A CLOSEDNESS THEOREM OVER HENSELIAN VALUED FIELDS WITH ANALYTIC STRUCTURE

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Abstract. We give a closedness theorem over Henselian valued fields K of equicharacteristic zero (possibly non algebraically closed) with separated analytic structure. It asserts that every projection with a projective fiber is a definably closed map. This remains valid for valued fields with analytic structures induced by strictly convergent Weierstrass systems, including the classical, complete rank one valued fields with the Tate algebras of strictly convergent power series.

1. Introduction

Throughout the paper, we shall deal with Henselian valued fields K with separated analytic structure, possibly non algebraically closed. We shall always assume that the ground field K is of equicharacteristic zero. A separated analytic structure is determined by a certain separated Weierstrass system $\mathcal{A}$ defined on an arbitrary commutative ring A with unit (cf. [3, 4]), and the involved analytic language $\mathcal{L}$ is the two sorted, semialgebraic language $\mathcal{L}_{Hen}$ augmented by the reciprocal function $1/x$ and the names of all functions of the system $\mathcal{A}$, construed via the analytic $\mathcal{A}$-structure on their natural domains and as zero outside them. For convenience, we remind the reader of these concepts in Section 2. The theory of valued fields with analytic structure was developed in the papers [7, 8, 9, 5, 3, 4].

Given a valued field K, denote by v, $\Gamma = \Gamma_K$, K°, $K^{\circ\circ}$ and $\widetilde{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. By the K-topology on K^n we mean the topology induced by the valuation v.

The main result of this article is the following closedness theorem.

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Theorem 1.1. *Given an $\mathcal{L}$-definable subset D of K^n, the canonical projection*

$$\pi : D \times (K^\circ)^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times (K^\circ)^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

It immediately yields five corollaries stated below. One of them, the descent property (Corollary 1.6), enables application of resolution of singularities and transformation to a normal crossing by blowing up in much the same way as over the locally compact ground field.

Corollary 1.2. *Let D be an $\mathcal{L}$-definable subset of K^n and $\mathbb{P}^m(K)$ stand for the projective space of dimension m over K. Then the canonical projection*

$$\pi : D \times \mathbb{P}^m(K) \longrightarrow D$$

is definably closed. □

Corollary 1.3. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or R^m. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.*

Corollary 1.4. *Let ϕ_i, $i = 0, \ldots, m$, be regular functions on K^n, D be an $\mathcal{L}$-definable subset of K^n and $\sigma : Y \longrightarrow K\mathbb{A}^n$ the blow-up of the affine space $K\mathbb{A}^n$ with respect to the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction*

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map.

Proof. Indeed, $Y(K)$ can be regarded as a closed algebraic subvariety of $K^n \times \mathbb{P}^m(K)$ and σ as the canonical projection. □

Corollary 1.5. *Let X be a smooth K-variety, ϕ_i, $i = 0, \ldots, m$, regular functions on X, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up of the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction*

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map. □

Corollary 1.6. *(Descent property) Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function*

$$g : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$$

that is constant on the fibers of the blow-up σ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \longrightarrow K$. □

The closedness theorem will be proven in Section 3. The strategy of proof in the analytic settings will generally follow the one in the algebraic case from my papers [10, 11]. We rely, in particular, on fiber shrinking and the local behavior of definable functions of one variable. Again, we make use of relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok [2]. But now we apply elimination of valued field quantifiers for the theory $T_{Hen,\mathcal{A}}$ and b-minimal cell decompositions with centers (cf. [6]).

Remark 1.7. The closedness theorem holds also for analytic structures induced by strictly convergent Weierstrass systems, because every such structure can be extended in a definitional way (extension by Henselian functions) to a separated analytic structure (cf. [4]). Examples of such structures are the classical, complete rank one valued fields with the Tate algebras of strictly convergent power series.

