal{E}_v)_{v \in V}$ is called the *skeleton* of (X, d, Γ). We often write $\mathcal{E}$ instead of $\mathcal{E}_X$.

Special case when Γ is totally ordered

(3.5) Lemma. If Γ is totally ordered, then $\equiv(X)$ is totally ordered.

Proof. Let $\alpha, \beta \in \equiv(X)$ with $\alpha \not\subseteq \beta$. Let $x, y \in X$ be such that $x\,\alpha\,y$ but x (not $\beta)y$. Let $x', y' \in X$ be such that $x'\,\beta\,y'$, then $d(x,y) \not\leq d(x',y')$.

Hence $d(x',y') < d(x,y)$, so $x'\,\alpha\,y'$ and this proves that $\beta \subseteq \alpha$. $\square$

For every $\gamma \in \Gamma^\bullet$ we define the binary relation $(\equiv_\gamma^-)$ as follows: $x(\equiv_\gamma^-)y$ whenever $d(x,y) < \gamma$.

(3.6) Lemma. Let Γ be totally ordered. For every $\gamma \in \Gamma^\bullet$, $(\equiv_\gamma^-) \in \equiv(X)$ and $(\equiv_\gamma)$ covers $(\equiv_\gamma^-)$.

Proof. We show that $(\equiv_\gamma^-)$ is a transitive relation. If $x(\equiv_\gamma^-)y$ and $y(\equiv_\gamma^-)z$ then $d(x,y) < \gamma$ and $d(y,z) < \gamma$. Hence $d(x,z) \leq \max\{d(x,y), d(y,z)\} < \gamma$, because Γ is totally ordered; thus $x(\equiv_\gamma^-)z$. It is now obvious that $(\equiv_\gamma^-)$ is an equivalence relation which moreover is compatible with the distance. If there exists $\alpha \in \equiv(X)$ such that $\equiv_\gamma^- \subset \alpha \subset \equiv_\gamma$, let $x, y \in X$ be such that $x\,\alpha\,y$ but $d(x,y) \not< \gamma$, so $d(x,y) \geq \gamma$. But $\alpha \subset \equiv_\gamma$, so $d(x,y) \leq \gamma$, thus $d(x,y) = \gamma$. By (2.1) x (not $\alpha)y$, which is a contradiction. This shows that $\equiv_\gamma$ covers $\equiv_\gamma^-$. $\square$

We shall write (γ^-, γ) instead of $(\equiv_\gamma^-, \gamma)$ or $(\equiv_\gamma^-, \equiv_\gamma)$.

(3.7) Lemma. Let Γ be totally ordered. Then the sets PrV, V and $\widetilde{\Gamma} = \{(\gamma^-, \gamma) \mid \gamma \in \Gamma^\bullet\}$ coincide.

Proof. We have $\widetilde{\Gamma} \subseteq PrV \subseteq V$ and we show that if $(\alpha, \beta) \in V$ then there exists $\gamma \in \Gamma^\bullet$ such that $(\alpha, \beta) = (\gamma^-, \gamma)$.

Let $x, y \in X$ be such that $x\,\beta\,y$, x (not $\alpha)y$ and let $d(x,y) = \gamma$. It follows that $(\equiv_\gamma) \subseteq \beta$. We show that $\alpha \subseteq (\equiv_\gamma^-)$. Indeed, let $t, u \in X$ be such that $t\,\alpha\,u$, if t (not $\equiv_\gamma^-)u$ then $d(t,u) \geq \gamma = d(x,y)$ hence $x\,\alpha\,y$, which is a contradiction. From $\alpha \subseteq (\equiv_\gamma^-) \subset (\equiv_\gamma) \subseteq \beta$ and since α is covered by β, it follows that $(\alpha, \beta) = (\gamma^-, \gamma)$. $\square$

(C) The Immersion into a Hahn Product

Our purpose is to define a Hahn product H associated to the ultrametric space X and to prove that there is an immersion from X into H.

§4. Construction of a Hahn Product

We shall construct a Hahn product which depends on the choice of a set $\sum$ as described below. For each equivalence relation $\beta \in \equiv (X)$, let S_β be a set of representatives of X/β. We write S_γ instead of $S_{\equiv_\gamma}$.

