KÄHLER DIFFERENTIALS OF EXTENSIONS OF VALUATION RINGS AND DEEPLY RAMIFIED FIELDS

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Abstract. Assume that (L, v) is a finite Galois extension of a valued field (K, v). We give an explicit construction of the valuation ring $\mathcal{O}_L$ of L as an $\mathcal{O}_K$-algebra, and an explicit description of the module of relative Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ when $L|K$ is a Kummer extension of prime degree or an Artin-Schreier extension, in terms of invariants of the valuation and field extension. The case when this extension has nontrivial defect was solved in a recent paper by the authors with Anna Rzepka. The present paper deals with the complementary (defectless) case. The results are known classically for (rank 1) discrete valuations, but our systematic approach to non-discrete valuations (even of rank 1) is new.

Using our results from the prime degree case, we characterize when $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ holds for an arbitrary finite Galois extension of valued fields. As an application of these results, we give a simple proof of a theorem of Gabber and Ramero, which characterizes when a valued field is deeply ramified. We further give a simple characterization of deeply ramified fields with residue fields of characteristic $p > 0$ in terms of the Kähler differentials of Galois extensions of degree p.

1. Introduction

The main goal of this paper is to study for algebraic extensions of valued fields the relation between their properties and the vanishing of the Kähler differentials of the extensions of their valuation rings.

All of our results are for arbitrary valuations; in particular, we have no restrictions on their rank or value groups. Ranks higher than 1 appear in a natural way when local uniformization, the local form of resolution of singularities, is studied. Deeply ramified fields of infinite rank appear in model theoretic investigations of the tilting construction, as presented by Jahnke and Kartas in [9]. Therefore, we do not restrict our computations to rank 1, thereby indicating how Kähler differentials can be computed in higher rank.

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The notation we use is mostly standard in valuation theory or commutative algebra. We review notation and some main notions in Section 2.1.

Our principal result is the following Theorem 1.1, which deals with extensions $L|K$ which are Kummer extensions of prime degree or Artin-Schreier extensions. In this paper we compute the Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ for such extensions when they are **unibranched** and **defectless**, which means that the extension of v from K to L is unique and $[L:K]=(vL:vK)[Lv:Kv]$ holds. For the complementary case of such extensions with **nontrivial defect**, which in this special case means that $(vL:vK)=1=[Lv:Kv]$, see Theorems 4.5 and 4.6 in the recent paper [2] by the authors with Anna Rzepka. The description of these Kähler differentials is known classically for (rank 1) discrete valuations, but our systematic and detailed description is new, even for arbitrary valuations of rank 1. By $\Omega_{B|A}$ we denote the Kähler differentials, i.e., the module of relative differentials, when A is a ring and B is an A-algebra.

Theorem 1.1. *Let $(L|K,v)$ be a finite Galois extension of valued fields where $L|K$ is a Kummer extension of prime degree or an Artin-Schreier extension. Then there is an explicit description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in terms of invariants of the valuation v and field extension $L|K$. This gives a characterization of when $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.*

The proof of Theorem 1.1 is given in Section 5, after Proposition 5.7. The analysis of the cases in Theorem 1.1 begins with explicit constructions of the extensions $\mathcal{O}_L|\mathcal{O}_K$ of valuation rings, as a chain of simple ring extensions. This construction depends strongly on the type of extension. For the case of defectless extensions it is given in Section 3.1; to the best of our knowledge, it is new and of independent interest. A result from [2], stated in Proposition 4.2 of the present paper, is then used to give the explicit description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in Sections 4.3 to 4.7.

Annihilators of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$, differents $\mathcal{D}_{\mathcal{O}_L|\mathcal{O}_K}$ and traces of the maximal ideal $\mathcal{M}_L$ of $\mathcal{O}_L$ for the extensions appearing in Theorem 1.1 have been determined in [2] in the case of nontrivial defect. (Note that before [2, Theorem 1.6] we meant to write "We denote the annihilator of an $\mathcal{O}_{\mathcal{E}}$-module M by $\operatorname{ann} M$".) The case of defectless extensions will be addressed in [12].

As an application of Theorem 1.1, we prove in Section 5 a criterion for the vanishing of the Kähler differentials for arbitrary finite Galois extensions; see part 2) of Theorem 5.3. Finally, in Section 6 all of these results are combined into the proof of the next theorem.

Take a valued field (K,v) with valuation ring $\mathcal{O}_K$. Choose any extension of v to the separable-algebraic closure K^{sep} of K and denote the valuation ring of K^{sep} with respect to this extension by $\mathcal{O}_{K^{sep}}$. Note that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K}$ does not depend on the choice of the extension of v since all of the possible extensions are conjugate. Gabber and Ramero prove the following result (see [7, Theorem 6.6.12 (vi)]):

Theorem 1.2. *For a valued field (K,v),*

$$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0 \tag{1}$$

holds if and only if it satisfies the following:

(DRvg) *whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);*

(DRvr) *if* $\operatorname{char} Kv = p > 0$*, then the homomorphism*

$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \tag{2}$$

is surjective, where $(\hat{K}, \hat{v})$ *is the completion of* (K, v) *for the valuation topology and* $\mathcal{O}_{\hat{K}}$ *denotes its valuation ring.*

Theorem 1.2 and the papers [29, 30] of Thatte were the motivation for our work in the present paper and in [2].

For the purpose of the proof of Theorem 1.2, we define (as we have done in [13]) a nontrivially valued field (K, v) to be a **deeply ramified field** if the conditions (DRvg) and (DRvr) hold. In [13], related classes of valued fields are introduced by weakening or strengthening condition (DRvg).

Note that by [26, Definition 3.1] a perfectoid field is a complete nondiscrete rank 1 valued field of positive residue characteristic such that the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. In rank 1, condition (DRvg) just says that the value group is not discrete. Consequently, when using (DRvg) and (DRvr) for the definition of deeply ramified fields, it is immediately seen that every perfectoid field is a deeply ramified field.

The proof of Theorem 1.2 in [7] is a demonstration of the power of the techniques of almost ring theory, and uses a large part of the theory developed in [7]. The proof is by reduction to the rank 1 case, where the techniques of almost ring theory are most applicable.

Our alternative proof of Theorem 1.2 in the present paper uses only methods from valuation theory and commutative algebra, and does not rely on techniques or results from almost ring theory. We hope that our proof makes this beautiful theorem accessible to a wider audience. Further, our proof yields the following additional new result. A criterion for a valued field (K, v) to be deeply ramified that only works with extensions of prime degree $p = \operatorname{char} Kv$ appears to be more easily accessible than the criterion $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$, in particular from the model theoretic point of view.

Theorem 1.3. *Let* (K, v) *be a valued field of residue characteristic* $p > 0$*. If* K *has characteristic* 0*, then assume in addition that it contains all* p*-th roots of unity. Then* (K, v) *is a deeply ramified field if and only if* $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ *for all unibranched Galois extensions* $(L|K, v)$ *of prime degree* p*.*

Let us mention two main ingredients of the proof. Theorem 1.10 (1) of [13] implies that if (K, v) is a deeply ramified field with $\operatorname{char} Kv = p > 0$, then each of its Galois defect extensions of degree p has independent defect. Hence we can infer the following result from [2, Theorem 1.4]:

Theorem 1.4. *Take a deeply ramified field* (K, v) *with* $\operatorname{char} Kv = p > 0$*; if* $\operatorname{char} K = 0$*, then assume that* K *contains all* p*-th roots of unity. Then every Galois extension* $(L|K, v)$ *of degree* p *with nontrivial defect satisfies* $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$*.*

This result will be complemented in the present paper by showing that for a deeply ramified field (K, v), every unibranched defectless Galois extension $(L|K, v)$ of prime degree p satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. Then Section 5 connects our results for Galois extensions of prime degree with $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K}$. There, the main approach is the study of Kähler differentials of towers of Galois extensions. In order to go upward through such towers, we make use of the following fact, which Gabber and Ramero deduce from Theorem 1.2 (see [7, Corollary 6.6.16 (i)]). However, as we want to prove Theorem 1.2, we refer the reader to Theorem 1.5 of [13] whose proof is done by a direct valuation theoretical computation not involving any Kähler differentials.

Theorem 1.5. *Every algebraic extension of a deeply ramified field is again a deeply ramified field.*

It should be noted that Theorem 1.5 also holds for the roughly deeply ramified and the semitame fields that are introduced in [13].

In [22], Novacoski and Spivakovsky use the theory of key polynomials to derive a presentation of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ for finite pure extensions $(L|K,v)$ under the condition $vL = vK$. Applying this presentation to Artin-Schreier and Kummer extensions, they derive results similar to our results presented in [2] and in this paper. Recently they also dealt with the case of $vL \neq vK$ by a different approach, not based on the use of key polynomials. See also [18, 19, 21].

To conclude this introduction, let us give some interesting examples. Let ζ_p denote a primitive p-th root of unity.

Example 1.6. Choose a prime $p > 2$. The field $K = \mathbb{Q}_p(\zeta_p, p^{1/p^n} \mid n \in \mathbb{N})$, equipped with the unique extension of the p-adic valuation of $\mathbb{Q}_p$, is known to be a deeply ramified field. The Kummer extension $(K(\sqrt{p})|K, v_p)$ is tamely ramified, as $(v_pK(\sqrt{p}) : v_pK) = 2 \neq p$. By an application of Theorem 4.8 below, $\Omega_{\mathcal{O}_{K(\sqrt{p})}|\mathcal{O}_K} = 0$. The fact that this holds in spite of the ramification is due to the value group v_pK being dense, as it is p-divisible.

Analoguously, we can consider the field $K = \mathbb{F}_p((t))(t^{1/p^n} \mid n \in \mathbb{N})$, equipped with the unique extension of the t-adic valuation of $\mathbb{F}_p((t))$. This field is a deeply ramified field since it is perfect of positive characteristic. Again, the extension $(K(\sqrt{t})|K, v_t)$ is tamely ramified as $(v_tK(\sqrt{t}) : v_tK) = 2 \neq p$, and v_tK is dense. By Theorem 4.8 below, $\Omega_{\mathcal{O}_{K(\sqrt{t})}|\mathcal{O}_K} = 0$.

Finally, here is an example of a Kummer extension $(L|K,v)$ with wild ramification and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

Example 1.7. Take a prime $p > 2$ and set $K = \mathbb{Q}(\zeta_p)(t^{1/2^n} \mid n \in \mathbb{N})$. Let v_p denote the p-adic valuation on $\mathbb{Q}(\zeta_p)$ and v_t the t-adic valuation on K. Now consider the valuation $v := v_t \circ v_p$ on K, where "$v_t \circ v_p$" denotes the valuation associated with the composition of the t-adic place on K and the p-adic place on $\mathbb{Q}(\zeta_p)$. Set $L = K(t^{1/p})$ and extend v to L. Then $(L|K,v)$ is a Kummer extension of degree p with ramification index $p = \operatorname{char} Kv$. Nevertheless, Theorem 4.8 shows that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

2. Preliminaries

2.1. **Notation.**

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the unique maximal ideals of $\mathcal{O}_L$ and $\mathcal{O}_K$. The value group of the valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av. In order to simplify notation by reducing the use of brackets, our convention will be that $v\ldots$ denotes the value of the term following "v", and $\ldots v$ denotes the residue of the term preceding "v"; for example, $vxy = v(xy)$ and $xyv = (xy)v$. A **final segment** of vL is a subset $S \subseteq vL$ such that $\gamma \leq \delta$ with $\gamma, \delta \in vL$ and $\gamma \in S$ implies that $\delta \in S$.

The **rank** of a valued field (K,v) is the order type of the chain of proper convex subgroups of its value group vK. We say that $(L|K,v)$ is unibranched if the extension of v from K to L is unique.

2.2. **Convex subgroups and archimedean components.**

Take an ordered abelian group Γ. Two elements $\alpha, \beta \in \Gamma$ are **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|\alpha| \geq |\beta|$ and $n|\beta| \geq |\alpha|$, where $|\alpha| := \max\{\alpha, -\alpha\}$. Note that if $0 < \alpha < \beta < n\alpha$ for some $n \in \mathbb{N}$, then α, β and $n\alpha$ are (mutually) archimedean equivalent. If every two nonzero elements of Γ are archimedean equivalent, then we say that Γ is **archimedean ordered**. This holds if and only if Γ admits an order preserving embedding in the ordered additive group of the real numbers.

We call Γ **discretely ordered** if every element in Γ has an immediate successor; this holds if and only if Γ contains a smallest positive element. In contrast, Γ is called **dense** if $\Gamma \neq \{0\}$ and for every two elements $\alpha < \gamma$ in Γ there is $\beta \in \Gamma$ such that $\alpha < \beta < \gamma$. If Γ is archimedean ordered and dense, then for every $i \in \mathbb{N}$ there is even some $\beta_i \in \Gamma$ such that $\alpha < i\beta_i < \gamma$; this can be easily proven via an embedding of Γ in the real numbers. Every ordered abelian group is discrete if and only if it is not dense.

For $\gamma \in \Gamma$, we define $\mathcal{C}_\Gamma(\gamma)$ to be the smallest convex subgroup of Γ containing γ, and for $\gamma \neq 0$, $\mathcal{C}_\Gamma^+(\gamma)$ to be the largest convex subgroup of Γ not containing γ. Note that $\mathcal{C}_\Gamma(0) = \{0\}$. The convex subgroups of Γ form a chain under inclusion, and the union and intersection of any collection of convex subgroups are again convex subgroups; this guarantees the existence of $\mathcal{C}_\Gamma(\gamma)$ and $\mathcal{C}_\Gamma^+(\gamma)$.

We have that $\mathcal{C}_\Gamma^+(\gamma) \subsetneq \mathcal{C}_\Gamma(\gamma)$ and that $\mathcal{C}_\Gamma^+(\gamma)$ and $\mathcal{C}_\Gamma(\gamma)$ are consecutive, that is, there is no convex subgroup of Γ lying properly between them. As a consequence,

$$\mathcal{A}_\Gamma(\gamma) \;:=\; \mathcal{C}_\Gamma(\gamma)/\mathcal{C}_\Gamma^+(\gamma)$$

for $\gamma \neq 0$ is an archimedean ordered group; we call it the **archimedean component of** Γ **associated with** γ. Two elements $\alpha, \beta \in \Gamma$ are archimedean equivalent if and only if

$$\mathcal{C}_\Gamma(\alpha) \;=\; \mathcal{C}_\Gamma(\beta)\,,$$

and then it follows that $\mathcal{A}_\Gamma(\alpha) = \mathcal{A}_\Gamma(\beta)$. In particular, $\mathcal{C}_\Gamma(\alpha) = \mathcal{C}_\Gamma(n\alpha)$ and $\mathcal{A}_\Gamma(\alpha) = \mathcal{A}_\Gamma(n\alpha)$ for all $\alpha \in \Gamma$ and all $n \in \mathbb{Z} \setminus \{0\}$.

Assume now that Γ is an ordered abelian group containing a subgroup $\Delta \neq \{0\}$. We say that Δ **is dense in** Γ if for every two elements $\alpha < \gamma$ in Γ there is $\beta \in \Delta$ such that $\alpha < \beta < \gamma$; this implies that Γ and Δ are dense. If Γ is archimedean ordered, then so is Δ, and Δ is dense in Γ if and only if it is dense.

For every $\gamma \in \Gamma$, $\mathcal{C}_\Gamma(\gamma) \cap \Delta$ and $\mathcal{C}_\Gamma^+(\gamma) \cap \Delta$ are convex subgroups of Δ; the quotient $\mathcal{C}_\Gamma(\gamma) \cap \Delta \,/\, \mathcal{C}_\Gamma^+(\gamma) \cap \Delta$ is either trivial or archimedean ordered. If γ is archimedean equivalent to $\delta \in \Delta$, then this quotient is equal to $\mathcal{A}_\Delta(\delta)$.

For each $\delta \in \Delta$ the function given by

$$\mathcal{A}_\Delta(\delta) \ni \alpha + \mathcal{C}_\Delta^+(\delta) \;\mapsto\; \alpha + \mathcal{C}_\Gamma^+(\delta) \in \mathcal{A}_\Gamma(\delta)$$

is an injective order preserving homomorphism. This follows from the fact that the kernel of the homomorphism $\mathcal{C}_\Delta(\delta) \ni \alpha \mapsto \alpha + \mathcal{C}_\Gamma^+(\delta) \in \mathcal{A}_\Gamma(\delta)$ is the convex subgroup $\mathcal{C}_\Delta^+(\delta) = \mathcal{C}_\Gamma^+(\delta) \cap \Delta$. In abuse of notation, we write $\mathcal{A}_\Delta(\delta) = \mathcal{A}_\Gamma(\delta)$ if this homomorphism is surjective.

2.3. **Artin-Schreier and Kummer extensions.**

We say that a valued field (K, v) has **equal characteristic** if $\operatorname{char} K = \operatorname{char} Kv$, and **mixed characteristic** if $\operatorname{char} K = 0$ and $\operatorname{char} Kv > 0$. Every Galois extension of degree p of a field K of characteristic $p > 0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** $X^p - X - b$ with $b \in K$. For every $c \in K$, also $\vartheta - c$ is an Artin-Schreier generator as its minimal polynomial is $X^p - X - b + c^p - c$. Every Galois extension of prime degree q of a field K of characteristic different from q which contains all q-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^q \in K$. For these facts, see [15, Chapter VI, §6].

A **1-unit** in a valued field (K, v) is an element of the form $u = 1 + b$ with $b \in \mathcal{M}_K$; in other words, u is a unit in $\mathcal{O}_K$ with residue 1. We note that if u is a 1-unit, then also u^{-1} is a 1-unit, and if $v(u - c) > vu = 0$ for some $c \in K$, then also c is a 1-unit. Conversely, if u and c are 1-units, then $v(u - c) > 0$.

Remark 2.1. Take a Kummer extension $(L|K, v)$ of degree p with any Kummer generator η. Assume that $v\eta \in vK$, so that there is $c_1 \in K$ such that $vc_1 = -v\eta$, whence $vc_1\eta = 0$. Assume further that $c_1\eta v \in Kv$, so that there is $c_2 \in K$ such that $c_2 v = (c_1\eta v)^{-1}$. Then $vc_2c_1\eta = 0$ and $c_2c_1\eta v = 1$. Furthermore, $K(c_2c_1\eta) = K(\eta)$ and $(c_2c_1\eta)^p = c_2^p c_1^p \eta^p \in K$. Hence $c_2c_1\eta$ is a Kummer generator of $(L|K, v)$ and a 1-unit. Therefore $v(c_2c_1\eta - 1) > 0$, whence $v(\eta - (c_2c_1)^{-1}) > v(c_2c_1)^{-1} = v\eta$. Consequently, for $c := (c_2c_1)^{-1} \in K$ we have $v(\eta - c) > v\eta$.

We will need the following facts. If $(L|K, v)$ is a unibranched defectless extension of prime degree p, then either $\mathrm{e}\,(L|K, v) = 1$ and $\mathrm{f}\,(L|K, v) = p$, or $\mathrm{f}\,(L|K, v) = 1$ and $\mathrm{e}\,(L|K, v) = p$. For $q \in \mathbb{N}$ let ζ_q denote a primitive q-th root of unity. We note that if $L|K$ is a Kummer extension of degree q, then K contains all q-th roots of unity. For a proof of the next well known results, see [2, Lemma 2.5].

