KÄHLER DIFFERENTIALS OF EXTENSIONS OF VALUATION RINGS AND DEEPLY RAMIFIED FIELDS

STEVEN DALE CUTKOSKY AND FRANZ-VIKTOR KUHLMANN

Abstract. Assume that (L,v) is a finite Galois extension of a valued field (K,v). We give an explicit construction of the valuation ring $\mathcal{O}_L$ of L as an $\mathcal{O}_K$-algebra, and an explicit description of the module of relative Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ when $L|K$ is a Kummer extension of prime degree or an Artin-Schreier extension. The case when this extension has nontrivial defect was solved in a recent paper by the authors with Anna Rzepka. Using this description, we characterize when $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ holds for an arbitrary finite Galois extension of valued fields. As an application of these results, we give a simple proof of a theorem of Gabber and Ramero, which characterizes when a valued field is deeply ramified. We further give a simple characterization of deeply ramified fields with residue fields of characteristic $p > 0$ in terms of the Kähler differentials of Galois extensions of degree p.

1. Introduction

The main goal of this paper is to study for algebraic extensions of valued fields the relation between their properties and the vanishing of the Kähler differentials of the extensions of their valuation rings.

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the unique maximal ideals of $\mathcal{O}_L$ and $\mathcal{O}_K$. The value group of the valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av. The **rank** of a valued field (K,v) is the order type of the chain of proper convex subgroups of its value group vK. All of our results are for arbitrary valuations; in particular, we have no restrictions on their rank or value groups. Ranks higher than 1 appear in a natural way when local uniformization, the local form of resolution of singularities, is studied. Deeply ramified fields of infinite rank appear in model theoretic investigations of the tilting construction, as presented by Jahnke and Kartas in [10]. Therefore, we do not restrict our computations to

Date: 22. 1. 2024.

2010 *Mathematics Subject Classification.* 12J10, 12J25.

Key words and phrases. Kähler differentials, deeply ramified fields, Artin-Schreier extension, Kummer extension.

The first author was partially supported by grant DMS 2054394 from NSF of the United States.

The second author was partially supported by Opus grant 2017/25/B/ST1/01815 from the National Science Centre of Poland. He was also supported by a Miller Fellowship during a four weeks visit to the University of Missouri Mathematics Department in 2022; he would like to thank the people at that department for their hospitality.

The authors thank Sylvy Anscombe, Arno Fehm, Hagen Knaf and Josnei Novacoski for inspiring discussions and helpful feedback, and Anna Rzepka for careful proofreading.

rank 1, thereby indicating how Kähler differentials and their annihilators can be computed in higher rank.

By $\Omega_{B|A}$ we denote the Kähler differentials, i.e., the module of relative differentials, when A is a ring and B is an A-algebra. In [5], we prove:

Theorem 1.1. *Take an algebraic field extension $L|K$ of degree n, a normal domain A with quotient field K and a domain B with quotient field L such that $B|A$ is an integral extension. Assume that there exist generators $b_\alpha \in B$ of $L|K$, which are indexed by a totally ordered set S, such that $A[b_\alpha] \subset A[b_\beta]$ if $\alpha \leq \beta$ and*

$$\bigcup_{\alpha \in S} A[b_\alpha] \;=\; B. \tag{1}$$

Further assume that there exist $a_\alpha, a_\beta \in A$ such that $a_\beta \mid a_\alpha$ if $\alpha \leq \beta$ and for $\alpha \leq \beta$, there exist $c_{\alpha,\beta} \in A$ and expressions

$$b_\alpha \;=\; \frac{a_\alpha}{a_\beta} b_\beta + c_{\alpha,\beta}\,. \tag{2}$$

Let h_α be the minimal polynomial of b_α over K. Take U and V to be the B-ideals

$$U \;=\; (a_\alpha \mid \alpha \in S) \quad \text{and} \quad V \;=\; (h'_\alpha(b_\alpha) \mid \alpha \in S)\,.$$

Then we have a B-module isomorphism

$$\Omega_{B|A} \;\cong\; U/UV\,. \tag{3}$$

For the case where $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$, for arbitrary $\gamma \in S$ the isomorphism (3) yields a B-module isomorphism

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; U/b_\gamma^\dagger U^n \quad \text{with} \quad b_\gamma^\dagger := \frac{h'_\gamma(b_\gamma)}{a_\gamma^{n-1}}\,. \tag{4}$$

The annihilator of U/UV and thus of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is determined in [5, Proposition 4.2]. If $b \in L$ and h is its minimal polynomial over K, then $h'(b)$ is called the **different of** b. Hence V is the $\mathcal{O}_L$-ideal generated by the differents of all b_α. It is shown in [5, Proposition 4.4] that it is in fact the $\mathcal{O}_L$-ideal generated by the differents of all elements in $\mathcal{O}_L$.

Take a Galois extension $(L|K, v)$ of prime degree p. If $\operatorname{char} K = p$, then $L|K$ is an Artin-Schreier extension, otherwise it is a Kummer extension if K contains all p-th roots of unity. In [5] and the present paper, for all Artin-Schreier extensions and Kummer extensions of prime degree, we give explicit computations of $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra in the form of (1) and use Theorem 1.1 to determine the Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ as well as their corresponding annihilators. For these types of extensions the computation of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is achieved in [5, Theorems 4.5 and 4.6] under the assumption that they have **nontrivial defect**, which in this special case means that $(vL : vK) = 1 = [Lv : Kv]$. In Sections 4.3 to 4.7 of the present paper, the Kähler differentials and their annihilators are computed under the assumption that the extensions are **unibranched** and **defectless**, which means that the extension of v from K to L is unique and $[L : K] = (vL : vK)[Lv : Kv]$ holds. Traces and Dedekind differents for the extensions have been determined in [5] in the case of nontrivial defect, and will be determined in a subsequent paper [16] for the case of defectless extensions. In order to obtain equation (1) of Theorem 1.1, the results of Section 3.1 are used, in which we determine how in the cases considered in the present paper the valuation ring $\mathcal{O}_L$ can be generated as an $\mathcal{O}_K$-module. For the computation of the ideals V in Sections 4.3 to 4.7, the differents of the generators b_α we will use there are computed in Section 3.2.

If the extension $(L|K,v)$ is not unibranched, then it is an inertial extension since it is Galois of prime degree. Proposition 5.6 of this paper (which relies on [25, Chapter X, Theorem 1]) then shows that $\mathcal{O}_L|\mathcal{O}_K$ is locally étale. Hence, as explained in the paragraph before Proposition 5.6, it follows that $\mathcal{O}_L$ is a localization of a finitely presented étale $\mathcal{O}_K$-algebra and that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.

After these preparations, we prove in Section 5 a criterion for the vanishing of the Kähler differentials for arbitrary finite Galois extensions; see part 2) of Theorem 5.2. Finally, in Section 6 all of these results are combined into the proof of Theorem 1.2 that we will present now.

Take a valued field (K,v) with valuation ring $\mathcal{O}_K$. Choose any extension of v to the separable-algebraic closure K^{sep} of K and denote the valuation ring of K^{sep} with respect to this extension by $\mathcal{O}_{K^{sep}}$. Note that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K}$ does not depend on the choice of the extension. Gabber and Ramero prove the following result (see [8, Theorem 6.6.12 (vi)]):

Theorem 1.2. *For a valued field (K,v),*

$$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \;=\; 0 \tag{5}$$

holds if and only if it satisfies the following:

(DRvg) *whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);*

(DRvr) *if $\operatorname{char} Kv = p > 0$, then the homomorphism*

$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \tag{6}$$

is surjective, where $\mathcal{O}_{\hat{K}}$ denotes the valuation ring of the completion $\hat{K}$ of (K,v).

We define a nontrivially valued field (K,v) to be a **deeply ramified field** if the equivalent conditions of the theorem hold. In [15], related classes of valued fields are introduced by weakening or strengthening condition (DRvg).

Theorem 1.2 and the papers [31, 32] of Thatte were the motivation for our work in the present paper and in [5].

Note that by definition, perfectoid fields have rank 1, meaning that their value groups admit an order preserving embedding in the ordered additive group of the reals. In this case, condition (DRvg) just says that the value group has no smallest element. Consequently, when using (DRvg) and (DRvr) for the definition of deeply ramified fields, it is immediately seen that every perfectoid field is a deeply ramified field.

The proof of Theorem 1.2 in [8] is a demonstration of the power of the techniques of "almost mathematics", and uses most of the theory developed in [8]. Their proof is by reduction to the rank 1 case, where the techniques of "almost mathematics" are most applicable.

Our alternative proof of Theorem 1.2 in the present paper uses only methods from valuation theory and commutative algebra, and does not rely on techniques or results from “almost mathematics”. We make no assumptions on rank in our proof. We hope that our proof makes this beautiful theorem accessible to a wider audience. Further, our proof yields the following additional result:

Theorem 1.3. *Let (K,v) be a valued field of residue characteristic $p>0$. If K has characteristic 0, then assume in addition that it contains all p-th roots of unity. Then (K,v) is a deeply ramified field if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ for all unibranched Galois extensions $(L|K,v)$ of prime degree p.*

For the purpose of the proof of Theorem 1.2, by “deeply ramified field” we will mean valued fields that satisfy the conditions (DRvg) and (DRvr). From [15, part (1) of Theorem 1.10] together with [5, Theorem 1.4] we obtain:

Theorem 1.4. *Take a deeply ramified field (K,v) with $\mathrm{char}\, Kv = p > 0$; if $\mathrm{char}\, K = 0$, then assume that K contains all p-th roots of unity. Then every Galois extension $(L|K,v)$ of degree p with nontrivial defect satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.*

This result will be complemented in the present paper by showing that for a deeply ramified field (K,v), every unibranched defectless Galois extension $(L|K,v)$ of prime degree p satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$. Then Section 5 connects our results for Galois extensions of prime degree with $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K}$. There, the main approach is the study of Kähler differentials of towers of Galois extensions. In order to go upward through such towers, we make use of [15, Theorem 1.5]:

Theorem 1.5. *Every algebraic extension of a deeply ramified field is again a deeply ramified field.*

It should be noted that while in [8], Theorem 1.5 is derived from Theorem 1.2, the proof presented in [15] is different and purely valuation theoretical. Further, Theorem 1.5 also holds for the roughly deeply ramified and the semitame fields that are introduced in [15].

In [24], Novacoski and Spivakovsky use the theory of key polynomials to derive a presentation of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ for finite pure extensions $(L|K,v)$ under the condition $vL=vK$. Applying this presentation to Artin-Schreier and Kummer extensions, they derive results similar to ours in [5] and in this paper. Recently they also dealt with the case of $vL\neq vK$ by a different approach, not based on the use of key polynomials. See also [21, 22].

To conclude this introduction, let us give some interesting examples.

Example 1.6. Choose a prime $p>2$. The field $K=\mathbb{Q}_p(p^{1/p^n}\mid n\in\mathbb{N})$, equipped with the unique extension of the p-adic valuation of $\mathbb{Q}_p$, is known to be a deeply ramified field. The extension $(K(\sqrt{p})|K,v_p)$ is tamely ramified, as $(v_pK(\sqrt{p}):v_pK)=2$. This construction is mentioned in [28, §4] as an example for an almost étale extension.

By an application of Theorem 4.6 below, $\Omega_{\mathcal{O}_{K(\sqrt{p})}|\mathcal{O}_K}=0$. The fact that this holds in spite of the ramification is due to the value group v_pK being dense, as it is p-divisible.

Analoguously, we can consider the field $K=\mathbb{F}_p((t))(t^{1/p^n}\mid n\in\mathbb{N})$, equipped with the unique extension of the t-adic valuation of $\mathbb{F}_p((t))$. This field is a deeply ramified field since it is perfect of positive characteristic. Again, the extension

$(K(\sqrt{t})|K, v_t)$ is tamely ramified as $(v_tK(\sqrt{t}) : v_tK) = 2$, and v_tK is dense. By Theorem 4.6 below, $\Omega_{\mathcal{O}_{K(\sqrt{t})}|\mathcal{O}_K} = 0$.

Finally, here is an example of a Kummer extension $(L|K, v)$ with wild ramification and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

Example 1.7. Take a prime $p > 2$ and set $K = \mathbb{Q}(\zeta_p)(t^{1/2^n} \mid n \in \mathbb{N})$, where ζ_p is a primitive p-th root of unity. Let v_p denote the p-adic valuation on $\mathbb{Q}(\zeta_p)$ and v_t the t-adic valuation on K. Now consider the composition $v := v_t \circ v_p$ on K. Set $L = K(t^{1/p})$ and extend v to L. Then $(L|K, v)$ is a Kummer extension of degree p with ramification index $p = \operatorname{char} Kv$. Nevertheless, Theorem 4.6 shows that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

2. Preliminaries

2.1. Convex subgroups and archimedean components.

We take an ordered abelian group Γ. Two elements $\alpha, \beta \in \Gamma$ are **archimedean equivalent** if there is some $n \in \mathbb{N}$ such that $n|\alpha| \geq |\beta|$ and $n|\beta| \geq |\alpha|$, where $|\alpha| := \max\{\alpha, -\alpha\}$. Note that if $0 < \alpha < \beta < n\alpha$ for some $n \in \mathbb{N}$, then α, β and $n\alpha$ are (mutually) archimedean equivalent. If any two nonzero elements of Γ are archimedean equivalent, then we say that Γ is **archimedean ordered**. This holds if and only if Γ admits an order preserving embedding in the ordered additive group of the real numbers.

We call Γ **discretely ordered** if every element in Γ has an immediate successor; this holds if and only if Γ contains a smallest positive element. In contrast, Γ is called **dense** if for any two elements $\alpha < \gamma$ in Γ there is $\beta \in \Gamma$ such that $\alpha < \beta < \gamma$. If Γ is archimedean ordered and dense, then for every $i \in \mathbb{N}$ there is even some $\beta_i \in \Gamma$ such that $\alpha < i\beta_i < \gamma$; this can be easily proven via an embedding of Γ in the real numbers. Every ordered abelian group is discrete if and only if it is not dense.

