Krull-Tropical Hypersurfaces.*

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Abstract

The concepts of tropical semiring and tropical hypersurface, are extended to the case of an arbitrary ordered group. Then, we define the tropicalization of a polynomial with coefficients in a Krull-valued field.

After a close study of the properties of the operator "tropicalization" we conclude with an extension of Kapranov's theorem to algebraically closed fields together with a valuation over an ordered group.

Introduction

The *tropical semi-ring* is the set $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ together with the operations $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. A *tropical hypersurface* is a subset of $\mathbb{R}^N$ defined by a polynomial with coefficients in $\mathbb{T}$. A valuation of a field into the real numbers is used to *tropicalize* algebraic geometry propositions. A naturally real-valued algebraically closed field is the field of Puiseux series.

Let $\mathbb{K}$ be an algebraically closed real-valued field. In [2] M. Einsieder, M. Kapranov and D. Lind show that the image of an algebraic hypersurface via a valuation into the reals coincides with the non-linearity locus of its tropical map.

Valuations into the real numbers are just a particular type of valuations called *classical* (see for example [8]). In 1932 W. Krull extended the classical definition considering valuations with values in an arbitrary ordered group [7]. Krull's definition is the one currently used in most articles and reference texts (see for example [12, 3, 11]).

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Replacing $\mathbb{R}$ by another totally ordered group Γ, the tropical semi-ring $\mathbb{G} := \Gamma \cup \{\infty\}$ may be defined naturally. The same happens with the concept of tropical hypersurface and the tropicalization of a polynomial. A first step in this direction has been done in [1] where an example is given.

In this note we extend these concepts and prove some properties of the tropicalization map. Using these properties we extend the so called Kapranov's theorem. Our proof is not just an extension of an existing proof in the classical case but it is essentially different.

In [2], a tropical hypersurface is defined as the closure in $\mathbb{R}^N$ of the image, via valuation, of an algebraic hypersurface. Defining the tropical hypersurface as a subset of Γ^N has the advantage (even when $\Gamma \subset \mathbb{R}$) that we do not need to deal with topological arguments. This idea is already present in [6].

Sections 1 and 2 are devoted to extending the definitions of tropical semi-ring and tropical hypersurface. In sections 3 and 4 we recall the definition of Krull valuation and extend the definition of tropicalization and tropical hypersurface of a polynomial with coefficients in a valued field.

In section 5 we prove that the hypersurface associated to the tropicalization of a product is the union of the hypersurfaces of the tropicalization of its factors. Kapranov's theorem in one variable is a consequence of this fact.

Sections 6 and 7 are devoted to finding polynomials f for which the value $val(f(x))$ of f evaluated at a point x equals the tropicalization of f, evaluated at the point $val(x)$.

In section 8 we give a proof of the extension of Kapranov's theorem.

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1 Ordered groups, tropical semi-rings and tropical polynomials.

A **totally ordered group** is an abelian group $(\Gamma, +)$ equipped with a total order such that for all $x, y, z \in \Gamma$ if $x \leq y$ then $x + z \leq y + z$. For $a > 0$ we have $a + a > 0 + a$, therefore a totally ordered group is torsion free.

The following definition is an extension of a classical definition for the ordered group $(\mathbb{R}, +, \leq)$ [5, 9, 4].

Definition 1.1. *A totally ordered group* $(\Gamma, +, \leq)$ *induces an idempotent semi-ring* $\mathbb{G} := (\Gamma \cup \{\infty\}, \oplus, \odot)$*. Here*

- $a \oplus b := \min\{a, b\}$ *and* $a \oplus \infty := a$ *for* $a, b \in \Gamma$.

- $a \odot b := a + b$ *and* $a \odot \infty := \infty$ *for* $a, b \in \Gamma$.

This semiring is called the **min-plus algebra** *induced by* Γ *or the* **tropical semi-ring**.

A non-zero Laurent polynomial $F \in \mathbb{G}[x^*] := \mathbb{G}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ is an expression of the form

$$F = \bigoplus_{\alpha \in \mathcal{E}(F) \subset \mathbb{Z}^N} a_\alpha \odot x^\alpha, \qquad a_\alpha \in \Gamma, \quad \#\mathcal{E}(F) < \infty. \tag{1.1}$$

These polynomials are called **tropical polynomials**.