2. Fields with analytic structure

In this section we recall the concept of an analytic structure (cf. [6, Section 4.1]). Let A be a commutative ring with unit and with a fixed proper ideal $I \subsetneq A$. A *separated* (A, I)*-system* is a certain system $\mathcal{A}$ of A-subalgebras $A_{m,n} \subset A[[\xi, \rho]]$, $m, n \in \mathbb{N}$; here $A_{0,0} = A$. Two kinds of variables, ξ and ρ, play different roles. Roughly speaking, the variables ξ vary over the valuation ring (or the closed unit disc) K° of a valued field K, and the variables ρ vary over the maximal ideal (or the open unit disc) $K^{\circ\circ}$ of K. $\mathcal{A}$ is called a *separated pre-Weierstrass system* if two usual Weierstrass division theorems hold in each $A_{m,n}$. When, in addition, such a pre-Weierstrass system $\mathcal{A}$ satisfies a condition referring to the so-called rings of $\mathcal{A}$-fractions, it is called a *separated Weierstrass system* (loc. cit.). This condition may be regarded as a kind of weak Noetherian property, because it implies, in particular, that if

$$f = \sum_{\mu,\nu} a_{\mu\nu}\, \xi^\mu \rho^\nu \in A_{m,n},$$

then the ideal of A generated by the $a_{\mu\nu}$ is finitely generated.

Let $\mathcal{A}$ be a separated Weierstrass system and K be a valued field. A *separated analytic* $\mathcal{A}$*-structure* on K (loc. cit.) is a collection of homomorphisms $\sigma_{m,n}$ from $A_{m,n}$ to the ring of K°-valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$, $m, n \in \mathbb{N}$, such that

1) $\sigma_{0,0}(I) \subset K^{\circ\circ}$;

2) $\sigma_{m,n}(\xi_i)$ and $\sigma_{m,n}(\rho_j)$ are the i-th and $(m+j)$-th coordinate functions on $(K^\circ)^m \times (K^{\circ\circ})^n$, respectively;

3) $\sigma_{m+1,n}$ and $\sigma_{m,n+1}$ extend $\sigma_{m,n}$, where functions on $(K^{\circ})^m \times (K^{\circ\circ})^n$ are identified with those functions on

$$(K^{\circ})^{m+1} \times (K^{\circ\circ})^n \quad \text{or} \quad (K^{\circ})^m \times (K^{\circ\circ})^{n+1}$$

which do not depend on the coordinate ξ_{m+1} or ρ_{n+1}, respectively.

Further, consider a separated pre-Weierstrass (A,I)-system $\mathcal{A}$ and assume that $A = F^{\circ}$ and $I = F^{\circ\circ}$ for a valued field F. Then $\mathcal{A}$ is a Weierstrass system iff for every $f \in A_{m,n}$, $f \neq 0$, $m,n \in \mathbb{N}$, there is an element $c \in F$ such that $cf \in A_{m,n}$ and the Gauss norm $\|cf\| = 1$ (loc. cit.).

Now let us recall some properties of analytic structures. Analytic $\mathcal{A}$-structures preserve composition (op. cit., Proposition 4.5.3). If the ground field K is non-trivially valued, then the function induced by a power series from $A_{m,n}$, $m,n \in \mathbb{N}$, is the zero function iff the image in K of each of its coefficients is zero (op. cit., Proposition 4.5.4).

Remark 2.1. When considering a particular field K with analytic $\mathcal{A}$-structure, one may assume that $\ker \sigma_{0,0} = (0)$. Indeed, replacing A by $A/\ker \sigma_{0,0}$ yields an equivalent analytic structure on K with this property. Then $A = A_{0,0}$ can be regarded as a subring of K°. Moreover, by extension of parameters, one can get a separated Weierstrass system $\mathcal{A}(K)$ over $(K^{\circ}, K^{\circ\circ})$ and K has separated analytic $\mathcal{A}(K)$-structure. A similar extension of parameters can be performed for any subfield $F \subset K$ of parameters (op. cit., Theorem 4.5.7 f.f.). Finally, every valued field with separated analytic structure is Henselian (op. cit., Proposition 4.5.10). The forgoing properties remain valid in the case of strictly convergent Weierstrass systems too.