For $\gamma \in \Gamma$, we define $\mathcal{C}_\Gamma(\gamma)$ to be the smallest convex subgroup of Γ containing γ, and for $\gamma \neq 0$, $\mathcal{C}^+_\Gamma(\gamma)$ to be the largest convex subgroup of Γ not containing γ. Note that $\mathcal{C}_\Gamma(0) = \{0\}$. The convex subgroups of Γ form a chain under inclusion, and that the union and the intersection of any collection of convex subgroups are again convex subgroups; this guarantees the existence of $\mathcal{C}_\Gamma(\gamma)$ and $\mathcal{C}^+_\Gamma(\gamma)$.

We have that $\mathcal{C}^+_\Gamma(\gamma) \subsetneq \mathcal{C}_\Gamma(\gamma)$, and that $\mathcal{C}^+_\Gamma(\gamma)$ and $\mathcal{C}_\Gamma(\gamma)$ are consecutive, that is, there is no convex subgroup of Γ lying properly between them. As a consequence,

$$\mathcal{A}_\Gamma(\gamma) := \mathcal{C}_\Gamma(\gamma)/\mathcal{C}^+_\Gamma(\gamma)$$

for $\gamma \neq 0$ is an archimedean ordered group; we call it the **archimedean component of Γ associated with** γ. Two elements $\alpha, \beta \in \Gamma$ are archimedean equivalent if and only if

$$\mathcal{C}_\Gamma(\alpha) = \mathcal{C}_\Gamma(\beta)\,,$$

and then it follows that $\mathcal{A}_\Gamma(\alpha) = \mathcal{A}_\Gamma(\beta)$. In particular, $\mathcal{C}_\Gamma(\alpha) = \mathcal{C}_\Gamma(n\alpha)$ and $\mathcal{A}_\Gamma(\alpha) = \mathcal{A}_\Gamma(n\alpha)$ for all $\alpha \in \Gamma$ and all $n \in \mathbb{Z}$.

Assume now that Γ is an ordered abelian group containing a subgroup Δ. We say that Δ **is dense in** Γ if for any two elements $\alpha < \gamma$ in Γ there is $\beta \in \Delta$ such that $\alpha < \beta < \gamma$. If Γ is archimedean ordered, then so is Δ, and if in addition Δ is dense, then it is dense in Γ, provided that $\Delta \neq \{0\}$.

For every $\gamma \in \Gamma$, $\mathcal{C}_\Gamma(\gamma) \cap \Delta$ and $\mathcal{C}_\Gamma^+(\gamma) \cap \Delta$ are convex subgroups of Δ; the quotient $\mathcal{C}_\Gamma(\gamma) \cap \Delta \,/\, \mathcal{C}_\Gamma^+(\gamma) \cap \Delta$ is either trivial or archimedean ordered. If γ is archimedean equivalent to $\delta \in \Delta$, then this quotient is equal to $\mathcal{A}_\Delta(\delta)$.

For each $\delta \in \Delta$ the function given by

$$\mathcal{A}_\Delta(\delta) \ni \alpha + \mathcal{C}_\Delta^+(\delta) \;\mapsto\; \alpha + \mathcal{C}_\Gamma^+(\delta) \in \mathcal{A}_\Gamma(\delta)$$

is an injective order preserving homomorphism. This follows from the fact that the kernel of the homomorphism $\mathcal{C}_\Delta(\delta) \ni \alpha \mapsto \alpha + \mathcal{C}_\Gamma^+(\delta) \in \mathcal{A}_\Gamma(\delta)$ is the convex subgroup $\mathcal{C}_\Delta^+(\delta) = \mathcal{C}_\Gamma^+(\delta) \cap \Delta$. In abuse of notation, we write $\mathcal{A}_\Delta(\delta) = \mathcal{A}_\Gamma(\delta)$ if this homomorphism is surjective.

We show that the two properties mentioned in condition (DRvg) are equivalent:

Lemma 2.1. *The following statements are equivalent:*

a) no archimedean component of the ordered abelian group Γ is discrete,

b) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of Γ, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$.

Proof. a)$\Rightarrow$b): Assume that $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of Γ such that $\Gamma_2/\Gamma_1 \cong \mathbb{Z}$. Let $\gamma + \Gamma_1$ be the smallest positive element of Γ_2/Γ_1, where $\gamma \in \Gamma_2$. Then $\Gamma_1 = \mathcal{C}_\Gamma^+(\gamma)$ and $\mathcal{C}_\Gamma(\gamma)/\mathcal{C}_\Gamma^+(\gamma)$ has smallest positive element $\gamma+\Gamma_1$, hence is discrete. So assertion a) does not hold.

b)$\Rightarrow$a): Suppose that there is an archimedean component $\mathcal{C}_\Gamma(\gamma)/\mathcal{C}_\Gamma^+(\gamma)$ of Γ which is discrete. Take $\Gamma_2 = \mathcal{C}_\Gamma(\gamma)$ and $\Gamma_1 = \mathcal{C}_\Gamma^+(\gamma)$ to find that assertion b) does not hold. $\square$

2.2. Artin-Schreier and Kummer extensions.

We say that a valued field (K, v) has **equal characteristic** if $\mathrm{char}\, K = \mathrm{char}\, Kv$, and **mixed characteristic** if $\mathrm{char}\, K = 0$ and $\mathrm{char}\, Kv > 0$. Every Galois extension of degree p of a field K of characteristic $p > 0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** $X^p - X - b$ with $b \in K$. For every $c \in K$, also $\vartheta - c$ is an Artin-Schreier generator as its minimal polynomial is $X^p - X - b + c^p - c$. Every Galois extension of prime degree n of a field K of characteristic different from n which contains all n-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^n \in K$. For these facts, see [18, Chapter VI, §6].

A **1-unit** in a valued field (K, v) is an element of the form $u = 1 + b$ with $b \in \mathcal{M}_K$; in other words, u is a unit in $\mathcal{O}_K$ with residue 1. We note that if u is a 1-unit and if $v(u - c) > vu = 0$ for some $c \in K$, then also c and c^{-1} are 1-units. Conversely, if c is a 1-unit, then $v(u - c) > 0$.

Remark 2.2. Take a Kummer extension $(L|K, v)$ of degree p of fields of characteristic 0 with any Kummer generator η. Assume that $v\eta \in vK$, so that there is $c_1 \in K$ such that $vc_1 = -v\eta$, whence $vc_1\eta = 0$. Assume further that $c_1\eta v \in Kv$, so that there is $c_2 \in K$ such that $c_2 v = (c_1\eta v)^{-1}$. Then $vc_2c_1\eta = 0$ and $c_2c_1\eta v = 1$.

Furthermore, $K(c_2c_1\eta) = K(\eta)$ and $(c_2c_1\eta)^p = c_2^p c_1^p \eta^p \in K$. Hence $c_2c_1\eta$ is a Kummer generator of $(L|K,v)$ and a 1-unit.

Further, it follows that $v(c_2c_1\eta-1) > 0$, whence $v(\eta-(c_2c_1)^{-1}) = v(c_2c_1)^{-1} = v\eta$. Consequently, for $c := (c_2c_1)^{-1} \in K$ we have $v(\eta - c) > v\eta$.

We will need the following facts. If $(L|K,v)$ is a unibranched defectless extension of prime degree p, then either $\mathrm{e}\,(L|K,v) = 1$ and $\mathrm{f}\,(L|K,v) = p$, or $\mathrm{f}\,(L|K,v) = 1$ and $\mathrm{e}\,(L|K,v) = p$. For the next lemma, see e.g. [17, Lemma 2.1] and the proof of [13, Theorem 2.19].

Lemma 2.3. *If $(L|K,v)$ is a finite unibranched defectless extension, then for every element $x \in L$ the set*

$$v(x-K) \;:=\; \{v(x-c) \mid c \in K\}$$

admits a maximal element. If $c \in K$ is such that $v(x-c)$ is maximal, then $v(x-c) \notin vK$ or there is some $\tilde{c} \in K$ such that $v\tilde{c}(x-c) = 0$ and $\tilde{c}(x-c)v \notin Kv$.

For the proof of the next lemma, see [5].

Lemma 2.4. *Take a valued field (K,v), $n \in \mathbb{N}$, and a primitive n-th root of unity $\zeta_n \in K$. Then*

$$\prod_{i=1}^{n-1}(1-\zeta_n^i) \;=\; n\,. \tag{7}$$

If in addition n is prime, then

$$v(1-\zeta_n) \;=\; \frac{vn}{n-1}\,. \tag{8}$$

Take a unibranched Kummer extension $(L|K,v)$ of prime degree q with Kummer generator η. Further, take $c \in K$ and $\sigma \neq \mathrm{id}$ in the Galois group $\mathrm{Gal}\,L|K$. Then

$$\eta - c - \sigma(\eta - c) \;=\; \eta - \sigma\eta \;=\; \eta(1-\zeta_q)\,, \tag{9}$$

where ζ_q is a primitive q-th root of unity. Hence if $v(\eta - c) > v\eta(1-\zeta_q)$, then

$$v\sigma(\eta-c) \;=\; v(\eta-c-\eta(1-\zeta_q)) \;=\; \min\{v(\eta-c), v\eta(1-\zeta_q)\} \;=\; v\eta(1-\zeta_q) \;<\; v(\eta-c)\,,$$

which shows that the extension is not unibranched. We have proved:

Lemma 2.5. *Take a Kummer extension $(L|K,v)$ of prime degree q with Kummer generator η. If the extension is unibranched, then for all $c \in K$,*

$$v(\eta - c) \;\leq\; v\eta(1-\zeta_q)\,. \tag{10}$$

Lemma 2.6. *Take a unibranched Kummer extension $(L|K,v)$ of degree $p = \mathrm{char}\,Kv$ with Kummer generator η. Then for all $c \in K$,*

$$v(\eta - c) \;\leq\; v\eta + \frac{vp}{p-1}\,. \tag{11}$$

Assume in addition that $\mathrm{f}\,(L|K,v) = p$ and $c, \tilde{c} \in K$ are such that $v\tilde{c}(\eta - c) = 0$ and $\tilde{c}(\eta - c)v$ generates the residue field extension $Lv|Kv$. Then $Lv|Kv$ is inseparable if $v(\eta - c) < v\eta + \frac{vp}{p-1}$, and it is separable if and only if $v(\eta - c) = v\eta + \frac{vp}{p-1}$.

Proof. The first assertion follows from Lemma 2.5 together with equation (8).

Now assume the situation as in the second part of the lemma. Since $L|K$ is a Galois extension, $Lv|Kv$ is a normal extension, with its automorphisms induced by those of $L|K$. Take σ to be a generator of $\mathrm{Gal}\, L|K$. Via the residue map, its action on $\mathcal{O}_L^\times$ induces a generator $\bar{\sigma}$ of the automorphism group of $Lv|Kv$. From (9) we infer that

$$\tilde{c}(\eta - c) - \sigma\tilde{c}(\eta - c) \;=\; \tilde{c}\eta(1 - \zeta_p)\,.$$

It follows that $\bar{\sigma}$ is the identity and hence $Lv|Kv$ is inseparable if and only if $v\tilde{c}\eta(1-\zeta) > 0$. This is equivalent to

$$v(\eta - c) \;=\; -v\tilde{c} \;<\; v\eta(1-\zeta) \;=\; v\eta + \frac{vp}{p-1}\,.$$

Since $v(\eta - c) > v\eta + \frac{vp}{p-1}$ is impossible according to (11), we can conclude that the residue field extension is separable if and only if $v(\eta - c) = v\eta + \frac{vp}{p-1}$. □

Proposition 2.7. *Take a Kummer extension $(L|K, v)$ of prime degree $q \neq \mathrm{char}\, Kv$.*

1) If $\mathrm{f}\,(L|K, v) = q$, then there is a Kummer generator $\eta \in \mathcal{O}_L^\times$ such that ηv is a Kummer generator of $Lv|Kv$.

2) If $\mathrm{e}\,(L|K, v) = q$, then there is a Kummer generator $\eta \in L$ such that $v\eta$ generates the value group extension, that is, $vL = vK + \mathbb{Z}v\eta$.

Proof. 1): Take a Kummer generator η. Since $\mathrm{f}\,(L|K, v) = q$, we have that $vL = vK$. Therefore, as shown in Remark 2.2, we can assume that $v\eta = 0$. The reduction of the minimal polynomial of η over K to the residue field is $X^q - \eta^q v$ with $\eta^q v \neq 0$. Suppose that this polynomial has a root in Kv. Since $\mathrm{Gal}\, Lv|Kv$ is cyclic of prime degree (generated by the reduction of a generator of $\mathrm{Gal}\, L|K$), it follows that $X^q - \eta^q v$ splits. Hence its root ηv lies in Kv and there is $c \in K$ such that $cv = \eta v$. It follows that $v(\eta - c) > 0 = v\eta(1 - \zeta_q)$, so by Lemma 2.5, $(L|K, v)$ is not unibranched. As this contradicts our assumption, $X^q - \eta^q v$ must be irreducible, which means that ηv generates the extension $Lv|Kv$. Since $\eta^q \in K$, we have that $(\eta v)^q \in Kv$, i.e., ηv is a Kummer generator of $Lv|Kv$.

2) Take a Kummer generator η. We will show that $v\eta \notin vK$; as q is prime, it then follows that $vL = vK + \mathbb{Z}v\eta$. Suppose that $v\eta \in vK$. Since $\mathrm{e}\,(L|K, v) = q = [L : K]$, we have that $Lv = Kv$. Thus as shown in Remark 2.2, there is some $c \in K$ such that $v(\eta - c) > v\eta = v\eta(1 - \zeta_q)$. As in the proof of part 1), this leads to a contradiction. Hence $v\eta \notin vK$, as asserted. □

Using Lemma 2.3, we prove:

Proposition 2.8. *1) Take a valued field (K, v) of equal positive characteristic p and a unibranched defectless Artin-Schreier extension $(L|K, v)$.*

If $\mathrm{f}\,(L|K, v) = p$, then the extension has an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that for some $\tilde{c} \in K$, $v\tilde{c}\vartheta = 0$ and $Lv = Kv(\tilde{c}\vartheta v)$; the extension $Lv|Kv$ is separable if and only if $v\vartheta = 0$ (in which case we can take $\tilde{c} = 1$).