Lemma 2.2. *Take $q \in \mathbb{N}$ and a valued field (K, v) containing ζ_q. Then*

$$\prod_{i=1}^{q-1} (1 - \zeta_q^i) \;=\; q\,. \tag{3}$$

If in addition q is prime, then

$$v(\zeta_q - 1) \;=\; \frac{vq}{q-1}\,. \tag{4}$$

Lemma 2.3. *Take a unibranched Kummer extension $(L|K,v)$ of prime degree q with Kummer generator η. Then for all $c\in K$,*

$$v(\eta-c)\;\leq\;v\eta(\zeta_q-1)\;=\;v\eta+\frac{vq}{q-1}\;. \tag{5}$$

Assume in addition that $\mathrm{f}\,(L|K,v)=q=\mathrm{char}\,Kv$ and $c,\tilde{c}\in K$ are such that $v\tilde{c}(\eta-c)=0$ and $\tilde{c}(\eta-c)v$ generates the residue field extension $Lv|Kv$. Then $Lv|Kv$ is inseparable if and only if $v(\eta-c)<v\eta(\zeta_q-1)$, and it is separable if and only if $v(\eta-c)=v\eta(\zeta_q-1)$.

Proof. Take $c\in K$ and $\sigma\in\mathrm{Gal}\,L|K$ such that $\sigma\eta=\zeta_q\eta$. Then

$$\eta-c-\sigma(\eta-c)\;=\;\eta-\sigma\eta\;=\;\eta(1-\zeta_q)\;. \tag{6}$$

Hence if $v(\eta-c)>v\eta(1-\zeta_q)$, then

$$v\sigma(\eta-c)\;=\;v(\eta-c-\eta(1-\zeta_q))\;=\;\min\{v(\eta-c),v\eta(1-\zeta_q)\}\;=\;v\eta(1-\zeta_q)\;<\;v(\eta-c),$$

which shows that $v\sigma\neq v$, i.e., the extension is not unibranched. This contradiction proves the first assertion.

Now assume the situation as in the second part of the lemma. Since $L|K$ is a Galois extension, $Lv|Kv$ is a normal extension, with its automorphisms induced by those of $L|K$. Take σ to be a generator of $\mathrm{Gal}\,L|K$. Via the residue map, its action on $\mathcal{O}_L^\times$ induces a generator $\bar{\sigma}$ of the automorphism group of $Lv|Kv$. From (6) we infer that

$$\tilde{c}(\eta-c)-\sigma\tilde{c}(\eta-c)\;=\;\tilde{c}\eta(1-\zeta_q)\;.$$

It follows that $\bar{\sigma}$ is the identity, i.e, $Lv|Kv$ is inseparable, if and only if $v\tilde{c}\eta(1-\zeta_q)>0$. This is equivalent to $v(\eta-c)=-v\tilde{c}<v\eta(1-\zeta_q)$. Since $v(\eta-c)>v\eta(1-\zeta_q)$ is impossible according to (5), we can conclude that the residue field extension is separable if and only if $v(\eta-c)=v\eta(1-\zeta_q)$. □

Proposition 2.4. *Take a Kummer extension $(L|K,v)$ of prime degree $q\neq\mathrm{char}\,Kv$.*

1) If $\mathrm{f}\,(L|K,v)=q$, then there is a Kummer generator $\eta\in\mathcal{O}_L^\times$ such that ηv is a Kummer generator of $Lv|Kv$.

2) If $\mathrm{e}\,(L|K,v)=q$, then there is a Kummer generator $\eta\in L$ such that $v\eta$ generates the value group extension, that is, $vL=vK+\mathbb{Z}v\eta$.

Proof. Since $q\neq\mathrm{char}\,Kv$, we have $vq=0$ and thus $v(1-\zeta_q)=0$.

1): Take a Kummer generator η. Since $\mathrm{f}\,(L|K,v)=q$, we have that $vL=vK$. Therefore, as shown in Remark 2.1, we can assume that $v\eta=0$. The reduction of the minimal polynomial of η over K to the residue field is $X^q-\eta^qv$ with $\eta^qv\neq 0$. Suppose that this polynomial has a root in Kv. Since $\mathrm{Gal}\,Lv|Kv$ is cyclic (generated by the reduction of a generator of $\mathrm{Gal}\,L|K$), it follows that $X^q-\eta^qv$ splits. Hence its root ηv lies in Kv and there is $c\in K$ such that $cv=\eta v$. It follows that $v(\eta-c)>0=v\eta(1-\zeta_q)$, so by Lemma 2.3, $(L|K,v)$ is not unibranched. As this contradicts our assumption, $X^q-\eta^qv$ must be irreducible (cf. [27]), which means that ηv generates the extension $Lv|Kv$. Since $\eta^q\in K$, we have that $(\eta v)^q\in Kv$, i.e., ηv is a Kummer generator of $Lv|Kv$.

2) Take a Kummer generator η. We will show that $v\eta\notin vK$; as q is prime, it then follows that $vL=vK+\mathbb{Z}v\eta$. Suppose that $v\eta\in vK$. Since $\mathrm{e}\,(L|K,v)=q=[L:K]$, we have that $Lv=Kv$. Thus as shown in Remark 2.1, there is some $c\in K$

such that $v(\eta - c) > v\eta = v\eta(1-\zeta_q)$. As in the proof of part 1), this leads to a contradiction. Hence $v\eta \notin vK$, as asserted. □

For the next lemma, see [14, Lemma 2.1] and the proof of [10, Theorem 2.19].

Lemma 2.5. *If $(L|K,v)$ is a finite unibranched defectless extension, then for every element $x \in L$ the set*

$$v(x-K) \;:=\; \{v(x-c) \mid c \in K\}$$

admits a maximal element. If $c \in K$ is such that $v(x-c)$ is maximal, then $v(x-c) \notin vK$ or otherwise, for every $\tilde{c} \in K$ such that $v\tilde{c}(x-c) = 0$ we have $\tilde{c}(x-c)v \notin Kv$.

Using this lemma, we prove:

Proposition 2.6. *1) Take a valued field (K,v) of equal positive characteristic p and a unibranched defectless Artin-Schreier extension $(L|K,v)$.*

If $\mathrm{f}\,(L|K,v) = p$, then the extension has an Artin-Schreier generator ϑ of value $v\vartheta \le 0$ such that $Lv = Kv(\tilde{c}\vartheta v)$ for every $\tilde{c} \in K$ with $v\tilde{c}\vartheta = 0$; the extension $Lv|Kv$ is separable if and only if $v\vartheta = 0$.

If $\mathrm{e}\,(L|K,v) = p$, then the extension has an Artin-Schreier generator ϑ such that $vL = vK + \mathbb{Z}v\vartheta$. Every such ϑ satisfies $v\vartheta < 0$.

2) Take a valued field (K,v) of mixed characteristic and a unibranched defectless Kummer extension $(L|K,v)$ of degree $p = \mathrm{char}\, Kv$. Then the extension has a Kummer generator η such that:

a) if $\mathrm{f}\,(L|K,v) = p$, then either ηv generates the residue field extension, in which case it is inseparable, or η is a 1-unit and for some $\tilde{c} \in K$, $\tilde{c}(\eta-1)v$ generates the residue field extension;

b) if $\mathrm{e}\,(L|K,v) = p$, then either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta-1)$ generates the value group extension.

Proof. 1): Take any Artin-Schreier generator y of $(L|K,v)$. Then by Lemma 2.5 there is $c \in K$ such that either $v(y-c) \notin vK$, or for every $\tilde{c} \in K$ such that $v\tilde{c}(x-c) = 0$ we have $\tilde{c}(y-c)v \notin Kv$. Since p is prime, in the first case it follows that $\mathrm{e}\,(L|K,v) = p$ and that $v(y-c)$ generates the value group extension. In the second case it follows that $\mathrm{f}\,(L|K,v) = p$ and that $\tilde{c}(y-c)v$ generates the residue field extension. In both cases, $\vartheta = y - c$ is an Artin-Schreier generator. Let $\vartheta^p - \vartheta = b \in K$.

Assume that $\mathrm{f}\,(L|K,v) = p$. If $v\vartheta < 0$, then $v(\vartheta^p - b) = v\vartheta > pv\vartheta = v\vartheta^p$, whence $v((\tilde{c}\vartheta)^p - \tilde{c}^p b) = v\tilde{c}^p\vartheta > v(\tilde{c}\vartheta)^p$ for $\tilde{c} \in K$ with $v\tilde{c}\vartheta = 0$ and therefore, $(\tilde{c}\vartheta)^p v = \tilde{c}^p bv \in Kv$. In this case, the residue field extension is inseparable. Now assume that $v\vartheta \ge 0$ and hence also $vb \ge 0$. The reduction of $X^p - X - b$ to $Kv[X]$ is a separable polynomial, so $Lv|Kv$ is separable. The polynomial $X^p - X - bv$ cannot have a zero in Kv, since otherwise the p distinct roots of this polynomial give rise to p distinct extensions of v from K to L, contradicting our assumption that $(L|K,v)$ is unibranched. Consequently, $bv \ne 0$, whence $vb = 0$ and $v\vartheta = 0$.

Assume that $\mathrm{e}\,(L|K,v) = p$. If $v\vartheta \ge 0$, then $vb \ge 0$ and ϑv is a root of $X^p - X - bv$. If this polynomial does not have a zero in Kv, then ϑv generates a nontrivial residue field extension, contradicting our assumption that $\mathrm{e}\,(L|K,v) = p$. If the polynomial has a zero in Kv, then similarly as before one deduces that $(L|K,v)$ is not unibranched, contradiction. Hence $v\vartheta < 0$.

2): Take any Kummer generator y of $(L|K,v)$. If there is a Kummer generator η such that $v\eta\notin vK$, then it follows as before that $\mathrm{e}\,(L|K,v)=p$ and that $v\eta$ generates the value group extension. Now assume that there is no such η.

If there is a Kummer generator y and some $\tilde{c}\in K$ such that $v\tilde{c}y=0$ and $\tilde{c}yv\notin Kv$, then it follows as before that $\mathrm{f}\,(L|K,v)=p$ and that $\tilde{c}yv$ generates the residue field extension. We set $\eta=\tilde{c}y$ and observe that also η is a Kummer generator. Since $(\eta v)^p\in Kv$, $Lv|Kv$ is purely inseparable in this case.

Now assume that the above cases do not appear, and choose an arbitrary Kummer generator y of $(L|K,v)$. Consequently, we have that $vy\in vK$ and $\tilde{c}yv\in Kv$ for all $\tilde{c}\in K$ with $v\tilde{c}y=0$. Then as described in Remark 2.1, there are $c_1,c_2\in K$ such that c_2c_1y is a Kummer generator of $(L|K,v)$ which is a 1-unit. We replace y by c_2c_1y.

By Lemma 2.5 there is $c\in K$ such that $v(y-c)$ is maximal in $v(y-K)$ and either $v(y-c)\notin vK$ or there is some $\tilde{c}\in K$ such that $v\tilde{c}(y-c)=0$ and $\tilde{c}(y-c)v\notin Kv$. Since y is a 1-unit, we know that $v(y-1)>0$, hence also $v(y-c)>0=vy$, showing that also c is a 1-unit. Then $\eta:=c^{-1}y$ is again a Kummer generator of $(L|K,v)$ which is a 1-unit. Since $vc=0$, we know that $v(\eta-1)=vc(\eta-1)=v(y-c)$. Hence if $v(y-c)\notin vK$, then $v(\eta-1)$ generates the value group extension.

Now assume that there is $\tilde{c}\in K$ such that $v\tilde{c}(y-c)=0$ and $\tilde{c}(y-c)v\notin Kv$. Since c is a 1-unit, it follows that $v\tilde{c}(\eta-1)=v\tilde{c}c(\eta-1)=v\tilde{c}(y-c)=0$ and $\tilde{c}(\eta-1)v=\tilde{c}c(\eta-1)v=\tilde{c}(y-c)v$. We find that $\tilde{c}(\eta-1)v$ generates the residue field extension. □

2.4. **Ramification ideals.**

Take a unibranched Galois extension $\mathcal{E}=(L|K,v)$ and let $G=\mathrm{Gal}\,L|K$ denote its Galois group. An $\mathcal{O}_L$-ideal

$$\left(\frac{\sigma b-b}{b}\;\Big|\;\sigma\in H\,,\;b\in L^\times\right)\,, \tag{7}$$

where H is a nontrivial subgroup of G, is called a **ramification ideal** of $\mathcal{E}$. Hence if $\mathcal{E}$ is of prime degree, then it has a unique ramification ideal, which we denote by $I_{\mathcal{E}}$. For further background on ramification ideals, see [2, 13, 11]. In [11], the following is shown:

Proposition 2.7. *Take a unibranched defectless Galois extension $\mathcal{E}=(L|K,v)$ of prime degree q.*

1) Let $(L|K,v)$ be an Artin-Schreier extension and ϑ an Artin-Schreier generator as in part 1) of Proposition 2.6. Then

$$I_{\mathcal{E}}\;=\;\left(\frac{1}{\vartheta}\right)\,. \tag{8}$$

We have $I_{\mathcal{E}}=\mathcal{O}_L$ if and only if $v\vartheta=0$, and this holds if and only if $Lv|Kv$ is separable of degree q.

2) Let $(L|K,v)$ be a Kummer extension. Then there are two cases:

a) Let η be a Kummer generator as in part 2)a) of Proposition 2.6. Then

$$I_{\mathcal{E}}\;=\;(\zeta_q-1)\,. \tag{9}$$

b) Let η be a Kummer generator as in part 2) b) of Proposition 2.6. Then

$$I_{\mathcal{E}} \;=\; \left(\frac{\zeta_q - 1}{\eta - 1}\right) . \tag{10}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v(\eta-1) = v(\zeta_q - 1)$, and this holds if and only if $Lv|Kv$ is separable of degree q.

Also the ramification ideals of Artin-Schreier defect extensions and Kummer defect extensions of prime degree are computed in [11].

3. Generation of extensions of valuation rings

In this section we will assume that $\mathcal{E} = (L|K,v)$ is a finite unibranched defectless extension and develop the groundwork needed for the computation of the Kähler differential of $\mathcal{E}$ in Sections 4.4 to 4.7.

3.1. Generating the $\mathcal{O}_K$-algebra $\mathcal{O}_L$.

In order to use Proposition 4.2 below to compute $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$, we need to present $\mathcal{O}_L$ as a union over a chain of simple ring extensions of $\mathcal{O}_K$. We consider finite extensions $\mathcal{E} = (L|K,v)$ of degree q that satisfy

$$[L:K] \;=\; [Lv:Kv] \quad \text{or} \quad [L:K] \;=\; (vL:vK)\,.$$

Such extensions are unibranched and defectless. We distinguish the following two cases:

Case (DL1): $[L:K] = [Lv:Kv]$. In this case, we can choose elements $a_1,\dots,a_q \in \mathcal{O}_L^\times$ such that $a_1v,\dots,a_qv$ form a basis of $Lv|Kv$. Then $a_1,\dots,a_q$ form a valuation basis of $(L|K,v)$, which by definition means that every element of $z \in L$ can be written as

$$z \;=\; c_1a_1 + \ldots + c_qa_q \quad \text{with} \quad vz \;=\; \min_i vc_ia_i\,, \tag{11}$$

and we have that $vc_ia_i = vc_i$. Consequently, $z \in \mathcal{O}_L$ if and only if $c_1,\dots,c_q \in \mathcal{O}_K$. This shows that $\mathcal{O}_L$ is a free $\mathcal{O}_K$-module with basis $a_1,\dots,a_q$.

In the case where $Lv|Kv$ is simple, that is, there is $\xi \in Lv$ such that $Lv = Kv(\xi)$, we can choose $x \in L$ such that $xv = \xi$; then $1,x,\dots,x^{q-1}$ form a valuation basis of $(L|K,v)$. In this special case (which by the Primitive Element Theorem always appears when $Lv|Kv$ is separable), we have

$$\mathcal{O}_L \;=\; \mathcal{O}_K[x]\,. \tag{12}$$

Case (DL2): $[L:K] = (vL:vK)$. We assume in addition that q is a prime. In this case we define $H_{\mathcal{E}}$ to be the largest convex subgroup of vL which is also a convex subgroup of vK; it exists since unions over arbitrary collections of convex subgroups are again convex subgroups. The subgroup $H_{\mathcal{E}}$ defined here has important similarities with the convex subgroup $H_{\mathcal{E}}$ defined in the defect case in [2]. We will discuss them in detail in [12]. In case (DL1) we set $H_{\mathcal{E}} := \{0\}$.

Now we diivide (DL2) into three mutually exclusive cases:

(DL2a): there is no smallest convex subgroup of vL that properly contains $H_{\mathcal{E}}$;

(DL2b): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is dense;

(DL2c): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is discrete.

We will freely use the facts outlined in Section 2.2.

Pick any $x \in L$ such that $vx \notin vK$. Then $vL = vK + \mathbb{Z}vx$ with $qvx \in vK$, and we have that $1, x, \ldots, x^{q-1}$ form a valuation basis of $(L|K, v)$. This means that every element of L can be written as a K-linear combination of these elements and for every choice of $c_0, \ldots, c_{q-1} \in K$,

$$v\sum_{i=0}^{q-1} c_i x^i \;=\; \min_i vc_i x^i \,.$$

Again, the sum is an element of $\mathcal{O}_L$ if and only if all summands $c_i x^i$ are, but the latter does not necessarily imply that $c_i \in \mathcal{O}_K$. We set

$$A_x \;:=\; \{c_i x^i \mid c_i \in K^\times \text{ and } 1 \le i < q \text{ such that } vc_i x^i > 0\}$$

and

$$vA_x \;:=\; \{va \mid a \in A_x\} \,.$$

(Note that $vc_i x^i = 0$ is impossible for $1 \le i < q$.) We obtain that

$$\mathcal{O}_L \;=\; \mathcal{O}_K[A_x] \,. \tag{13}$$

However, we wish to derive a much more useful representation of $\mathcal{O}_L$. Our goal is to find an element x as above such that

$$\mathcal{O}_L \;=\; \bigcup_{c \in K \text{ with } vcx > 0} \mathcal{O}_K[cx] \,. \tag{14}$$

If $c, c' \in K$ with $vc \ge vc'$, then $cx = \frac{c}{c'} c'x \in \mathcal{O}_K[c'x]$, hence $\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x]$. So the right hand side is an increasing union of rings and thus is itself a ring. For (14) to hold, it suffices that

$$A_x \;\subseteq\; \bigcup_{c \in K \text{ with } vcx > 0} \mathcal{O}_K[cx] \,. \tag{15}$$

This in turn will hold if

$$\left\{ \begin{array}{l} \text{for every element } c_i x^i \in A_x \text{ there is } c \in K \text{ with } cx \in A_x \\ \text{such that } c_i x^i \in (cx)^i \mathcal{O}_K \,. \end{array} \right. \tag{16}$$

Lemma 3.1. *The convex subgroup $H_{\mathcal{E}}$ of vL is the largest that has empty intersection with vA_x.*

Proof. From (13) it follows that the positve values in $vL \setminus vK$ all lie in the smallest final segment of vL generated by vA_x. On the other hand, from the definition of $H_{\mathcal{E}}$ it follows that it is the largest convex subgroup of vL that does not contain elements of $vL \setminus vK$. This proves our assertion. □

As a preparation for what follows, let us prove two useful facts.

(F1) For each $c_m x^m \in A_x$, there is $c \in K$ such that $cx \in A_x$ and $\mathcal{C}_{vL}(vc_m x^m) = \mathcal{C}_{vL}(vcx)$.

Proof. As q is prime, there is $k \in \mathbb{N}$ such that $mk = 1 + rq$ for some $r \in \mathbb{Z}$. Taking $c := c_m^k b^r \in K$ where $b \in K$ with $vb = qvx$, we obtain $vcx = v(c_m x^m)^k > 0$ and $\mathcal{C}_{vL}(vcx) = \mathcal{C}_{vL}(kv(c_m x^m)) = \mathcal{C}_{vL}(vc_m x^m)$. □

(F2) If c_ix^i, $\tilde{c}x \in A$ and $vc_ix^i \notin \mathcal{C}_{vL}(v\tilde{c}x)$, then $c_ix^i \in (\tilde{c}x)^i\mathcal{O}_K \subseteq \mathcal{O}_K[\tilde{c}x]$.