The family $\sum = (S_\beta)_{\beta \in \equiv(X)}$ is said to be *coherent* when $S_\beta \subseteq S_{\beta'}$ for $\beta' \subseteq \beta$.

(4.1) Lemma. For each well-ordering of the set X there exists a coherent family $\sum$.

Proof. (See [4]):

Let $\{x_i \mid i < \rho\}$ be a well-ordering on X. For each $\beta \in \equiv (X)$ we define S_β. If $C \in X/\beta$ let $\lambda_C = \min\{\lambda < \rho \mid x_\lambda \in C\}$ and let $S_\beta = \{x_{\lambda_C} \mid C \in X/\beta\}$, so S_β is a set of representatives of X/β. Let $\beta, \beta' \in \equiv (X)$ and $\beta' \subseteq \beta$. If $x_{\lambda_C} \in S_\beta$, where $C \in X/\beta$, then there exists an equivalence class $C' \subseteq C$. This implies that $\lambda_C = \lambda_{C'}$, so $x_{\lambda_C} = x_{\lambda_{C'}} \in S_{\beta'}$. Thus $S_\beta \subseteq S_{\beta'}$ proving that $\sum$ is coherent. $\square$

Henceforth we choose a coherent family $\sum$.

For every $\beta \in \equiv (X)$ and $x \in X$ let $s_\beta(x)$ be the unique element in S_β such that $x \,\beta\, s_\beta(x)$.

If $x \in S_\beta$ then $s_\beta(x) = x$. If $x, x' \in X$ and $x \,\beta\, x'$ then $s_\beta(x) = s_\beta(x')$. So for each $D \in X/\beta$ we define $s_\beta(D) = s_\beta(x)$ for any $x \in D$. For each $v = (\alpha, \beta) \in V$ let $\Omega_v = \{[s_\beta(D)]_\alpha \mid D \in X/\beta\} \subseteq \mathcal{E}_v$. For each $v \in V$ let 0_v be a new symbol such that $0_v \notin \mathcal{E}_v \backslash \Omega_v$ and let $E_v = (\mathcal{E}_v \backslash \Omega_v) \cup \{0_v\}$. Let $0 = (0_v)_{v \in V}$.

Now we define the Hahn product H of the family $(E_v)_{v \in V}$ with respect to 0. For each $h \in \prod_{v \in V} E_v$ let the *support* of h be defined by $\operatorname{supp}(h) = \{v \in V \mid h_v \neq 0_v\}$.

Let H be the set of all $h \in \prod_{v \in V} E_v$ such that $\operatorname{supp}(h)$ is a noetherian subset of V.

If $h, h' \in H$ and $h \neq h'$ the set $D(h, h') = \{v \in V \mid h_v \neq h'_v\}$ is non-empty and noetherian. The set of maximal elements of $D(h, h')$ is an antichain of V.

Let $d_H \colon H \times H \to \mathcal{A}(V)$ be defined as follows for any $h, h' \in H$: $d_H(h, h) = \varnothing$ (the empty antichain), if $h \neq h'$ let $d_H(h, h') = \operatorname{Max} D(h, h') \in \mathcal{A}(V)$. Often, we write d instead of d_H.

We verify that d is an ultrametric distance. Clearly, it suffices to verify the property (d3). Let $d(h, h') \leq A$ and $d(h', h'') \leq A$ (where $A \in \mathcal{A}(V)$). We show that $d(h, h'') \leq A$ and it suffices to consider the case where h, h' and h'' are distinct. Let $v \in d(h, h'')$, so either $h_v \neq h'_v$ or $h'_v \neq h''_v$, say $h_v \neq h'_v$; hence there exists $\bar{v} \in d(h, h')$ such that $v \leq \bar{v}$; again there exists $\tilde{v} \in A$ such that $\bar{v} \leq \tilde{v}$. This shows that $d(h, h'') \leq A$.

The ultrametric space $(H, d_H, \mathcal{A}(V))$ is called the *Hahn product of* $(E_v)_{v \in V}$ *with respect to* 0. It is also called the *Hahn space associated to*X (with respect to $\sum$). We shall often write

H instead of $(H, d_H, \mathcal{A}(V))$.