If $\mathrm{e}\,(L|K, v) = p$, then the extension has an Artin-Schreier generator ϑ such that $v\vartheta$ generates the value group extension, that is, $vL = vK + \mathbb{Z}v\vartheta$. Every such ϑ satisfies $v\vartheta < 0$.

2) Take a valued field (K,v) of mixed characteristic and a unibranched defectless Kummer extension $(L|K,v)$ of degree $p = \operatorname{char} Kv$. Then the extension has a Kummer generator η such that:

a) if $\mathrm{f}(L|K,v) = p$, then either ηv generates the residue field extension, in which case it is inseparable, or η is a 1-unit and for some $\tilde{c} \in K$, $\tilde{c}(\eta - 1)v$ generates the residue field extension;

b) if $\mathrm{e}(L|K,v) = p$, then either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta - 1)$ generates the value group extension.

Proof. 1): Take any Artin-Schreier generator y of $(L|K,v)$. Then by Lemma 2.3 there is $c \in K$ such that $v(y-c) \notin vK$ or for some $\tilde{c} \in K$, $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since p is prime, in the first case it follows that $\mathrm{e}(L|K,v) = p$ and that $v(y-c)$ generates the value group extension. In the second case, it follows that $\mathrm{f}(L|K,v) = p$ and that $\tilde{c}(y-c)v$ generates the residue field extension. In both cases, $\vartheta = y - c$ is an Artin-Schreier generator.

Assume that $\mathrm{f}(L|K,v) = p$ and let $\vartheta^p - \vartheta = b \in K$. If $v\vartheta < 0$, then $v(\vartheta^p - b) = v\vartheta > pv\vartheta = v\vartheta^p$, whence $v((\tilde{c}\vartheta)^p - \tilde{c}^p b) = v\tilde{c}^p\vartheta > v(\tilde{c}\vartheta)^p$ and therefore, $(\tilde{c}\vartheta)^p v = \tilde{c}^p bv \in Kv$. In this case, the residue field extension is inseparable. Now assume that $v\vartheta \geq 0$ and hence also $vb \geq 0$. The reduction of $X^p - X - b$ to $Kv[X]$ is a separable polynomial, so $Lv|Kv$ is separable. The polynomial $X^p - X - bv$ cannot have a zero in Kv, since otherwise the p distinct roots of this polynomial give rise to p distinct extensions of v from K to L, contradicting our assumption that $(L|K,v)$ is unibranched. Consequently, $bv \neq 0$, whence $vb = 0$ and $v\vartheta = 0$.

Assume that $\mathrm{e}(L|K,v) = p$ and let $\vartheta^p - \vartheta = b \in K$. If $v\vartheta \geq 0$, then $vb \geq 0$, and ϑv is a root of $X^p - X - bv$. If this polynomial is irreducible, then ϑv generates a residue field extension of degree p, contradicting our assumption that $\mathrm{e}(L|K,v) = p$. If the polynomial is not irreducible, then it has a zero in Kv and similarly as before, one deduces a contradiction. Hence we must have $v\vartheta < 0$.

2): Take any Kummer generator y of $(L|K,v)$. If there is a Kummer generator η such that $v\eta \notin vK$, then it follows as before that $\mathrm{e}(L|K,v) = p$ and that $v\eta$ generates the value group extension. Now assume that there is no such η.

If there is a Kummer generator y and some $\tilde{c} \in K$ such that $v\tilde{c}y = 0$ and $\tilde{c}yv \notin Kv$, then it follows as before that $\mathrm{f}(L|K,v) = p$ and that $\tilde{c}yv$ generates the residue field extension. We set $\eta = \tilde{c}y$ and observe that also η is a Kummer generator. Since $(\eta v)^p \in Kv$, $Lv|Kv$ is purely inseparable in this case.

Now assume that the above cases do not appear, and choose an arbitrary Kummer generator y of $(L|K,v)$. Consequently, we have that $vy \in vK$ and $cyv \in Kv$ for all $c \in K$ with $vcy = 0$. Then as described in Remark 2.2, there are $c_1, c_2 \in K$ such that c_2c_1y is a Kummer generator of $(L|K,v)$ which is a 1-unit. We replace y by c_2c_1y.

By Lemma 2.3 there is $c \in K$ such that $v(y-c)$ is maximal in $v(y-K)$ and either $v(y-c) \notin vK$ or there is some $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since y is a 1-unit, we know that $v(y-1) > 0$, hence also $v(y-c) > 0 = vy$, showing that also c is a 1-unit. Then $\eta = c^{-1}y$ is again a Kummer generator of $(L|K,v)$ which is a 1-unit. Since $vc = 0$, we know that $v(\eta - 1) = vc(\eta - 1) = v(y-c)$. Hence if $v(y-c) \notin vK$, then $v(\eta - 1)$ generates the value group extension and we are done.

Now assume that there is some $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since c is a 1-unit, it follows that $v\tilde{c}(\eta-1) = v\tilde{c}c(\eta-1) = v\tilde{c}(y-c) = 0$ and $\tilde{c}(\eta-1)v = \tilde{c}c(\eta-1)v = \tilde{c}(y-c)v$. We find that $\tilde{c}(\eta-1)v$ generates the residue field extension. □

3. Generation of extensions of valuation rings

In this section we will assume that $(L|K,v)$ is a finite unibranched defectless extension and in various cases determine generators for the valuation ring $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra, and their properties.

3.1. Generating the $\mathcal{O}_K$-algebra $\mathcal{O}_L$.

We will consider finite extensions $(L|K,v)$ of degree n that satisfy

$$[L:K] \;=\; [Lv:Kv] \quad \text{or} \quad [L:K] \;=\; (vL:vK)\,.$$

Such extensions are unibranched and defectless.

We consider the following two cases:

Case (DL1): $[L:K] = [Lv:Kv]$. In this case, we can choose elements $a_1,\ldots,a_n \in \mathcal{O}_L^\times$ such that $a_1v,\ldots,a_nv$ form a basis of $Lv|Kv$. Then $a_1,\ldots,a_n$ form a valuation basis of $(L|K,v)$, that is, every element of $z \in L$ can be written as

$$z \;=\; c_1a_1 + \ldots + c_na_n \quad \text{with} \quad vz \;=\; \min_i vc_ia_i\,, \tag{12}$$

and we have that $vc_ia_i = vc_i$. Consequently, $z \in \mathcal{O}_L$ if and only if $c_1,\ldots,c_n \in \mathcal{O}_K$. This shows that

$$\mathcal{O}_L \;=\; \mathcal{O}_K[a_1,\ldots,a_n]\,.$$

In the case where $Lv|Kv$ is simple, that is, there is $\xi \in Lv$ such that $Lv = Kv(\xi)$, we can choose $x \in L$ such that $xv = \xi$; then $1,x,\ldots,x^{n-1}$ form a valuation basis of $(L|K,v)$. In this special case (which by the Primitive Element Theorem always appears when $Lv|Kv$ is separable),

$$\mathcal{O}_L \;=\; \mathcal{O}_K[x]\,. \tag{13}$$

Case (DL2): $[L:K] = (vL:vK)$. In this case, we can choose elements $a_1,\ldots,a_n \in L$ such that $va_1,\ldots,va_n$ form a system of representatives for the distinct cosets of vL modulo vK. Then again, every $z \in L$ can be written in the form (12). Consequently, $z \in \mathcal{O}_L$ if and only if $c_1a_1,\ldots,c_na_n \in \mathcal{O}_L$. However, in this case, $c_ia_i \in \mathcal{O}_L$ does not imply that $c_i \in \mathcal{O}_K$.

In what follows, we will analyze the special case where vL/vK is cyclic (which always appears when n is a prime). This means that there is some $x_0 \in L$ such that $nvx_0 \in vK$ and $vL = vK + \mathbb{Z}vx_0$. In this special case, as in case (DL1), $1,x_0,\ldots,x_0^{n-1}$ form a valuation basis of $(L|K,v)$, hence every element of L can be written as a K-linear combination of these elements, and for every choice of $c_0,\ldots,c_{n-1} \in K$,

$$v\sum_{i=0}^{n-1} c_ix_0^i \;=\; \min_i vc_ix_0^i\,.$$

Again, the sum is an element of $\mathcal{O}_L$ if and only if all summands $c_i x_0^i$ are, but the latter does not necessarily imply that $c_i \in \mathcal{O}_L$. We set

$$A_0 \;:=\; \{c x_0^i \mid c \in K^\times \text{ and } 1 \le i < n \text{ such that } vcx_0^i > 0\}\,.$$

(Note that $vc = -vx_0^i$ is impossible for $1 \le i < n$.) We obtain that

$$\mathcal{O}_L \;=\; \mathcal{O}_K[A_0]\,.$$

However, we wish to derive a much more useful representation of $\mathcal{O}_L$.

We consider $vA_0 := \{va \mid a \in A_0\} \subseteq vL^{>0} \setminus vK$. Since the set $\{1,\ldots,n-1\}$ is finite, it contains at least one j such that $\{vcx_0^j \mid c \in K^\times \text{ such that } vcx_0^j > 0\}$ is coinitial in vA_0. We do not know whether j can always be chosen to be equal to 1; we will now present two cases where it can.

(DL2a): vK is i-divisible for all $i \in \{2,\ldots,n-1\}$. Take $vc_i x_0^i \in A_0$. Since vK is i-divisible, there is $c \in K$ such that $vc_i = ivc$. We obtain that $vc^i x_0^i = vc_i x_0^i > 0$, hence also $vcx_0 > 0$. Consequently, $c_i x_0^i \in \mathcal{O}_K[cx_0]$. It follows that for $x = x_0$,

$$\mathcal{O}_L \;=\; \bigcup_{c \in K \text{ with } vcx > 0} \mathcal{O}_K[cx]\,. \tag{14}$$

For all remaining cases, we assume that n is prime.

(DL2b): The convex subgroups of vK are well-ordered under inclusion and all archimedean components of vK are dense; since $L|K$ is finite, the same is true for vL. From the former assumption it follows that there is a smallest convex subgroup of vL that contains some element of A_0. Let $c_{i_0} x_0^{i_0}$ be such an element. Then $\mathcal{C}_{vL}^+(c_{i_0} x_0^{i_0}) \cap A_0 = \emptyset$. As n is prime, there is $k \in \mathbb{N}$ such that $i_0 k = 1 + rn$ for some $r \in \mathbb{Z}$, whence $vcx_0 = v(c_{i_0} x_0^{i_0})^k > 0$ for $c = c_{i_0}^k b^r$ where $b \in K$ with $vb = nvx_0$. Since $0 < c_{i_0} x_0^{i_0} < v(c_{i_0} x_0^{i_0})^k = vcx_0$, we have that $\mathcal{C}_{vL}(vc_{i_0} x_0^{i_0}) = \mathcal{C}_{vL}(kv(c_{i_0} x_0^{i_0})) = \mathcal{C}_{vL}(vcx_0)$.

Take any element $c_i x_0^i \in A_0$. If $vc_i x_0^i \notin \mathcal{C}_{vL}(vcx_0)$, then $v(cx_0)^i = ivcx_0 < vc_i x_0^i$, whence $c_i x_0^i \in \mathcal{O}_K[cx_0]$. Hence assume that $vc_i x_0^i \in \mathcal{C}_{vL}(vcx_0)$. By assumption, $\mathcal{A}_{vK}(nvcx_0)$ is dense, so it is dense in $\mathcal{A}_{vL}(vcx_0)$. Denote by α the image of vcx_0, and by β the image of $vc_i x_0^i$ in $\mathcal{A}_{vL}(vcx_0)$. Note that both of them are positive, and $0 < \beta$ implies that

$$-i\alpha \;<\; \beta - i\alpha\,.$$

By the density of $\mathcal{A}_{vK}(nvcx_0)$ in $\mathcal{A}_{vL}(vcx_0)$ there is $c_0 \in K$ such that the image γ of vc_0 in $\mathcal{A}_{vL}(vcx_0)$ satisfies

$$-i\alpha \;<\; i\gamma \;<\; \beta - i\alpha\,.$$

This leads to $0 < vc_0^i c^i x_0^i < vc_i x_0^i$. Setting $\tilde{c} = c_0 c$, we obtain that $0 < v\tilde{c}^i x_0^i < vc_i x_0^i$, whence $c_i x_0^i \in \mathcal{O}_K[\tilde{c}x_0]$ with $v\tilde{c}x_0 > 0$ as desired. We have proved that also in this case, (14) holds for $x = x_0$.

Now assume that the above two cases do not apply; so we do not know whether j can be chosen to be equal to 1. As n is prime, we know that $L = K(x_0) = K(x_0^j)$ for all $j \in \{1,\ldots,n-1\}$. By assumption, $vx_0 > 0$, hence we also have that $vx_0^j > 0$. Thus we may replace x_0 by $x_1 := x_0^j$. (We set $x_1 = x_0$ if j can be chosen to be equal to 1.) We note that if $x_0^n \in K$, then replacing x_0 by x_0^j in the definition of A_0 does not change A_0 because if $1 \le i < n$, then $ij = k + rn$ with $1 \le k < n$ and

$r \in \mathbb{Z}$, so that $(x_0^j)^i = (x_0^n)^r x_0^k$ with $(x_0^n)^r \in K$. (A similar argument shows that in this case, A_0 is also closed under multiplication.) Even if $x_0^n \notin K$, we still have that $vx_0^n \in vK$, and our argument can be adapted to show that replacing x_0 by x_0^j in the definition of A_0 does not change the value set vA_0. We just choose some $b \in K$ such that $vb = vx_0^n$ and replace $(x_0^n)^r x_0^k$ by $b^r x_0^k$.