The set of tropical polynomials is a semi-ring with the natural operations: Given F as above and $G = \bigoplus_{\beta \in \mathcal{E}(G) \subset \mathbb{Z}^N} b_\beta \odot x^\beta$, we define

$$F \odot G := \bigoplus_{\eta \in \mathcal{E}(F) + \mathcal{E}(G)} \left(\bigoplus_{\alpha + \beta = \eta} a_\alpha \odot b_\beta \right) \odot x^\eta$$

and

$$F \oplus G := \bigoplus_{\eta \in \mathcal{E}(F) \cup \mathcal{E}(G)} a_\eta \oplus b_\eta \odot x^\eta$$

where $a_\eta := \infty$ for all $\eta \in \mathcal{E}(G) \setminus \mathcal{E}(F)$ and $b_\eta := \infty$ for all $\eta \in \mathcal{E}(F) \setminus \mathcal{E}(G)$.

2 Tropical maps and non-linearity locus.

Let $\mathbb{G}$ be the min-plus algebra induced by the group $(\Gamma, \leq)$.

Given $g \in \mathbb{G}$ and a natural number k, we will use the standard notation

$$g^k := \overbrace{g \odot \cdots \odot g}^{k \text{ times}} \quad \text{and} \quad g^{-k} = \left(g^{-1}\right)^k;$$

and, for $\gamma \in \Gamma^N$ and $\alpha \in \mathbb{Z}^N$ we will denote

$$\gamma^\alpha := {\gamma_1}^{\alpha_1} \odot \cdots \odot {\gamma_N}^{\alpha_N}.$$

A tropical polynomial $F = \bigoplus_{\alpha \in \mathcal{E}(F) \subset \mathbb{Z}^N} a_\alpha \odot x^\alpha$ induces a map $F : \Gamma^N \longrightarrow \Gamma$ given by

$$F : \gamma \mapsto \bigoplus_{\alpha \in \mathcal{E}(F)} a_\alpha \odot \gamma^\alpha.$$

A map induced by a tropical polynomial is called a **tropical map**.

For each $\gamma \in \Gamma^N$ there exists at least one $\alpha \in \mathcal{E}(F)$ such that $F(\gamma) = a_\alpha \odot \gamma^\alpha$. The set of α's with this property will be denoted by $\mathcal{D}_\gamma(F)$. That is

$$\mathcal{D}_\gamma(F) := \{\alpha \in \mathcal{E}(F) \mid F(\gamma) = a_\alpha \odot \gamma^\alpha\}. \tag{2.1}$$

Definition 2.1. *The* **hypersurface associated to** F *is the subset of* Γ^N *given by*

$$\mathcal{V}(F) := \{\gamma \in \Gamma \mid \#\mathcal{D}_\gamma(F) > 1\}. \tag{2.2}$$

For $\alpha \in \mathcal{E}(F)$, the restriction $F|_{\{\gamma\in\Gamma^N | \alpha\in\mathcal{D}_\gamma(F)\}}$ is given by an *affine linear* function $\gamma \mapsto a_\alpha \odot \gamma^\alpha$. We say that F defines a *piecewise linear* function on Γ^N. The hypersurface associated to F is called the *non-linearity locus*.

3 Valuations.

Let $(\Gamma, \leq, +)$ be a totally ordered group and let $(\mathbb{G}, \oplus, \odot)$ be its min-plus algebra. A **valuation** of a field $(\mathbb{K}, +_\mathbb{K}, \cdot)$ with values in $(\Gamma, \leq, +)$ is a surjective map $val : \mathbb{K} \longrightarrow \mathbb{G}$ such that

1. $val(x) = \infty \Leftrightarrow x = 0$,
2. $val(x \cdot y) = val(x) \odot val(y)$ for all $x, y \in \mathbb{K}$, and
3. $val(x +_\mathbb{K} y) \geq val(x) \oplus val(y)$.

We say that $\mathbb{K}$ has **values** in Γ. A field together with a valuation is called a **valued field** and $(\Gamma, \leq, +)$ is called **the group of values**.