Proof. Since $v\tilde{c}^ix^i = iv\tilde{c}x \in \mathcal{C}_{vL}(v\tilde{c}x)$ and $vc_ix^i \notin \mathcal{C}_{vL}(v\tilde{c}x)$, we have $v\tilde{c}^ix^i < vc_ix_0^i$. Thus $v\tilde{c}^i < vc_i$ and therefore, $c_ix^i \in (\tilde{c}x)^i\mathcal{O}_K$. □

Inspired by case (DL1) we ask whether (15) will hold with $x = x_0$ for any $x_0 \in L$ such that $vx_0 \notin vK$. We choose such an x_0 and set $A_0 := A_{x_0}$. It can be shown that the element x we are looking for cannot always be chosen to be equal to x_0. However, we will show that in cases (DL2a) and (DL2b) it can.

(DL2a): Take any $c_ix_0^i \in A_0$. By assumption, $\mathcal{C}^+_{vL}(vc_ix_0^i)$ properly contains $H_{\mathcal{E}}$. By Lemma 3.1, this means that $\mathcal{C}^+_{vL}(vc_ix_0^i) \cap vA_0 \neq \emptyset$, so take some $c_mx_0^m \in A_0$ such that $vc_mx_0^m \in \mathcal{C}^+_{vL}(vc_ix_0^i) \cap vA_0$. By (F1), there is $\tilde{c} \in K$ such that $\tilde{c}x_0 \in A_0$ and $\mathcal{C}_{vL}(v\tilde{c}x_0) = \mathcal{C}_{vL}(vc_mx_0^m) \subseteq \mathcal{C}^+_{vL}(vc_ix_0^i)$. Hence $vc_ix_0^i \notin \mathcal{C}_{vL}(v\tilde{c}x_0)$ and by (F2), $c_ix_0^i \in (\tilde{c}x_0)^i\mathcal{O}_K \subseteq \mathcal{O}_K[\tilde{c}x_0]$. Hence in this case, (16) and thus also (15) and (14) hold for $x = x_0$.

(DL2b): By Lemma 3.1, $\tilde{H}_{\mathcal{E}}$ is the smallest convex subgroup of vL that contains some element of vA_0, say $vc_mx_0^m$. The archimedean component $\mathcal{A}_{vL}(vc_mx_0^m)$ is equal to $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$, which is dense. The archimedean component $\mathcal{A}_{vK}(vqc_mx_0^m)$ is equal to $(\tilde{H}_{\mathcal{E}} \cap vK)/H_{\mathcal{E}}$. Since $(vL : vK)$ is finite, so is this quotient. This shows that also $\mathcal{A}_{vK}(vqc_mx_0^m)$ is dense, so it is dense in $\mathcal{A}_{vL}(vc_mx_0^m)$. We have $\mathcal{C}^+_{vL}(c_mx_0^m) \cap vA_0 = H_{\mathcal{E}} \cap vA_0 = \emptyset$. From (F1) we know that there is $\tilde{c} \in K$ such that $\tilde{c}x_0 \in A$ and $\mathcal{C}_{vL}(v\tilde{c}x_0) = \mathcal{C}_{vL}(vc_mx_0^m)$.

Take any element $c_ix_0^i \in A_0$. If $vc_ix_0^i \notin \mathcal{C}_{vL}(v\tilde{c}x_0)$, then $c_ix_0^i \in \mathcal{O}_K[\tilde{c}x_0]$ by (F2). So let us assume that $vc_ix_0^i \in \mathcal{C}_{vL}(v\tilde{c}x_0)$. Denote by α the image of $v\tilde{c}x_0$ and by β the image of $vc_ix_0^i$ in $\mathcal{A}_{vL}(v\tilde{c}x_0)$. Note that both of them are positive, so

$$-i\alpha \;<\; \beta - i\alpha\,.$$

By the density of $\mathcal{A}_{vK}(qv\tilde{c}x_0) = \mathcal{A}_{vK}(qvc_mx_0^m)$ in $\mathcal{A}_{vL}(vc_mx_0^m) = \mathcal{A}_{vL}(v\tilde{c}x_0)$ there is $c_0 \in K$ such that the image γ of vc_0 in $\mathcal{A}_{vL}(v\tilde{c}x_0)$ satisfies

$$-i\alpha \;<\; i\gamma \;<\; \beta - i\alpha\,,$$

whence $0 < i\gamma + i\alpha < \beta$. This leads to $0 < vc_0^i\tilde{c}^ix_0^i < vc_ix_0^i$. Setting $c = c_0\tilde{c}$, we obtain that $0 < vc^ix_0^i < vc_ix_0^i$, whence $c_ix_0^i \in (cx_0)^i\mathcal{O}_K \subseteq \mathcal{O}_K[cx_0]$ with $vcx_0 > 0$. We have proved that also in this case, (16), (15) and (14) hold for $x = x_0$.

In case (DL2a), fact (F2) shows that (14) holds for $x = x_0$ because for every $c_ix_0^i \in A_0$ there is some $\tilde{c}x_0 \in A_0$ with $\mathcal{C}_{vL}(v\tilde{c}x_0) \subseteq \mathcal{C}^+_{vL}(vc_ix_0^i)$. In case (DL2b) the latter is not true, but using density we were able to show $c_ix_0^i \in \mathcal{O}_K[cx_0]$ for some $c \in K$ with $vcx_0 > 0$ even when $vc_ix_0^i \in \mathcal{C}_{vL}(v\tilde{c}x_0)$. In cases (DL2a) and (DL2b) we set $x := x_0$. The next case treats the instance where we do not have density at hand.

(DL2c): In this case, $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is discrete. Choose the element $c_mx_0^m$ as in case (DL2b). Now we have that $\mathcal{A}_{vL}(vc_mx_0^m)$ and $\mathcal{A}_{vK}(qvc_mx_0^m)$ are discrete. From (F1) we know that there is $\tilde{c} \in K$ such that $\tilde{c}x_0 \in A_0$ and $\mathcal{C}_{vL}(v\tilde{c}x_0) = \mathcal{C}_{vL}(vc_mx_0^m)$. The image α of $v\tilde{c}x_0$ in $\mathcal{A}_{vL}(v\tilde{c}x_0)$ may not be its smallest positive element, which creates the problem that not all elements $c_ix_0^i \in A_0$ with $vc_ix_0^i \in \mathcal{C}_{vL}(v\tilde{c}x_0)$ may lie in $\mathcal{O}_L[\tilde{c}x_0]$. So take any $c_jx_0^j \in A_0$ ($j \in \{1,\ldots,q-1\}$) with $vc_jx_0^j \in \mathcal{C}_{vL}(v\tilde{c}x_0)$ whose image γ in $\mathcal{A}_{vL}(v\tilde{c}x_0)$ is its smallest positive element. Since $j \in \{1,\ldots,q-1\}$, also

$1, x, \dots, x^{q-1}$ form a valuation basis of $(L|K,v)$. Hence we may set $x := x_0^j$ and from now on work with A_x in place of A_0.

Now we have that vc_jx is the smallest positive element in $\mathcal{A}_{vL}(vc_jx) = \mathcal{A}_{vL}(v\tilde{c}x_0)$ and qvc_jx is the smallest positive element in $\mathcal{A}_{vK}(qvc_jx)$. Further, the only elements strictly between 0 and $q\gamma$ are $\gamma, 2\gamma, \dots, (q-1)\gamma$.

Take any element $c_ix^i \in A$. If $vc_ix^i \notin \mathcal{C}_{vL}(vc_jx)$, then $c_ix^i \in (c_jx)^i\mathcal{O}_K \subseteq \mathcal{O}_K[c_jx]$ by (F2). So let us assume that $vc_ix^i \in \mathcal{C}_{vL}(vc_jx)$, and denote the image of vc_ix^i in $\mathcal{A}_{vL}(vc_jx)$ by β. Write $c_ix^i = dc_j^ix^i$ with $d = c_ic_j^{-i}$, and denote by δ the image of d in $\mathcal{A}_{vL}(vc_jx)$, so that $0 \leq \beta = \delta + i\gamma$. Suppose that $\delta < 0$; then $\delta + i\gamma = k\gamma$ for some $k \in \{0, \dots, i-1\}$, but as $\delta \in \mathcal{A}_{vK}(qvc_jx)$, this is impossible. Hence $\delta \geq 0$. If $\delta > 0$, then $vd > 0$, whence $c_ix^i \in (c_jx)^i\mathcal{O}_K \subseteq \mathcal{O}_K[c_jx]$.

Now assume that $\delta = 0$. Then $vd \in \mathcal{C}^+_{vL}(vc_jx)$. If $d \in \mathcal{O}_K$, then we are done again. So assume that $vd < 0$ and write $c_ix^i = dc_j^ix^i = d^{1-i}(dc_jx)^i$. Then $d^{1-i} \in \mathcal{O}_K$, hence for $c := dc_j$, $c_ix^i \in (cx)^i\mathcal{O}_K \subseteq \mathcal{O}_K[cx]$. As $vd \in \mathcal{C}^+_{vL}(vc_jx)$ and $vc_jx > 0$, we have $vcx = vd + vc_jx > 0$.

We have proved that in this case, (16), (15) and (14) hold for $x = x_0^j$.

Remark 3.2. Assume that vK is i-divisible for all $i \in \{2, \dots, q-1\}$, with q not necessarily prime. Take $c_ix_0^i \in A_0$. Then there is $c \in K$ such that $vc_i = ivc$. We obtain that $vc^ix_0^i = vc_ix_0^i > 0$, hence also $vcx_0 > 0$. Consequently, $c_ix_0^i \in (cx_0)^i\mathcal{O}_K^\times \subseteq \mathcal{O}_K[cx_0]$. It follows that (16), (14) and (16) hold for $x = x_0$.

This case appears when $q = \operatorname{char} Kv > 0$ and (K,v) is equal to its own absolute ramification field, since then vK is divisible by all primes other than q.

Assume that $\mathcal{E}$ is of type (DL2c) and, using the notation of that case, that $\mathcal{C}^+_{vL}(v\tilde{c}x_0) = \{0\}$ or equivalently, $H_\mathcal{E} = \{0\}$. Then in the case of $\delta = 0$ we have $vd = 0$, whence $vc_ix^i = vc_j^ix^i$ and $c_ix^i \in (c_jx)^i\mathcal{O}_K^\times \subseteq \mathcal{O}_K[c_jx]$. This shows that $\mathcal{O}_L = \mathcal{O}_K[c_jx]$. The assumption $H_\mathcal{E} = \{0\}$ holds in case (DL2c) if and only if $[L:K] = (vL:vK)$ equals the initial index of the extension $(L|K,v)$, which is the number of nonnegative values of vL that are smaller than any positive element in vK. Therefore, our result is a proof of Knaf's conjecture about essentially finite generation of $\mathcal{O}_L$ over $\mathcal{O}_K$ for the case of extensions of prime degree. The formulation and the (considerably more involved) full proof of Knaf's conjecture is given in [4]. See also [3, 21] for proofs of important special cases.

We summarize what we have shown in case (DL2):

Theorem 3.3. *Take an extension $\mathcal{E} = (L|K,v)$ of prime degree $q = \mathrm{e}\,(L|K,v)$, with $x_0 \in L$ such that $vx_0 \notin vK$.*

1) If $\mathcal{E}$ is of type (DL2a) or (DL2b), then (14) - (16) hold for $x = x_0$.

2) If $\mathcal{E}$ is of type (DL2c), then (14) - (16) hold for $x = x_0^j$ with suitable $j \in \{1, \dots, q-1\}$. If in addition $H_\mathcal{E} = \{0\}$, then $\mathcal{O}_L = \mathcal{O}_K[cx]$ for suitable $c \in K$.

The assumption of part 1) holds in particular when every archimedean component of vK is dense, and this in turn holds for every deeply ramified field (K,v).

In all cases, $1, x, \dots, x^{q-1}$ form a valuation basis of $(L|K,v)$, and for all $c, c' \in K$,

$$\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x] \Leftrightarrow vc \leq vc' .$$

Proof. Only the implication "$\Rightarrow$" of the last assertion needs a proof. Take $c, c' \in K$. If $\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x]$, then $cx \in \mathcal{O}_K[c'x]$. Since the elements $1, c'x, \ldots, (c'x)^{n-1}$ form a valuation basis of $(L|K, v)$, it follows that $cx = \frac{c}{c'}c'x$ with $\frac{c}{c'} \in \mathcal{O}_K$, whence $vc \leq vc'$. □

3.2. **Valuation rings and ideals associated with the generation of $\mathcal{O}_L|\mathcal{O}_K$.**

The convex subgroup $H_\mathcal{E}$ of vL and the associated valuation ring and maximal ideal turn out to be important invariants of the extension $\mathcal{E}$. We take a closer look at them in this section.

The convex subgroups H of vL are in one-to-one correspondence with the coarsenings v_H of v on L in such a way that $v_H L = vL/H$. The valuation ring of v_H on L is $\mathcal{O}_{v_H} = \{a \in L \mid \exists \gamma \in H : va \geq \gamma\}$, and its maximal ideal is $\mathcal{M}_{v_H} = \{a \in L \mid va > H\}$. We write $\mathcal{O}_\mathcal{E}$ for $\mathcal{O}_{v_{H_\mathcal{E}}}$, $\mathcal{M}_\mathcal{E}$ for $\mathcal{M}_{v_{H_\mathcal{E}}}$, and $v_\mathcal{E}$ for $v_{H_\mathcal{E}}$.

We note that $\mathcal{M}_\mathcal{E}$ is a nonprincipal $\mathcal{O}_\mathcal{E}$-ideal if $\mathcal{E}$ is of type (DL2a) or (DL2b), and a principal $\mathcal{O}_\mathcal{E}$-ideal if $\mathcal{E}$ is of type (DL2c). Indeed, the value group $v_\mathcal{E}L$ is (up to equivalence) the quotient $vL/H_\mathcal{E}$. In case (DL2a), this does not have a smallest convex subgroup and thus no smallest positive element. In case (DL2b) the quotient has a smallest convex subgroup. As it is dense, it does not have a smallest positive element, and therefore the same holds for $v_\mathcal{E}L$. Also in case (DL2c) the quotient has a smallest convex subgroup. As now it is discrete, it has a smallest positive element, and the same holds for $v_\mathcal{E}L$.

Proposition 3.4. *Take an extension $\mathcal{E} = (L|K, v)$ of prime degree $q = \mathrm{e}\,(L|K, v)$, with x determined by Theorem 3.3. Then for every $a \in L$ such that $va > H_\mathcal{E}$ there is $c \in K$ with $0 < vcx \leq va$. Further, $\mathcal{M}_\mathcal{E}$ is equal to the $\mathcal{O}_L$-ideal*

$$I_x := (cx \mid c \in K \text{ with } vcx > 0)\,. \tag{17}$$

Proof. Take $a \in L$ such that $va > H_\mathcal{E}$ and write $a = \sum_{i=0}^{q-1} c_i x^i$ with $c_i \in K$. Since $1, x, \ldots, x^{q-1}$ form a valuation basis, we have $vc_i x^i \geq va > 0$ for $0 \leq i \leq q-1$ with $va = \min_i vc_i x^i$. In particular, $c_i x^i \in A_x$ for $1 \leq i \leq q-1$. Hence it follows from (16) that for each such i there is $d_i \in K$ with $0 < vd_i x \leq v(d_i x)^i \leq vc_i x^i$. It remains to consider the case of $i = 0$. We have $vc_0 \geq va > H_\mathcal{E}$.

If $H_\mathcal{E} \subsetneq \mathcal{C}_{vL}^+(vc_0)$, then there is some $c_\ell x^\ell \in A_x$ with $vc_\ell x^\ell \in \mathcal{C}_{vL}^+(vc_0)$. By (16) there is $d_0 \in K$ with $0 < vd_0 x \leq v(d_0 x)^\ell \leq vc_\ell x^\ell < vc_0$.

Now assume that $H_\mathcal{E} = \mathcal{C}_{vL}^+(vc_0)$, so $\mathcal{E}$ is of type (DL2b) or (DL2c). We will use the notation as in the computations for these two cases in Section 3.1. With the element $c_m x_0^m$ appearing in these cases, we have $\mathcal{C}_{vL}(vc_0) = \mathcal{C}_{vL}(vc_m x_0^m)$ and $\mathcal{A}_{vL}(vc_0) = \mathcal{A}_{vL}(vc_m x_0^m)$.

In case (DL2b), $\mathcal{A}_{vK}(qvc_m x_0^m)$ is dense in $\mathcal{A}_{vL}(vc_m x_0^m) = \mathcal{A}_{vL}(vc_0)$. Denote by α the image of $vc_m x_0^m$ and by γ the image of vc_0 in $\mathcal{A}_{vL}(vc_0)$. By density, there is $b \in K$ such that $\alpha - \gamma < \beta < \alpha$ with β the image of vb in $\mathcal{A}_{vL}(vc_0)$. This leads to $vc_m x_0^m c_0^{-1} < vb < vc_m x_0^m$, that is, $vc_0 > vc_m x_0^m - vb > 0$. Since $x = x_0$ in the present case, we obtain that $b^{-1}c_m x_0^m = b^{-1}c_m x^m \in A_x$. Hence by (16) there is $d_0 \in K$ with $0 < vd_0 x \leq v(d_0 x)^m \leq vb^{-1}c_m x^m < vc_0$.

In case (DL2c), $\mathcal{A}_{vK}(qvc_m x_0^m)$ is not dense in $\mathcal{A}_{vL}(vc_m x_0^m) = \mathcal{A}_{vL}(vc_0)$. With $c_j x_0^j \in A_0$ chosen as in case (DL2c), the image γ of $vc_j x_0^j$ in $\mathcal{A}_{vL}(vc_m x_0^m)$ is its smallest positive element. Then the image $q\gamma$ of $qvc_j x_0^j$ is the smallest positive element of $\mathcal{A}_{vK}(qvc_m x_0^m) = \mathcal{A}_{vK}(vc_0)$. Since $x = x_0^j$ in the present case, we obtain that $vc_0 \geq qvc_j x_0^j > vc_j x_0^j = vc_j x$, and we set $d_0 = c_j$.

We have now proved that in all cases there is $d_0 \in K$ such that $0 < vd_0 x < vc_0$. We choose some $i_0 \in \{0, \ldots, q-1\}$ such that $vd_{i_0} = \min\{vd_i \mid 0 \leq i \leq q-1\}$ and set $c := d_{i_0}$. Then

$$vcx \;\leq\; vd_i x \;\leq\; vc_i x^i \quad \text{for } 0 \leq i \leq q-1\,. \tag{18}$$

Hence $0 < vcx \leq va$ as required.

Now we prove the second assertion. All elements cx as in (17) lie in A_x and therefore have value $> H_{\mathcal{E}}$. It follows that all elements in I_x have value $> H_{\mathcal{E}}$ and thus lie in $\mathcal{M}_{\mathcal{E}}$. This proves the inclusion $I_x \subseteq \mathcal{M}_{\mathcal{E}}$.

For the converse, take $a \in \mathcal{M}_{\mathcal{E}}$, so $va > H_{\mathcal{E}}$. By the first assertion of our proposition, there is $c \in K$ with $0 < vcx \leq va$. This implies $a \in cx\mathcal{O}_L \subseteq I_x$. □

We give an application of this proposition.