Now we can describe the analytic language $\mathcal{L}$ of an analytic structure K determined by a separated Weierstrass system $\mathcal{A}$. We begin by defining the semialgebraic language $\mathcal{L}_{Hen}$. It is a two sorted language with the main, valued field sort K, and the auxiliary RV-sort

$$RV = RV(K) := RV^* \cup \{0\}, \quad RV^*(K) := K^{\times}/(1 + K^{\circ\circ});$$

here $A^{\times}$ denotes the set of units of a ring A. The language of the valued field sort is the language of rings $(0, 1, +, -, \cdot)$. The language of the auxiliary sort is the so-called inclusion language (op. cit., Section 6.1). The only map connecting the sorts is the canonical map

$$rv : K \to RV(K), \quad 0 \mapsto 0.$$

Since

$$\widetilde{K}^{\times} \simeq (K^{\circ})^{\times}/(1 + K^{\circ\circ}) \quad \text{and} \quad \Gamma \simeq K^{\times}/(K^{\circ})^{\times},$$

we get the canonical exact sequence

$$1 \to \widetilde{K} \to RV(K) \to \Gamma \to 0.$$

This sequence splits iff the valued field K has an angular component map.

The analytic language $\mathcal{L} = \mathcal{L}_{Hen,\mathcal{A}}$ is the semialgebraic language $\mathcal{L}_{Hen}$ augmented on the valued field sort K by the reciprocal function $1/x$ (with $1/0 := 0$) and the names of all functions of the system $\mathcal{A}$, together with the induced language on the auxiliary sort RV (op. cit., Section 6.2). A power series $f \in A_{m,n}$ is construed via the analytic $\mathcal{A}$-structure on their natural domains and as zero outside them. More precisely, f is interpreted as a function

$$\sigma(f) : (K^{\circ})^m \times (K^{\circ\circ})^n \to K^{\circ},$$

extended by zero on $K^{m+n} \setminus (K^{\circ})^m \times (K^{\circ\circ})^n$.

In the equicharacteristic case, however, the induced language on the auxiliary sort RV further coincides with the semialgebraic inclusion language. It is so because then [6, Lemma 6.3.12] can be strengthen as follows, whereby [6, Lemma 6.3.14] can be directly reduced to its algebraic analogue. Consider a strong unit on the open ball $B = K^{\circ\circ}_{alg}$. Then $rv(E^{\sigma})(x)$ is constant when x varies over B. This is no longer true in the mixed characteristic case, where the weaker conclusion asserts that the functions $rv_n(E^{\sigma})(x)$ depend only on $rv_n(x)$ when x varies over B. Under the circumstances, the residue field $\widetilde{K}$ is orthogonal to the value group Γ_K, whenever the ground field K has an angular component map or, equivalently, the auxiliary sort RV splits (in a non-canonical way):

$$RV(K) \simeq \widetilde{K} \times \Gamma_K.$$

This means that every definable set in the auxiliary sort $RV(K)$ is a finite union of the Cartesian products of some sets definable in the residue field sort $\widetilde{K}$ (in the language of rings) and in the value group sort Γ_K (in the language of ordered groups). The orthogonality property will often be used in the paper, similarly as it was in the algebraic case treated in our papers [10, 11].

Remark 2.2. Not all valued fields K have an angular component map, but it exists if K has a cross section, which happens whenever K is $\aleph_1$-saturated (cf. [1, Chap. II]). Moreover, a valued field K has an angular component map whenever its residue field $\Bbbk$ is $\aleph_1$-saturated (cf. [13, Corollary 1.6]). In general, unlike for p-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. Since the K-topology is $\mathcal{L}$-definable, the closedness theorem is a first order property. Therefore it can be proven using elementary extensions, and thus one may assume that an angular component map exists.