Note that

- $val(1) = val(1 \cdot 1) = val(1) \odot val(1) \overset{\Gamma \text{ is torsion free}}{\Longrightarrow} val(1) = 0.$
- $0 = val((-1)(-1)) = val(-1) \odot val(-1) \overset{\Gamma \text{ is torsion free}}{\Longrightarrow} val(-1) = 0.$
- $val(-b) = val((-1)b) = val(-1) \odot val(b) = val(b).$

Lemma 3.1. *Let* $E \subset \mathbb{K}$ *be a finite set. If* $val\left(\sum_{\varphi\in E} \varphi\right) > \oplus_{\varphi\in E} val\varphi$ *then the set of elements in* E *where the valuation attains its minimum has at least two elements.*

Proof. Let $E_{\min}$ be the subset of E consisting of elements where the valuation attains its minimum:

$$E_{\min} = \{\varphi \in E \mid val\varphi = \oplus_{\varphi\in E} val\varphi\}.$$

Suppose that $E_{\min} = \{a\}$ and set $b := \sum_{\varphi\in E\setminus\{a\}} \varphi$. We have $val(b) > val(a)$ and

$$val\left(\sum_{\varphi\in E} \varphi\right) > \oplus_{\varphi\in E} val\varphi \Longleftrightarrow val(a + b) > val(a).$$

Then $val(a) = val((a + b) - b) \geq val(a + b) \oplus val(b) > val(a)$ which is a contradiction. ∎

4 The tropicalization.

Let $(\mathbb{K}, val)$ be a valued field with values in a group Γ and let $\mathbb{G}$ be the min-plus algebra induced by Γ.

A non-zero Laurent polynomial in N variables with coefficients in $\mathbb{K}$, $f \in \mathbb{K}[x^*]$, is written in the form:

$$f = \sum_{\alpha \in \mathcal{E}(f) \subset \mathbb{Z}^N} \varphi_\alpha x^\alpha \qquad \varphi_\alpha \in \mathbb{K} \setminus \{0\}, \quad \#\mathcal{E}(f) < \infty. \tag{4.1}$$

The polynomial f via the valuation val induces an element of $\mathbb{G}[x^*]$

$$\mathcal{T}f := \bigoplus_{\alpha \in \mathcal{E}(f) \subset \mathbb{Z}^N} val(\varphi_\alpha) \odot x^\alpha$$

this polynomial is called **the tropicalization of** f.

Remark 4.1. *Since* $val(a+b) \geq val(a) \oplus val(b)$ *and* $val(ab) = val(a) \odot val(b)$, *we have*

$$val(f(x)) \geq \mathcal{T}f(val(x)) \quad \textit{for all} \quad x \in \mathbb{K}^N.$$

Given a Laurent polynomial in N variables with coefficients in $\mathbb{K}$, $f \in \mathbb{K}[x^*] := \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$, the set of **zeroes** of f is defined as

$$\mathbf{V}(f) := \{x \in (\mathbb{K} \setminus \{0\})^N \mid f(x) = 0\}.$$

The **tropical hypersurface** associated to f is the set of values of $\mathbf{V}(f)$. That is:

$$\mathbf{TV}f := val(\mathbf{V}(f)).$$

Proposition 4.2. *Let f be a non-zero polynomial in $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$. If $\phi \in \mathbb{K}^N$ is a zero of f, then $val\phi$ is in the hypersurface associated to the tropicalization of f. That is:*

$$\mathbf{TV}f \subset \mathcal{V}\mathcal{T}f.$$

Proof. For $f = \sum_{\alpha \in \mathcal{E}(f)} \varphi_\alpha x^\alpha$, we have

$$\mathcal{T}f = \oplus_{\alpha \in \mathcal{E}(f)} val(\varphi_\alpha) \odot x^\alpha.$$

Since $\sum_{\alpha \in \mathcal{E}(f)} \varphi_\alpha \phi^\alpha = 0$, by lemma 3.1, the set

$$E_{\min} := \{\alpha_0 \in \mathcal{E}(f) \mid val(\varphi_{\alpha_0} \phi^{\alpha_0}) = \oplus_{\alpha \in \mathcal{E}(f)} val(\varphi_\alpha \phi^\alpha)\}$$

has at least two elements.