(4.2) Proposition. $(H, d, \mathcal{A}(V))$ is a solid and spherically complete ultrametric space. In particular, the distance is surjective.

Proof. This proposition has been proved in [4]. $\square$

§5. The Immersion Theorem

We shall define mappings $\lambda\colon X \to H$ and $\underline{\lambda}\colon \Gamma \to \mathcal{A}(V)$. For each $v \in V$ and $x \in X$ let

$$\lambda(x)_v = \begin{cases} [x]_\alpha \text{ if } x \notin [s_\beta(x)]_\alpha \\ 0_v \text{ if } x \in [s_\beta(x)]_\alpha \end{cases}$$

and $\lambda(x) = (\lambda(x)_v)_{v\in V} \in \prod\limits_{v\in V} E_v$. We define $\underline{\lambda}\colon \Gamma \to \mathcal{A}(V)$ as follows: $\underline{\lambda}(0) = \varnothing$ and if $\gamma \in \Gamma^\bullet$ then $\underline{\lambda}(\gamma) = V(\gamma)$, so $\underline{\lambda}(\gamma)$ is an antichain of V (by (3.4)).

We are ready to prove the Immersion Theorem:

(5.1) Theorem.

1) $\lambda(X) \subseteq H$.

2) $(\lambda, \underline{\lambda})$ is an immersion from X into H.

Proof. 1) We show that for every $x \in X$, $\lambda(x)$ has noetherian support, so $\lambda(x) \in H$. We assume the contrary, let $v_0 < v_1 < v_2 < \dots$ with $v_i = (\alpha_i, \beta_i) \in V$ and assume that each v_i is in the support of $\lambda(x)$. Let $\delta = \bigcup\limits_{i=0}^{\infty} \beta_i$ so $\delta \in \equiv(X)$; let $t \in S_\delta$ be such that $x\,\delta\,t$. Then there exists $i \geq 0$ such that $x\,\beta_i\,t$. Hence $x\,\alpha_{i+1}\,t$ because $\beta_i \subseteq \alpha_{i+1}$. From $S_\delta \subseteq S_{\beta_i+1} \subseteq S_{\beta_i}$ then $t \in S_{\beta_{i+1}}$, so $t = s_{\beta_i+1}(t)$. Therefore $[x]_{\alpha_{i+1}} = [t]_{\alpha_{i+1}} = \big[s_{\beta_{i+1}}(t)\big]_{\alpha_{i+1}} = \big[s_{\beta_{i+1}}(x)\big]_{\alpha_{i+1}}$. This means that $\lambda(x)_{v_{i+1}} = 0_{v_{i+1}}$ which is a contradiction.

2) First we show that $\underline{\lambda}(d(x,y)) \leq d(\lambda(x), \lambda(y))$ for all $x, y \in X$. It is trivial if $x = y$, so we assume that $d(x,y) = \gamma > 0$. Let $v = (\alpha, \beta) \in \underline{\lambda}(\gamma) = V(\gamma)$, so x (not α)y and $x\,\beta\,y$. Hence $[x]_\alpha \neq [y]_\alpha$ and $[x]_\beta = [y]_\beta$, hence $s_\beta(x) = s_\beta(y)$. This implies that $\lambda(x)_v \neq \lambda(y)_v$. Hence there exists $v' \in d(\lambda(x), \lambda(y))$ such that $v \leq v'$; this implies the required relation for the antichains, that is, $\underline{\lambda}(d(x,y)) \leq d(\lambda(x), \lambda(y))$.