Let $\mathcal{T}_{Hen,\mathcal{A}}$ be the theory of all Henselian valued fields of characteristic zero with analytic $\mathcal{A}$-structure. The crucial result about analytic structures is the following [6, Theorem 6.3.7].

Theorem 2.3. *The theory $\mathcal{T}_{Hen,\mathcal{A}}$ eliminates valued field quantifiers, is b-minimal with centers and preserves all balls. Moreover, $\mathcal{T}_{Hen,\mathcal{A}}$ has the Jacobian property.* □

Therefore the theory $\mathcal{T}_{Hen,\mathcal{A}}$ admits b-minimal cell decompositions with centers (cf. [6]).

3. Proof of the closedness theorem

From now on we shall assume that the ground field K with separated analytic structure $\mathcal{A}$ is of equicharacteristic zero, and that K has an angular component map. In the algebraic case, the proofs of the closedness theorem given in our papers [10, 11]) make use of the following three main tools: the theorem on existence of the limit ([10, Proposition 5.2] and [11, Theorem 5.1]), fiber shrinking ([10, 11, Proposition 6.1]) and cell decomposition in the sense of Pas.

Fiber shrinking was reduced, by means of elimination of valued field quantifiers, to Lemma 3.1 below ([11, Lemma 6.2]), which, in turn, was obtained via relative quantifier elimination for ordered abelian groups. That approach can be repeated verbatim in the analytic settings.

Lemma 3.1. *Let Γ be an ordered abelian group and P be a definable subset of Γ^n. Suppose that $(\infty, \dots, \infty)$ is an accumulation point of P, i.e. for any $\delta \in \Gamma$ the set*

$$\{x \in P : x_1 > \delta, \dots, x_n > \delta\} \neq \emptyset$$

is non-empty. Then there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \dots, r_n t + \gamma_n) : \, t \in \Gamma, \; t \geq 0\} \quad \textit{with} \;\; r_1, \dots, r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that $(\infty, \dots, \infty)$ is an accumulation point of the intersection $P \cap L$ too. □

Similarly, one can obtain the following

Lemma 3.2. *Let P be a definable subset of Γ^n and*

$$\pi : \Gamma^n \to \Gamma, \quad (x_1, \dots, x_n) \mapsto x_1$$

be the projection onto the first factor. Suppose that ∞ is an accumulation point of $\pi(P)$. Then there is an affine semi-line

$$L = \{(r_1 t+\gamma_1, \dots, r_n t+\gamma_n) : \, t \in \Gamma, \; t \geq 0\} \ \text{with} \ r_1, \dots, r_n \in \mathbb{N}, \, r_1 > 0,$$

passing through a point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that ∞ is an accumulation point of $\pi(P \cap L)$ too.

In this paper, however, a suitable analytic version of the theorem on existence of the limit and application of b-minimal cell decompositions require some new ideas and work. The proof of the former relies on the theorem on term structure ([6, Theorem 6.3.8]), which we recall below. In further reasonings, we shall often make use of Lemmas 3.1 and 3.2.

Denote by $\mathcal{L}^*$ the analytic language $\mathcal{L}$ augmented by all Henselian functions

$$h_m : K^{m+1} \times RV(K) \to K, \quad m \in \mathbb{N},$$

which are defined by means of a version of Hensel's lemma (cf. [6], Section 6.1).

Theorem 3.3. *Let K be a Henselian field with analytic $\mathcal{A}$-structure. Let $f : X \to K$, $X \subset K^n$, be an $\mathcal{L}(B)$-definable function for some set of parameters B. Then there exist an $\mathcal{L}(B)$-definable function $g : X \to S$ with S auxiliary and an $\mathcal{L}^*(B)$-term t such that*

$$f(x) = t(x, g(x)) \quad \text{for all} \;\; x \in X.$$

$\square$

We turn to the following analytic version of the theorem on existence of the limit, which also may be regarded as a version of Puiseux's theorem.