Now $val(\varphi_\alpha \phi^\alpha) = val(\varphi_\alpha) \odot (val\phi)^\alpha$, then $E_{\min} = \mathcal{D}_{val\phi}(\mathcal{T}f)$ and we have the result. □

5 The tropicalization of a product.

The map $\mathcal{T} : \mathbb{K}[x_1, \ldots, x_N] \longrightarrow \mathbb{G}[x_1, \ldots, x_N]$ may not preserve sum nor product. Nevertheless, the tropical variety of the product may be described.

Lemma 5.1. *Let $\mathbb{K}$ be a valued field and let Γ be its group of values. Given $\omega \in \mathbb{R}^N$ with rationally independent coordinates, $f, g \in \mathbb{K}[x^*]$ and $\gamma \in \Gamma^N$; set $\alpha_0 \in \mathcal{D}_\gamma(\mathcal{T}f)$ and $\beta_0 \in \mathcal{D}_\gamma(\mathcal{T}g)$ such that*

$$\omega \cdot \alpha_0 = \min_{\alpha \in \mathcal{D}_\gamma(\mathcal{T}f)} \omega \cdot \alpha \quad \text{and} \quad \omega \cdot \beta_0 = \min_{\beta \in \mathcal{D}_\gamma(\mathcal{T}g)} \omega \cdot \beta. \tag{5.1}$$

Set $\eta_0 := \alpha_0 + \beta_0$. We have:

$$\eta_0 \in \mathcal{D}_\gamma(\mathcal{T}(fg)) \quad \text{and} \quad \omega \cdot \eta_0 = \min_{\eta \in \mathcal{D}_\gamma(\mathcal{T}fg)} \omega \cdot \eta.$$

Proof. Write $f = \sum_{\alpha \in \mathcal{E}(f)} \varphi_\alpha x^\alpha$ and $g = \sum_{\beta \in \mathcal{E}(g)} \varphi'_\beta x^\beta$. Then

$$fg = \sum_{\eta \in \mathcal{E}(f) \cup \mathcal{E}(g)} \left(\sum_{\alpha + \beta = \eta} \varphi_\alpha \varphi'_\beta \right) x^\eta.$$

By (5.1), we have

$$\omega \cdot \eta_0 = \min_{\eta \in \mathcal{D}_\gamma(\mathcal{T}f) + \mathcal{D}_\gamma(\mathcal{T}g)} \omega \cdot \eta. \tag{5.2}$$

Since $\alpha_0 \in \mathcal{D}_\gamma(\mathcal{T}f)$ and $\beta_0 \in \mathcal{D}_\gamma(\mathcal{T}g)$, by definition (2.1), we have

$$\left\{ \begin{array}{c} val(\varphi_{\alpha_0}) \odot \gamma^{\alpha_0} \leq val(\varphi_\alpha) \odot \gamma^\alpha, \quad \forall \alpha \in \mathcal{E}(f) \\ \text{and} \\ val(\varphi'_{\beta_0}) \odot \gamma^{\beta_0} \leq val(\varphi'_\beta) \odot \gamma^\beta, \quad \forall \beta \in \mathcal{E}(g). \end{array} \right. \tag{5.3}$$

and

$$\left\{ \begin{array}{c} val(\varphi_{\alpha_0}) \odot \gamma^{\alpha_0} < val(\varphi_\alpha) \odot \gamma^\alpha, \quad \forall \alpha \in \mathcal{E}(f) \setminus \mathcal{D}_\gamma(\mathcal{T}f) \\ \text{and} \\ val(\varphi'_{\beta_0}) \odot \gamma^{\beta_0} < val(\varphi'_\beta) \odot \gamma^\beta, \quad \forall \beta \in \mathcal{E}(g) \setminus \mathcal{D}_\gamma(\mathcal{T}g). \end{array} \right. \tag{5.4}$$

Let $\alpha \in \mathcal{E}(f)$ and $\beta \in \mathcal{E}(f)$ be such that $\eta_0 = \alpha + \beta$. If $\alpha_0 \neq \alpha$ then either $\omega \cdot \alpha < \omega \cdot \alpha_0$ or $\omega \cdot \beta < \omega \cdot \beta_0$. Then, by (5.1), $\alpha \notin \mathcal{D}_\gamma(\mathcal{T}f)$ or $\beta \notin \mathcal{D}_\gamma(\mathcal{T}g)$, and then

$$\left\{ \begin{array}{c} \alpha + \beta = \eta_0 \\ \text{and} \\ \alpha \neq \alpha_0 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} val(\varphi_{\alpha_0}) \odot \gamma^{\alpha_0} < val(\varphi_\alpha) \odot \gamma^\alpha \\ \text{or} \\ val(\varphi'_{\beta_0}) \odot \gamma^{\beta_0} < val(\varphi'_\beta) \odot \gamma^\beta. \end{array} \right. \tag{5.5}$$