Corollary 3.5. *Take an extension $\mathcal{E} = (L|K, v)$ of prime degree $q = \mathrm{e}\,(L|K, v)$, with $x \in L$ determined by Theorem 3.3. If $\mathcal{E}$ is of type (DL2a) or (DL2b), then for every $a \in \mathcal{M}_{\mathcal{E}}$ there is $c \in K$ such that $a\,\mathcal{O}_L \subseteq \mathcal{O}_K[cx]$.*

Proof. Since $a \in \mathcal{M}_{\mathcal{E}}$, we have $va > H_{\mathcal{E}}$. In case (DL2a), $H_{\mathcal{E}} \subsetneq \mathcal{C}^+_{vL}(va)$. Then there is an element $c_\ell x^\ell \in A_x \cap \mathcal{C}^+_{vL}(va)$. By Proposition 3.4, there is $c \in K$ such that $0 < vcx \leq vc_\ell x^\ell < va$. It follows that $vcx \in \mathcal{C}^+_{vL}(va)$, hence $qvcx \in \mathcal{C}^+_{vL}(va)$ and therefore, $qvcx \leq va$.

In case (DL2b), $H_{\mathcal{E}} = \mathcal{C}^+_{vL}(va)$ and $\mathcal{A}_{vL}(va)$ is dense. Hence there is $b \in L$ such that $H_{\mathcal{E}} = \mathcal{C}^+_{vL}(va) < vb$ (so $b \in \mathcal{M}_{\mathcal{E}}$) and $qvb \leq va$. By Proposition 3.4 there is $c \in K$ such that $0 < vcx \leq vb$, whence again, $qvcx \leq va$.

Take any $a' \in a\,\mathcal{O}_L$, so $va' \geq va$. Write $a' = \sum_{i=0}^{q-1} c_i x^i$. Then $vc_i x^i \geq va' \geq va \geq qvcx \geq vc^i x^i$ for $0 < i < q-1$, hence $c_i x^i \in c^i x^i \mathcal{O}_K \subseteq \mathcal{O}_K[cx]$. Since also $c_0 \in \mathcal{O}_K \subseteq \mathcal{O}_K[cx]$, we obtain that $a' \in \mathcal{O}_K[cx]$, which shows that $a\,\mathcal{O}_L \subseteq \mathcal{O}_K[cx]$. □

Note that the assertions of this corollary are trivially satisfied if $q = \mathrm{f}\,(L|K, v)$. Moreover, the last assertion also holds if $\mathcal{E}$ is of type (DL2c) with $H_{\mathcal{E}} = \{0\}$.

The ideal $\mathcal{M}_{\mathcal{E}}$ will be useful in the computation of the Kähler differentials in Theorems 4.6 and 4.8. In preparation, we need a small technical lemma.

Lemma 3.6. *Take a valuation ring $\mathcal{O}$ with maximal ideal $\mathcal{M}$. Whenever $2 \leq n \in \mathbb{N}$ and $a \in \mathcal{O}_L$, then*

1) $a\mathcal{M} = \mathcal{M}$ if and only if $a \notin \mathcal{M}$,

2) $\mathcal{M}^n = \mathcal{M}$ if and only if $\mathcal{M}$ is a nonprincipal $\mathcal{O}$-ideal,

3) $(a\mathcal{M})^n = a\mathcal{M}$ if and only if $a \notin \mathcal{M}$ and $\mathcal{M}$ is a nonprincipal $\mathcal{O}$-ideal.

Proof. Denote by w the valuation associated with $\mathcal{O}$.

1): We have $a \notin a\mathcal{M}$, hence if $a \in \mathcal{M}$, then $a\mathcal{M} \neq \mathcal{M}$. If $a \notin \mathcal{M}$, then a is a unit in $\mathcal{O}$, so $a\mathcal{M} = \mathcal{M}$.

2): The value group of w is not discrete, and hence dense, if and only if $\mathcal{M}$ is a nonprincipal $\mathcal{O}$-ideal. If it is discrete and γ is its smallest positive element, then $\mathcal{M} = \{b \in K \mid wb \geq \gamma\}$ and $\mathcal{M}^n = \{c \in K \mid wc \geq n\gamma\} \subsetneq \mathcal{M}$ since $n\gamma > \gamma$. If it is dense, then for every $b \in \mathcal{M}$ there is $c \in K$ such that $0 < nwc < wb$, whence $b \in \mathcal{M}^n$; therefore, $\mathcal{M}^n \subseteq \mathcal{M} \subseteq \mathcal{M}^n$ and consequently, $\mathcal{M}^n = \mathcal{M}$.

3): If $a \notin \mathcal{M}$ and $\mathcal{M}$ is a nonprincipal $\mathcal{O}$-ideal, then by parts 1) and 2), $(a\mathcal{M})^n = \mathcal{M}^n = \mathcal{M} = a\mathcal{M}$. If $a \in \mathcal{M}$, then $wa > 0$, whence $a\mathcal{M} = \{c \in K \mid wc > wa\}$ and $(a\mathcal{M})^n \subseteq a^n\mathcal{M} = \{c \in K \mid wc > nwa\} \subsetneq a\mathcal{M}$ since $nwa > wa$. If $\mathcal{M}$ is a principal $\mathcal{O}$-ideal, say $\mathcal{M} = b\mathcal{O}$ with $b \in \mathcal{M}$, then $a\mathcal{M} = ab\mathcal{O} = \{c \in K \mid wc \geq wab\}$ and $(a\mathcal{M})^n = (ab\mathcal{O})^n = \{c \in K \mid wc \geq nwab\} \subsetneq a\mathcal{M}$ since $nwab > wab$ and $ab \in a\mathcal{M}$. □

3.3. **Differents of generators for Artin-Schreier and Kummer extensions.** The proofs in Sections 4.3 to 4.7 make use of the differents of the chosen generators for $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra. In this section we compute those differents.

If $b \in L$ and h_b is its minimal polynomial over K, then $\delta(b) := h_b'(b)$ is called the **different of** b. The $\mathcal{O}_L$-ideal

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;:=\; (h_b'(b) \mid b \in \mathcal{O}_L \setminus \mathcal{O}_K)\,. \tag{19}$$

generated by the differents of all elements in $\mathcal{O}_L \setminus \mathcal{O}_K$ will be called the **naive different ideal**.

Proposition 3.7. *Assume that $(L|K,v)$ is a nontrivial finite unibranched Galois extension and that*

$$\mathcal{O}_L \;=\; \bigcup_{\alpha\in S} \mathcal{O}_K[b_\alpha]$$

for some (possibly finite) index set S and elements $b_\alpha \in \mathcal{O}_L\setminus\mathcal{O}_K$. Then $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)$ is equal to the $\mathcal{O}_L$-ideal $(\delta(b_\alpha) \mid \alpha \in S)$.

Proof. In the proof of [2, Proposition 4.1] it is shown that $b \in \mathcal{O}_K[b_\alpha]$ implies $v\delta(b) \geq v\delta(b_\alpha)$. Hence,

$$(\delta(b) \mid b \in \mathcal{O}_L\setminus\mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(\delta(b) \mid b \in \mathcal{O}_K[b_\alpha]\setminus\mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(\delta(b_\alpha)) \;=\; (\delta(b_\alpha) \mid \alpha \in S)\,.$$

□

In the case of Artin-Schreier and Kummer extensions $(L|K,v)$ with Galois group G we have sufficient information about the minimal polynomials f of the various generators x we have worked with in the previous sections, and about their conjugates, to work out the values $vf'(x)$ of their differents $f'(x)$. In order to do this, we can either compute f', or we can use the formula

$$f'(x) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (x - \sigma x)\,. \tag{20}$$

We keep the notations from the previous sections.

3.3.1. *Artin-Schreier extensions.*

Take an Artin-Schreier polynomial f with ϑ as its root. Then its minimal polynomial is $f(X) = X^p - X - \vartheta^p + \vartheta$ with $f'(X) = -1$, whence

$$f'(\vartheta) = -1\,. \tag{21}$$

For $c \in K^\times$, denote by f_c the minimal polynomial of $c\vartheta$. Then

$$f_c'(c\vartheta) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (c\vartheta - \sigma c\vartheta) \;=\; c^{p-1} f'(\vartheta) \;=\; -c^{p-1}\,. \tag{22}$$

Lemma 3.8. *Take an Artin-Schreier extension $\mathcal{E} = (L|K,v)$ of prime degree $p = \mathrm{f}\,(L|K,v)$. If the extension $Lv|Kv$ is purely inseparable, then $\mathcal{E}$ admits an Artin-Schreier generator ϑ of value $v\vartheta < 0$ and $\tilde{c} \in K$ such that $v\tilde{c}\vartheta = 0$, $Lv = Kv(\tilde{c}\vartheta v)$, $\mathcal{O}_L = \mathcal{O}_K[\tilde{c}\vartheta]$ and*

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; f_{\tilde{c}}'(\tilde{c}\vartheta)\mathcal{O}_L \;=\; \tilde{c}^{p-1}\mathcal{O}_L \;=\; I_{\mathcal{E}}^{p-1}\,. \tag{23}$$

Proof. The first assertions follow from part 1) of Proposition 2.6 and case (DL1). Applying Proposition 3.7 with $S = \{1\}$ and $b_1 = \tilde{c}\vartheta$, we obtain the first equality of (23). Since $v\tilde{c} = -v\vartheta$, we have $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) = f_{\tilde{c}}'(\tilde{c}\vartheta)\mathcal{O}_L = \tilde{c}^{p-1}\mathcal{O}_L = (\vartheta^{-1})^{p-1} = I_{\mathcal{E}}^{p-1}$ by (22) and part 1) of Proposition 2.7. This proves (23). □

Lemma 3.9. *Take an Artin-Schreier extension $\mathcal{E} = (L|K,v)$ of prime degree $p = \mathrm{e}\,(L|K,v)$. Then $\mathcal{E}$ admits an Artin-Schreier generator ϑ of value $v\vartheta < 0$ such that $vL = vK + \mathbb{Z}v\vartheta$, (14) holds for $x = \vartheta^j$ with suitable $j \in \{1,\ldots,p-1\}$, and we have the equality of $\mathcal{O}_L$-ideals*

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (\vartheta^{-1}I_{\vartheta^j})^{p-1} \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}\,. \tag{24}$$

Proof. The existence of such ϑ and j follows from part 1) of Proposition 2.6 together with Theorem 3.3. Since (14) holds for $x = \vartheta^j$, we can apply Proposition 3.7 with $S = \{c \in K^\times \mid vc\vartheta^j > 0\}$ and $b_c = c\vartheta^j$ to obtain:

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (h_{j,c}'(c\vartheta^j) \mid c \in K^\times \text{ with } vc\vartheta^j > 0)\,,$$

where $h_{j,c}$ denotes the minimal polynomial of $c\vartheta^j$. Now we compute:

$$c\vartheta^j - \sigma c\vartheta^j \;=\; c(\vartheta^j - (\sigma\vartheta)^j) \;=\; c(\vartheta^j - (\vartheta+k)^j) \;=\; -c\sum_{i=1}^{j}\binom{j}{i}\vartheta^{j-i}k^i$$

for suitable $k \in \mathbb{F}_p^\times$. The summand of least value in the sum on the right hand side is the one for $i = 1$. Using (20), we obtain:

$$vh_{j,c}'(c\vartheta^j) \;=\; (p-1)(vc\vartheta^{j-1})\,. \tag{25}$$

Hence,

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (c\vartheta^{j-1} \mid vc\vartheta^j > 0)^{p-1} \;=\; (\vartheta^{-1}I_{\vartheta^j})^{p-1} \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{p-1}\,,$$

where the last two equalities follow from part 1) of Proposition 2.7 and Proposition 3.4. This proves (24). □

3.3.2. *Kummer extensions.*

In what follows, ζ_p will denote a primitive p-th root of unity. If $L|K$ is a Kummer extension, then $\zeta_p \in K$. Take a Kummer polynomial f of degree q with η as its root. Then $f(X) = X^q - \eta^q$ and $f'(X) = qX^{q-1}$, whence

$$f'(\eta) \;=\; q\eta^{q-1}\,. \tag{26}$$

Lemma 3.10. *Take a Kummer extension $\mathcal{E} = (L|K,v)$ of degree $p = \operatorname{char} Kv$. Assume that $\mathrm{f}\,(L|K,v) = p$. Then there exists a Kummer generator $\eta \in L$ such that one of the following cases holds:*

i) $v\eta = 0$, $Lv = Kv(\eta v)$ with $Lv|Kv$ inseparable, and $\mathcal{O}_L = \mathcal{O}_K[\eta]$,

ii) η is a 1-unit, $v\tilde{c}(\eta-1) = 0$, $Lv = Kv(\tilde{c}(\eta-1)v)$ and $\mathcal{O}_L = \mathcal{O}_K[\tilde{c}(\eta-1)]$ for suitable $\tilde{c} \in K^\times$.

In case i), for f the minimal polynomial of η,

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; f'(\eta)\mathcal{O}_L \;=\; p\mathcal{O}_L \;=\; I_{\mathcal{E}}^{p-1}\,. \tag{27}$$

In case ii), for $h_{\tilde{c}}$ the minimal polynomial of $\tilde{c}(\eta-1)$,

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; h'_{\tilde{c}}(\tilde{c}(\eta-1))\mathcal{O}_L \;=\; p\tilde{c}^{p-1}\mathcal{O}_L \;=\; I_{\mathcal{E}}^{p-1}\,. \tag{28}$$

Proof. The existence of such η and $\tilde{c}$ follows from part 2)a) of Proposition 2.6. The presentation of $\mathcal{O}_L$ follows from case (DL1). Applying Proposition 3.7 with $S = \{1\}$ and setting $b_1 = \eta$ and $b_1 = \tilde{c}(\eta-1)$, respectively, we obtain the first equalities of (27) and (28).

In case i), the second equality of (27) follows from (26) since $v\eta = 0$. The third equality holds since $vp = (p-1)v(\zeta_p - 1)$ by (4), whence $p\mathcal{O}_L = I_{\mathcal{E}}^{p-1}$ by part 2)a) of Proposition 2.7.

For case ii) we compute with σ a generator of $\operatorname{Gal} L|K$, using (20):

$$h'_{\tilde{c}}(\tilde{c}(\eta-1)) \;=\; \prod_{i=1}^{p-1} \tilde{c}(\eta - \sigma^i\eta) \;=\; (\tilde{c}\eta)^{p-1}\prod_{i=1}^{p-1}(1-\zeta_p^i)\,,$$

whence by (3),

$$vh'_{\tilde{c}}(\tilde{c}(\eta-1)) \;=\; vp(\tilde{c}\eta)^{p-1}\,. \tag{29}$$

This yields the second equality of (28) since $v\eta = 0$. The third holds as $v\tilde{c} = -v(\eta-1)$ yields $vp\tilde{c}^{p-1} = (p-1)(v(\zeta_p-1) - v(\eta-1))$, whence $p\tilde{c}^{p-1}\mathcal{O}_L = I_{\mathcal{E}}^{p-1}$ by part 2)b) of Proposition 2.7. □

Lemma 3.11. *Take a Kummer extension $\mathcal{E} = (L|K,v)$ of prime degree $q = \mathrm{e}\,(L|K,v)$. Then there are two possible cases.*

i) There is a Kummer generator $\eta \in L$ such that $vL = vK + \mathbb{Z}v\eta$, (14) holds for $x = \eta$, and we have the equality

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; qI_\eta^{q-1} \;=\; q\mathcal{M}_{\mathcal{E}}^{q-1} \tag{30}$$

of $\mathcal{O}_L$-ideals. If $q = \operatorname{char} Kv$, then

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{q-1}\,. \tag{31}$$

If $q \neq \operatorname{char} Kv$, then always this case i) holds, and the factor q can be dropped in (30) since $vq = 0$.

ii) There is a Kummer generator $\eta \in L$ which is a 1-unit such that for

$$\xi \;:=\; \frac{\eta - 1}{\zeta_q - 1}\,, \tag{32}$$

we have that $v\xi < 0$, $vL = vK + \mathbb{Z}v\xi$, (14) holds for $x = \xi^j$ with suitable $j \in \{1,\ldots,q-1\}$, and we have the equality of $\mathcal{O}_L$-ideals

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (\xi^{-1}I_{\xi^j})^{q-1} \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{q-1}\,. \tag{33}$$

Proof. By part 2) of Proposition 2.4 and part 2)b) of Proposition 2.6, the extension admits a Kummer generator η such that either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta - 1)$ generates the value group extension; moreover, the first case always holds if $q \neq \mathrm{char}\,Kv$.

Let us consider the first case. Applying Theorem 3.3 with $x_0 = \eta$, we find that (14) holds for $x = \eta^j$ with suitable $j \in \{1,\ldots,q-1\}$. Since η^j is again a Kummer generator and also $v\eta^j$ generates the value group extension as j is prime to q, we may replace η by η^j. As now (14) holds for $x = \eta$, we can apply Proposition 3.7 with $S = \{c \in K^\times \mid vc\eta > 0\}$ and $b_c = c\eta$ to obtain:

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (f_c'(c\eta) \mid c \in K^\times \text{ with } vc\eta > 0)\,,$$

where f_c denotes the minimal polynomial of $c\eta$.

As also $c\eta$ is a Kummer generator, we can apply equation (26) to obtain that $f_c'(c\eta) = q(c\eta)^{q-1}$. Hence,

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; q(c\eta \mid c \in K \text{ with } vc\eta > 0)^{q-1} \;=\; qI_\eta^{q-1} \;=\; q\mathcal{M}_{\mathcal{E}}^{q-1}$$

where the last equation follows from Proposition 3.4. This proves (30).

If $q = \mathrm{char}\,Kv$, then $q\mathcal{M}_{\mathcal{E}}^{q-1} = ((\zeta_q - 1)\mathcal{M}_{\mathcal{E}})^{q-1} = (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{q-1}$ since $vq = (q-1)v(\zeta_q - 1)$ by (4) and the last equality follows from part 2)a) of Proposition 2.7. This proves (31).

Now we consider the second case. Since $L|K$ is a Kummer extension, K contains ζ_q. By Lemma 2.3, $v(\eta - 1) \leq v(\zeta_q - 1) \in vK$ because $v\eta = 0$. Since $v(\eta - 1) \notin vK$, inequality must hold. Hence with ξ defined by (32), we have $v\xi < 0$. Further, applying Theorem 3.3 with $x_0 = \xi$, we find that (14) holds for $x = \xi^j$ with suitable $j \in \{1,\ldots,q-1\}$. We apply Proposition 3.7 with $S = \{c \in K^\times \mid vc\xi^j > 0\}$ and $b_c = c\xi^j$ to obtain:

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (h_{j,c}'(c\xi^j) \mid c \in K^\times \text{ with } vc\xi^j > 0)\,,$$

where $h_{j,c}$ denotes the minimal polynomial of $c\xi^j$.

We note that $v(1 - \zeta_q) = v(1 - \zeta)$ for each primitive q-th root of unity ζ. We set $a := \eta - 1$. Then for every $\sigma \in G$, $v(a - \sigma a) = v(\eta - \sigma\eta) = v(1 - \zeta_q) > va$, hence

$$a^j - \sigma a^j \;=\; a^j - (\sigma a)^j \;=\; a^j - (a + \sigma a - a)^j \;=\; -\sum_{i=0}^{j-1} \binom{j}{i} a^i(\sigma a - a)^{j-i}\,.$$

Since $va < v(\sigma a - a)$, the summand of least value in the sum on the right hand side is the one for $i = j-1$. Consequently,

$$\begin{aligned} v(\xi^j - \sigma\xi^j) &= v(a^j - \sigma a^j) - jv(1-\zeta_q) \;=\; (j-1)va + v(a-\sigma a) - jv(1-\zeta_q) \\ &= (j-1)va + v(1-\zeta_q) - jv(1-\zeta_q) \;=\; (j-1)(va - v(1-\zeta_q)) \\ &= v\xi^{j-1}\,. \end{aligned}$$

Hence, equation (20) shows that

$$vh'_{j,c}(c\xi^j) \;=\; (q-1)vc\xi^{j-1}\,. \tag{34}$$

Hence,

$$\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) \;=\; (c\xi^{j-1} \mid vc\xi^j > 0)^{q-1} \;=\; (\xi^{-1}I_{\xi^j})^{q-1} \;=\; (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^{q-1}$$

where the last two equalities follow from by part 2)b) of Proposition 2.7 and Proposition 3.4. This proves equation (33). □

4. Kähler differentials for Galois extensions of prime degree

4.1. **Motivation.**

We prove a proposition that will be a main tool for our handling of Kähler differentials in the subsequent sections. It will provide a motivation for the calculation of the Kähler differentials for Artin-Schreier extensions and Kummer extensions of prime degree which will be dealt with in this section.