Theorem 3.4. *Let $f : E \to K$ be an $\mathcal{L}$-definable function on a subset E of K and suppose 0 is an accumulation point of E. Then there is an $\mathcal{L}$-definable subsets $F \subset E$ with accumulation point 0 and a point $w \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \to 0} f|F\,(x) = w.$$

Moreover, we can require that

$$\{(x, f(x)) : x \in F\} \subset \{(x^r, \phi(x)) : x \in G\},$$

where r is a positive integer and ϕ is a definable function, a composite of some functions induced by series from $\mathcal{A}$ and of some algebraic power series (coming, in a certain way, from Henselian functions h_m). Then, in particular, the definable set

$$\{(v(x), v(f(x))) : \; x \in (F \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$$

is contained in an affine line with rational slope

$$l = \frac{p}{q} \cdot k + \beta,$$

with $p, q \in \mathbb{Z}$, $q > 0$, $\beta \in \Gamma$, *or in* $\Gamma \times \{\infty\}$.

Proof. In view of Remark 2.1, we may assume that K has separated analytic $\mathcal{A}(K)$-structure. We apply Theorem 3.3 and proceed with induction with respect to the complexity of the term t. Since an angular component map exists, the sorts $\widetilde{K}$ and Γ are orthogonal in

$$RV(K) \simeq \widetilde{K} \times \Gamma_K.$$

Therefore, after shrinking F, we can assume that $\overline{ac}\,(F) = \{1\}$ and the function g goes into $\{\xi\} \times \Gamma^s$ with a $\xi \in \widetilde{K}^s$, and next that $\xi = (1, \dots, 1)$; similar reductions were considered in our papers [10, 11]. For simplicity, we look at g as a function into Γ^s. We shall briefly explain the most difficult case where

$$t(x, g(x)) = h_m(a_0(x), \dots, a_m(x), g_0(x)),$$

assuming that the theorem holds for the terms $a_0, \dots, a_m$; here g_0 is one of the components of g. By Lemma 3.2, we can assume that

$$pv(x) + qg_0(x) + v(a) = 0 \tag{3.1}$$

for some $p, q \in \mathbb{Z}$, $a \in K \setminus \{0\}$. By the induction hypothesis, we get

$$\{(x, a_i(x)) : x \in F\} \subset \{(x^r, \alpha_i(x)) : x \in G\}, \quad i = 0, 1, \dots, m.$$

Put

$$P(x, T) := \sum_{i=0}^{m} a_i(x) T^i.$$

By the very definition of h_m and since we are interested in the vicinity of zero, we may assume that there is $i_0 = 0, \dots, m$ such that

$$\forall\, x \in F \;\exists\, u \in K \quad v(u) = g_0(x), \quad \overline{ac}\, u = 1,$$

$$v(a_{i_0}(x)u^{i_0}) = \min\{v(a_i(x)u^i),\ i = 1, \dots, m\}, \tag{3.2}$$

$$v(P(x, u)) > v(a_{i_0}(x)u^{i_0}), \quad v\left(\frac{\partial P}{\partial T}(x, u)\right) = v(a_{i_0}(x)u^{i_0}).$$