Inequalities (5.5) together with (5.3) give

$$\left\{\begin{array}{cc} & \alpha+\beta=\eta_0 \\ \text{and} & \\ & \alpha\neq\alpha_0 \end{array}\right. \Rightarrow val(\varphi_{\alpha_0}\varphi'_{\beta_0})\odot\gamma^{\eta_0} < val(\varphi_\alpha\varphi'_\beta)\odot\gamma^{\eta_0}.$$

Therefore, by lemma 3.1,

$$val\left(\sum_{\alpha+\beta=\eta_0}\varphi_\alpha\varphi'_\beta\right) = val(\varphi_{\alpha_0}\varphi'_{\beta_0}). \tag{5.6}$$

Inequalities (5.3) give

$$val\left(\varphi_{\alpha_0}\varphi'_{\beta_0}\right)\odot\gamma^{\eta_0} \leq val\left(\varphi_\alpha\varphi'_\beta\right)\odot\gamma^{\alpha+\beta}, \quad \forall\alpha\in\mathcal{E}(f),\beta\in\mathcal{E}(g). \tag{5.7}$$

Equality (5.6) together with (5.7) gives

$$val\left(\sum_{\alpha+\beta=\eta_0}\varphi_\alpha\varphi'_\beta\right)\odot\gamma^{\eta_0} \leq val\left(\sum_{\alpha+\beta=\eta}\varphi_\alpha\varphi'_\beta\right)\odot\gamma^{\eta},\ \forall\eta\in\mathcal{E}(f)+\mathcal{E}(g).$$

In other words:

$$\eta_0\in\mathcal{D}_\gamma(\mathcal{T}fg) \tag{5.8}$$

and

$$\mathcal{T}fg(\gamma) = val\left(\sum_{\alpha+\beta=\eta_0}\varphi_\alpha\varphi'_\beta\right)\odot\gamma^{\eta_0} \ = val(\varphi_{\alpha_0}\varphi'_{\beta_0})\odot\gamma^{\eta_0}. \tag{5.9}$$

By (5.4) and (5.9), we have

$$\mathcal{D}_\gamma(\mathcal{T}fg)\subset\mathcal{D}_\gamma(\mathcal{T}f)+\mathcal{D}_\gamma(\mathcal{T}g) \tag{5.10}$$

(5.2), (5.8) and (5.10) give

$$\omega\cdot\eta_0 = \min_{\eta\in\mathcal{D}_\gamma(\mathcal{T}fg)}\omega\cdot\eta. \tag{5.11}$$

□

Proposition 5.2. *The hypersurface associated to the tropicalization of a finite product of polynomials is equal to the union of the hypersurfaces associated to the tropicalization of each polynomial. That is*

$$\mathcal{V}\mathcal{T}(fg) = \mathcal{V}(\mathcal{T}f)\cup\mathcal{V}(\mathcal{T}g).$$

Proof. Take $\omega \in \mathbb{R}^N$ with rationally independent coordinates. Set $\alpha_0 \in \mathcal{D}_\gamma(\mathcal{T}f)$ and $\beta_0 \in \mathcal{D}_\gamma(\mathcal{T}g)$ such that

$$\omega \cdot \alpha_0 = \min_{\alpha \in \mathcal{D}_\gamma(\mathcal{T}f)} \omega \cdot \alpha \quad \text{and} \quad \omega \cdot \beta_0 = \min_{\beta \in \mathcal{D}_\gamma(\mathcal{T}g)} \omega \cdot \beta.$$

Now take $\alpha_1 \in \mathcal{D}_\gamma(\mathcal{T}f)$ and $\beta_1 \in \mathcal{D}_\gamma(\mathcal{T}g)$ such that

$$(-\omega) \cdot \alpha_1 = \min_{\alpha \in \mathcal{D}_\gamma(\mathcal{T}f)} (-\omega) \cdot \alpha \quad \text{and} \quad (-\omega) \cdot \beta_1 = \min_{\beta \in \mathcal{D}_\gamma(\mathcal{T}g)} (-\omega) \cdot \beta.$$