Since n is assumed prime and $(j,n) = 1$, also $vx_1 = vx_0^j$ is a generator of the value group extension and therefore, $1, x_1, \ldots, x_1^{n-1}$ form again a valuation basis of $(L|K,v)$. Thus we can replace A_0 by

$$A_1 \;:=\; \{cx_1^i \mid c \in K^\times \text{ and } 1 \le i < n \text{ such that } vcx_1^i > 0\}\,,$$

and we again have that

$$\mathcal{O}_L \;=\; \mathcal{O}_K[A_1]\,.$$

Claim: $\{vcx_1 \mid c \in K^\times \text{ such that } vcx_1 > 0\}$ is coinitial in vA_1. Indeed, take any $cx_1^i \in A_1$. Then by our above computation,

$$vcx_1^i \;=\; vc(x_0^j)^i \;=\; vcb^r x_0^k$$

with $b^r \in K$ and $1 \le k < n$. Since $cb^r x_0^k \in A_0$, by our choice of j, there is $c' \in K$ such that $0 < vc'x_1 = vc'x_0^j \le vcb^r x_0^k = vcx_1^i$. This proves our claim.

Let us first assume the following case.

(DL2c): For all $c \in K^\times$, if $\mathcal{A}_{vK}(nvcx_1)$ is discrete, then $\mathcal{A}_{vL}(vcx_1) = \mathcal{A}_{vK}(nvcx_1)$. Then there is $b \in K$ such that the image of vb in $\mathcal{A}_{vL}(vcx_1)$ is the same as that of vcx_1. Consequently, $vcx_1 - vb \in \mathcal{C}^+_{vL}(vcx_1)$. If $vcx_1 - vb < 0$, we choose $b' \in K^\times$ with $vb' \in \mathcal{C}^+_{vK}(nv(vcx_1 - vb))$ large enough such that $vcb'b^{-1}x_1 = vcx_1 - vb + vb' > 0$. Setting $\tilde{c} = cb'b^{-1}$, we can then assume that $0 < v\tilde{c}x_1 \in \mathcal{C}^+_{vL}(vcx_1)$.

Assume as before that $vc_jx_1^j > 0$. By our claim, we know that there exists $c \in K$ such that $0 < vcx_1 < vc_jx_1^j$. If $v(cx_1)^j \le vc_jx_1^j$, then $c_jx_1^j \in \mathcal{O}_K[cx_1]$ and we are done. Otherwise, we proceed as follows.

Assume first that $\mathcal{A}_{vK}(nvcx_1)$ is discrete. Then we can apply our above procedure to replace c by $\tilde{c} \in K^\times$ such that $0 < v\tilde{c}x_1 \in \mathcal{C}^+_{vL}(vcx_1)$. On the other hand, $vc_jx_1^j \notin \mathcal{C}^+_{vL}(vcx_1)$. It follows that $v\tilde{c}^j x_1^j = jv\tilde{c}x_1 < vc_jx_1^j$, whence $c_jx_1^j \in \mathcal{O}_K[\tilde{c}x_1]$.

It remains to consider the case where the archimedean component of vcx is dense. Then we can apply the same argument as in case (DL2b) to find $\tilde{c} \in K$ such that $c_jx_1^j \in \mathcal{O}_K[\tilde{c}x_1]$. We have proved that in this case, (14) holds for $x = x_1$.

Now we consider the case where the assumption of (DL2c) does not hold, i.e., there is $c \in K^\times$ such that $\mathcal{A}_{vK}(nvcx_1)$ is discrete and $\mathcal{A}_{vL}(vcx_1) \ne \mathcal{A}_{vK}(nvcx_1)$. Since n is assumed to be prime, we must have that $(\mathcal{A}_{vL}(vcx_1) : \mathcal{A}_{vK}(nvcx_1)) = n$. Denote by α the smallest positive element of $\mathcal{A}_{vK}(nvcx_1)$; then $\frac{\alpha}{n}$ is the smallest positive element of $\mathcal{A}_{vL}(vcx_1)$.

We will show that for some $\tilde{c} \in K$, the image of $v\tilde{c}x_1$ in $\mathcal{A}_{vL}(vcx_1)$ is equal to $\frac{\alpha}{n}$. Since vx_1 generates the value group extension, there must be some $c' \in K$ and $i \in \mathbb{N}$ with $1 \le i < n$ such that $\frac{\alpha}{n}$ is the image of $v\tilde{c}x_1^i$ in $\mathcal{A}_{vL}(vcx_1)$. From our claim we know that $0 < v\tilde{c}x_1 \le v\tilde{c}x_1^i$ for some $\tilde{c} \in K$. If equality holds, then we are done. Suppose not; then it follows that $v\tilde{c}x_1 \in \mathcal{C}^+_{vL}(vcx_1) = \mathcal{C}^+_{vL}(v\tilde{c}x_1^i)$, so that also

$$v\frac{\tilde{c}^i}{c'} + v\tilde{c}x_1^i \;=\; v\tilde{c}^i x_1^i \;=\; iv\tilde{c}x_1 \;\in\; \mathcal{C}^+_{vL}(vcx_1)\,.$$

However, this implies that the image $\frac{\alpha}{n}$ of $v\tilde{c}x_1^i$ in $\mathcal{A}_{vL}(vcx_1)$ is the same as that of $-v\frac{\tilde{c}^i}{c'}$ and thus lies in $\mathcal{A}_{vK}(nvcx_1)$, contradiction. Consequently, the image of $v\tilde{c}x_1$ in $\mathcal{A}_{vL}(vcx_1)$ is equal to $\frac{\alpha}{n}$.

Remark: Since the image of $v\tilde{c}x_1$ in $\mathcal{A}_{vL}(vcx_1)$ is $\frac{\alpha}{n}$, for $1 \leq i \leq n-1$ the image of $v(\tilde{c}x_1)^i$ in $\mathcal{A}_{vL}(vcx_1)$ is less than α. As a consequence, every $\beta \in vK$ with $0 \leq \beta \leq v(\tilde{c}x_1)^i$ must lie in $\mathcal{C}^+_{vL}(vcx_1) = \mathcal{C}^+_{vL}(v\tilde{c}x_1)$.

Now we distinguish two further subcases of (DL2).

(DL2d): $\mathcal{C}^+_{vL}(vcx_1) = 0$. Take $c_i(\tilde{c}x_1)^i \in A_1$. By our above remark and our assumption $\mathcal{C}^+_{vL}(vcx_1) = 0$, there are no positive elements in vK smaller than $v(\tilde{c}x_1)^i$; therefore, $c_i(\tilde{c}x_1)^i$ lies in $\mathcal{O}_L$ if and only if $c_i \in \mathcal{O}_K$. It follows that (13) holds for $x = \tilde{c}x_1$. This case appears when vK has a smallest element γ and x can be chosen such that $nvx = \gamma$. In particular, it always appears when (K,v) is discretely valued.

(DL2e): $\mathcal{C}^+_{vL}(vcx_1) \neq 0$, hence also $\mathcal{C}^+_{vL}(vcx_1) \cap vK \neq 0$. For simplicity, we may from now on write x_1 for $\tilde{c}x_1$ since also the powers of the latter generate a valuation basis and it also satisfies our above claim.

Take $c_i x_1^i \in A_1$. If $vc_i \geq 0$, then $c_i x_1^i \in \mathcal{O}_K[x_1]$, and we are done. Thus we assume that $vc_i < 0$. Since $vc_i x_1^i > 0$, it follows that $0 < -vc_i < ivx_1$. By our above remark, $\pm vc_i \in \mathcal{C}^+_{vL}(vcx_1)$. Choose $c_+ \in K$ such that $vc_+ \in \mathcal{C}^+_{vL}(vx_1)$ and $ivc_+ \leq vc_i$. Then $vc_+^i x_1^i \leq vc_i x_1^i$, showing that $c_i x_1^i \in \mathcal{O}_K[c_+x_1]$. On the other hand, as $vc_+ \in \mathcal{C}^+_{vL}(vx_1)$, we have that $-vc_+ < vx_1$ and therefore, $vc_+x_1 > 0$. This proves that (14) holds for $x = x_1$.

We summarize what we have shown in case (DL2):

Theorem 3.1. *Take a unibranched defectless extension $(L|K,v)$ of degree $n = \mathrm{e}\,(L|K,v)$, with $x_0 \in L$ a generator of $L|K$ such that $vL = vK + \mathbb{Z}vx_0$.*

1) Assume that vK is i-divisible for all $i \in \{2,\ldots,n-1\}$. Then (14) holds for $x = x_0$.

From now on, assume that n is prime.

2) Assume that the convex subgroups of vK are well-ordered under inclusion and that all archimedean components of vK are dense. Then (14) holds for $x = x_0$.

3) Assume that if $\alpha \in vL \setminus vK$ such that $\mathcal{A}_{vK}(n\alpha)$ is discrete, then $\mathcal{A}_{vL}(\alpha) = \mathcal{A}_{vk}(n\alpha)$. Then (14) holds for $x = x_0^j$ with suitable $j \in \{1,\ldots,n-1\}$.

4) Assume that there is $\alpha \in vL \setminus vK$ such that $\mathcal{A}_{vK}(n\alpha)$ is discrete and $\mathcal{A}_{vL}(\alpha) \neq \mathcal{A}_{vK}(n\alpha)$. If $\mathcal{C}^+_{vL}(\alpha) = 0$, then $\mathcal{O}_L = \mathcal{O}_K[x]$ holds for $x = \tilde{c}x_0^j$ with suitable $\tilde{c} \in K^\times$ and $j \in \{1,\ldots,n-1\}$. If $\mathcal{C}^+_{vL}(\alpha) \neq 0$, then (14) holds with $x = x_0^j$ for some $j \in \{1,\ldots,n-1\}$.

Summarizing, in all cases there is some $j \in \{1,\ldots,n-1\}$ such that

$$\mathcal{O}_L \;=\; \bigcup_{c\in K \textit{ with } vcx_0^j>0} \mathcal{O}_K[cx_0^j]\,. \tag{15}$$

Remark 3.2. If (K,v) is the absolute ramification field of some valued field (K_0,v) of residue characteristic $p > 0$ and if $n = p$, then the divisibility condition of the lemma is satisfied because vK is divisible by all primes other than p.

Let us give an example which shows what may happen in case 4) of the theorem when $\mathcal{C}^+_{vL}(\alpha) = 0$.

Example 3.3. Take a valued field (K, v) of characteristic $p > 0$ and assume that $\pi \in K$ has smallest positive value in vK. For $1 \le m < p$, take ϑ_m to be a root of

$$X^p - X - \frac{1}{\pi^m}\,.$$

Then $v\vartheta_m = -\frac{m}{p} v\pi$ and $(K(\vartheta_m)|K, v)$ is unibranched with $(vK(\vartheta_m) : vK) = p$ and $vK(\vartheta_m) = vK + \mathbb{Z}v\vartheta_m$. We have that $-pv\vartheta_m = mv\pi$. Since $(m, p) = 1$, there are $k, \ell \in \mathbb{Z}$ such that $-\ell m = 1 - kp$. Here we can choose ℓ to be the least positive inverse of $-m$ modulo p. We obtain:

$$kpv\pi + \ell pv\vartheta_m \;=\; kpv\pi - \ell mv\pi \;=\; v\pi\,,$$

whence

$$v\pi^k\vartheta_m^\ell \;=\; kv\pi + \ell v\vartheta_m \;=\; \frac{v\pi}{p}\,.$$

This shows that the set A contains $\pi^k\vartheta_m^\ell$ as the element with smallest value, and j must be equal to ℓ. We see that $\ell = 1$ occurs only for $m = p - 1$.

The rings $\mathcal{O}_K[cx]$ form a chain under inclusion with the following property.

Proposition 3.4. *For any $x \in L$ and all $c_1, c_2 \in K$,*

$$\mathcal{O}_K[c_1x] \;\subset\; \mathcal{O}_K[c_2x] \;\Leftrightarrow vc_2 \le vc_1\,.$$

Proof. Take $c_1, c_2 \in K$. If $vc_2 \le vc_1$, then $c_1x = \frac{c_1}{c_2}c_2x \in \mathcal{O}_K[c_2x]$, hence $\mathcal{O}_K[c_1x] \subseteq \mathcal{O}_K[c_2x]$. Conversely, if the latter holds, then $c_1x \in \mathcal{O}_K[c_2x]$. Since the elements $1, c_2x, \ldots, (c_2x)^{n-1}$ form a valuation basis of $(L|K, v)$, it follows that $c_1x = \frac{c_1}{c_2}c_2x$ with $\frac{c_1}{c_2} \in \mathcal{O}_K$, whence $vc_2 \le vc_1$. □

From our results in this section we also obtain:

Corollary 3.5. *Take a unibranched defectless extension $(L|K, v)$ of prime degree n. Then there is $x \in L$ such that (14) holds, $1, x, \ldots, x^{n-1}$ is a valuation basis, and for every $a \in \mathcal{M}_L$ there is $c \in K$ such that $a\,\mathcal{O}_L \subseteq \mathcal{O}_K[cx]$.*

Proof. The existence of an element x satisfying the first two assertions follows from our computations leading up to Theorem 3.1. We prove the last assertion. Write $a = \sum_{i=0}^{n-1} c_ix^i$. Since $1, x, \ldots, x^{n-1}$ is a valuation basis, $vc_ix^i \ge va$ for all i. We know that there is some $c(i) \in K$ with $c_ix^i \in \mathcal{O}_K[c(i)x]$, which is equivalent to $vc(i)^ix^i \le vc_ix^i$. Then every term $c_i'x^i$ of value at least va lies in $K[c(i)x]$. Now choose c to be the element $c(i_0)$ for which $vc(i_0) = \min_i vc(i)$.