Given a Galois extension $(L|K,v)$, we denote by $(L|K,v)^{in}$ its inertia field (cf. [6, Section 19]).

Proposition 4.1. *Let $(L|K,v)$ be a finite Galois extension. Then the following assertions hold.*

1) There exists a tower of field extensions

$$K \;\subset\; K^{in} \;=\; K_0 \;\subset\; K_1 \;\subset\; \cdots \;\subset\; K_\ell \;=\; L \tag{35}$$

where $K^{in} = (L|K,v)^{in}$ and each extension $K_{i+1}|K_i$ is a Galois extension of prime degree. Note that if K is henselian, then the extension $K^{in}v|Kv$ is separable of degree equal to $[K^{in} : K]$.

2) Further, $(L|K,v)$ can be embedded in a finite Galois extension $(M|K,v)$ having the following properties:

$$\left\{\begin{array}{l} \textit{there exists a tower of field extensions} \\ \qquad\qquad K \subset M_0 \subset M_1 \subset \cdots \subset M_m = M, \\ \textit{where } M_0 = (M|K,v)^{in} \\ \textit{and each extension } M_{i+1}|M_i \textit{ is a Kummer extension of prime degree,} \\ \textit{or an Artin-Schreier extension if the extension is of degree } p = \operatorname{char} K. \end{array}\right. \tag{36}$$

Proof. 1): Set $K_0 := K^{in} := (L|K,v)^{in}$. Since the extension $L|K^{in}$ is solvable (cf. Theorems 24 and 25 on pages 77 and 78 of [32]), there exists a tower (35) of Galois extensions such that each extension $K_{i+1}|K_i$ is Galois of prime degree. The assertions about the extension $K^{in}v|Kv$ are part of the general properties of inertia fields.

2): This proof is essentially the same argument as in the Galois characterization of sovability by radicals. If an extension $K_{i+1}|K_i$ in the tower (35) is of degree $p = \operatorname{char} K$, then it is an Artin-Schreier extension. If it is of prime degree $q \neq \operatorname{char} K$, it is a Kummer extension if K_i contains a primitive q-th root of unity. We will now explain how to enlarge the extension $(L|K,v)$ so that this will be the case for each extension of prime degree $q \neq \operatorname{char} K$ in a resulting new tower.

Assume that (K,v) is of characteristic 0 with $\operatorname{char} Kv = p > 0$ and that some extension $K_{i+1}|K_i$ is Galois of degree p, but K does not contain a primitive p-th root of unity. In this case we will have to replace tower (35) by a larger one. Let ζ_p denote a primitive p-th root of unity. Then $K(\zeta_p)|K$ is a Galois extension, and so is $L(\zeta_p)|K$ since $L|K$ is assumed to be Galois.

Set $K_0' := (L(\zeta_p)|K,v)^{in}$; then $K_0 = K^{in} \subset K_0'$. As before, $K_0'|K$ is Galois, hence so are $K_0'(\zeta_p)|K$ and $K_0'(\zeta_p)|K_0'$. By part 1) of our proposition, there exists a tower of Galois extensions $K_0' \subset K_1' \subset \cdots \subset K_{r'}' = K_0'(\zeta_p)$ such that each extension $K_{i+1}'|K_i'$ is Galois of prime degree. Since $[K_0'(\zeta_p) : K_0'] < p$, none of the Galois extensions $K_{i+1}'|K_i'$ is of degree p.

We replace the tower (35) by the tower

$$K_0' \subset K_1' \subset \cdots \subset K_{r'}' = K_0'(\zeta_p) \subset K_1(\zeta_p) \subset \cdots \subset K_\ell(\zeta_p) = L(\zeta_p)\,. \tag{37}$$

Now we have that if in mixed characteristic any extension in the tower (35) is Galois of degree $p = \operatorname{char} Kv$, then it is a Kummer extension.

We now return to the general case, with no restriction on the characteristic of K, first making the above change if necessary.

In order to also make sure that all Galois extensions of prime degree $q \neq p$ in the tower are Kummer extensions, we take Q to be the set consisting of all such primes q. For every $q \in Q$, we choose a primitive q-th root of unity ζ_q and set $M := L(\zeta_q \mid q \in Q)$. Every extension $K(\zeta_q)|K$ is Galois, so $M|K$ is also a Galois extension.

Let us show that for every $q \in Q$, ζ_q lies in the inertia field of $(M|K,v)$. This is a standard fact, but we give a proof for completeness. The reduction of $X^q - 1$ modulo v is $X^q - 1v$ with $1v$ being the 1 in Kv. Since $q \neq \operatorname{char} Kv$, the polynomial $X^q - 1v$ has q distinct roots. The minimal polynomial f of ζ_q over K divides $X^q - 1$, so its reduction fv divides $X^q - 1v$ and has therefore only simple roots. It follows that if $\sigma \in \operatorname{Gal} M|K$ with $\sigma\zeta_q \neq \zeta_q$, then $(\sigma\zeta_q)v \neq \zeta_q v$, whence $v(\sigma\zeta_q - \zeta_q) = 0$. Hence every automorphism in the inertia group $\{\sigma \in \operatorname{Gal} M|K \mid \forall x \in \mathcal{O}_M : v(\sigma x - x) = 0\}$ must fix ζ_q, which proves our claim. It follows that $M_0 := K_0(\zeta_q \mid q \in Q)$ is the inertia field of $(M|K,v)$. Finally, we set $M_i := K_i(\zeta_q \mid q \in Q)$. By our construction, now also all extensions of prime degree $q \neq p$ in the tower are Kummer extensions. So we have obtained a tower as described in (36). □

4.2. **Some calculations of Kähler differentials.**

Let $L|K$ be an algebraic field extension. Let $A \subseteq K$ be a normal domain whose quotient field is K. Assume that $z \in L$ is integral over A and let $f(X)$ be the minimal polynomial of z over K. Then $f(X) \in A[X]$ (see [31, Theorem 4, page 260]). Since $f(X)$ is monic, $(f(X)K[X]) \cap A[X] = f(X)A[X]$, so $A[z] \cong A[X]/(f(X))$.

Thus,

$$\Omega_{A[z]|A} \cong [A[X]/(f(X),f'(X))]dX \cong [A[z]/(f'(z))]dX \tag{38}$$

by [16, Example 26.J, page 189] and [16, Theorem 58, page 187]. There is a universal derivation $d_{A[z]|A} : A[z] \to \Omega_{A[z]|A}$ defined by

$$g(z) \mapsto [g'(z)]dX \text{ for } g(X) \in A[X], \tag{39}$$

where $[g'(z)]$ is the class of $g'(z)$ in $A[z]/(f'(z))$.

We will also require the following theorem to calculate Kähler differentials.

Proposition 4.2. *(*[2, Theorem 1.1]*) Take an algebraic field extension $L|K$ of degree n, a normal domain A with quotient field K and a domain B with quotient field L such that $B|A$ is an integral extension. Assume that there exist generators $b_\alpha \in B$ of $L|K$, which are indexed by a totally ordered set S, such that $A[b_\alpha] \subset A[b_\beta]$ if $\alpha \le \beta$ and*

$$\bigcup_{\alpha\in S} A[b_\alpha] = B. \tag{40}$$

Further assume that there exist $a_\alpha, a_\beta \in A$ such that $a_\beta \mid a_\alpha$ if $\alpha \le \beta$ and for $\alpha \le \beta$, there exist $c_{\alpha,\beta} \in A$ and expressions

$$b_\alpha = \frac{a_\alpha}{a_\beta} b_\beta + c_{\alpha,\beta} . \tag{41}$$

Let h_α be the minimal polynomial of b_α over K. Take U and V to be the B-ideals

$$U = (a_\alpha \mid \alpha \in S) \quad \text{and} \quad V = (h'_\alpha(b_\alpha) \mid \alpha \in S) . \tag{42}$$

Then we have a B-module isomorphism

$$\Omega_{B|A} \cong U/UV . \tag{43}$$

For the case where $(L|K,v)$ is a valued field extension and $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$, for arbitrary $\gamma \in S$ the isomorphism (43) yields an $\mathcal{O}_L$-module isomorphism

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/b_\gamma^\dagger U^n \quad \text{with} \quad b_\gamma^\dagger := \frac{h'_\gamma(b_\gamma)}{a_\gamma^{n-1}} . \tag{44}$$

Identifying the B-module $\Omega_{B/A}$ with U/UV in the above theorem, the universal derivation $d_{B|A} : B \to \Omega_{B|A}$ is defined by

$$d_{B|A}(z) = [a_\alpha g'_\alpha(b_\alpha)] \in U/UV \quad \text{for } z = g_\alpha(b_\alpha) \in A[b_\alpha] \tag{45}$$

where $[a_\alpha g'_\alpha(b_\alpha)]$ is the class of $a_\alpha g'_\alpha(b_\alpha)$ in U/UV.

By definition, V is the B-ideal generated by the differents of all b_α. For the case where $(L|K,v)$ is a valued field extension, we obtain from Proposition 3.7:

Lemma 4.3. *Under the assumptions of Proposition 4.2, the $\mathcal{O}_L$-ideal V defined in (42) is equal to the $\mathcal{O}_L$-ideal $\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)$.*

This will be applied in the proofs of Theorems 4.6 and 4.8.

4.3. **Finite extensions $(L|K,v)$ of degree $[L:K]=\mathrm{f}(L|K)$ with separable residue field extension.**

Theorem 4.4. *Take a finite extension $(L|K,v)$ with $Lv|Kv$ separable of degree $[Lv:Kv]=[L:K]$. Then $\mathcal{O}_L=\mathcal{O}_K[x]$ for some $x\in L$ with $vx=0$ and $Lv=Kv(xv)$, and we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0\,.$$

Proof. By (12), $\mathcal{O}_L=\mathcal{O}_K[x]$ where x is a lift of a generator χ of Lv over Kv. Let $f(X)\in K[X]$ be the minimal polynomial of x over K. Since $vx=0$ and the extension is unibranched, also the conjugates of x have value 0 and thus, f has coefficients in $\mathcal{O}_K$. As $\deg f=[L:K]=[Lv:Kv]$, the reduction $\bar{f}$ of f in $Kv[X]$ is the minimal polynomial of χ over Kv. We have that $f'(x)v=\bar{f}'(\chi)$ which is nonzero since χ is separable over Kv. Thus $f'(x)$ is a unit in $\mathcal{O}_L$. By (38), $\Omega_{\mathcal{O}_L|\mathcal{O}_K}\cong\mathcal{O}_L/(f'(x))=0$. □

We note that this theorem always applies when $(L|K,v)$ is a Kummer extension of prime degree $q=\mathrm{f}(L|K)\neq\operatorname{char}Kv$ since then $Lv|Kv$ is separable.

4.4. **Artin-Schreier extensions $(L|K,v)$ of degree p with $\mathrm{f}(L|K)=p$ and inseparable residue field extension.**

Theorem 4.5. *Take an Artin-Schreier extension $\mathcal{E}=(L|K,v)$ of degree $p=\mathrm{f}(L|K)=\operatorname{char}K$ with $Lv|Kv$ inseparable. Then there exists an Artin-Schreier generator ϑ as in Lemma 3.8, and we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\;\cong\;\mathcal{O}_L/(\tilde{c}^{p-1})\;\cong\;I_{\mathcal{E}}/I_{\mathcal{E}}^p \tag{46}$$

as $\mathcal{O}_L$-modules. Consequently, $\Omega_{\mathcal{O}_L|\mathcal{O}_K}\neq 0$.

Proof. The first isomorphism in (46) follows from (38) together with Lemma 3.8. Since $v\tilde{c}=-v\vartheta$, we have $v\tilde{c}\neq 0$, whence $(\tilde{c}^{p-1})\neq\mathcal{O}_L$, as well as $\mathcal{O}_L/(\tilde{c}^{p-1})\cong(\tilde{c})/(\tilde{c})^p=I_{\mathcal{E}}/I_{\mathcal{E}}^p$ by part 1) of Proposition 2.7. This proves the second isomorphism in (46). □

With the notation of the statement and proof of Theorem 4.5, we have that for $z\in\mathcal{O}_L$, $z=g(\tilde{c}\vartheta)$ for some $g(X)\in\mathcal{O}_K[X]$, and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z)\;=\;[g'(\tilde{c}\vartheta)]\in\mathcal{O}_L/(\tilde{c}^{p-1})$$

by equation (39).

4.5. **Artin-Schreier extensions $(L|K,v)$ of degree p with $\mathrm{e}(L|K,v)=p$.**

Theorem 4.6. *Take an Artin-Schreier extension $\mathcal{E}=(L|K,v)$ of degree $p=\mathrm{e}(L|K)$. Then there exists an Artin-Schreier generator ϑ as in Lemma 3.9, and we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\;\cong\;\vartheta^{-1}\mathcal{M}_{\mathcal{E}}/(\vartheta^{-1}\mathcal{M}_{\mathcal{E}})^p\;=\;I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^p \tag{47}$$

as $\mathcal{O}_L$-modules; in particular, $\Omega_{\mathcal{O}_L|\mathcal{O}_K}\neq 0$.

Proof. We will apply Proposition 4.2 with $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$. We set $S = \{\alpha \in vK \mid \alpha + v\vartheta^j > 0\}$, endowed with the reverse ordering of vK. For each $\alpha \in S$ we choose $c_\alpha \in K$ such that $vc_\alpha = \alpha$. We set $b_\alpha = c_\alpha \vartheta^j$, $a_\alpha = c_\alpha$, and $c_{\alpha,\beta} = 0$. Then $a_\beta | a_\alpha$ and $A[b_\alpha] \subseteq A[b_\beta]$ if $\alpha \leq \beta$, and we have that $c_1\vartheta^j = \frac{c_1}{c_2} c_2 \vartheta^j$. We denote by h_α the minimal polynomial of $b_\alpha = c_\alpha \vartheta^j$ over K. Thus in the notation of Lemma 3.9, $h_\alpha = h_{j,\alpha}$ so that $h'_\alpha(b_\alpha) = h'_{j,\alpha}(c_\alpha \vartheta^j)$ and $V = \mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) = (\vartheta^{-1} I_{\vartheta^j})^{p-1}$ by equation (24) of Lemma 3.9. Further, $U = (a_\alpha \mid \alpha \in S) = (c_\alpha \mid \alpha \in S) = \vartheta^{-j} I_{\vartheta^j}$. Hence by Proposition 4.2,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/UV \;\cong\; \vartheta^{-j} I_{\vartheta^j} / \vartheta^{-j} I_{\vartheta^j} \, (\vartheta^{-1} I_{\vartheta^j})^{p-1} \;\cong\; \vartheta^{-1} I_{\vartheta^j} / (\vartheta^{-1} I_{\vartheta^j})^p \,.$$

Together with part 1) of Proposition 2.7 and Proposition 3.4, this proves (47).

Since $0 < v\vartheta^{-1} \notin vK$, we have $v\vartheta^{-1} > H_\mathcal{E}$ and therefore, $\vartheta^{-1} \in \mathcal{M}_\mathcal{E}$. By part 3) of Lemma 3.6 it follows that $(\vartheta^{-1}\mathcal{M}_\mathcal{E})^p \subsetneq \vartheta^{-1}\mathcal{M}_\mathcal{E}$, which shows that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. □

With the notation of the statement and proof of Theorem 4.6, we have that for $z \in \mathcal{O}_L$, $z = g(c_\alpha \vartheta^j)$ for some $g(X) \in \mathcal{O}_K[X]$, where $c_\alpha \in K$ is such that $vc_\alpha \vartheta^j > 0$ and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z) \;=\; [c_\alpha g'(\tilde{c}_\alpha \vartheta^j)] \in I_\mathcal{E}\mathcal{M}_\mathcal{E} / (I_\mathcal{E}\mathcal{M}_\mathcal{E})^p$$

by equation (45).

4.6. Kummer extensions $(L|K, v)$ of degree $p = \operatorname{char} Kv$ with $\mathrm{f}(L|K) = p$.

Theorem 4.7. *Let $(L|K, v)$ be a Kummer extension of degree $p = \mathrm{f}(L|K) = \operatorname{char} Kv$.*

In case i) of Lemma 3.10,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \mathcal{O}_L/(p) \;\cong\; I_\mathcal{E}/I_\mathcal{E}^p \tag{48}$$

as $\mathcal{O}_L$-modules. Consequently, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

In case ii) of Lemma 3.10,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \mathcal{O}_L/(p\tilde{c}^{p-1}) \;\cong\; I_\mathcal{E}/I_\mathcal{E}^p \tag{49}$$

as $\mathcal{O}_L$-modules, and

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0 \textit{ if and only if } Lv|Kv \textit{ is separable.} \tag{50}$$

Proof. The first isomorphisms in (48) and (49) follow from Lemma 3.10 and (38).

In case i),

$$\mathcal{O}_L/(p) \;=\; \mathcal{O}_L/((\zeta_p - 1)^{p-1}) \;\cong\; (\zeta_p - 1)/(\zeta_p - 1)^p \;=\; I_\mathcal{E}/I_\mathcal{E}^p$$

by part 2)a) of Proposition 2.7.

In case ii), where $v\tilde{c} = -v(\eta - 1)$,

$$\mathcal{O}_L/(p\tilde{c}^{p-1}) \;=\; \mathcal{O}_L/(\tilde{c}(\zeta_p - 1))^{p-1} \;\cong\; (\tilde{c}(\zeta_p - 1))/(\tilde{c}(\zeta_p - 1))^p \;=\; I_\mathcal{E}/I_\mathcal{E}^p$$

by part 2)b) of Proposition 2.7.

By Lemma 2.3, $Lv|Kv$ is separable if and only if $-v\tilde{c} = v(\eta - 1) = \frac{vp}{p-1}$, i.e., $vp\tilde{c}^{p-1} = 0$. This is equivalent to $(p\tilde{c}^{p-1}) = \mathcal{O}_L$, and thus to $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. □

Let the notation be as in the statement and proof of Theorem 4.7. In case i), we have that for $z \in \mathcal{O}_L$, $z = g(\eta)$ for some $g(X) \in \mathcal{O}_K[X]$ and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z) \;=\; [g'(\eta)] \in \mathcal{O}_L/(p)$$

by equation (39).

In case ii), we have that for $z \in \mathcal{O}_L$, $z = g(\tilde{c}(\eta-1))$ for some $g(X) \in \mathcal{O}_K[X]$ and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z) \;=\; [g'(\tilde{c}(\eta-1))] \in \mathcal{O}_L/(p\tilde{c})^{p-1}$$

by equation (39).