By lemma 5.1 we have $\eta_0 := \alpha_0 + \beta_0, \eta_1 := \alpha_1 + \beta_1 \in \mathcal{D}_\gamma(\mathcal{T}(fg))$. And

$$\omega \cdot \eta_0 = \min_{\eta \in \mathcal{D}_\gamma(\mathcal{T}fg)} \omega \cdot \eta. \quad \text{and} \quad \omega \cdot \eta_1 = \max_{\eta \in \mathcal{D}_\gamma(\mathcal{T}fg)} \omega \cdot \eta. \tag{5.12}$$

Now

$$\left\{ \begin{array}{ll} & \gamma \in \mathcal{VT}f \\ \text{or} & \\ & \gamma \in \mathcal{VT}g \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} & \alpha_0 \neq \alpha_1 \\ \text{or} & \\ & \beta_0 \neq \beta_1 \end{array} \right. \Leftrightarrow \eta_0 \neq \eta_1 \Leftrightarrow \gamma \in \mathcal{VT}fg.$$

□

Corollary 5.3. *Let $(\mathbb{K}, val)$ be an algebraically closed valued field. For $N = 1$ and $f \in \mathbb{K}[x]$ we have*

$$\mathcal{VT}f = \mathbf{TV}f.$$

Proof. $f = \prod_{a \in \mathbf{V}(f)}(x - a)$ then $\mathcal{VT}f = \cup_{a \in \mathbf{V}(f)} \mathcal{VT}(x - a) = \{val(a) \mid a \in \mathbf{V}(f)\}$. □

6 Valuation ring and residue field.

The set

$$A_{val} := \{a \in \mathbb{K} \mid val(a) \geq 0\}$$

is a ring called the **valuation ring**. The valuation ring has only one maximal ideal given by

$$\mathfrak{m}_{val} := \{a \in \mathbb{K} \mid val(a) > 0\},$$

the group of units of A_{val} is given by:

$$U_{val} := \{a \in \mathbb{K} \mid val(a) = 0\}.$$

Its **residue field** is defined as

$$R_{val} := A_{val}/\mathfrak{m}_{val}.$$

There is a natural map

$$\begin{array}{ccc} A_{val} & \longrightarrow & R_{val} \\ \varphi & \mapsto & \bar{\varphi} = \varphi \mod \mathfrak{m}_{val}. \end{array} \tag{6.1}$$

Lemma 6.1. *If $\mathbb{K}$ is algebraically closed, then its residue field is algebraically closed.*

Proof. Given $P(x) \in R_{val}[x] \setminus R_{val}$ let $Q(x) \in A_{val}[x] \setminus A_{val}$ be a pre-image of $P(x)$ via the map (6.1). Since $A_{val} \subset \mathbb{K}$, the polynomial Q has a root $k \in \mathbb{K}$.

Write $Q = \sum_{j=0}^{d} u_j x^j \in U_{val}[x]$ with $u_0, u_d \neq 0$. We have

$$val(u_j k^j) = j\, val(k).$$

Since $\sum_{j=0}^{d} u_j k^j = 0$, by lemma 3.1, there exists $j \neq j'$ such that $j\, val(k) = j'\, val(k)$. Then, $val(k) = 0$ or $val(k) = \infty$. This implies that k is an element of A_{val}.

The image of k under the map (6.1), is a root of P. □

Remark 6.2. *As a consequence of lemma 6.1 we have: If $\mathbb{K}$ is algebraically closed then R_{val} is infinite.*

7 The value of a polynomial at a point.

As we noted in remmark 4.1, we have

$$val(f(x)) \geq \mathcal{T}f(val(x)) \quad \text{for all} \quad x \in \mathbb{K}^N,$$

in this section we will see that, for each $\gamma \in \Gamma^N$, there exist $x \in val^{-1}(\gamma) \in \mathbb{K}^N$ for which the equality holds.