Next we show that $d(\lambda(x), \lambda(y)) \leq \underline{\lambda}(d(x,y))$ for all $x, y \in X$. It is trivial if $\lambda(x) = \lambda(y)$. Now let $\lambda(x) \neq \lambda(y)$ and let $v = (\alpha, \beta) \in V$ be maximal such that $\lambda(x)_v \neq \lambda(y)_v$.

i) If x (not $\beta)y$, since $x(\mathrm{tr})y$ by (3.1) there exists $v' = (\alpha', \beta') \in V$ such that $\beta \subseteq \alpha'$, x (not $\alpha')y$, $x\,\beta'\,y$. So $[x]_{\beta'} = [y]_{\beta'}$, $[x]_{\alpha'} \neq [y]_{\alpha'}$, therefore $\lambda(x)_{v'} \neq \lambda(y)_{v'}$, with $v < v'$; this is a contradiction.

ii) Hence $x\,\beta\,y$, and from $\lambda(x)_v \neq \lambda(y)_v$ then $[x]_\alpha \neq [y]_\alpha$, so x (not $\alpha)y$, hence $d(x,y) = \gamma \neq 0$ and $v = (\alpha, \beta) \in V(\gamma) = \underline{\lambda}(\gamma)$. This proves that $d(\lambda(x), \lambda(y)) \subseteq \underline{\lambda}(d(x,y))$ so $d(\lambda(x), \lambda(y)) \leq \underline{\lambda}(d(x,y))$. We conclude that $\underline{\lambda}(d(x,y)) = d(\lambda(x), \lambda(y))$ for all $x, y \in X$.

It was seen in (3.3) and (3.4) that $\underline{\lambda}$ is an injective order preserving mapping. From the relation $\underline{\lambda}(d(x,y)) = d(\lambda(x), \lambda(y))$ it follows that λ is also injective.

This shows that $(\lambda, \underline{\lambda})$ is an immersion form X into H. □

As a corollary, we obtain:

(5.2) Corollary. Every ultrametric space (X, d, Γ) is a subspace of a solid and spherically complete ultrametric space $(\widehat{X}, \hat{d}, \widehat{\Gamma})$. Moreover, if Γ is totally ordered, then $\widehat{\Gamma} = \Gamma$.

Proof. By (5.1) $(\lambda, \underline{\lambda})$ is an immersion from (X, d, Γ) into $(H, d_H, \mathcal{A}(V))$. By (1.6) there exists $(\widehat{X}, \hat{d}, \widehat{\Gamma})$, isomorphic to $(H, d_H, \mathcal{A}(V))$ which is an extension of (X, d, Γ). By (4.2) $(H, d_H, \mathcal{A}(V))$ is solid and spherically complete, hence $(\widehat{X}, \hat{d}, \widehat{\Gamma})$ has the same properties. If Γ is totally ordered, then by (3.7) V is totally ordered and order isomorphic to $\Gamma^\bullet$. Hence $\mathcal{A}(V)$ is order isomoprhic to Γ, therefore $\widehat{\Gamma} = \Gamma$. □

(D) The Special Case of Ultrametric Groups and Vector Spaces

The results presented above for arbitrary ultrametric spaces may be phrased more specifically for ultrametric groups and vector spaces.

§6. Ultrametric Groups and Vector Spaces

We shall restrict our attention to abelian additive groups G. The ultrametric space (G, d, Γ) is said to be an *ultrametric group* when the following condition is satisfied:

$$d(z + x, z + y) = d(x, y) \text{ for all } x, y, z \in G.$$

As easily seen an ultrametric group with surjective distance is solid. There are numerous examples of ultrametric groups, which may be found in §1.

Let K be a commutative field, let (G, d, Γ) be an ultrametric abelian additive group and assume that G is a K-vector space and $d(qx, qy) = d(x, y)$ for all $x, y \in G$ and $q \in K \backslash \{0\}$. This special type of ultrametric K-vector space is the only one which will be considered here.

If G is a torsion-free abelian additive group, then G is a subgroup of $G^* = \mathbb{Q} \underset{\mathbb{Z}}{\otimes} G$ which is a $\mathbb{Q}$-vector space. If (G, d, Γ) is an ultrametric group and $d(nx, ny) = d(x, y)$ for all $x, y \in G$ and integers $n > 0$, then the mapping $d^* \colon G^* \times G^* \to \Gamma$ defined by $d^*\left(\frac{1}{m}\, x, \frac{1}{n}\, y\right) = d(nx, my)$ is well-defined and it is an ultrametric distance function extending d. Moreover (G^*, d^*, Γ) is an ultrametric $\mathbb{Q}$-vector space. We leave to the reader the verification of these assertions.