4.7. Kummer extensions $(L|K,v)$ of prime degree q with $\mathrm{e}\,(L|K)=q$.

Theorem 4.8. *Let $\mathcal{E}=(L|K,v)$ be a Kummer extension of prime degree q with $\mathrm{e}\,(L|K)=q$.*

In case i) of Lemma 3.11,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \mathcal{M}_\mathcal{E}/q\mathcal{M}_\mathcal{E}^q \tag{51}$$

as $\mathcal{O}_L$-modules. If $q\neq \mathrm{char}\,Kv$, then

$$\mathcal{M}_\mathcal{E}/q\mathcal{M}_\mathcal{E}^q \;=\; \mathcal{M}_\mathcal{E}/\mathcal{M}_\mathcal{E}^q\,. \tag{52}$$

If $q=\mathrm{char}\,Kv$, then

$$\mathcal{M}_\mathcal{E}/q\mathcal{M}_\mathcal{E}^q \;\cong\; (\zeta_q-1)\mathcal{M}_\mathcal{E}/((\zeta_q-1)\mathcal{M}_\mathcal{E})^q \;=\; I_\mathcal{E}\mathcal{M}_\mathcal{E}/(I_\mathcal{E}\mathcal{M}_\mathcal{E})^q\,. \tag{53}$$

We have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ if and only if $q\notin\mathcal{M}_\mathcal{E}$ and $\mathcal{M}_\mathcal{E}$ is a nonprincipal $\mathcal{O}_\mathcal{E}$-ideal. The condition $q\notin\mathcal{M}_\mathcal{E}$ always holds when $q\neq\mathrm{char}\,Kv$.

In case ii) of Lemma 3.11,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \xi^{-1}\mathcal{M}_\mathcal{E}/(\xi^{-1}\mathcal{M}_\mathcal{E})^q \;=\; I_\mathcal{E}\mathcal{M}_\mathcal{E}/(I_\mathcal{E}\mathcal{M}_\mathcal{E})^q \tag{54}$$

as $\mathcal{O}_L$-modules; in particular, $\Omega_{\mathcal{O}_L|\mathcal{O}_K}\neq 0$.

Proof. Assume that case i) holds. We will apply Proposition 4.2 with $A=\mathcal{O}_K$ and $B=\mathcal{O}_L$. We set $S=\{\alpha\in vK \mid \alpha+v\eta>0\}$, endowed with the reverse ordering of vK. For each $\alpha\in S$ we choose $c_\alpha\in K$ such that $vc_\alpha=\alpha$. We set $b_\alpha=c_\alpha\eta$, $a_\alpha=c_\alpha$, and $c_{\alpha,\beta}=0$. Then $a_\beta|a_\alpha$ and $A[b_\alpha]\subseteq A[b_\beta]$ if $\alpha\leq\beta$, and we have that $c_1\eta=\frac{c_1}{c_2}c_2\eta$. We denote by h_α the minimal polynomial of $b_\alpha=c_\alpha\eta$ over K. Thus in the notation of Lemma 3.11, $h_\alpha=f_{c_\alpha}$ so that $h'_\alpha(b_\alpha)=f'_{c_\alpha}(c_\alpha\eta)$ and $V=\mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K)=qI_\eta^{q-1}$ by equation (30) of Lemma 3.11. Further, $U=(a_\alpha\mid\alpha\in S)=(c_\alpha\mid\alpha\in S)=\eta^{-1}I_\eta$. Hence by Proposition 4.2,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\cong U/UV \;\cong\; \eta^{-1}I_\eta/\eta^{-1}I_\eta\,qI_\eta^{q-1} \;\cong\; I_\eta/qI_\eta^q\,.$$

From Proposition 3.4 we know that $I_\eta=\mathcal{M}_\mathcal{E}$. This proves (51).

Assume that $q\neq\mathrm{char}\,Kv$. Then $vq=0$, hence $q\notin\mathcal{M}_\mathcal{E}$ and by part 1) of Lemma 3.6, $q\mathcal{M}_\mathcal{E}^q=(q\mathcal{M}_\mathcal{E})\mathcal{M}_\mathcal{E}^{q-1}=\mathcal{M}_\mathcal{E}\mathcal{M}_\mathcal{E}^{q-1}=\mathcal{M}_\mathcal{E}^q$. This proves (52).

Assume that $q=\mathrm{char}\,Kv$. Then

$$\mathcal{M}_\mathcal{E}/q\mathcal{M}_\mathcal{E}^q\cong(\zeta_q-1)\mathcal{M}_\mathcal{E}/(\zeta_q-1)q\mathcal{M}_\mathcal{E}^q=(\zeta_q-1)\mathcal{M}_\mathcal{E}/(\zeta_q-1)^q\mathcal{M}_\mathcal{E}^q=I_\mathcal{E}\mathcal{M}_\mathcal{E}/I_\mathcal{E}^q\mathcal{M}_\mathcal{E}^q,$$

where we have used that $vq = (q-1)v(\zeta_q - 1)$ and that $I_{\mathcal{E}} = (\zeta_q - 1)$ by part 2)a) of Proposition 2.7. This proves (53).

Now we determine when $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ holds in the present case. If $q \in \mathcal{M}_{\mathcal{E}}$, then $q\mathcal{M}_{\mathcal{E}}^q \subseteq q\mathcal{M}_{\mathcal{E}} \subsetneq \mathcal{M}_{\mathcal{E}}$ by part 1) of Lemma 3.6, and if $\mathcal{M}_{\mathcal{E}}$ is a principal $\mathcal{O}_{\mathcal{E}}$-ideal, then $q\mathcal{M}_{\mathcal{E}}^q \subseteq \mathcal{M}_{\mathcal{E}}^q \subsetneq \mathcal{M}_{\mathcal{E}}$ by part 2) of Lemma 3.6; hence in both cases, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. On the other hand, if $q \notin \mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal, then by parts 1) and 2) of Lemma 3.6, $q\mathcal{M}_{\mathcal{E}}^q = q\mathcal{M}_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$, whence $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. If $q \neq \operatorname{char} Kv$, then $vq = 0$, hence $q \notin \mathcal{M}_{\mathcal{E}}$.

Assume that case ii) holds. Again we will apply Proposition 4.2 with $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$. We set $S = \{\alpha \in vK \mid \alpha + v\xi^j > 0\}$, endowed with the reverse ordering of vK. For each $\alpha \in S$ we choose $c_\alpha \in K$ such that $vc_\alpha = \alpha$. We set $b_\alpha = c_\alpha \xi^j$, $a_\alpha = c_\alpha$, and $c_{\alpha,\beta} = 0$. Then $a_\beta | a_\alpha$ and $A[b_\alpha] \subseteq A[b_\beta]$ if $\alpha \leq \beta$; we have that $c_1\xi^j = \frac{c_1}{c_2} c_2 \xi^j$. We denote by h_α the minimal polynomial of $b_\alpha = c_\alpha \xi^j$ over K. Thus in the notation of Lemma 3.11, $h_\alpha = h_{j,\alpha}$ so that $h'_\alpha(b_\alpha) = h'_{j,\alpha}(c_\alpha \xi^j)$ and $V = \mathcal{D}_0(\mathcal{O}_L|\mathcal{O}_K) = (\xi^{-1} I_{\xi^j})^{q-1}$ by equation (33) of Lemma 3.11. Further, $U = (a_\alpha \mid \alpha \in S) = (c_\alpha \mid \alpha \in S) = \xi^{-j} I_{\xi^j}$. Hence by Proposition 4.2, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/UV \cong \xi^{-j} I_{\xi^j} / \xi^{-j} I_{\xi^j} (\xi^{-1} I_{\xi^j})^{q-1} \cong \xi^{-1} I_{\xi^j} / \xi^{-1} I_{\xi^j} (\xi^{-1} I_{\xi^j})^{q-1} = \xi^{-1} I_{\xi^j} / (\xi^{-1} I_{\xi^j})^q$. Again from Proposition 3.4 we know that $I_{\xi^j} = \mathcal{M}_{\mathcal{E}}$. Further, $(\xi^{-1}) = I_{\mathcal{E}}$ by part 2)b) of Proposition 2.7. This proves (54).

Since $0 < v\xi^{-1} \notin vK$, we have $v\xi^{-1} > H_{\mathcal{E}}$ and therefore, $\xi^{-1} \in \mathcal{M}_{\mathcal{E}}$. By part 3) of Lemma 3.6 it follows that $(\xi^{-1}\mathcal{M}_{\mathcal{E}})^q \subsetneq \xi^{-1}\mathcal{M}_{\mathcal{E}}$, which shows that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. □

Let the notation be as in the statement and proof of Theorem 4.8. In case i), we have that for $z \in \mathcal{O}_L$, $z = g(c_\alpha \eta)$ for some $c_\alpha \in K$ such that $vc_\alpha \vartheta > 0$ and $g(X) \in \mathcal{O}_K[X]$ and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z) \;=\; [c_\alpha g'(c_\alpha \eta)] \in \mathcal{M}_{\mathcal{E}} / q\mathcal{M}_{\mathcal{E}}^q$$

by equation (45).

In case ii), we have that for $z \in \mathcal{O}_L$, $z = g(c_\alpha \xi^j)$ for some $c_\alpha \in K$ such that $vc_\alpha \xi^j > 0$ and $g(X) \in \mathcal{O}_K[X]$ and the universal derivation $d_{\mathcal{O}_L|\mathcal{O}_K}$ is defined by

$$d_{\mathcal{O}_L|\mathcal{O}_K}(z) \;=\; [c_\alpha g'(c_\alpha \xi^j)] \in I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}} / (I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^p$$

by equation (45).

5. Kähler differentials of towers of Galois extensions

In this section, our goal is the proof of the following two theorems, which will be given in Subsection 5.2. We begin by preparing the ingredients for the proofs.

We first state the "first fundamental exact sequence" of Kähler differentials.

Theorem 5.1. *(*[16, Theorem 25.1]*) A composite $k \to A \to B$ of ring homomorphisms leads to a natural exact sequence of B-modules*

$$\Omega_{A|k} \otimes_A B \to \Omega_{B|k} \to \Omega_{B|A} \to 0.$$

We will verify that in relevant situations, the left most homomorphism is injective, giving a short exact sequence. The following theorem is a consequence of the more general Theorem 6.3.32 of [7]. However, we will give an alternate proof in Section 5.2.

Theorem 5.2. *Assume that $L|K$ and $M|L$ are towers of finite Galois extensions of valued fields. Then there is a natural short exact sequence*

$$0 \to \Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M \to \Omega_{\mathcal{O}_M|\mathcal{O}_K} \to \Omega_{\mathcal{O}_M|\mathcal{O}_L} \to 0.$$

In particular, $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_M|\mathcal{O}_L} = 0$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

Theorem 5.3. *Let (K, v) be a valued field. Then*

1) *$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions $L|K$ of K^{sep}.*
2) *Let $L|K$ be a finite Galois subextension of K^{sep} and assume that*

$$K \subset K^{in} = K_0 \subset K_1 \subset \cdots \subset K_\ell = L$$

is a tower of field extensions factoring $L|K$ such that K^{in} is the inertia field of $(L|K, v)$ and $K_{i+1}|K_i$ is Galois of prime degree for all i. Then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_{K_{i+1}}|\mathcal{O}_{K_i}} = 0$ for $0 \le i \le \ell - 1$.

Lemma 5.4. *Assume that $(L|K, v)$ is a valued field extension. Then $\mathcal{O}_L$ is a faithfully flat $\mathcal{O}_K$-module.*

Proof. We have that $\mathcal{O}_L$ is a flat $\mathcal{O}_K$-module by [24, Theorem 4.33] (see also [25, Theorem 4.35]), since $\mathcal{O}_K$ is a valuation ring and $\mathcal{O}_L$ is a torsion free $\mathcal{O}_K$-module. Further, $\mathcal{O}_L$ is a faithfully flat $\mathcal{O}_K$-module by Theorem 7.2 [16], since $\mathcal{M}_K\mathcal{O}_L \ne \mathcal{O}_L$. □

Lemma 5.5. *Let $(L|K, v)$ be a finite valued field extension which is unibranched and such that there is a tower of field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_\ell = L$ such that for $1 \le i \le \ell$ one of the following holds:*

1) *$K_i|K_{i-1}$ is Galois of prime degree or*
2) *$[K_i : K_{i-1}] = [K_iv : K_{i-1}v]$ and K_iv is separable over $K_{i-1}v$.*

Then for $2 \le i \le \ell$, we have natural short exact sequences

$$0 \to (\Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K}) \otimes_{\mathcal{O}_{K_{i-1}}} \mathcal{O}_{K_i} \to \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_K} \to \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_{i-1}}} \to 0. \tag{55}$$

Proof. By Theorem 4.4, Theorem 3.3 for unibranched defectless extensions of prime degree and [2, Lemma 2.3, Lemma 3.1, Lemma 3.2 and Proposition 3.3] for extensions of prime degree with nontrivial defect for $1 \le i \le \ell$ there exist directed sets S_i with associated $\alpha(i)_j \in K_i$ for $j \in S_i$ such that $\mathcal{O}_{K_{i-1}}[\alpha(i)_j] \subset \mathcal{O}_{K_{i-1}}[\alpha(i)_k]$ if $j \le k$ and $\mathcal{O}_{K_i} = \cup_{j \in S_i} \mathcal{O}_{K_{i-1}}[\alpha(i)_j]$. Further, $\mathcal{O}_{K_i}[\alpha(i)_j] \cong \mathcal{O}_{K_i}[X]/(f_i^j(X))$ where $f_i^j(X)$ is the minimal polynomial $\alpha(i)_j$ over K_{i-1}.

Let T_i be the set of $(k_1, k_2, \dots, k_{i-1}, k_i) \in S_1 \times S_2 \times \cdots \times S_i$ such that $f_n^{k_n}(x) \in \mathcal{O}_K[\alpha(1)_{k_1}, \alpha(2)_{k_2}, \dots, \alpha(n-1)_{k_{n-1}}][x]$ for $2 \le n \le i$. We define a partial order on T_i by the rule $(k_1, \dots, k_i) \le (l_1, \dots, l_i)$ if $k_m \le l_m$ for $1 \le m \le i$. The T_i are directed sets since the S_i are, and setting

$$A_{k_1,\dots,k_i} = \mathcal{O}_K[\alpha(1)_{k_1}, \alpha(2)_{k_2}, \dots, \alpha(i-1)_{k_{i-1}}, \alpha(i)_{k_i}]$$

for $(k_1, \dots, k_i) \in T_i$, we have inclusions

$$A_{k_1,\dots,k_i} \subset A_{l_1,\dots,l_i} \text{ for } (k_1, \dots, k_i) \le (l_1, l_2, \dots, l_i) \text{ in } T_i.$$

By our construction, for $2 \le m \le i$, there exist

$$g_m^{k_m}(X_1, \ldots, X_{m-1}, X_m) \in \mathcal{O}_K[X_1, X_2, \ldots, X_{m-1}, X_m]$$

such that $g_m^{k_m}(\alpha(1)_{k_1}, \cdots, \alpha(m-1)_{k_{m-1}}, X_m) = f_m^{k_m}(X_m)$.

By [28, Theorem 1], we have that

$$A_{k_1,\ldots,k_i} \cong \mathcal{O}_K[X_1, \ldots, X_i]/(g_1^{k_1}(X_1), g_2^{k_2}(X_1, X_2), \ldots, g_i^{k_i}(X_1, \ldots, X_i)).$$

By [16, Theorem 25.2], $\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K} \cong (A_{k_1,\ldots,k_i}dX_1 \oplus \cdots \oplus A_{k_1,\ldots,k_i}dX_i)/U_{k_1,\ldots,k_i}$, where $U_{k_1,\ldots,k_i}$ is the $A_{k_1,\ldots,k_i}$-submodule of $A_{k_1,\ldots,k_i}dX_1 \oplus \cdots \oplus A_{k_1,\ldots,k_i}dX_i$ generated by

$$\left[\frac{\partial f_1^{k_1}}{\partial X_1}(\alpha(1)_{k_1})\right] dX_1 \tag{56}$$

and

$$\left[\frac{\partial g_m^{k_m}}{\partial X_1}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] dX_1 + \cdots + \left[\frac{\partial g_m^{k_m}}{\partial X_m}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] dX_m \tag{57}$$

for $2 \le m \le i$. We further have that

$$\left[\frac{\partial f_1^{k_1}}{\partial X_1}(\alpha(1)_{k_1})\right] = (f_1^{k_1})'(\alpha(1)_{k_1}) \tag{58}$$

and

$$\left[\frac{\partial g_m^{k_m}}{\partial X_m}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] = (f_m^{k_m})'(\alpha(m)_{k_m}) \tag{59}$$

for $2 \le m \le i$.

By Theorem 5.1, we have a natural exact sequence of $A_{k_1,\ldots,k_i}$-modules

$$\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} A_{k_1,\ldots,k_i} \to \Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K} \to \Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}} \to 0. \tag{60}$$

For $(k_1, \ldots, k_i) \in T_i$, let

$$\begin{aligned} L_{k_1,\ldots,k_i} &= \Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_i}, \\ M_{k_1,\ldots,k_i} &= \Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_i}} \mathcal{O}_{K_i}, \\ N_{k_1,\ldots,k_i} &= \Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}} \otimes_{A_{k_1,\ldots,k_i}} \mathcal{O}_{K_i}. \end{aligned}$$

Applying the right exact functor $\otimes_{A_{k_1,\ldots,k_i}}\mathcal{O}_{K_i}$ to (60), we have an exact sequence of $\mathcal{O}_{K_i}$-modules

$$L_{k_1,\ldots,k_i} \xrightarrow{u} M_{k_1,\ldots,k_i} \to N_{k_1,\ldots,k_i} \to 0. \tag{61}$$

Now $\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_i}$ is the quotient of $\mathcal{O}_{K_i}dX_1 \oplus \cdots \oplus \mathcal{O}_{K_i}dX_{i-1}$ by the relations (56) and (57) for $2 \le m \le i-1$ and $\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_i}} \mathcal{O}_{K_i}$ is the quotient of $\mathcal{O}_{K_i}dX_1 \oplus \cdots \oplus \mathcal{O}_{K_i}dX_i$ by the relations (56) and (57) for $2 \le m \le i$. Since $(f_i^{k_i})'(\alpha(i)_{k_i}) \ne 0$ (as K_i is separable over K_{i-1}) we have by (59) with $m = i$ that u is injective, so that (61) is actually short exact.

Let $(k_1, \ldots, k_i)$ and $(l_1, \ldots, l_i)$ in T_i be such that $(k_1, \ldots, k_i) \le (l_1, \ldots, l_i)$. Then we have a natural commutative diagram of $\mathcal{O}_{K_i}$-modules with short exact rows

$$\begin{array}{ccccccccc} 0 & \to & L_{k_1,\ldots,k_i} & \to & M_{k_1,\ldots,k_i} & \to & N_{k_1,\ldots,k_i} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_{l_1,\ldots,l_i} & \to & M_{l_1,\ldots,l_i} & \to & N_{l_1,\ldots,l_i} & \to & 0 \end{array} \tag{62}$$

where the vertical arrows are the natural maps determined by the differentials of the inclusions of $A_{k_1,\ldots,k_{i-1}}$ into $A_{l_1,\ldots,l_{i-1}}$ and of $A_{k_1,\ldots,k_i}$ into $A_{l_1,\ldots,l_i}$. By [24, Theorem 2.18] (see also [25, Proposition 5.33]), we have a short exact sequence of $\mathcal{O}_K$-modules

$$0 \to \lim_{\to} L_{k_1,\ldots,k_i} \to \lim_{\to} M_{k_1,\ldots,k_i} \to \lim_{\to} N_{k_1,\ldots,k_i} \to 0. \tag{63}$$

By our construction of T_i, we have that $\cup A_{k_1,\ldots,k_i} = \mathcal{O}_{K_i}$, where the union is over all $(k_1,\ldots,k_i) \in T_i$. Thus $\lim_{\to} M_{k_1,\ldots,k_i} \cong \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_K}$ by [5, Theorem 16.8]. We also have that $\cup A_{k_1,\ldots,k_{i-1}} = \mathcal{O}_{K_{i-1}}$, where the union is over all $(k_1,\ldots,k_{i-1})$ such that $(k_1,\ldots,k_i) \in T_i$. Thus

$$\lim_{\to} \left(\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_{i-1}} \right) \cong \Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K}$$

again by [5, Theorem 16.8]. Now

$$\begin{array}{rcl} \lim_{\to} L_{k_1,\ldots,k_i} & = & \lim_{\to} \left(\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_i} \right) \\ & \cong & \lim_{\to} \left((\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_{i-1}}) \otimes_{\mathcal{O}_{K_{i-1}}} \mathcal{O}_{K_i} \right) \\ & \cong & \left(\lim_{\to} (\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K} \otimes_{A_{k_1,\ldots,k_{i-1}}} \mathcal{O}_{K_{i-1}}) \right) \otimes_{\mathcal{O}_{K_{i-1}}} \mathcal{O}_{K_i} \\ & \cong & \Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K} \otimes_{\mathcal{O}_{K_{i-1}}} \mathcal{O}_{K_i} \end{array}$$

where the equality of the third row is by [24, Corollary 2.20].