Lemma 7.1. *Let $f_1, \ldots, f_k$ be a finite set of non-zero Laurent polynomials in N variables with coefficients in R_{val}. There exists an N-tuple of non-zero elements $r \in (R_{val} \setminus \{0\})^N$ such that $f_i(r)$ is non-zero for each $i \in 1, \ldots, k$.*

Proof. Set $g := \prod_{i=1}^{k} f_i$, then g is a Laurent polynomial

$$g = \sum_{\alpha=(\alpha_1,\ldots,\alpha_N)\in\Lambda\subset\mathbb{Z}^N} r_\alpha x^\alpha, \quad r_\alpha \in R_{val},\ \#\Lambda < \infty$$

set $\beta := (1, \ldots, 1) - (\min_{\alpha\in\Lambda} \alpha_1, \ldots, \min_{\alpha\in\Lambda} \alpha_N)$. We have $x^\beta g \in < x^{(1,\ldots,1)} > R_{val}[x]$.

The set of zeroes of $f := x^\beta g - 1$ is a hypersurface of ${R_{val}}^N$ that doesn't intersect the coordinate hyperplanes. Since R_{val} is an algebraically closed field (lemma 6.1), there exists a point $r \in (R_{val} \setminus \{0\})^N$ where f vanishes.

We have:

$$f(r) = r^\beta g(r) - 1 = 0 \Rightarrow r^\beta \prod_{i=0}^{k} f_i(r) = 1 \Rightarrow f_i(r) \neq 0 \,\forall, i = 1 \dots k.$$

□

Lemma 7.2. *Let $f_1, \dots, f_k$ be a finite set of Laurent polynomials in N variables with coefficients in A_{val}. If one on the coefficients of each f_i is a unit, then there exists an N-tuple of units $u \in {U_{val}}^N$ such that $f_i(u)$ is a unit for each $i \in \{1, \dots, k\}$.*

Proof. Let $\bar{f}_i$ be the image of f_i in $R_{val}[x^*]$ via the natural morphism. That is

$$\begin{array}{cccc} \Phi : & A_{val}[x^*] & \longrightarrow & R_{val}[x^*] \\ & \sum_\alpha \varphi_\alpha x^\alpha & \mapsto & \sum_\alpha \bar{\varphi_\alpha} x^\alpha \end{array}$$

where $\bar{\varphi_\alpha}$ is the image of φ_α via the map (6.1).

Since at least one of the coefficients of f_i is a unit $\bar{f}_i$ is not zero. By lemma 7.1, there exists an N-tuple of non-zero elements $r \in (R_{val} \setminus \{0\})^N$ such that $\bar{f}_i(r)$ is non-zero for each $i \in \{1, \dots, k\}$. Take $x \in {A_{val}}^N$ such that $\bar{x} = r$ via the natural map (6.1).

We have $x \in {U_{val}}^N$ and $\Phi(f_i(x)) = \bar{f}_i(r) \neq 0$ implies $f_i(x) \in U_{val}$. □

Proposition 7.3. *Let $f_1, \dots, f_k$ be Laurent polynomials in N variables with coefficients in $\mathbb{K}$. Given an N-tuple $\gamma \in \Gamma^N$ there exists $x \in \mathbb{K}^N$ such that*

$$val(x) = \gamma \quad and \quad val(f_i(x)) = \mathcal{T} f_i(val(x))$$

for all $i \in \{1, \dots, k\}$.

Proof. Take $\phi \in \mathbb{K}^N$ and $\psi_i \in \mathbb{K}$ such that $val\phi = \gamma$ and $val\psi_i = \mathcal{T} f_i(\gamma)$. Set

$$g_i(x_1, \dots, x_N) := \frac{1}{\psi_i} f_i(\phi_1 x_1, \dots, \phi_N x_N).$$

Write $g_i = \sum_\alpha \varphi_{i,\alpha} x^\alpha$. We have $\mathcal{T} g_i(0, \dots, 0) = \bigoplus_\alpha val(\varphi_{i,\alpha}) = 0$. Then, for each i, there exists $\alpha_0^{(i)}$ such that $val(\varphi_{(i,\alpha_0^{(i)})}) = 0$ and $val(\varphi_{i,\alpha}) \geq 0$ for all α. That is $g_i \in A_{val}[x]$ and one of the coefficients is a unit. By lemma 7.2, there exists $u = (u_1, \dots, u_N) \in {U_{val}}^N$ such that $val(g_i(u)) = 0$. Then

$$val\left(\frac{1}{\psi_i} f_i(\phi_1 u_1, \dots, \phi_N u_N)\right) = 0 \Rightarrow val(f_i(\phi_1 u_1, \dots, \phi_N u_N)) = val(\psi_i) = \mathcal{T} f_i(\gamma).$$

Since $val((\phi_1 u_1, \dots, \phi_N u_N)) = \gamma$, we have the result.