§7. The Skeleton of an Ultrametric Group

We describe the virtuals of (G, d, Γ) in terms of convex subgroups of G.

A subgroup C of G is said to be *convex* when the following property is satisfied. If $g \in C$, $h \in G$ and $d(h, 0) \leq d(g, 0)$ then $h \in C$.

The set $C = C(G)$ of convex subgroups of G contains $\{0\}$ and G. The intersection of any family of convex subgroups of G is a convex subgroup. In particular, for every $g \in G$ there exists the smallest convex subgroup of G containing g. It is denoted by $C(g)$ and called the *principal convex subgroup generated by g*.

We say that the convex subgroup D *covers* the convex subgroup C, or that C is *covered* by D when $C \subset D$ and there does not exist any convex subgroup C' such that $C \subset C' \subset D$.

(7.1) Lemma. 1) If $\beta \in \equiv(G)$ then $[0]_\beta$ is a convex subgroup of G. The mapping $\beta \mapsto [0]_\beta$ is an order isomorphism from $\equiv(G)$ onto $C(G)$.

2) If $g \in G$ and $d(g, 0) = \gamma$ then $\equiv_\gamma$ corresponds to $[0]_{\equiv_\gamma} = C(g)$.

3) If $\alpha, \beta \in \equiv(G)$ then β covers α if and only if $[0]_\beta$ covers $[0]_\alpha$.

4) If $C, D \in C(G)$, $C \subset D$ and $g \in D \backslash C$, there exist $C_0, D_0 \in C(G)$ such that $C \subseteq C_0 \subset D_0 \subseteq D$, D_0 covers C_0, $g \in D_0 \backslash C_0$.

5) If $C \in C(G)$, $g \in G$ and $C \subset C(g)$ there exists $C_0 \in C(G)$ such that $C \subseteq C_0 \subset C(g)$ and $C(g)$ covers C_0.

Proof. The proof is left to the reader, who may also consult [5]. □

The virtuals $(\alpha, \beta) \in V(\gamma)$ where $\gamma \in \Gamma^\bullet$ correspond to the pair (C, D) of convex subgroups of G such that D covers C and there exists $g \in D \backslash C$ such that $d(g, 0) = \gamma$.

(7.2) Lemma. Let $\beta \in \equiv (G)$. Then $[x]_\beta = x + [0]_\beta$, $[x]_\beta + [y]_\beta = [x+y]_\beta$ for all $x, y \in G$. With above operation G/β is an abelian additive group which coincides with $G/[0]_\beta$.

Proof. The proof is left to the reader, who may consult [5]. □

Now we define the local skeleton of G. For every $v = (\alpha, \beta) \in V$ let $E_v^0 = [0]_\beta/[0]_\alpha$, so E_v^0 is an abelian additive group, which is a subgroup of $G/[0]_\alpha$. The family $\mathcal{E}_G^0 = (E_v^0)_{v \in V}$ is called the *local skeleton of* G.

(7.3) Lemma. Let K be a field, let G be an ultrametric K-vector space.

1) Every convex subgroup of G is a K-subspace of G.

2) For every $v = (\alpha, \beta) \in V$, $[0]_\beta$ and $E_v^0 = [0]_\beta/[0]_\alpha$ are K-vector spaces.

Proof. The proof is left to the reader, who may consult [5]. □

For each $h \in \prod\limits_{v \in V} E_v^0$ let $\operatorname{supp}(h) = \{v = (\alpha, \beta) \in V \mid h(v) \neq [0]_\alpha\}$ and let $H^0 = \big\{h \in \prod\limits_{v \in V} E_v^0 \mid \operatorname{supp}(h) \text{ is noetherian}\big\}$.

We define $d^0 \colon H^0 \times H^0 \to \mathcal{A}(V)$ as follows.

If $h, g \in H^0$ let $d^0(h,h) = \varnothing$ and if $h \neq g$ let $d^0(h,g) = \operatorname{Max} D^0(h,g)$ where $D^0(h,g) = \{v \in V \mid h(v) \neq g(v)\}$.

(7.4) Lemma. If G is an ultrametric group, respectively an ultrametric K-vector space, then $(H^0, d^0, \mathcal{A}(V))$ is an ultrametric group, respectively an ultrametric K-vector space and $(H^0, d^0, \mathcal{A}(V))$ is solid and spherically complete.