Take any element $b \in a\,\mathcal{O}_L$, that is, $vb \ge va$. Write $b = \sum_{i=0}^{n-1} c_i'x^i$. Then $vc_i'x^i \ge vb \ge va$ for all i. By our choice of c, $c_i'x^i \in \mathcal{O}_K[cx]$ for all i. This shows that $a\,\mathcal{O}_L \subseteq \mathcal{O}_K[cx]$. □

3.2. **Differents of generators for Artin-Schreier and Kummer extensions.**

In the case of Artin-Schreier and Kummer extensions $(L|K,v)$ with Galois group G we have sufficient information about the minimal polynomials f of the various generators x we have worked with in the previous sections, or equivalently, about their conjugates, to work out the values $vf'(x)$ of their differents $f'(x)$. In order to do this, we can either compute f', or we can use the formula

$$f'(x) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (x-\sigma x)\,. \tag{16}$$

We keep the notations from the previous sections.

3.2.1. *Artin-Schreier extensions.*

Take an Artin-Schreier polynomial f with ϑ as its root. Then its minimal polynomial is $f(X)=X^p-X-\vartheta^p+\vartheta$ with $f'(X)=-1$, whence

$$f'(\vartheta)=-1\,. \tag{17}$$

For $c\in K^\times$, denote by f_c the minimal polynomial of $c\vartheta$. Then

$$f_c'(c\vartheta) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (c\vartheta-\sigma c\vartheta) \;=\; c^{p-1}f'(\vartheta) \;=\; -c^{p-1}\,. \tag{18}$$

Lemma 3.6. *Take a unibranched Artin-Schreier extension $(L|K,v)$ of prime degree p. Assume that $\mathrm{f}\,(L|K,v)=p$. The extension $Lv|Kv$ is purely inseparable if and only if $(L|K,v)$ admits an Artin-Schreier generator ϑ of value $v\vartheta<0$. In this case, for each element $\tilde c\in K^\times$ such that $v\tilde c\vartheta=0$, we have that $\mathcal{O}_L=\mathcal{O}_K[\tilde c\vartheta]$ and*

$$f_{\tilde c}'(\tilde c\vartheta)\mathcal{O}_L \;=\; \tilde c^{p-1}\mathcal{O}_L\,. \tag{19}$$

Proof. All assertions follow from part 1) of Proposition 2.8, case (DL1) and (18). □

Lemma 3.7. *Take a unibranched Artin-Schreier extension $(L|K,v)$ of prime degree p. Assume that $\mathrm{e}\,(L|K,v)=p$. Then $(L|K,v)$ admits an Artin-Schreier generator ϑ of value $v\vartheta<0$ such that $vL=vK+\mathbb{Z}v\vartheta$, and there is $j\in\{1,\ldots,p-1\}$ such that*

$$\mathcal{O}_L \;=\; \bigcup_{c\in K^\times \;\textit{with}\; vc\vartheta^j>0} \mathcal{O}_K[c\vartheta^j]\,. \tag{20}$$

Denote the minimal polynomial of $c\vartheta^j$ by $h_{j,c}$. Then we have the equality

$$(h_{j,c}'(c\vartheta^j)\mid c\in K^\times \;\textit{with}\; vc\vartheta^j>0) \;=\; I^{p-1} \tag{21}$$

of $\mathcal{O}_L$-ideals, where

$$I \;=\; (c\vartheta^{j-1}\mid c\in K^\times \;\textit{with}\; vc\vartheta^j>0)\,.$$

Proof. The existence of such ϑ and j follows from part 1) of Proposition 2.8 together with Theorem 3.1. We compute:

$$c\vartheta^j-\sigma c\vartheta^j \;=\; c(\vartheta^j-(\sigma\vartheta)^j) \;=\; c(\vartheta^j-(\vartheta+k)^j) \;=\; -c\sum_{i=1}^{j}\binom{j}{i}\vartheta^{j-i}k^i$$

for suitable $k \in \mathbb{F}_p^\times$. The summand of least value in the sum on the right hand side is the one for $i = 1$. Using (16), we obtain:

$$vh'_{j,c}(c\vartheta^j) \;=\; (p-1)(vc\vartheta^{j-1})\,, \tag{22}$$

which for $j = 1$ coincides with (18). This proves (21). □

3.2.2. *Kummer extensions.*

Take a Kummer polynomial f of degree q with η as its root. Then $f(X) = X^q - \eta^q$ and $f'(X) = qX^{q-1}$, whence

$$f'(\eta) \;=\; q\eta^{q-1}\,. \tag{23}$$

Lemma 3.8. *Take a unibranched Kummer extension $(L|K,v)$ of degree $p = \mathrm{char}\, Kv$. Assume that $\mathrm{f}\,(L|K,v) = p$. Then there exists a Kummer generator $\eta \in L$ such that $v\eta = 0$, $Lv = Kv(\eta v)$ and $\mathcal{O}_L = \mathcal{O}_K[\eta]$, or $v\tilde{c}(\eta-1) = 0$, $Lv = Kv(\tilde{c}(\eta-1)v)$ and $\mathcal{O}_L = \mathcal{O}_K[\tilde{c}(\eta-1)]$ for suitable $\tilde{c} \in K^\times$. In the first case, $v\eta = 0$, $Lv|Kv$ is inseparable, and*

$$f'(\eta)\mathcal{O}_L \;=\; p\mathcal{O}_L\,. \tag{24}$$

holds for f the minimal polynomial of η.

In the second case, η is a 1-unit, $v\tilde{c}(\eta-1) = 0$, and

$$h'_{\tilde{c}}(\tilde{c}(\eta-1))\mathcal{O}_L \;=\; p\tilde{c}^{p-1}\mathcal{O}_L \tag{25}$$

holds for $h_{\tilde{c}}$ the minimal polynomial of $\tilde{c}(\eta-1)$. If $Lv|Kv$ is separable, then always the second case holds.

Proof. The existence of such η and $\tilde{c}$, as well as the last assertion, follow from part 2) b) of Proposition 2.8. The equality (24) follows from (23). For the second case, we compute, using (16) with σ a generator of $\mathrm{Gal}\, L|K$, together with (7),

$$h'_{\tilde{c}}(\tilde{c}(\eta-1)) \;=\; \prod_{i=1}^{p-1} \tilde{c}(\eta - \sigma^i\eta) \;=\; p(\tilde{c}\eta)^{p-1}\,,$$

which yields equation (25). □

Lemma 3.9. *Take a unibranched Kummer extension $(L|K,v)$ of prime degree q. Assume that $\mathrm{e}\,(L|K,v) = q$. Then there are two possible cases.*

a) There is a Kummer generator $\eta \in L$ such that $v\eta < 0$, $vL = vK + \mathbb{Z}v\eta$, and

$$\mathcal{O}_L \;=\; \bigcup_{c\in K^\times \text{ with } vc\eta>0} \mathcal{O}_K[c\eta]\,. \tag{26}$$

Denote the minimal polynomial of $c\eta$ by f_c. Then we have the equality

$$(f'_c(c\eta) \mid c \in K^\times \text{ with } vc\eta > 0) \;=\; qI^{q-1} \tag{27}$$

of $\mathcal{O}_L$-ideals, where

$$I \;=\; (c\eta \mid c \in K^\times \text{ with } vc\eta > 0)\,.$$

If $q \neq \mathrm{char}\, Kv$, then always this case a) holds, and the factor q can be dropped in (27) since $vq = 0$.

b) We have $\mathrm{char}\, K = 0$, $p = \mathrm{char}\, Kv = q$, *and there is a Kummer generator* $\eta \in L$, *which is a* 1*-unit, and* $j \in \{1, \ldots, p-1\}$ *such that for*

$$\xi \;:=\; \frac{\eta - 1}{1 - \zeta_p}\,, \tag{28}$$

where ζ_p *a primitive* p*-th root of unity, we have that* $v\xi < 0$, $vL = vK + \mathbb{Z}v\xi^j$, *and*

$$\mathcal{O}_L \;=\; \bigcup_{c\in K^\times \textit{ with } vc\xi^j>0} \mathcal{O}_K[c\xi^j]\,. \tag{29}$$

Denote the minimal polynomial of $c\xi^j$ *by* $h_{j,c}$. *Then we have the equality*

$$(h'_{j,c}(c\xi^j) \mid c \in K^\times \textit{ with } vc\xi^j > 0) \;=\; I^{p-1} \tag{30}$$

of $\mathcal{O}_L$*-ideals, where*

$$I \;=\; (c\xi^{j-1} \mid c \in K^\times \textit{ with } vc\xi^j > 0)\,.$$

If vK *is* k*-divisible for all* $k \in \{2, \ldots, n-1\}$, *or if the convex subgroups of* vK *are well-ordered under inclusion and all archimedean components of* vK *are dense, then* j *can be taken equal to* 1.

Proof. By part 2) of Proposition 2.7 and part 2) b) of Proposition 2.8, the extension admits a Kummer generator η such that either $v\eta < 0$ and $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta - 1)$ generates the value group extension; moreover, the first case always holds if $q \neq \mathrm{char}\, Kv$. Let us first consider this case.

Applying Theorem 3.1 with $x_0 = \eta$, we find that (26) holds with η^j in place of η for some $j \in \{1, \ldots, p-1\}$. However, since also η^j is a Kummer generator with $v\eta^j < 0$ and also $v\eta^j$ generates the value group extension, as j is prime to q, we may replace η by η^j.

As also $c\eta$ is a Kummer generator, we can apply equation (23) to obtain that

$$vf'_c(c\eta) \;=\; vq + (q-1)(vc\eta)\,, \tag{31}$$

which proves (27).

Now we consider the second case. Since $L|K$ is a Kummer extension, K contains a primitive p-th root of unity ζ_p. By Lemma 2.6,

$$v(\eta - 1) \;\le\; v\eta + \frac{vp}{p-1} \;\le\; \frac{vp}{p-1} \;=\; v(1 - \zeta_p) \;\in\; vK\,.$$

Since $v(\eta - 1) \notin vK$, inequality must hold. Hence with ξ defined by (28), we have $v\xi < 0$. Further, applying Theorem 3.1 with $x_0 = \xi$, we find that (29) holds for some $j \in \{1, \ldots, p-1\}$.

We note that $v(1 - \zeta_p) = v(1 - \zeta)$ for every primitive p-th root of unity ζ. We set $a := \eta - 1$. Then for every $\sigma \in G$, $v(a - \sigma a) = v(\eta - \sigma\eta) = v(1 - \zeta_p) > va$. We compute:

$$a^j - \sigma a^j \;=\; a^j - (\sigma a)^j \;=\; a^j - (a + \sigma a - a)^j \;=\; -\sum_{i=0}^{j-1} \binom{j}{i} a^i (\sigma a - a)^{j-i}\,.$$

Since $va < v(\sigma a - a)$, the summand of least value in the sum on the right hand side is the one for $i = j-1$. Consequently,

$$\begin{aligned} v(\xi^j - \sigma\xi^j) &= v(a^j - \sigma a^j) - jv(1-\zeta_p) \;=\; (j-1)va + v(a-\sigma a) - jv(1-\zeta_p) \\ &= (j-1)va + v(1-\zeta_p) - jv(1-\zeta_p) \;=\; (j-1)(va - v(1-\zeta_p)) \\ &= v\xi^{j-1}\,. \end{aligned}$$

Hence, equation (16) shows that

$$vh'_{j,c}(c\xi^j) \;=\; (p-1)vc\xi^{j-1}\,. \tag{32}$$

This proves equation (30). We note that I is an $\mathcal{O}_L$-ideal because $vc\xi^{j-1} > vc\xi^j$ as $v\xi < 0$.

The last assertion follows from Theorem 3.1. □

4. Kähler differentials for Galois extensions of prime degree

4.1. **Motivation.**

We prove a proposition that will be a main tool for our handling of Kähler differentials in the subsequent sections. It will provide a motivation for the calculation of the Kähler differentials for Artin-Schreier extensions and Kummer extensions of prime degree which will be dealt with in this section.

Given a Galois extension $(L|K,v)$, we denote by $(L|K,v)^{in}$ its inertia field (cf. [7, Section 19]).

Proposition 4.1. *Let $(L|K,v)$ be a finite Galois extension. Then the following assertions hold.*

1) There exists a tower of field extensions

$$K \;\subset\; K^{in} \;=\; K_0 \;\subset\; K_1 \;\subset\; \cdots \;\subset\; K_\ell \;=\; L \tag{33}$$

where $K^{in} = (L|K,v)^{in}$ and each extension $K_{i+1}|K_i$ is a Galois extension of prime degree. Note that if K is henselian then the extension $K^{in}v|Kv$ is separable of degree equal to $[K^{in}:K]$.

2) Further, $(L|K,v)$ can be embedded in a finite Galois extension $(M|K,v)$ having the following properties:

$$\left\{\begin{array}{l} \textit{there exists a tower of field extensions} \\ \qquad\qquad\qquad K \subset M_0 \subset M_1 \subset \cdots \subset M_m = M, \\ \textit{where } M_0 = (M|K,v)^{in} \\ \textit{and each extension } M_{i+1}|M_i \textit{ is a Kummer extension of prime degree,} \\ \textit{or an Artin-Schreier extension if the extension is of degree } p = \operatorname{char} K. \end{array}\right. \tag{34}$$

Proof. 1): Set $K_0 := K^{in} := (L|K,v)^{in}$. Since the extension $L|K^{in}$ is solvable (cf. Theorems 24 and 25 on pages 77 and 78 of [34]), there exists a tower (33) of Galois extensions such that each extension $K_{i+1}|K_i$ is Galois of prime degree. The assertions about the extension $K^{in}v|Kv$ are part of the general properties of inertia fields.