We have that

$$A_{k_1,\ldots,k_i} = A_{k_1,\ldots,k_{i-1}}[\alpha(i)_{k_i}] \cong A_{k_1,\ldots,k_{i-1}}[X_i]/(f_i^{k_i}),$$

so

$$\Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}} \cong \left(A_{k_1,\ldots,k_i}/(f_i^{k_i})'(\alpha(i)_{k_i}) \right) dX_i$$

by equation (38). Since $f_i^{k_i}$ is the minimal polynomial of $\alpha(i)_{k_i}$ over K_{i-1}, we have that

$$\Omega_{\mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]|\mathcal{O}_{K_{i-1}}} \cong \mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]/((f_i^{k_i})'(\alpha(i)_{k_i}))dX_i$$

also by (38). Thus

$$\begin{array}{rcl} N_{k_1,\ldots,k_i} & = & \Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}} \otimes_{A_{k_1,\ldots,k_i}} \mathcal{O}_{K_i} \cong \left(\mathcal{O}_{K_i}/(f_i^{k_i})'(\alpha(i)_{k_i}) \right) dX_i \\ & \cong & \left(\Omega_{\mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]|\mathcal{O}_{K_{i-1}}} \right) \otimes_{\mathcal{O}_{K_{i-1}[\alpha(i)_{k_i}]}} \mathcal{O}_{K_i}. \end{array}$$

Since $\cup \mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}] = \mathcal{O}_{K_i}$, we have that $\lim_{\to} N_{k_1,\ldots,k_i} \cong \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_{i-1}}}$ by [5, Theorem 16.8].

In conclusion, for $1 \le i \le r$, the sequence (55) is isomorphic to the short exact sequence (63). ☐

In Definition 2 of Chapter I, page 11 [23], an étale algebra is defined. Let A be a ring and B be an A-algebra. B is étale over A if

1) B is an A-algebra of finite presentation and
2) For all A-algebra D and ideals J of D such that $J^2 = 0$, the natural map $\mathrm{Hom}_{A\text{-alg}}(B, D) \to \mathrm{Hom}_{A\text{-alg}}(B, D/J)$ is a bijection.

In Definition IV.17.3.1 [8], an étale morphism of schemes is defined. After the definition, it is shown that a morphism of affine schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale if and only if B is étale over A.

Proposition 5.6. *Let $(L|K,v)$ be a finite Galois extension of valued fields. Let G be the Galois group of $L|K$ and let H be a subgroup of G which contains the inertia group of $L|K$. Denote the fixed field of H in L by L_0. Then $\Omega_{\mathcal{O}_{L_0}|\mathcal{O}_K}=0$.*

Proof. Let $A=\mathcal{O}_K$, C be the integral closure of A in $\mathcal{O}_L$ and $B=C^H$ be the integral closure of A in $L_0=L^H$. There exists a maximal ideal r of C such that $C_r=\mathcal{O}_L$. Let $n=r\cap B$, the maximal ideal of B, so that $\mathcal{O}_{L_0}=B_n$. By Theorem 1 of Chapter X, page 103 [23], there exists $f\in B\setminus n$ such that $B'=B_f$ is an étale A-algebra. We have that $(B')_{n_f}=\mathcal{O}_{L_0}$. $\operatorname{Spec}(B')\to\operatorname{Spec}(A)$ is an étale morphism, so the map is formally unramified (Definition IV.17.1.1 [8]). Thus $\Omega_{B'|A}=0$ by Proposition IV.17.2.1 [8]. Thus $0=(\Omega_{B'|A})\otimes_{B'}(B')_{n_f}=\Omega_{\mathcal{O}_{L_0}|\mathcal{O}_K}$ by [5, Proposition 16.9]. □

Proposition 5.7. *Assume that $(L|K,v)$ is a finite Galois extension. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\cong\Omega_{\mathcal{O}_L|\mathcal{O}_{K^{in}}}$$

where K^{in} is the inertia field of $(L|K,v)$.

Proof. This follows from Proposition 5.6 and the exact sequence of Theorem 5.1. □

We now give the proof of Theorem 1.1. Let p be the characteristic of the residue field Kv and $q=[L:K]$ a prime number. The description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ and the characterization of vanishing of this module depend, among other information, on the invariants of the valued field extension that appear in the following product:

$$q\;=\;[L:K]\;=\;\mathrm{d}\,(L|K)\,\mathrm{e}\,(L|K)\,\mathrm{f}\,(L|K)\,\mathrm{g}\,(L|K)$$

where $\mathrm{e}\,(L|K)=(vL:vK)$, $\mathrm{f}\,(L|K)=[Lv:Kv]$, $\mathrm{g}\,(L|K)$ is the number of distinct extensions of $v|K$ to L and $\mathrm{d}\,(L|K)$ is the defect of the extension, which is a power of p. Since q is a prime, exactly one of the factors will be equal to q, and the others equal to 1. The description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ also depends on the rank and the structure of the value group of (K,v) if $\mathrm{d}\,(L|K)\neq 1$ or $\mathrm{e}\,(L|K)\neq 1$, and on whether $Lv|Kv$ is separable or inseparable if $\mathrm{f}\,(L|K)\neq 1$.

In the case of $\mathrm{d}\,(L|K)=p$, our results are proven in [2, Theorem 1.2]. In the case of $\mathrm{e}\,(L|K)=q$, they are obtained in Theorem 4.6 for Artin-Schreier extensions and Theorem 4.8 for Kummer extensions. If $\mathrm{f}\,(L|K)=q$, then they are obtained in Theorems 4.4 and 4.5 for Artin-Schreier extensions and Theorems 4.4 and 4.7 for Kummer extensions.

In the remaining case when $\mathrm{g}\,(L|K)=q$, the extension $(L|K,v)$ is an inertial extension. Thus $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ by Proposition 5.6.

5.1. **Henselization.**

We now recall some facts about henselization of fields and rings. A valued field (K,v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, all of its algebraic extensions are unibranched (cf. [6, Section 16]).

An extension (K^h,v^h) of a valued field (K,v) is called a **henselization** of (K,v) if (K^h,v^h) is henselian and for all henselian valued fields (L,ω) and all embeddings $\lambda:(K,v)\to(L,\omega)$, there exists a unique embedding $\tilde{\lambda}:(K^h,v^h)\to(L,\omega)$ which extends λ.

A henselization (K^h, v^h) of (K, v) can be constructed by choosing an extension v^s of v to a separable closure K^{sep} of K and letting K^h be the fixed field of the decomposition group

$$G^d(K^{sep}|K) = \{\sigma \in G(K^{sep}|K) \mid v^s \circ \sigma = v^s\}$$

of v^s, and defining v^h to be the restriction of v^s to K^h ([6, Theorem 17.11]). The valuation ring $\mathcal{O}_{K^h}$ of v^H is then

$$\mathcal{O}_{K^h} = \mathcal{O}_{v^s} \cap K^h = \tilde{A}_{\tilde{m}} \tag{64}$$

where $\tilde{A}$ is the integral closure of $\mathcal{O}_v$ in K^h and $\tilde{m} = \mathcal{M}_{K^{\rm sep}} \cap K^h$.

The definition of a henselian local ring is given in Definition 1, Chapter I, page 1 of [23]. A local ring A is henselian if all finite A-algebras B are a product of local rings.

Assume that A is a local ring and $g(X) \in A[X]$ is a polynomial. Let $\bar{g}(X) \in A/m_A[X]$ be the polynomial obtained by reducing the coefficients of $g(X)$ modulo m_A.

By Proposition 5, Chapter I, page 2 [23], a local ring A is a henselian local ring if and only if it has the following property: Let $f(X) \in A[X]$ be a monic polynomial of degree n. If $\alpha(X)$ and $\beta(X)$ are relatively prime monic polynomials in $A/m_A[X]$ of degrees r and $n-r$ respectively such that $\bar{f}(X) = \alpha(X)\beta(X)$, then there exist monic polynomials $g(X)$ and $h(X)$ in $A[X]$ of degrees r and $n-r$ respectively such that $\bar{g}(X) = \alpha(X)$, $\bar{h}(X) = \beta(X)$ and $f(X) = g(X)h(X)$.

Henselization of a local ring is defined in Definition 1, Chapter VIII, page 80 [23]. If A is a local ring, a local ring A^h which dominates A is called a henselization of A if any local homomorphism from A to a henselian local ring can be uniquely extended to A^h. A henselization always exists, as is shown in [23, Theorem 1, Chapter VIII, page 87]. The construction is particularly nice when A is a normal local ring, as shown in [23, Theorem 2, Chapter X, page 110] (cf. [17, Theorem 43.5]). We now explain this construction. Let K be the quotient field of A and Let K^{sep} be a separable closure of K. Let $\bar{A}$ be the integral closure of A in K^{sep} and let $\bar{m}$ be a maximal ideal of $\bar{A}$.

Let H be the decomposition group

$$H = G^d(\bar{A}_{\bar{m}}|A) = \{\sigma \in G(K^{sep}|K) \mid \sigma(\bar{A}_{\bar{m}}) = \bar{A}_{\bar{m}}\}.$$

Then

$$A^h = (\tilde{A})_{\bar{m}\cap\tilde{A}} \tag{65}$$

where $\tilde{A}$ is the integral closure of A in $(K^{\rm sep})^H$.

Lemma 5.8. *Assume that (K, v) is a valued field and (K^h, v^h) is a henselization of K. Then there is a natural isomorphism*

$$\mathcal{O}_{K^h} \cong \mathcal{O}_K^h.$$

Proof. Let v^s be an extension of v to K^{sep} and

$$H = \{\sigma \in \mathrm{Gal}(K^{sep}|K) \mid v^s \circ \sigma = v^s\},$$

so that K^h is the fixed field of H in $K^{\rm sep}$. Let $\bar{V}$ be the integral closure of $\mathcal{O}_K$ in K^{sep}, and let $m = \bar{V} \cap \mathcal{M}_{K^{sep}}$, a maximal ideal in $\bar{V}$. Since K^{sep} is algebraic over

K, we have that $\mathcal{O}_{K^{sep}} = \bar{V}_m$ by [32, Theorem 12, page 27]. Now, as is shown on the bottom of page 68 of [32], H is the decomposition group

$$H = G^d(\mathcal{O}_{K^{sep}}|\mathcal{O}_K) = \{\sigma \in G(K^{sep}|K) \mid \sigma(\mathcal{O}_{K^{sep}}) = \mathcal{O}_{K^{sep}}\},$$

so that

$$\mathcal{O}_K^h = \mathcal{O}_{K^h}$$

by (64) and (65), establishing the lemma. □

Lemma 5.9. *Let K be a valued field and L be a field such that $K \subset L \subset K^h$. Then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.*

Proof. Let v^s be an extension of v to $K^{\rm sep}$. The field $K^{\rm sep}$ is the directed union $K^{\rm sep} = \cup_i M_i$ of the finite Galois extensions M_i of K in $K^{\rm sep}$. If M is a finite Galois extension of K in $K^{\rm sep}$, then restriction induces a surjection of Galois groups $G(K^{\rm sep}|K) \to G(M|K)$, and an isomorphism $G(M|K) \cong G(K^{\rm sep}|K)/G(K^{\rm sep}|M)$. We have an isomorphism of profinite groups ([20, Example 1, page 271] or [15, Theorem VI.14.1, page 313])

$$G(K^{\rm sep}|K) \cong \lim_{\leftarrow} G(M_i|K).$$

Let $G^d(M|K)$ be the decomposition group of the valued field extension $M|K$, for M a Galois extension of K which is contained in $K^{\rm sep}$ (where the valuation of M is $v^s|M$). For M a finite Galois extension of K, restriction induces a homomorphism

$$G^d(K^{\rm sep}|K) \to G^d(M|K). \tag{66}$$

Let $\sigma \in G^d(M|K)$. If N is a finite Galois extension of M contained in $K^{\rm sep}$, then there exists $\bar\sigma \in G(N|K)$ such that $\bar\sigma|_M = \sigma$. Let A be the integral closure of $\mathcal{O}_M$ in N. There exists a maximal ideal p of A such that $A_p \cong \mathcal{O}_N$. Let $q = \bar\sigma(p)$, a maximal ideal of A. The group $G(N|M)$ acts transitively on the maximal ideals of A ([1, Lemma 21.8]) so there exists $\tau \in G(N|M)$ such that $\tau(q) = p$. Thus $\tau\bar\sigma(\mathcal{O}_N) = \mathcal{O}_N$ and $\tau\bar\sigma|_M = \sigma$ and so the homomorphism (66) is surjective with Kernel $G^d(K^{\rm sep}|K) \cap G(K^{\rm sep}|M)$. We have that

$$K^h = (K^{\rm sep})^{G^d(K^{\rm sep}|K)} = \cup M_i^{G^d(M_i|K)}.$$

Thus

$$L = L \cap (\cup_i M_i^{G^d(M_i|K)}) = \cup L_i$$

where $L_i = L \cap M_i^{G^d(M_i|K)}$. We have that $\Omega_{\mathcal{O}_{L_i}|\mathcal{O}_K} = 0$ for all i by Proposition 5.6. Thus

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \lim_{\rightarrow}(\Omega_{\mathcal{O}_{L_i}|\mathcal{O}_K} \otimes_{\mathcal{O}_{L_i}} \mathcal{O}_L) = 0$$

by [5, Theorem 16.8]. □

Let K be a valued field. Fix an extension v^s of v to the separable closure K^{sep} of K. The field K^{sep} is henselian (for instance by the construction before Lemma 5.8); that is, the henselization $(K^{sep})^h = K^{sep}$ and $\mathcal{O}_{(K^{sep})^h} = \mathcal{O}_{K^{sep}}$.

Proposition 5.10. *Let (K, v) be a valued field. Then $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \cong \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}}$.*

Proof. We may embed K^h into K^{sep} (by the construction before Lemma 5.8) giving a tower of valued field extensions $K \subset K^h \subset K^{sep}$. By Theorem 5.1, we have an exact sequence $\Omega_{\mathcal{O}_{K^h}|\mathcal{O}_K} \otimes_{\mathcal{O}_{K^h}} \mathcal{O}_{K^{sep}} \to \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \to \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}} \to 0$. The proposition now follows from Lemma 5.9. □

Lemma 5.11. *Assume that $(L|K, v)$ is a finite separable extension of valued fields. Then*

$$\Omega_{\mathcal{O}_L^h|\mathcal{O}_K^h} \cong (\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{L^h} \,. \tag{67}$$

In particular, by Lemma 5.4, we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} = 0$.

Proof. We have that

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \cong \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_K} \tag{68}$$

by Lemma 5.9 and the exact sequence of Theorem 5.1. By [23, Theorem 1, page 87], there exist étale extensions $A_i|\mathcal{O}_L$ and maximal ideals m_i of A_i such that $\mathcal{O}_{L^h} = \varinjlim (A_i)_{m_i}$. We have the exact sequences

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} A_i \overset{\alpha}{\to} \Omega_{A_i|\mathcal{O}_K} \to \Omega_{A_i|\mathcal{O}_L} \to 0$$

of Theorem 5.1. Since $A_i|\mathcal{O}_L$ is étale, we have that this map is formally étale ([8, Definition IV.17.3.1]) and is thus formally unramified and formally smooth ([8, Definition IV.17.1.1]). Thus $\Omega_{A_i|\mathcal{O}_L} = 0$ by [8, Proposition IV.17.2.1] and α is injective by [8, Proposition IV.17.2.3]. By this calculation and [5, Proposition 16.9],

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} (A_i)_{m_i} \cong (\Omega_{A_i|\mathcal{O}_K}) \otimes_{A_i} (A_i)_{m_i} \cong \Omega_{(A_i)_{m_i}|\mathcal{O}_K}. \tag{69}$$

By Theorem 16.8 [5] and equations (68) and (69),

$$\begin{aligned} \Omega_{\mathcal{O}_L^h|\mathcal{O}_K^h} &\cong \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_K} \cong \varinjlim [(\Omega_{(A_i)_{m_i}|\mathcal{O}_K}) \otimes_{(A_i)_{m_i}} \mathcal{O}_{L^h}] \\ &\cong \varinjlim [(\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{L^h}] \cong (\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{L^h}. \end{aligned} \tag{70}$$

□

5.2. **Proofs of Theorems 5.2 and 5.3.**

We first prove Theorem 5.2.

The natural sequence of $\mathcal{O}_M$-modules

$$0 \to \Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M \to \Omega_{\mathcal{O}_M|\mathcal{O}_K} \to \Omega_{\mathcal{O}_M|\mathcal{O}_L} \to 0 \tag{71}$$

computed from the extensions of rings $\mathcal{O}_K \subset \mathcal{O}_L \subset \mathcal{O}_M$ is right exact (but the first map might not be injective) by Theorem 5.1. Tensor this sequence with $\mathcal{O}_M^h$ over $\mathcal{O}_M$ to get a right exact sequence of $\mathcal{O}_M^h$-modules

$$0 \to (\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M) \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \to \Omega_{\mathcal{O}_M|\mathcal{O}_K} \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \to \Omega_{\mathcal{O}_M|\mathcal{O}_L} \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \to 0. \tag{72}$$

By (67), we have isomorphisms

$$\Omega_{\mathcal{O}_M|\mathcal{O}_L} \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \cong \Omega_{\mathcal{O}_M^h|\mathcal{O}_L^h},\ \Omega_{\mathcal{O}_M|\mathcal{O}_K} \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \cong \Omega_{\mathcal{O}_M^h|\mathcal{O}_K^h}$$

and

$$\begin{aligned} &(\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M) \otimes_{\mathcal{O}_M} \mathcal{O}_M^h \cong \Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M^h \\ &\cong (\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_L^h) \otimes_{\mathcal{O}_L^h} \mathcal{O}_M^h \cong \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_L^h} \mathcal{O}_M^h. \end{aligned}$$

Thus (72) is the right exact sequence

$$0 \to \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{M^h} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{L^h}} \to 0 \tag{73}$$

of Theorem 5.1. Since $\mathcal{O}_M^h$ is a faithfully flat $\mathcal{O}_M$-module, we have that (71) is exact if and only if (73) is exact.