Proof. See [4]. □

$(H^0, d^0, \mathcal{A}(V))$ is called the *local Hahn space associated to* (G, d, Γ).

§8. The Immersion Theorem for Ultrametric K-Vector Spaces

Let G be an ultrametric K-vector space, let H^0 be the local Hahn space associated to G. We shall define an immersion $(\lambda^0, \underline{\lambda}^0)$ from G into H^0, such that λ^0 is a K-linear mapping.

We keep the notations of §5. For every $\beta \in \equiv (G)$ we consider a system of representatives S_β of G/β such that $s_\beta(0) = 0$.

The following lemma is crucial in the proof of the Immersion Theorem (8.3). (This lemma is used in our paper [5] and it goes back to Banaschewski [1]).

(8.1) Lemma. There exists a coherent family of representatives $\sum = (S_\beta)_\beta \in \equiv (G)$ satisfying the following conditions:

a) S_β is a K-subspace of G.

b) $G = S_\beta \oplus [0]_\beta$.

Proof. For each subset M of G let $\langle M \rangle$ denote the K-vector space generated by M.

Let $B = \{x_\sigma \mid \sigma < \rho\}$ (where ρ is an ordinal number) be a basis of the K-vector space G. For each $\beta \in \equiv (G)$ let $R_\beta = \big\{x_\sigma \in B \mid x_\sigma \notin \langle [0]_\beta \cup \{x_\iota \mid \iota < \sigma\}\rangle\big\}$ and let $S_\beta = \langle R_\beta \rangle$. By (7.3), $[0]_\beta$ is a K-subspace of G.

1º) We show that $[0]_\beta \cap S_\beta = \{0\}$. We assume, on the contrary, that there exists $x \neq 0$ such that $x \in [0]_\beta \cap S_\beta$. Thus $x = \sum_{i=1}^{n} q_i\, x_{\sigma_i}$ with $q_i \in K$, $q_1 \neq 0$ and $x_{\sigma_i} \in R_\beta$ for all $i = 1, \ldots, n$. Moreover, we may assume that $\sigma_1, \ldots, \sigma_n$ are distinct, $\sigma_n = \max\{\sigma_1, \ldots, \sigma_n\}$. Hence $x_{\sigma_n} = q_n^{-1} x - q_n^{-1} \sum_{i=1}^{n-1} x_{\sigma_i} \in \langle [0]_\beta \cup \{x_\iota \mid \iota < \sigma_n\}\rangle$, hence $x_{\sigma_n} \notin R_\beta$ which is a contradiction.

2º) We show that $G = [0]_\beta + S_\beta$. It suffices to show that $x_\sigma \in [0]_\beta + S_\beta$ for all $\sigma < \rho$. If this is not true, then there exists $\sigma < \rho$ minimal such that $x_\sigma \notin [0]_\beta + S_\beta$. So by definition $x_\sigma \in R_\beta$. Hence $x_\sigma \in S_\beta$, and this is a contradiction. Thus for all $\sigma < \rho$, $x_\sigma \in [0]_\beta + S_\beta$.

3º) We show that S_β is a set of representatives of G/β, that is, for every $C \in G/\beta$ there exists a unique element $s \in S_\beta \cap C$, such that $C = [s]_\beta$. Indeed, let $c \in G$ be such that $C = [0]_\beta$. By (2º) there exist elements $g \in [0]_\beta$ and $s \in S_\beta$ such that $c = g + s$, hence $s = c - g \in c + [0]_\beta$ and so $[s]_\beta = s + [0]_\beta = c + [0]_\beta = C$.

If also $C = [s']_\beta$ with $s' \in S_\beta$ then $s' = s + g$ with $g \in [0]_\beta$. Hence by (1º) $s' - s \in S_\beta \cap [0]_\beta = \{0\}$, so $s' = s$. This proves the assertion (3º).