Then $h_m(a_0(x), \dots, a_m(x), g_0(x))$ is a unique $b(x) \in K$ such that

$$P(x, b(x)) = 0, \quad v(b(x)) = g_0(x), \quad \overline{ac}\, b(x) = 1.$$

By [11, Remarks 7.2, 7.3], the set F contains the set of points of the form $c^r t^{Nqr}$ for some $c \in K$ with $\overline{ac}\, c = 1$, a positive integer N and all $t \in K^\circ$ with $\overline{ac}\, t = 1$. Hence and by equation (3.1), we get

$$g_0(c^r t^{Nqr}) = g_0(c^r) - v(t^{Npr}).$$

Take $d \in K$ such that $g_0(c^r) = v(d)$ and $\overline{ac}\, d = 1$. Then

$$g_0(c^r t^{Nqr}) = v(dt^{-Npr}).$$

Thus the homothetic change of variable

$$Z = T/dt^{-Npr} = t^{Npr}T/d$$

transforms the polynomial

$$P(c^r t^{Nqr}, T) = \sum_{i=0}^{m} \alpha_i(ct^{Nq})T^i$$

into a polynomial $Q(t, Z)$ to which Hensel's lemma applies (cf. [12, Lemma 3.5]):

$$P(c^r t^{Nqr}, T) = P(c^r t^{Nqr}, dt^{-Npr}Z) = \alpha_{i_0}(ct^{Nq}) \cdot (dt^{-Npr})^{i_0} \cdot Q(t, Z). \tag{3.3}$$

Indeed, the formulas (3.2) imply that the coefficients of the polynomial Q are power series (of order ≥ 0) in the variable t, and that

$$v(Q(0,1)) > 0 \quad \text{and} \quad v\left(\frac{\partial Q}{\partial Z}(0,1)\right) = 0.$$

Therefore the conclusion of the theorem follows. $\square$

We still need the concept of fiber shrinking introduced in our paper [10]. Let A be an $\mathcal{L}$-definable subset of K^n with accumulation point

$$a = (a_1, \dots, a_n) \in K^n$$

and E an $\mathcal{L}$-definable subset of K with accumulation point a_1. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \dots, a_n),$$

i.e. for any neighbourhood U of $(a_2, \dots, a_n) \in K^{n-1}$, there is a neighbourhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, A is itself a fiber shrinking for the subset A of K at an accumulation point $a \in K$.

Proposition 3.5. *(Fiber shrinking) Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

By means of elimination of valued field quantifiers (Theorem 2.3), this proposition reduces easily to Lemma 3.1 (see [11]). Now we can readily proceed with the

Proof of the closedness theorem (Theorem 1.1). We must show that if B is an $\mathcal{L}$-definable subset of $D \times (K^{\circ})^n$ and a point a lies in the closure of $A := \pi(B)$, then there is a point b in the closure of B such that $\pi(b) = a$. As before (cf. [11, Section 8]), the theorem reduces easily to the case $m = 1$ and next, by means of fiber shrinking (Proposition 3.5), to the case $n = 1$. We may obviously assume that $a = 0 \notin A$.

By b-minimal cell decomposition, we can assume that the set B is a relative cell with center over A. It means that has a presentation of the form

$$\Lambda : B \ni (x, y) \to (x, \lambda(x, y)) \in A \times RV(K)^s,$$

where $\lambda : B \to RV(K)^s$ is an $\mathcal{L}$-definable function, such that for each $(x, \xi) \in \Lambda(B)$ the pre-image $\lambda_x^{-1}(\xi) \subset K$ is either a point or an open ball; here $\lambda_x(y) := \lambda(x, y)$. In the latter case, there is a center, i.e. an $\mathcal{L}$-definable map $\zeta : \Lambda(B) \to K$, and a (unique) map $\rho : \Lambda(B) \to RV(K) \setminus \{0\}$ such that

$$\lambda_x^{-1}(\xi) = \{y \in K : rv\,(y - \zeta(x, \xi)) = \rho(x, \xi)\}.$$

Again, since the sorts $\widetilde{K}$ and Γ are orthogonal in $RV(K) \simeq \widetilde{K} \times \Gamma_K$, we can assume, after shrinking the sets A and B, that

$$\lambda(B) \subset \{(1, \dots, 1)\} \times \Gamma^s \subset \widetilde{K}^s \times \Gamma_K^s;$$

let $\tilde{\lambda}(x, y)$ be the projection of $\lambda(x, y)$ onto Γ^s. By Lemma 3.2, we can assume once again, after shrinking the sets A and B, that the set

$$\{(v(x), v(y), \tilde{\lambda}(x, y)) : \ (x, y) \in B\} \subset \Gamma^{s+2}$$

is contained in an affine semi-line with integer coefficients. Hence $\lambda(x, y) = \phi(v(x)$ is a function of one variable x. We have two cases.