By assumption, $L|K$ and $M|L$ are towers of Galois extensions

$$K = K_0 \subset K_1 \subset \cdots \subset K_r = L \text{ and } L = L_0 \subset L_1 \subset \cdots \subset L_s = M$$

so

$$K^h = K_0^h \subset K_1^h \subset \cdots \subset K_r^h = L^h \text{ and } L^h = L_0^h \subset L_1^h \subset \cdots \subset L_s^h = M^h$$

are towers of Galois extensions. Since each $K_{i+1}^h|K_i^h$ is unibranched, there exist factorizations

$$K_i^h \subset U_i^1 \subset U_i^2 \subset \cdots \subset U_i^{m_i} = K_{i+1}^h$$

where U_i^1 is the inertia field of $K_{i+1}^h|K_i^h$ and $U_i^{j+1}|U_i^j$ is Galois of prime degree. These extensions are all necessarily unibranched, so $U_i^1|K_i^h$ satisfies 2) of Lemma 5.5 and $U_i^{j+1}|U_i^j$ satisfies 1) of Lemma 5.5 for $1 \le j$. Similarly, we have factorizations

$$L_i^h \subset V_i^1 \subset V_i^2 \subset \cdots \subset V_i^{n_i} = L_{i+1}^h$$

where $V_i^1|L_i^h$ satisfies 2) of Lemma 5.5 and $V_i^{j+1}|V_i^j$ satisfies 1) of Lemma 5.5 for $1 \le j$. By Lemma 5.5, we have exact sequences

$$\begin{array}{c}
0 \to \Omega_{\mathcal{O}_{U_0^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_0^1}} \mathcal{O}_{U_0^2} \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{U_0^1}} \to 0\\
0 \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_0^2}} \mathcal{O}_{U_0^3} \to \Omega_{\mathcal{O}_{U_0^3}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_0^3}|\mathcal{O}_{U_0^2}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{K_1^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{K_1^h}} \mathcal{O}_{U_1^1} \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K_1^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_1^1}} \mathcal{O}_{U_1^2} \to \Omega_{\mathcal{O}_{U_1^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_1^2}|\mathcal{O}_{U_1^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{V_0^1} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{L^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_0^1}} \mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_{V_0^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{L_1^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L_1^h}} \mathcal{O}_{V_1^1} \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{L_1^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_1^1}} \mathcal{O}_{V_1^2} \to \Omega_{\mathcal{O}_{V_1^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_1^2}|\mathcal{O}_{V_1^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{V_{s-1}^{n_s-1}}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_{s-1}^{n_s-1}}} \mathcal{O}_{M^h} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{V_{s-1}^{n_s-1}}} \to 0.
\end{array}$$

In particular, differentiation defines an injection of $\mathcal{O}_{V_0^1}$-modules

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{V_0^1} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_K^h}.$$

Since $\mathcal{O}_{V_0^2}$ is a flat $\mathcal{O}_{V_0^1}$-module, we have injections

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{V_0^2} \cong (\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{V_0^1}) \otimes_{\mathcal{O}_{V_0^1}} \mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_0^1}} \mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_K^h}$$

and continuing, we obtain that differentiation gives an injection of $\mathcal{O}_{M^h}$-modules

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{M^h} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}}$$

so that (73) is short exact and thus (71) is short exact.

Since $\mathcal{O}_M$ is a faithfully flat $\mathcal{O}_L$, module, we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M = 0$ if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$, and so $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_M|\mathcal{O}_L} = 0$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. □

We now prove Theorem 5.3. We first prove Statement 1). By [5, Theorem 16.8], we have an isomorphism of $\mathcal{O}_{K^{sep}}$-modules

$$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \cong \lim_{\rightarrow}[(\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{K^{sep}}].$$

where the limit is over finite Galois subextensions $L|K$ of K^{sep}.

If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions of K^{sep}, then it follows immediately from the above formula that $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0$.

Assume that $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0$ and $L|K$ is a finite Galois subextension of K^{sep}. If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$, then there exists $0 \neq x \in \Omega_{\mathcal{O}_L|\mathcal{O}_K}$ and a finite Galois extension N of K such that N contains L and the image of $x \otimes 1$ by the natural homomorphism

$$(\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{K^{\mathrm{sep}}} \rightarrow (\Omega_{\mathcal{O}_N|\mathcal{O}_K}) \otimes_{\mathcal{O}_N} \mathcal{O}_{K^{\mathrm{sep}}}$$

is zero. Since $\mathcal{O}_{K^{\mathrm{sep}}}$ is a faithfully flat $\mathcal{O}_N$-module (by Lemma 5.4) we have that the image of $x \otimes 1$ by the natural homomorphism

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_N \rightarrow \Omega_{\mathcal{O}_N|\mathcal{O}_K}$$

is zero, so that $x \otimes 1 = 0$ by Theorem 5.2. Thus $x = 0$ since $\mathcal{O}_N$ is a faithfully flat $\mathcal{O}_L$-module, giving a contradiction, and showing that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

We now prove Statement 2). We have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \Omega_{\mathcal{O}_L|\mathcal{O}_{K^{in}}}$ by Proposition 5.7. For $0 \leq i \leq \ell - 1$, $\Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_0}} = 0$ if and only if $\Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_0}} \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}} = 0$ since $\mathcal{O}_{K_{i+1}}$ is a faithfully flat $\mathcal{O}_{K_i}$-module by Lemma 5.4. Statement 2) now follows from Lemma 5.5 by induction on i in equation (55). □

6. Proof of Theorems 1.2 and 1.3

Take a valued field (K, v) and extend v to the separable closure K^{sep} of K. Recall that we call (K, v) a deeply ramified field if it satisfies (DRvg) and (DRvr).

Throughout we assume that $\mathrm{char}\, Kv = p > 0$. If $\mathrm{char}\, K = 0$, then we set $K' := K(\zeta_p)$ with ζ_p a primitive p-th root of unity and extend v to K'. If $\mathrm{char}\, K = p$, then we set $K' := K$. The next proposition will show that in our proof of Theorem 1.2 we can assume that $K = K'$.

Proposition 6.1. *1) If $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$, then $\Omega_{\mathcal{O}_L|\mathcal{O}_{K'}} = 0$ holds for every finite Galois extension $(L|K', v)$.*

2) If (K', v) is a deeply ramified field, then so is (K, v).

Proof. 1): Assume that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$. By part 1) of Theorem 5.3 this implies that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for every finite Galois extension $(L|K, v)$. In all cases, $(K'|K, v)$ is a finite Galois extension, possibly trivial. Take any finite Galois extension $(L|K', v)$, let N be the normal hull of $L|K$, and take any extension of v to N. Then $(N|K, v)$ is a finite Galois extension, so we have $\Omega_{\mathcal{O}_N|\mathcal{O}_K} = 0$. Since also $(N|K', v)$ and $(K'|K, v)$ are finite Galois extensions, Theorem 5.2 shows that $\Omega_{\mathcal{O}_N|\mathcal{O}_{K'}} = 0$. Finally, since $(N|L, v)$ and $(L|K', v)$ are finite Galois extensions, Theorem 5.2 shows that $\Omega_{\mathcal{O}_L|\mathcal{O}_{K'}} = 0$.

2): This follows from [13, Theorem 1.8]. □

We split Theorem 1.2 into the following two propositions, which we will prove separately. In view of Proposition 6.1 it suffices to prove them under the assumption that K contains a primitive p-th root of unity if $\operatorname{char} K = p > 0$, i.e., $K = K'$.

Proposition 6.2. *If $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$, then (K, v) is a deeply ramified field.*

Proposition 6.3. *If (K, v) is a deeply ramified field, then $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$.*

One of the implications of Theorem 1.3 will be proved in Proposition 6.5, and the other in Proposition 6.6.

6.1. **Proof of Proposition 6.2.**

We will need some preparations. If the valued field (K, v) is of characteristic 0 with residue characteristic $p > 0$, then we decompose $v = v_0 \circ v_p \circ \overline{v}$, where v_0 is the finest coarsening of v that has residue characteristic 0, v_p is a rank 1 valuation on Kv_0, and $\overline{v}$ is the valuation induced by v on the residue field of v_p (which is of characteristic $p > 0$). The valuations v_0 and $\overline{v}$ may be trivial. Note that while it makes no sense to compose the valuations as functions, in this notation the valuations are interpreted as their associated places (as we have done before by writing "Kv"): in this way, $Kv = K(v_0 \circ v_p \circ \overline{v}) = ((Kv_0)v_p)\overline{v}$. For simplicity, we will write v_0v_p for $v_0 \circ v_p$ and $v_p\overline{v}$ for $v_p \circ \overline{v}$. In our decomposition, the valuation v_p is at the center, so we define $\operatorname{crf}(K, v) := (Kv_0)v_p$ as one may call it the "central residue field". In the equal characteristic case, we set $\operatorname{crf}(K, v) := Kv$.

Now take any valued field (K, v) of residue characteristic $p > 0$. We will use the following observation; we note that $\mathcal{C}_{vK}(vp)$ was denoted by $(vK)_{vp}$ in [13].

Proposition 6.4. *If $K = K'$ and $\mathcal{C}_{vK}(vp)$ is p-divisible, Kv is perfect and all Galois extensions $(L|K, v)$ of prime degree p with nontrivial defect satisfy $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$, then (K, v) satisfies (DRvr).*

Proof. We will show that the assumptions imply that $\operatorname{crf}(K, v)$ is perfect. Then the assertion follows from [13, Proposition 4.13] since by [2, Theorem 1.4], all Galois extensions $(L|K, v)$ of prime degree p with nontrivial defect that satisfy $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ have independent defect in the sense of [13, 2].

In the equal characteristic case, $\operatorname{crf}(K, v) = Kv$ and there is nothing to show. So we assume that (K, v) has mixed characteristic. Take any nonzero element of $\operatorname{crf}(K, v)$; it can be written as bv_0v_p with $b \in K$. Consider the extension $K(\eta)|K$ with $\eta^p = b$. We have that ηv_0v_p is a p-th root of bv_0v_p in $\operatorname{crf}(K(\eta), v)$.

Suppose that bv_0v_p does not have a p-th root in $\operatorname{crf}(K, v)$, so $K(\eta)|K$ is a Kummer extension of degree p. Then $(K(\eta)v_0v_p|Kv_0v_p$ is purely inseparable of degree p. It follows that $v_0v_pK(\eta) = v_0v_pK$ and that $(K(\eta)|K, v_0v_p)$ and $(K(\eta)v_0v_p|Kv_0v_p, \overline{v})$ are unibranched. Consequently, $(K(\eta)|K, v)$ is unibranched. Further, as $\mathcal{C}_{vK}(vp)$ and thus also $\overline{v}(Kv_0v_p)$ is p-divisible, we have $\overline{v}(K(\eta)v_0v_p) = \overline{v}(Kv_0v_p)$ and therefore, $vK(\eta) = vK$. Moreover, $K(\eta)v = K(\eta)v_0v_p\bar{v}$ is a purely inseparable extension of $Kv = Kv_0v_p\bar{v}$ and since Kv is perfect, we find that $K(\eta)v = Kv$. Thus $(K(\eta)|K, v)$ is an extension with nontrivial defect. Since (K, v) is an independent defect field, the defect must be independent. Hence by [2, condition b) of Theorem 1.8],

$$v(b - K^p) \;=\; \frac{p}{p-1}vp - \{\alpha \in pvK \mid \alpha > H\}$$

for some convex subgroup H of vK that does not contain vp, so also does not contain $\frac{p}{p-1}vp$. It follows that there is some $a \in K$ such that $v(b - a^p) > vp$, whence $(b - a^p)v_0v_p = 0$. This shows that $(av_0v_p)^p = bv_0v_p$, so that bv_0v_p has a p-th root in $\mathrm{crf}\,(K, v)$, which contradicts our assumption.

We have now proved that $\mathrm{crf}\,(K, v)$ is perfect, as desired. □

Now we are ready to prove one part of Theorem 1.3:

Proposition 6.5. *If $K = K'$ and if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all unibranched Galois extensions $(L|K, v)$ of prime degree p, then (K, v) is a deeply ramified field.*

Proof. We first deal with the equal characteristic case. In this case, $\mathcal{C}_{vK}(vp) = vK$. Suppose that vK is not p-divisible and take some $a \in K$ such that $va \notin pvK$. We may assume that $va < 0$. Take $\vartheta \in K^{sep}$ such that $\vartheta^p - \vartheta = a$. Then $pv\vartheta = va$ and $(K(\vartheta)|K, v)$ is an Artin-Schreier extension with $\mathrm{e}\,(K(\vartheta)|K, v) = p$. Hence by Theorem 4.6, $\Omega_{\mathcal{O}_{K(\vartheta)}|\mathcal{O}_K} \neq 0$, contradiction. Thus $\mathcal{C}_{vK}(vp) = vK$ is p-divisible, and in particular, (DRvg) holds.

Suppose that Kv is not perfect, and take $b \in \mathcal{O}_K^\times$ such that bv does not have a p-th root in Kv. Take $c \in K$ such that $vc < 0$ and $\vartheta \in K^{sep}$ such that $\vartheta^p - \vartheta = c^p b$. Then $(K(\vartheta)|K, v)$ is an Artin-Schreier extension with $K(\vartheta)v = Kv(bv^{1/p})$. Hence by Theorem 4.5, $\Omega_{\mathcal{O}_{K(\vartheta)}|\mathcal{O}_K} \neq 0$, which again is a contradiction. Hence Kv is perfect. Now Proposition 6.4 shows that also (DRvr) holds and consequently, (K, v) is a deeply ramified field.

Now we deal with the mixed characteristic case. If we are able to show that (K, v) satisfies (DRvg), $\mathcal{C}_{vK}(vp)$ is p-divisible and Kv is perfect, then we can as before apply Proposition 6.4 to obtain again that (K, v) is a deeply ramified field.

Suppose that there is an archimedean component of vK which is discrete. Pick $a \in K$ such that $va < 0$ and $va + \mathcal{C}_{vK}^+(va)$ is the largest negative element in $\mathcal{A}_{vK}(va)$. Take $\eta \in K^{sep}$ such that $\eta^p \in K$ with $v\eta^p = va$. Then $v\eta + \mathcal{C}_{vL}^+(v\eta)$ is the largest negative element in $\mathcal{A}_{vL}(v\eta)$, not contained in $\mathcal{A}_{vK}(pv\eta)$, and $(K(\eta)|K, v)$ is a Kummer extension with $\mathrm{e}\,(K(\eta)|K, v) = p$. It follows that $(vK(\eta)/\mathcal{C}_{vL}^+(v\eta) : vK/\mathcal{C}_{vK}^+(pv\eta)) = p$, hence we must have $\mathcal{C}_{vL}^+(v\eta) = \mathcal{C}_{vK}^+(pv\eta)$. Therefore, $\mathcal{E}$ is of type (DL2c) with $H_\mathcal{E} = \mathcal{C}_{vL}^+(v\eta)$, so $\mathcal{M}_\mathcal{E}$ is a principal $\mathcal{O}_\mathcal{E}$-ideal. From case i) of Theorem 4.8 we now infer that $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K} \neq 0$, contradiction.

Suppose that $\mathcal{C}_{vK}(vp)$ is not p-divisible and take some $a \in K$ such that $va \in \mathcal{C}_{vK}(vp) \setminus p\,\mathcal{C}_{vK}(vp)$. We may assume that $va < 0$. Take $\eta \in K^{sep}$ such that $\eta^p = a$. Then $pv\eta = va$ and $(K(\eta)|K, v)$ is a Kummer extension with $\mathrm{e}\,(K(\eta)|K, v) = p$. We have that $vI_\eta \cap \mathcal{C}_{vL}(vp) \neq \emptyset$. This implies that $vp \notin H_\mathcal{E}$, whence $p \in \mathcal{M}_\mathcal{E}$. Again from case i) of Theorem 4.8 we conclude that $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K} \neq 0$, contradiction.

Suppose that Kv is not perfect, and take $b \in \mathcal{O}_K^\times$ such that bv does not have a p-th root in Kv. Take $\eta \in K^{sep}$ such that $\eta^p = b$. Then $(K(\eta)|K, v)$ is a Kummer extension with $K(\eta)v = Kv(bv^{1/p})$. Hence by Theorem 4.7, $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K} \neq 0$, which is again a contradiction. This finishes the proof that (K, v) is deeply ramified. □

Now Proposition 6.2 follows from Proposition 6.5 in conjunction with part 1) of Proposition 6.1.

6.2. **Proof of Proposition 6.3.**

We first observe:

Proposition 6.6. *Take a deeply ramified field* (K,v) *such that* $K = K'$, *and a unibranched Galois extension* $(L|K,v)$ *of prime degree. Then* $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

Proof. In view of Theorem 1.4, we only have to deal with the case of defectless extensions.

Assume that $\operatorname{char} K = p$ and $(L|K,v)$ is an Artin-Schreier extension of degree p. We have that vK is p-divisible and Kv is perfect by [13, Lemma 4.2]. Thus, the case of $\mathrm{e}\,(L|K) = p$ cannot appear and we must have that $\mathrm{f}\,(L|K) = p$ with the extension $Lv|Kv$ separable. Hence $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ by Theorem 4.4.

Assume that $(L|K,v)$ is a Kummer extension of prime degree $q = \mathrm{f}\,(L|K)$. Again, $Lv|Kv$ is separable, so $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ by Theorem 4.4.

Finally, assume that $\mathcal{E} = (L|K,v)$ is a Kummer extension of prime degree $q = \mathrm{e}\,(L|K)$. Since each archimedean component of the deeply ramified field (K,v) is dense, the same holds for all archimedean components of vL. This shows that $\mathcal{E}$ is not of type (DL2c), so $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal.

If $q \neq \operatorname{char} Kv$, then $vq = 0$ implies that $q \notin \mathcal{M}_L$ and hence $q \notin \mathcal{M}_{\mathcal{E}}$. From case i) of Theorem 4.8 we now obtain that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

If $q = \operatorname{char} Kv$, then necessarily $\operatorname{char} K = 0$. By [13, part (1) of Lemma 4.3], $\mathcal{C}_{vK}(vq)$ is q-divisible. If case ii) of Theorem 4.8 would apply, then by (4), $0 < v(\eta - 1) < v(\zeta_q - 1) = \frac{vq}{q-1}$ with $v(\eta - 1) \notin vK$, whence $v(\eta - 1) \in \mathcal{C}_{vL}(vq)$ and $(\mathcal{C}_{vL}(vq) : \mathcal{C}_{vK}(vq)) = q$. As this contradicts the fact that $\mathcal{C}_{vK}(vq)$ is q-divisible, case ii) cannot appear and moreover, $vq \in H_{\mathcal{E}}$ and thus $q \notin \mathcal{M}_{\mathcal{E}}$ since $\mathcal{C}_{vL}(vq) = \mathcal{C}_{vK}(vq)$. By case i) of Theorem 4.8 we conclude that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. □

Take any deeply ramified field (K,v). By [13, Corollary 1.7 (2)], also the henselization $(K,v)^h$ of (K,v) inside of (K^{sep},v), for any of the conjugate extensions from v from K to K^{sep}, is a deeply ramified field. By Proposition 5.10 it suffices to prove that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}} = 0$. We may therefore assume from the start that (K,v) is henselian.

Part 1) of Theorem 5.3 shows that in order to prove that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$ it suffices to prove that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions $(L|K,v)$ of $(K^{sep}|K,v)$. Proposition 4.1 shows that after enlarging $(L|K,v)$ to a finite Galois extension $(M|K,v)$ if necessary, there is a tower of field extensions

$$K \subset M_0 \subset M_1 \subset \cdots \subset M_m = M$$

where M_0 is the inertia field of $(M|K,v)$ and each extension $M_{i+1}|M_i$ is a Kummer extension of prime degree, or an Artin-Schreier extension if the extension is of degree $p = \operatorname{char} K$. By part 2) of Theorem 5.3, to prove that $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ it suffices to prove that $\Omega_{\mathcal{O}_{M_{i+1}}|\mathcal{O}_{M_i}} = 0$ for $0 \le i \le m-1$. By Theorem 1.5, (M_i,v) is a deeply ramified field for each i, hence $\Omega_{\mathcal{O}_{M_{i+1}}|\mathcal{O}_{M_i}} = 0$ by Proposition 6.6. We have shown that $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$.

Since $M|K$ is a Galois extension, so is $M|L$. Hence we can apply Theorem 5.2 to conclude that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. This completes our proof of Theorem 1.2.

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