2): If an extension $K_{i+1}|K_i$ in the tower (33) is of degree $p = \operatorname{char} K$, then it is an Artin-Schreier extension. If it is of prime degree $q \neq \operatorname{char} K$, it is a Kummer extension if K_i contains a primitive q-th root of unity. We will now explain how to enlarge the extension $(L|K,v)$ so that this will be the case for each extension of prime degree $q \neq \operatorname{char} K$ in a resulting new tower.

Assume that (K,v) is of characteristic 0 with $\operatorname{char} Kv = p > 0$ and that some extension $K_{i+1}|K_i$ is Galois of degree p, but K does not contain a primitive p-th root of unity. In this case we will have to replace tower (33) by a larger one. Let ζ_p denote a primitive p-th root of unity. Then $K(\zeta_p)|K$ is a Galois extension, and so is $L(\zeta_p)|K$ since $L|K$ is assumed to be Galois.

Set $K'_0 := (L(\zeta_p)|K,v)^{in}$; then $K_0 = K^{in} \subset K'_0$. As before, $K'_0|K$ is Galois, hence so are $K'_0(\zeta_p)|K$ and $K'_0(\zeta_p)|K'_0$. By part 1) of our proposition, there exists a tower of Galois extensions $K'_0 \subset K'_1 \subset \cdots \subset K'_{r'} = K'_0(\zeta_p)$ such that each extension $K'_{i+1}|K'_i$ is Galois of prime degree. Since $[K'_0(\zeta_p) : K'_0] < p$, none of the Galois extensions $K'_{i+1}|K'_i$ is of degree p. We observe that $K'_0 = (L(\zeta_p)|K,v)^{in}$.

We replace the tower (33) by the tower

$$K'_0 \subset K'_1 \subset \cdots \subset K'_{r'} = K'_0(\zeta_p) \subset K_1(\zeta_p) \subset \cdots \subset K_\ell(\zeta_p) = L(\zeta_p)\,. \tag{35}$$

Now we have that if in mixed characteristic any extension in the tower (33) is Galois of degree $p = \operatorname{char} Kv$, then it is a Kummer extension.

In order to make sure that also all Galois extensions of prime degree $q \neq p$ in the tower are Kummer extensions, we take Q to be the set consisting of all such primes q. For every $q \in Q$, we choose a primitive q-th root of unity ζ_q and set $M := L(\zeta_q \mid q \in Q)$. Every extension $K(\zeta_q)|K$ is Galois, so $M|K$ is also a Galois extension.

Let us show that for every $q \in Q$, ζ_q lies in the inertia field of $(M|K,v)$. The reduction of $X^q - 1$ modulo v is $X^q - 1v$ with $1v$ being the 1 in Kv. Since $q \neq \operatorname{char} Kv$, the polynomial $X^q - 1v$ has q distinct roots. The minimal polynomial f of ζ_q over K divides $X^q - 1$, so its reduction fv divides $X^q - 1v$ and has therefore only simple roots. It follows that if $\sigma \in \operatorname{Gal} M|K$ with $\sigma\zeta_q \neq \zeta_q$, then $(\sigma\zeta_q)v \neq \zeta_q v$, whence $v(\sigma\zeta_q - \zeta_q) = 0$. Hence every automorphism in the inertia group $\{\sigma \in \operatorname{Gal} M|K \mid \forall x \in \mathcal{O}_M : v(\sigma x - x) = 0\}$ must fix ζ_q, which proves our claim. It follows that $M_0 := K_0(\zeta_q \mid q \in Q)$ is the inertia field of $(M|K,v)$. Finally, we set $M_i := K_i(\zeta_q \mid q \in Q)$. By our construction, now also all extensions of prime degree $q \neq p$ in the tower are Kummer extensions. So we have obtained a tower as described in (34). □

4.2. **A basic calculation.**

Let $(L|K,v)$ be a unibranched algebraic extension. Let $A \subseteq K$ be a normal domain whose quotient field is K. Assume that $z \in L$ is integral over A and let $f(X)$ be the minimal polynomial of z over K. Then $f(X) \in A[X]$ (see [33, Theorem 4, page 260]). By the Gauss Lemma (see [30, Theorem A]), $A[z] \cong A[X]/(f(X))$. Thus,

$$\Omega_{A[z]|A} \;\cong\; [A[X]/(f(X),f'(X))]dX \;\cong\; [A[z]/(f'(z))]dX \tag{36}$$

by [19, Example 26.J, page 189] and [19, Theorem 58, page 187]. There is a canonical derivation $d_{A[z]|A} : A \to \Omega_{A[z]|A}$ defined by $g(z) \mapsto g'(z)dX$ for $g(X) \in A[X]$.

4.3. **Finite extensions $(L|K,v)$ of degree $[L:K] = \mathrm{f}(L|K)$ with separable residue field extension.**

We denote the annihilator of an $\mathcal{O}_L$-ideal I by $\operatorname{ann} I$.

Theorem 4.2. *Take a finite extension $(L|K,v)$ with $Lv|Kv$ separable of degree $[Lv:Kv] = [L:K]$. Then $\mathcal{O}_L = \mathcal{O}_K[x]$ for some $x \in L$ with $vx = 0$ and $Lv = Kv(xv)$, and we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0\,,$$

and $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$.

Proof. By (13), $\mathcal{O}_L = \mathcal{O}_K[x]$ where x is a lift of a generator ξ of Lv over Kv. Let $f(X) \in K[X]$ be the minimal polynomial of x over K. As $\deg f = [L:K] = [Lv:Kv]$, the reduction $\bar{f}$ of f in $Kv[X]$ is the minimal polynomial of ξ over Kv. We have that $f'(x)v = \bar{f}'(\xi)$ which is nonzero since ξ is separable over Kv. Thus $f'(x)$ is a unit in $\mathcal{O}_L$. By (36), $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{O}_L/(f'(\alpha)) = 0$. □

We note that this theorem always applies when $(L|K,v)$ is a Kummer extension of prime degree $q = \mathrm{f}(L|K) \neq \operatorname{char} Kv$ since then $Lv|Kv$ is separable.

4.4. **Artin-Schreier extensions $(L|K,v)$ of degree p with $\mathrm{f}(L|K) = p$ and inseparable residue field extension.**

By Lemma 3.6, there exists an Artin-Schreier generator ϑ of value $v\vartheta < 0$ and an element $\tilde{c} \in \mathcal{M}_K$ such that $v(\tilde{c}\vartheta) = 0$ and $\mathcal{O}_L = \mathcal{O}_K[\tilde{c}\vartheta]$. Let $f_{\tilde{c}}$ be the minimal polynomial of $\tilde{c}\vartheta$ over K. By (19), $f'_{\tilde{c}}(\tilde{c}\vartheta)\mathcal{O}_L = \tilde{c}^{p-1}\mathcal{O}_L \neq \mathcal{O}_L$.

Now we obtain from (36):

Theorem 4.3. *Take an Artin-Schreier extension $(L|K,v)$ of degree $p = \mathrm{f}(L|K) = \operatorname{char} K$ with $Lv|Kv$ inseparable. Then we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{O}_L/(\tilde{c}^{p-1})$$

as an $\mathcal{O}_L$-module. Consequently, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ and $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (\tilde{c}^{p-1})$. Hence, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ if and only if $p = 2$ and $\mathcal{M}_L = (\tilde{c})$.

4.5. **Artin-Schreier extensions $(L|K,v)$ of degree p with $\mathrm{e}(L|K,v) = p$.**

By Lemma 3.7, there exists an Artin-Schreier generator ϑ of value $v\vartheta < 0$ such that $vL = vK + \mathbb{Z}v\vartheta$, and $j \in \{1,\ldots,p-1\}$ such that equation (20) holds.

Theorem 4.4. *Take an Artin-Schreier extension $(L|K,v)$ of degree $p = \mathrm{e}(L|K)$, and a generator ϑ as described above. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I/I^p\,. \tag{37}$$

as an $\mathcal{O}_L$-module, where I is the $\mathcal{O}_L$-ideal

$$I = (c\vartheta^{j-1} \mid c \in K \text{ with } vc\vartheta^j > 0)\,.$$

In particular,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\neq\; 0$$

and $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq \mathcal{M}_L$.

Proof. By Proposition 3.4, the $\mathcal{O}_K$-algebras $\mathcal{O}_K[c\vartheta^j]$, for $c \in K$ with $vc\vartheta^j > 0$, form a chain such that $\mathcal{O}_k[c_1\vartheta^j] \subset \mathcal{O}_K[c_2\vartheta^j]$ if and only if $c_1, c_2 \in K^\times$ with $vc_1 \geq vc_2$; we have that $c_1\vartheta^j = \frac{c_1}{c_2}c_2\vartheta^j$. Let $h_{j,c}$ be the minimal polynomial of $c\vartheta^j$.

We will apply Proposition 1.1. Let $A = \mathcal{O}_K, B = \mathcal{O}_L$ and $S = \{\alpha \in K \mid v\alpha\vartheta^j > 0\}$ ordered by $\alpha \leq \beta$ if $v\alpha \geq v\beta$. For $\alpha \in S$, set $b_\alpha = \alpha\vartheta^j$, $a_\alpha = \alpha$, $c_{\alpha,\beta} = 0$. The polynomial h_α appearing in Proposition 1.1 is the minimal polynomial $h_{j,\alpha}$ of $\alpha\vartheta^j$ over K. With the notation of Proposition 1.1, $U = (\alpha \mid \alpha \in S) = \vartheta^{1-j}I$, and $V = (h'_{j,\alpha}(b_\alpha) \mid \alpha \in S) = I^{p-1}$ by equation (21). Hence by Proposition 1.1, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/UV \cong I/I^p$.

We turn to the proof of $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. We set $\tau := v\vartheta < 0$ and let $\mathcal{A}_{vL}(\tau)$ be the archimedean component of τ in vL. We have a natural order preserving inclusion $\mathcal{A}_{vK}(p\tau) \subset \mathcal{A}_{vL}(\tau)$ (see Section 2.1).

Assume first that $\mathcal{A}_{vK}(p\tau)$ is dense in $\mathcal{A}_{vL}(\tau)$, and write $\bar{\tau} := \tau + \mathcal{C}^+_{vL}(\tau) \in \mathcal{A}_{vL}(\tau)$. Since $\bar{\tau} < 0$, there exists $\gamma \in \mathcal{A}_{vK}(p\tau)$ such that

$$-p\bar{\tau} \;>\; \gamma + (j-1)\bar{\tau} \;>\; -\bar{\tau}\,. \tag{38}$$

Hence for $c \in K$ with $vc + \mathcal{C}^+_{vL}(\tau) = \gamma$,

$$-pv\vartheta \;>\; vc\vartheta^{j-1} \;>\; -v\vartheta\,. \tag{39}$$

We observe that $c\vartheta^{j-1} \in I$ if and only if $vc\vartheta^{j-1} > -v\vartheta$. Thus the right hand inequality in (39) shows that $c\vartheta^{j-1} \in I$, and the left hand inequality in (39) shows that $c\vartheta^{j-1} \notin I^p$. Hence $I \neq I^p$ in this case.

Now assume that $\mathcal{A}_{vK}(p\tau)$ is not dense in $\mathcal{A}_{vL}(\tau)$, so that $\mathcal{A}_{vK}(p\tau)$ and hence also $\mathcal{A}_{vL}(\tau)$ is discrete. Choose some $c \in K$ such that $vc = -pv\vartheta$. Then $vc\vartheta^j = (j-p)v\vartheta > 0$, so $c\vartheta^{j-1} \in I$ and $vc\vartheta^{j-1} = (j-1-p)v\vartheta \in \mathcal{C}_{vL}(\tau)$. Thus, the image of $vI \cap \mathcal{C}_{vL}(\tau)$ in $\mathcal{A}_{vL}(\tau)$ is a nonempty set of non-negative elements. Hence there must exist some $c_0 \in K$ such that $vc_0\vartheta^{j-1} + \mathcal{C}^+_{vL}(\tau)$ is its minimal element, with $c_0\vartheta^{j-1} \in I$. Suppose that there is some $c\vartheta^{j-1} \in I$ such that $pvc\vartheta^{j-1} \leq vc_0\vartheta^{j-1}$. By what we observed earlier, $vc\vartheta^{j-1} > -v\vartheta = -\tau > 0$; this together with the previous inequality yields that $vc\vartheta^{j-1} \in \mathcal{C}_{vL}(\tau)$ and that $vc\vartheta^{j-1} + \mathcal{C}^+_{vL}(\tau)$ is a positive element in $\mathcal{A}_{vL}(\tau)$. We obtain that

$$0 \;<\; vc\vartheta^{j-1} + \mathcal{C}^+_{vL}(\tau) \;<\; p(vc\vartheta^{j-1} + \mathcal{C}^+_{vL}(\tau)) \;\leq\; vc_0\vartheta^{j-1} + \mathcal{C}^+_{vL}(\tau)\,,$$

which contradicts our choice of c_0. This shows that $c_0\vartheta^{j-1} \notin I^p$, hence again, $I \neq I^p$.

Suppose that $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$. Applying [5, part 2) of Proposition 4.2] with $U = I$ and $V = I^{p-1}$ and observing that $I \neq \mathcal{O}_L$ since $v\vartheta \notin vK$, we deduce that $I = \mathcal{M}_L$. However, we observed already that $c\vartheta^{j-1} \in I$ if and only if $vc\vartheta^{j-1} > -v\vartheta > 0$. Thus, $-v\vartheta \notin vI$, showing that $I \neq \mathcal{M}_L$. This contradiction proves that $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq \mathcal{M}_L$. □

In [16], the following is proven. If $\mathcal{A}_{vK}(p\tau)$ is dense in $\mathcal{A}_{vL}(\tau)$, then

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;=\; (a \in \mathcal{O}_L \mid va \;\geq\; \alpha - (p-1)v\vartheta \;\text{ for some }\; \alpha \in \mathcal{C}^+_{vL}(\tau))\,,$$

which properly contains V. Otherwise, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = V$.