Case I. $\lambda_x^{-1}(\xi) \subset K^{\circ}$ is a point. Since each λ_x is a constant function, B is the graph of an $\mathcal{L}$-definable function. The conclusion of the theorem follows thus from Theorem 3.4.

Case II. $\lambda_x^{-1}(\xi) \subset K^{\circ}$ is a ball. Again, application of Lemma 3.2 makes it possible, after shrinking the sets A and B, to arrange the center

$$\zeta : \Lambda(B) \ni (x, k) \to \zeta(x, v(x)) = \zeta(x) \in K$$

and the function $\rho(x,k) = \rho(v(x))$ as functions of one variable x. Likewise as it was above, we can assume that the set

$$P := \{(v(x), \rho(v(x))) : x \in A\} \subset \Gamma^2$$

is contained in an affine line $pv(x) + q\rho(v(x)) + v(c) = 0$ with integer coefficients p, q, $q \neq 0$; furthermore, that P contains the set

$$Q := \{(v(ct^{qN}), \rho(v(ct^{qN}))) : \ t \in K^{\circ}\}$$

for a positive integer N. Then we easily get

$$\rho(v(ct^{qN})) = \rho(c) - pNv(t) = v(ct^{-pN}).$$

Hence the set B contains the graph

$$\{(ct^{qN}, \zeta(ct^{qN}) + ct^{-pN}) : \ t \in K^{\circ}\}.$$

As before, the conclusion of the theorem follows thus from Theorem 3.4, and the proof is complete.

Let us conclude with the following comment. We are currently preparing subsequent articles, which will provide several applications of the closedness theorem, possibly over non-algebraically closed ground fields, including i.al. the analytic, non-Archimedean versions of the Łojasiewicz inequalities and of curve selection. The algebraic versions of these results were established in our papers [10, 11].

References

[1] G. Cherlin, *Model Theoretic Algebra, Selected Topics*, Lect. Notes Math. **521**, Springer-Verlag, Berlin, 1976.

[2] R. Cluckers, E. Halupczok, *Quantifier elimination in ordered abelian groups*, Confluentes Math. **3** (4) (2011), 587–615.

[3] R. Cluckers, L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. **13** (2011), 1147–1223.

[4] R. Cluckers, L. Lipshitz, *Strictly convergent analytic structures*, J. Eur. Math. Soc. **19** (2017), 107–149.

[5] R. Cluckers, L. Lipshitz, Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Sci. École Norm. Sup. (4) **39** (2006), 535–568.

[6] R. Cluckers, F. Loeser, *b-minimality*, J. Math. Logic **7** (2) (2007), 195–227.

[7] L. van den Dries, *Analytic Ax–Kochen–Ershov theorems*, Contemporary Mathematics **131** (1992), 379–392.

[8] L. van den Dries, D. Haskell, D. Macpherson, *Onr dimensional p-adic subanalytic sets*, J. London Math. Soc. **56** (1999), 1–20.

[9] L. Lipshitz, Z. Robinson, *Uniform properties of rigid subanalytic sets*, Trans. Amer. Math. Soc. 357 (11) (2005), 4349–4377.

[10] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. **23** (2017), 455–495.

[11] K.J. Nowak, *A closedness theorem and applications in geometry of rational points over Henselian valued fields*, arXiv:1706.01774 [math.AG] (2017).

[12] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. **399** (1989), 137–172.
[13] J. Pas, *On the angular component map modulo p*, J. Symbolic Logic **55** (1990), 1125–1129.

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