□

8 The main theorem

Now we are ready to extend the theorem proved by Einsieder, Kapranov and Lind.

Theorem 8.1. *Let $\mathbb{K}$ be an algebraically closed valued field. The tropical hypersurface associated to a polynomial $f \in \mathbb{K}[x^*]$ is the hypersurface associated to the tropicalization of f. That is,*

$$\mathbf{TV}f = \mathcal{VT}f.$$

Proof. The inclusion $\mathbf{TV}f \subset \mathcal{VT}f$ is just proposition 4.2.

To see the other inclusion:
Given $\gamma \in \mathcal{VT}f$ we want to see that there exists $\phi = (\phi_1, \ldots, \phi_N)$ such that $val\phi = \gamma$ and $f(\phi) = 0$.

$\gamma \in \mathcal{VT}f$ if and only if there exist $\alpha^{(0)} \neq \alpha^{(1)} \in \mathcal{D}_\gamma(\mathcal{T}f)$. The vector $\alpha^{(0)}$ is different from $\alpha^{(1)}$ if and only if one of the coordinates is different. Let us suppose that ${\alpha^{(0)}}_N \neq {\alpha^{(1)}}_N$. Write f as in (4.1) and set $\Lambda := \{\alpha_N \in \mathbb{Z} \mid \alpha \in \mathcal{E}(f)\}$. The polynomial f may be rewritten in the form

$$f = \sum_{i\in\Lambda} h_i(x_1, \ldots, x_{N-1}){x_N}^i \text{ where } h_i = \sum_{(\beta,i)\in\mathcal{E}(f)} \varphi_{(\beta,i)} x^{(\beta,0)}.$$

Write $\gamma = (\mu, \eta) \in \Gamma^{N-1} \times \Gamma$, and choose $y \in \mathbb{K}^{N-1}$ such that $valy = \mu$ and $val(h_i(y)) = \mathcal{T}h_i(\mu)$ (proposition 7.3). Set

$$g := \sum_{i\in\Lambda} h_i(y){x_N}^i \in \mathbb{K}[x_N].$$

We have

$$\begin{aligned} \mathcal{T}f(\gamma) &= \oplus_{\alpha\in\mathcal{E}(f)} val(\varphi_\alpha) \odot \gamma^\alpha \\ &= \oplus_{i\in\Lambda} \left(\oplus_{(\beta,i)\in\mathcal{E}(f)} val(\varphi_{(\beta,i)}) \odot \mu^\beta\right) \odot \eta^i \\ &= \oplus_{i\in\Lambda} \mathcal{T}h_i(\mu) \odot \eta^i \\ &= \oplus_{i\in\Lambda} val(h_i(y)) \odot \eta^i \\ &= \mathcal{T}g(\eta). \end{aligned}$$

Write $\alpha^{(k)} = (\beta^{(k)}, j^{(k)}) \in \mathbb{Z}^{N-1} \times \mathbb{Z}$, $k = 0, 1$. We have

$$\begin{aligned} \mathcal{T}g(\eta) &= val(\varphi_{\alpha^{(k)}}) \odot \gamma^{\alpha^{(k)}} = val(\varphi_{(\beta^{(k)},j^{(k)})}) \odot \mu^{\beta^{(k)}} \odot \eta^{j^{(k)}} \\ &= \mathcal{T}h_i(\mu) \odot \eta^{j^{(k)}} = val(h_i(y)) \odot \eta^{j^{(k)}}. \end{aligned}$$

Since $j^{(0)} \neq j^{(1)}$, the element $\eta \in \Gamma$ is in the variety $\mathcal{VT}g$, then, by corollary 5.3, there exists $z \in \mathbb{K}$ such that $valz = \eta$ and $g(z) = 0$.

We have $\phi := (y, z) \in \mathbb{K}^N$, $val(y, z) = \gamma$ and $f(y, z) = g(z) = 0$. □

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