4.6. **Kummer extensions $(L|K,v)$ of degree $p = \mathrm{char}\, Kv$ with $\mathrm{f}\,(L|K) = p$.** By Lemma 3.8, we have two cases.

i) There exists a Kummer generator η of L such that $v\eta = 0$, $Lv = Kv(\eta v)$ and $\mathcal{O}_L = \mathcal{O}_K[\eta]$. In this case, $Lv|Kv$ is inseparable. Let f be the minimal polynomial of η over K. Then $f'(\eta)\mathcal{O}_L = p\mathcal{O}_L \neq \mathcal{O}_L$.

ii) There exists a Kummer generator η such that η is a 1-unit and $v\tilde{c}(\eta-1) = 0$, $Lv = Kv(\tilde{c}(\eta-1)v)$ and $\mathcal{O}_L = \mathcal{O}_K[\tilde{c}(\eta-1)]$ for some $\tilde{c} \in K$. Let $h_{\tilde{c}}$ be the minimal polynomial of $\tilde{c}(\eta-1)$ over K. Then $h'_{\tilde{c}}(\tilde{c}(\eta-1))\mathcal{O}_L = p\tilde{c}^{p-1}\mathcal{O}_L$.

Theorem 4.5. *Let $(L|K,v)$ be a Kummer extension of degree $p = \mathrm{f}\,(L|K) = \mathrm{char}\, Kv$. Then we have the following description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.*

In case i) above,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{O}_L/(p) \neq 0$$

as an $\mathcal{O}_L$-module, and $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (p)$. *Hence,* $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ *if and only if $\mathcal{M}_L = (p)$.*

In case ii) above,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{O}_L/(p\tilde{c}^{p-1})$$

as an $\mathcal{O}_L$-module, and

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0 \text{ if and only if } Lv|Kv \text{ is separable.} \tag{40}$$

We have $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (p\tilde{c}^{p-1})$*, which is equal to $\mathcal{O}_L$ if and only if $Lv|Kv$ is separable. Further,* $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ *if and only if $\mathcal{M}_L = (p\tilde{c}^{p-1})$.*

Proof. All assertions except for (40) follow from (36). In case ii), the minimal polynomial of $\tilde{c}(\eta-1)v$ over Kv is the reduction $h'_{\tilde{c}}v$. Hence $Kv(\tilde{c}(\eta-1)v)|Kv$ is separable if and only if $h'_{\tilde{c}}(\tilde{c}(\eta-1))v = (h_{\tilde{c}}v)'(\tilde{c}(\eta-1)v) \neq 0$, that is, $vp\tilde{c}^{p-1} = vp(\tilde{c}\eta)^{p-1} = vh'_{\tilde{c}}(\tilde{c}(\eta-1)) = 0$. This is equivalent to $(p\tilde{c}^{p-1}) = \mathcal{O}_L$, and this in turn is equivalent to $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. □

4.7. **Kummer extensions $(L|K,v)$ of prime degree q with $\mathrm{e}\,(L|K) = q$.** Let $(L|K,v)$ be a Kummer extension of prime degree q with $e(L|K) = q$. By Lemma 3.9, there is a Kummer generator $\eta \in L$ such that one of the following cases appears:

i) Equation (26) holds with $v\eta < 0$.

ii) We have $\mathrm{char}\, K = 0$, $p = \mathrm{char}\, Kv = q$, and equation (29) holds with $\xi = \frac{\eta-1}{1-\zeta_p}$, η a 1-unit, ζ_p a primitive p-th root of unity, $1 \leq j \leq p-1$ and $v\xi < 0$.

Theorem 4.6. *Let $(L|K,v)$ be a Kummer extension of prime degree q with* $\mathrm{e}\,(L|K) = q$. *Then we have the following description of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.*

In case i) above,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I/qI^q$$

as $\mathcal{O}_L$-modules, where I is the $\mathcal{O}_L$-ideal

$$I = (c\eta \mid c \in K \text{ and } vc\eta > 0)\,.$$

This case always occurs when $q \neq p$.

We have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ *if and only if* $vI \cap \mathcal{C}_{vL}(vq) = \emptyset$ *and whenever there is* $c \in K$ *such that* $vc\eta > 0$ *and* $vI \cap \mathcal{C}^+_{vL}(vc\eta) = \emptyset$*, then* $\mathcal{A}_{vK}(qvc\eta)$ *is not discrete. The condition* $vI \cap \mathcal{C}_{vL}(vq) = \emptyset$ *always holds when* $q \neq p$*.*

In this case, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ *holds if and only if* $I = \mathcal{M}_L$ *is principal and* $q = 2 \neq \operatorname{char} Kv$*.*

In case ii) above,

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I/I^p$$

as $\mathcal{O}_L$*-modules, where* I *is the* $\mathcal{O}_L$*-ideal*

$$I = (c\xi^{j-1} \mid c \in K \text{ and } vc\xi^j > 0).$$

In this case we always have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ *and* $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq \mathcal{M}_L$ *.*

Proof. Assume that case i) holds. By Proposition 3.4, the $\mathcal{O}_K$-algebras $\mathcal{O}_K[c\eta]$, for $c \in K^\times$ with $vc\eta > 0$, form a chain such that $\mathcal{O}_k[c_1\eta] \subset \mathcal{O}_K[c_2\eta]$ if and only if $c_1, c_2 \in K^\times$ with $vc_1 \geq vc_2$; we have that $c_1\eta = \frac{c_1}{c_2}c_2\eta$. Let f_c be the minimal polynomial of $c\eta$ over K.

We will apply Proposition 1.1. Let $A = \mathcal{O}_K, B = \mathcal{O}_L$ and $S = \{c \in K \mid vc\eta > 0\}$ ordered by $\alpha \leq \beta$ if $v\alpha \geq v\beta$. Let $b_\alpha = \alpha\eta$, $a_\alpha = \alpha$, $c_{\alpha,\beta} = 0$ and $h_\alpha = f_\alpha$. With the notation of Proposition 1.1, $U = (\alpha \mid \alpha \in S) = \eta^{-1}I$, and $V = (h'_\alpha(b_\alpha) \mid \alpha \in S) = qI^{q-1}$ by equation (27). Hence by Proposition 1.1, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/UV \cong I/qI^q$.

Now we determine when $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ holds in the present case. Assume first that the condition $vI \cap \mathcal{C}_{vL}(vq) = \emptyset$ does not hold. For every $c \in K$ with $c\eta \in I$ and $vc\eta \in \mathcal{C}_{vL}(vq)$, take $n_c \geq 0$ such that $n_c vq < vc\eta \leq (n_c+1)vq$ (n_c exists since $vc\eta > 0$). Choose $c_0\eta \in I$ with $vc_0\eta \in \mathcal{C}_{vL}(vq)$ and n_{c_0} minimal. Suppose that $c_0\eta \in qI^q$. Then there exists $c_1 \in K$ with $c_1\eta \in I$ such that $vq + v(c_1\eta)^q \leq vc_0\eta \leq (n_{c_0}+1)vq$, whence $vc_1\eta \in \mathcal{C}_{vL}(vq)$ and $0 < vc_1\eta < qv(c_1\eta) \leq n_{c_0}vq$, which contradicts our choice of c_0. Thus, $I \neq qI^q$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ in this case.

Now assume that $vI \cap \mathcal{C}_{vL}(vq) = \emptyset$ and choose any $c \in K$ such that $vc\eta > 0$. If there is $\tilde{c} \in K$ such that $v\tilde{c}\eta > 0$ and $v\tilde{c}\eta \in \mathcal{C}^+_{vL}(vc\eta)$, then $vq + qv\tilde{c}\eta < vc\eta$ since also $vq \in \mathcal{C}^+_{vL}(vc\eta)$; this yields that $c\eta \in qI^q$. Hence if there is no $c\eta \in I$ such that $vI \cap \mathcal{C}^+_{vL}(vc\eta) = \emptyset$, then $I = qI^q$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

In the remaining case, there exists $c \in K$ such that $vc\eta > 0$ and $vI \cap \mathcal{C}^+_{vL}(vc\eta) = \emptyset$. Set $\tau := vc\eta > 0$ and $\bar{\alpha} := \alpha + \mathcal{C}^+_{vL}(vc\eta) \in \mathcal{A}_{vL}(\tau)$ for $\alpha \in \mathcal{C}_{vL}(vc\eta)$. The image of $vI \cap \mathcal{C}_{vL}(\tau)$ in $\mathcal{A}_{vL}(\tau)$ is a nonempty set of positive elements. If $\mathcal{A}_{vK}(q\tau)$ and hence also $\mathcal{A}_{vL}(\tau)$ is discrete, then there is $c_0 \in K$ such that $\overline{vc_0\eta}$ is the minimal element of $\mathcal{A}_{vL}(\tau)$. Suppose that $c_0\eta \in qI^q$. Then there is $c_1\eta \in I$ such that

$$0 \;<\; \overline{vc_1\eta} \;<\; q\overline{vc_1\eta} \;=\; \overline{vq + qvc_1\eta} \;\leq\; \overline{vc_0\eta}$$

since $\overline{vq} = 0$, which contradicts our choice of c_0. Hence $I \neq qI^q$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ in this case.

If on the other hand, $\mathcal{A}_{vK}(q\tau)$ is not discrete, then there exists $\gamma \in \mathcal{A}_{vK}(q\tau)$ such that $-q\bar{\tau} < q\gamma < (1-q)\bar{\tau}$, so that $0 < q(\gamma + \bar{\tau}) < \bar{\tau}$. Choose $c_1 \in K$ such that $\overline{vc_1} = \gamma$. Then $0 < q\,\overline{v(c_1c\eta)} < \bar{\tau}$, whence $c_1c\eta \in I$ and $vq + v(c_1c\eta)^q < vc\eta$. This shows that $c\eta \in qI^q$. Hence if $\mathcal{A}_{vK}(qvc\eta)$ is not discrete whenever $vc\eta > 0$ and $vI \cap \mathcal{C}^+_{vL}(vc\eta) = \emptyset$, then $I = qI^q$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ in this case.

If $q \neq p$, then $vq = 0$ and $\mathcal{C}_{vL}(vq) = \{0\}$. It follows that $vI \cap \mathcal{C}_{vL}(vq) = \emptyset$ since otherwise there is $c \in K$ such that $vc\eta = 0$, whence $v\eta \in vK$, contradiction.

Assume that case ii) holds. By Proposition 3.4, the $\mathcal{O}_K$-algebras $\mathcal{O}_K[c\xi^j]$, for $c \in K^\times$ with $vc\xi^j > 0$, form a chain such that $\mathcal{O}_k[c_1\xi^j] \subset \mathcal{O}_K[c_2\xi^j]$ if and only if $c_1, c_2 \in K^\times$ with $vc_1 \geq vc_2$; we have that $c_1\xi^j = \frac{c_1}{c_2}c_2\xi^j$. Let $h_{j,c}$ be the minimal polynomial of $c\xi^j$ over K.

We will apply Proposition 1.1. Let $A = \mathcal{O}_K, B = \mathcal{O}_L$ and $S = \{c \in K \mid vc\xi^j > 0\}$ ordered by $\alpha \leq \beta$ if $v\alpha \geq v\beta$. Let $b_\alpha = \alpha\xi^j$, $a_\alpha = \alpha$, $c_{\alpha,\beta} = 0$ and $h_\alpha = h_{j,\alpha}$. With the notation of Proposition 1.1, $U = (\alpha \mid \alpha \in S) = \xi^{1-j}I$, and $V = (h'_{j,\alpha}(b_\alpha) \mid \alpha \in S) = I^{p-1}$ by equation (30). Hence by Proposition 1.1, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong U/UV \cong I/I^p$.

The fact that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ in case ii) is shown as in the proof of Theorem 4.4.

Suppose that $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$; in particular, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. In case ii), we see as in the proof of Theorem 4.4 that $I \neq \mathcal{O}_L$ and deduce that $I = \mathcal{M}_L$. However, $c\xi^{j-1} \in I$ if and only if $vc\xi^{j-1} > -v\xi > 0$. Thus, $-v\xi \notin vI$, showing that $I \neq \mathcal{M}_L$, contradiction.

In case i), we apply [5, part 2) of Proposition 4.2] with $U = I$ and $V = qI^{q-1}$. Since also in this case $I \neq \mathcal{O}_L$, we obtain that $\mathcal{M}_L$ is principal with $I = qI^{q-1} = \mathcal{M}_L$. Hence □

The annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ will be determined in detail in [16].

5. Kähler differentials of towers of Galois extensions

In this section, our goal is the proof of the following two theorems, which will be given in Subsection 5.2. We begin by preparing the ingredients for the proofs.

Theorem 5.1. *Assume that $L|K$ and $M|L$ are towers of finite Galois extensions of valued fields. Then there is a natural short exact sequence*

$$0 \to \Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M \to \Omega_{\mathcal{O}_M|\mathcal{O}_K} \to \Omega_{\mathcal{O}_M|\mathcal{O}_L} \to 0.$$

In particular, $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_M|\mathcal{O}_L} = 0$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

Theorem 5.2. *Let (K, v) be a valued field. Then*

1) *$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions $L|K$ of K^{sep}.*
2) *Let $L|K$ be a finite Galois subextension of K^{sep} and assume that*

$$K \subset K^{in} = K_0 \subset K_1 \subset \cdots \subset K_\ell = L$$

is a tower of field extensions factoring $L|K$ such that K^{in} is the inertia field of $(L|K, v)$ and $K_{i+1}|K_i$ is Galois of prime degree for all i. Then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_{K_{i+1}}|\mathcal{O}_{K_i}} = 0$ for $0 \leq i \leq \ell - 1$.