4º) We show that $\sum = (S_\beta)_\beta$ is a coherent family. Let $\beta, \beta' \in \equiv (G)$ be such that $\beta' \subseteq \beta$. Then $[0]_{\beta'} \subseteq [0]_\beta$ and therefore $R_\beta = \{x_\sigma \in B \mid x_\sigma \notin \langle [0]_\beta \cup \{x_\iota \mid \iota < \sigma\}\rangle\} \subseteq \{x_\sigma \in B \mid x_\sigma \notin \langle [0]_{\beta'} \cup \{x_\iota \mid \iota < \sigma\}\rangle\} = R_{\beta'}$, hence $S_\beta \subseteq S_{\beta'}$. This shows that $\sum$ is coherent.

The proof is complete. □

We deduce the following corollary: Let G, K be as above.

(8.2) Corollary. Let $\sum = (S_\beta)_\beta \in \equiv (G)$ be a coherent family as in the preceding lemma. For all $x, y \in G$ and $q \in K$, $s_\beta(x + y) = s_\beta(x) + s_\beta(y)$ and $s_\beta(qx) = q s_\beta(x)$.

Proof. From $s_\beta(x)\,\beta\, x$ it follows $s_\beta(x)-x \in [0]_\beta$. Similarly, $s_\beta(y)-y\in[0]_\beta$ and $s_\beta(x+y)-(x+y)\in[0]_\beta$. Since $[0]_\beta$ and S_β are subgroups of G we have $s_\beta(x+y)-s_\beta(x)-s_\beta(y)\in [0]_\beta \cap S_\beta=\{0\}$. So $s_\beta(x+y)=s_\beta(x)+s_\beta(y)$. The proof that $s_\beta(qx)=qs_\beta(x)$ is similar. □

If $v=(\alpha,\beta)\in V$ and $x\in G$ we define $\lambda^0 x_v=[x-s_\beta(x)]_\alpha \in [0]_\beta/[0]_\alpha = E^0_v$. We define $\lambda^0 x=(\lambda^0 x_v)_v \in V \in \prod\limits_{v\in V} E^0_v$.

Let $\underline{\lambda}^0=\varnothing$ (the empty antichain) and if $\gamma>0$ let $\underline{\lambda}^0\gamma = V(\gamma)$.

(8.3) Theorem. With the above assumptions, λ^0 is a K-linear mapping and $(\lambda^0,\underline{\lambda}^0)$ is an immersion from G into H^0.

Proof. We show that for every $x\in G$ $\operatorname{supp}(\lambda^0 x)\subseteq \operatorname{supp}(\lambda x)$.

Indeed, let $v=(\alpha,\beta)$ be such that $\lambda^0 x_v \neq [0]_\alpha$. By definition x (not α)$s_\beta(x)$, hence $\lambda x_v \neq 0_v$. By (5.1) $\operatorname{supp}(\lambda x)$ is noetherian, hence $\operatorname{supp}(\lambda^0 x)$ is noetherian, so $\lambda^0 x\in H^0$.

We show that λ^0 is a K-linear mapping. Let $v=(\alpha,\beta)\in V$ and $x,y\in G$. We have $\lambda^0(x+y)_v=[(x+y)-s_\beta(x+y)]_\alpha=[(x-s_\beta(x))+(y-s_\beta(x)]_\alpha=[x-s_\beta(x)]_\alpha+[y-s_\beta(y)]_\alpha=\lambda^0 x+\lambda^0 y$. Similarly, if $q\in K^\bullet$ then $\lambda^0(qx)_v=[qx-s_\beta(qx)]_\alpha=[qx-qs_\beta(x)]_\alpha=q[x-s_\beta x]_\alpha=q(\lambda^0 x_v)$. This shows that λ^0 is K-linear.

Now we show that if $x,y\in G$ and $v\in V$, if $\lambda^0 x_v\neq\lambda^0 y_v$ then $\lambda x_v\neq \lambda y_v$.

Indeed, if $x\,\beta\,y$ then $s_\beta(x)=s_\beta(y)$. From $\lambda^0 x_v\neq\lambda^0 y_v$, $[x-s_\beta(x)]_\alpha \neq [y-s_\beta(y)]_\alpha$, so $[x]_\alpha\neq[y]_\alpha$, hence $\lambda x_v\neq\lambda y_v$.

If x (not β)y then x (not α)y, so $[x]_\alpha\neq[y]_\alpha$. If $x\,\alpha\, s_\beta(x)$ and $y\,\alpha\, s_\beta(y)$ then $\lambda^0 x_v=\lambda^0 y_v=[0]_\alpha$, which is contrary to the assumption. So λx_v and λy_v are not both equal to 0_v; from $[x]_\alpha\neq[y]_\alpha$ then $\lambda x_v\neq\lambda y_v$.

We show that $d^0(\lambda^0 x,\lambda^0 y)\leq \underline{\lambda}^0(d(x,y))$. We may assume $\lambda^0 x\neq\lambda^0 y$. Let $v_0\in d^0(\lambda^0 x,\lambda^0 y)\subseteq D(\lambda x,\lambda y)=\{v\in V\mid \lambda x_v\neq\lambda y_v\}$, so there exists $v\in d(\lambda x,\lambda y)$ such that $v_0\leq v$. This shows, by applying (5.1) that $d^0(\lambda^0 x,\lambda^0 y)\leq d(\lambda x,\lambda y)=\underline{\lambda}(d(x,y))=V(\gamma)=\underline{\lambda}^0(d(x,y))$.

Now we show that $\underline{\lambda}^0(d(x,y))\leq d^0(\lambda^0 x,\lambda^0 y)$. We may assume $x\neq y$, let $d(x,y)=\gamma$, so $\underline{\lambda}^0(d(x,y))=V(\gamma)$.

Let $v=(\alpha,\beta)\in V(\gamma)$, so $x\,\beta\,y$ and x (not α)y. So $s_\beta(x)=s_\beta(y)$, hence $\lambda^0 x_v=[x-s_\beta(x)]_\alpha\neq[y-s_\beta(y)]_\alpha=\lambda^0 y_v$. Hence there exists $v'\in d^0(\lambda^0 x,\lambda^0 y)$ such that $v\leq v'$, thus proving that $\underline{\lambda}^0(d(x,y))\leq d^0(\lambda^0 x,\lambda^0 y)$. We deduce that $d^0(\lambda^0 x,\lambda^0 y)=\underline{\lambda}^0(d(x,y))$.

It was shown in (3.3) that $\underline{\lambda}^0$ is injective, then λ^0 is also injective. The proof is complete. □

$>$From the theorem we deduce:

(8.4) Corollary. Every ultrametric K-vector space (G, d, Γ) is a subspace of a solid and spherically complete ultrametric K-vector space $(\widehat{G}, \hat{d}, \widehat{\Gamma})$. Moreover, if Γ is totally ordered then $\widehat{\Gamma} = \Gamma$.

Proof. By (8.3) $(\lambda^0, \underline{\lambda}^0)$ is an immersion of (G, d, Γ) into the K-vector space $(H^0, d^0, \mathcal{A}(V))$ and λ^0 is a K-linear mapping. By (7.4) H^0 is solid and spherically complete. By (1.6) there exists $(\widehat{G}, \hat{d}, \widehat{\Gamma})$ which is an ultrametric space isomorphic to $(H^0, d^0, \mathcal{A}(V))$. Since H^0 is an K-vector space, it is straightforward to endow $\widehat{G}$ with the structure of a K-vector space, making $\widehat{G}$ into an ultrametric K-vector space isomorphic to H^0 and such that G is a K-subspace of $\widehat{G}$. Again, $\widehat{G}$ is solid and spherically complete, because H^0 has these properties.

Finally, if Γ is totally ordered, with natural identification, $\Gamma = V = \mathcal{A}(V) = \widehat{\Gamma}$. $\square$

NOTES

The special case of the Immersion Theorem when Γ is totally ordered was proved by Priess-Crampe and Ribenboim [3]. The notion of a virtual point and the skeleton, as defined here, were inspired by the work of Conrad on groups with valuations [1]. The definition of virtual and skeleton used here allowed to prove the much desired Immersion Theorem, without assuming that the set of distances is totally ordered.

We also obtain the corresponding immersion theorem for ultrametric K-vector spaces (with K having the trivial distance) which was proved in [5] under the assumption that the set of distances is totally ordered.

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