Lemma 5.3. *Assume that $(L|K, v)$ is a valued field extension. Then $\mathcal{O}_L$ is a faithfully flat $\mathcal{O}_K$-module.*

Proof. We have that $\mathcal{O}_L$ is a flat $\mathcal{O}_K$-module by [26, Theorem 4.33] (see also [27, Theorem 4.35]), since $\mathcal{O}_K$ is a valuation ring and $\mathcal{O}_L$ is a torsion free $\mathcal{O}_K$-module. Further, $\mathcal{O}_L$ is a faithfully flat $\mathcal{O}_K$-module by Theorem 7.2 [19], since $\mathcal{M}_K\mathcal{O}_L \neq \mathcal{O}_L$. □

Theorem 5.4. *Let $(L|K,v)$ be a finite valued field extension which is unibranched and such that there is a tower of field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_\ell = L$ such that for $1 \le i \le \ell$ one of the following holds:*

1) $K_i|K_{i-1}$ is Galois of prime degree or

2) $[K_i : K_{i-1}] = [K_iv : K_{i-1}v]$ and K_iv is separable over $K_{i-1}v$.

Then for $2 \le i \le \ell$, we have natural short exact sequences

$$0 \to (\Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K}) \otimes_{\mathcal{O}_{K_{i-1}}} \mathcal{O}_{K_i} \to \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_K} \to \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_{i-1}}} \to 0. \tag{41}$$

Proof. By Theorem 4.2, Theorem 3.1 for unibranched defectless extensions of prime degree and [5, Lemma 2.3, Lemma 3.1, Lemma 3.2 and Proposition 3.3] for extensions of prime degree with nontrivial defect for $1 \le i \le \ell$ there exist directed sets S_i with associated $\alpha(i)_j \in K_i$ for $j \in S_i$ such that $\mathcal{O}_{K_{i-1}}[\alpha(i)_j] \subset \mathcal{O}_{K_{i-1}}[\alpha(i)_k]$ if $j \le k$ and $\mathcal{O}_{K_i} = \cup_{j\in S_i}\mathcal{O}_{K_{i-1}}[\alpha(i)_j]$. Further, $\mathcal{O}_{K_i}[\alpha(i)_j] \cong \mathcal{O}_{K_i}[X]/(f_i^j(X))$ where $f_i^j(X)$ is the minimal polynomial $\alpha(i)_j$ over K_{i-1}.

Let T_i be the set of $(k_1, k_2, \ldots, k_{i-1}, k_i) \in S_1 \times S_2 \times \cdots \times S_i$ such that $f_n^{k_n}(x) \in \mathcal{O}_K[\alpha(1)_{k_1}, \alpha(2)_{k_2}, \ldots, \alpha(n-1)_{k_{n-1}}][x]$ for $2 \le n \le i$. We define a partial order on T_i by the rule $(k_1, \ldots, k_i) \le (l_1, \ldots, l_i)$ if $k_m \le l_m$ for $1 \le m \le i$. The T_i are directed sets since the S_i are, and setting

$$A_{k_1,\ldots,k_i} \;=\; \mathcal{O}_K[\alpha(1)_{k_1}, \alpha(2)_{k_2}, \ldots, \alpha(i-1)_{k_{i-1}}, \alpha(i)_{k_i}]$$

for $(k_1, \ldots, k_i) \in T_i$, we have inclusions

$$A_{k_1,\ldots,k_i} \subset A_{l_1,\ldots,l_i} \text{ for } (k_1, \ldots, k_i) \le (l_1, l_2, \ldots, l_i) \text{ in } T_i.$$

By our construction, for $2 \le m \le i$, there exist

$$g_m^{k_m}(X_1, \ldots, X_{m-1}, X_m) \in \mathcal{O}_K[X_1, X_2, \ldots, X_{m-1}, X_m]$$

such that $g_m^{k_m}(\alpha(1)_{k_1}, \cdots, \alpha(m-1)_{k_{m-1}}, X_m) = f_m^{k_m}(X_m)$.

By [30, Theorem 1], we have that

$$A_{k_1,\ldots,k_i} \cong \mathcal{O}_K[X_1, \ldots, X_i]/(g_1^{k_1}(X_1), g_2^{k_2}(X_1, X_2), \ldots, g_i^{k_i}(X_1, \ldots, X_i)).$$

By [19, Theorem 25.2], $\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K} \cong (A_{k_1,\ldots,k_i}dX_1 \oplus \cdots \oplus A_{k_1,\ldots,k_i}dX_i)/U_{k_1,\ldots,k_i}$, where $U_{k_1,\ldots,k_i}$ is the $A_{k_1,\ldots,k_i}$-submodule of $A_{k_1,\ldots,k_i}dX_1 \oplus \cdots \oplus A_{k_1,\ldots,k_i}dX_i$ generated by

$$\left[\frac{\partial f_1^{k_1}}{\partial X_1}(\alpha(1)_{k_1})\right] dX_1 \tag{42}$$

and

$$\left[\frac{\partial g_m^{k_m}}{\partial X_1}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] dX_1 + \cdots + \left[\frac{\partial g_m^{k_m}}{\partial X_m}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] dX_m \tag{43}$$

for $2 \le m \le i$. We further have that

$$\left[\frac{\partial f_1^{k_1}}{\partial X_1}(\alpha(1)_{k_1})\right] = (f_1^{k_1})'(\alpha(1)_{k_1}) \tag{44}$$

and

$$\left[\frac{\partial g_m^{k_m}}{\partial X_m}(\alpha(1)_{k_1}, \ldots, \alpha(m)_{k_m})\right] = (f_m^{k_m})'(\alpha(m)_{k_m}) \tag{45}$$

for $2 \le m \le i$.

By [19, Theorem 25.1], we have a natural exact sequence of $A_{k_1,\ldots,k_i}$-modules

$$\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}A_{k_1,\ldots,k_i}\rightarrow\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K}\rightarrow\Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}}\rightarrow 0. \tag{46}$$

For $(k_1,\ldots,k_i)\in T_i$, let

$$L_{k_1,\ldots,k_i}=\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_i},$$
$$M_{k_1,\ldots,k_i}=\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_i}}\mathcal{O}_{K_i},$$
$$N_{k_1,\ldots,k_i}=\Omega_{A_{k_1,\ldots,k_i}|A_{k_1,\ldots,k_{i-1}}}\otimes_{A_{k_1,\ldots,k_i}}\mathcal{O}_{K_i}.$$

Applying the right exact functor $\otimes_{A_{k_1,\ldots,k_i}}\mathcal{O}_{K_i}$ to (46), we have an exact sequence of $\mathcal{O}_{K_i}$-modules

$$L_{k_1,\ldots,k_i}\stackrel{u}{\rightarrow}M_{k_1,\ldots,k_i}\rightarrow N_{k_1,\ldots,k_i}\rightarrow 0. \tag{47}$$

Now $\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_i}$ is the quotient of $\mathcal{O}_{K_i}dX_1\oplus\cdots\oplus\mathcal{O}_{K_i}dX_{i-1}$ by the relations (42) and (43) for $2\le m\le i-1$ and $\Omega_{A_{k_1,\ldots,k_i}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_i}}\mathcal{O}_{K_i}$ is the quotient of $\mathcal{O}_{K_i}dX_1\oplus\cdots\oplus\mathcal{O}_{K_i}dX_i$ by the relations (42) and (43) for $2\le m\le i$. Since $(f_i^{k_i})'(\alpha(i)_{k_i})\neq 0$ (as K_i is separable over K_{i-1}) we have by (45) with $m=i$ that u is injective, so that (47) is actually short exact.

Let $(k_1,\ldots,k_i)$ and $(l_1,\ldots,l_i)$ in T_i be such that $(k_1,\ldots,k_i)\le(l_1,\ldots,l_i)$. Then we have a natural commutative diagram of $\mathcal{O}_{K_i}$-modules with short exact rows

$$\begin{array}{ccccccccc}
0&\rightarrow&L_{k_1,\ldots,k_i}&\rightarrow&M_{k_1,\ldots,k_i}&\rightarrow&N_{k_1,\ldots,k_i}&\rightarrow&0\\
&&\downarrow&&\downarrow&&\downarrow&&\\
0&\rightarrow&L_{l_1,\ldots,l_i}&\rightarrow&M_{l_1,\ldots,l_i}&\rightarrow&N_{l_1,\ldots,l_i}&\rightarrow&0
\end{array} \tag{48}$$

where the vertical arrows are the natural maps determined by the differentials of the inclusions of $A_{k_1,\ldots,k_{i-1}}$ into $A_{l_1,\ldots,l_{i-1}}$ and of $A_{k_1,\ldots,k_i}$ into $A_{l_1,\ldots,l_i}$. By [26, Theorem 2.18] (see also [27, Proposition 5.33]), we have a short exact sequence of $\mathcal{O}_K$-modules

$$0\rightarrow\lim_{\rightarrow}L_{k_1,\ldots,k_i}\rightarrow\lim_{\rightarrow}M_{k_1,\ldots,k_i}\rightarrow\lim_{\rightarrow}N_{k_1,\ldots,k_i}\rightarrow 0. \tag{49}$$

By our construction of T_i, we have that $\cup A_{k_1,\ldots,k_i}=\mathcal{O}_{K_i}$, where the union is over all $(k_1,\ldots,k_i)\in T_i$. Thus $\lim\limits_{\rightarrow}M_{k_1,\ldots,k_i}\cong\Omega_{\mathcal{O}_{K_i}|\mathcal{O}_K}$ by [6, Theorem 16.8]. We also have that $\cup A_{k_1,\ldots,k_{i-1}}=\mathcal{O}_{K_{i-1}}$, where the union is over all $(k_1,\ldots,k_{i-1})$ such that $(k_1,\ldots,k_i)\in T_i$. Thus

$$\lim_{\rightarrow}\left(\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_{i-1}}\right)\cong\Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K}$$

again by [6, Theorem 16.8]. Now

$$\begin{array}{rcl}
\lim_{\rightarrow}L_{k_1,\ldots,k_i}&=&\lim_{\rightarrow}\left(\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_i}\right)\\
&\cong&\lim_{\rightarrow}\left((\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_{i-1}})\otimes_{\mathcal{O}_{K_{i-1}}}\mathcal{O}_{K_i}\right)\\
&\cong&\left(\lim_{\rightarrow}(\Omega_{A_{k_1,\ldots,k_{i-1}}|\mathcal{O}_K}\otimes_{A_{k_1,\ldots,k_{i-1}}}\mathcal{O}_{K_{i-1}})\right)\otimes_{\mathcal{O}_{K_{i-1}}}\mathcal{O}_{K_i}\\
&\cong&\Omega_{\mathcal{O}_{K_{i-1}}|\mathcal{O}_K}\otimes_{\mathcal{O}_{K_{i-1}}}\mathcal{O}_{K_i}
\end{array}$$

where the equality of the third row is by [26, Corollary 2.20].

We have that

$$A_{k_1,\dots,k_i} = A_{k_1,\dots,k_{i-1}}[\alpha(i)_{k_i}] \cong A_{k_1,\dots,k_{i-1}}[X_i]/(f_i^{k_i}),$$

so

$$\Omega_{A_{k_1,\dots,k_i}|A_{k_1,\dots,k_{i-1}}} \cong \left(A_{k_1,\dots,k_i}/(f_i^{k_i})'(\alpha(i)_{k_i})\right) dX_i$$

by equation (36). Since $f_i^{k_i}$ is the minimal polynomial of $\alpha(i)_{k_i}$ over K_{i-1}, we have that

$$\Omega_{\mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]|\mathcal{O}_{K_{i-1}}} \cong \mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]/((f_i^{k_i})'(\alpha(i)_{k_i}))dX_i$$

also by (36). Thus

$$\begin{aligned} N_{k_1,\dots,k_i} &= \Omega_{A_{k_1,\dots,k_i}|A_{k_1,\dots,k_{i-1}}} \otimes_{A_{k_1,\dots,k_i}} \mathcal{O}_{K_i} \cong \left(\mathcal{O}_{K_i}/(f_i^{k_i})'(\alpha(i)_{k_i})\right) dX_i \\ &\cong \left(\Omega_{\mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}]|\mathcal{O}_{K_{i-1}}}\right) \otimes_{\mathcal{O}_{K_{i-1}[\alpha(i)_{k_i}]}} \mathcal{O}_{K_i}. \end{aligned}$$

Since $\cup\mathcal{O}_{K_{i-1}}[\alpha(i)_{k_i}] = \mathcal{O}_{K_i}$, we have that $\varinjlim N_{k_1,\dots,k_i} \cong \Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_{i-1}}}$ by [6, Theorem 16.8].

In conclusion, for $1 \le i \le r$, the sequence (41) is isomorphic to the short exact sequence (49). □

Proposition 5.5. *Let $(L|K, v)$ be a finite Galois extension of valued fields. Let G be the Galois group of $L|K$ and let H be a subgroup of G which contains the inertia group of $L|K$. Denote the fixed field of H in L by L_0. Then $\Omega_{\mathcal{O}_{L_0}|\mathcal{O}_K} = 0$.*

Proof. By [25, Theorem 1 of Chapter X, page 103], $\mathcal{O}_{L_0} = B_n$ where B is an étale $\mathcal{O}_K$-algebra and n is a maximal ideal of B. Now B is an étale $\mathcal{O}_K$-algebra if and only if it is unramified and flat ([9, Corollary IV.17.6.2]). Since B is unramified over $\mathcal{O}_K$ we have that $\Omega_{B|\mathcal{O}_K} = 0$ ([9, Theorem IV.17.4.2]) and so $\Omega_{\mathcal{O}_{L_0}|\mathcal{O}_K} \cong (\Omega_{B|\mathcal{O}_K}) \otimes_B B_n = 0$ by [6, Proposition 16.9]. □

To put Proposition 5.5 into context, we give in Proposition 5.6 a simple consequence of [25, Theorem 1 of Chapter X, page 103]. The definitions of unramified (it is also called net there) and étale morphisms are given in [9, Definition IV.17.31]. The definition of locally étale is given in Definition VIII.2 on page 80 of [25] and using more geometric language, "étale at a point", in Definition IV.17.3.7 [9]. Let (A, m_A) be a local ring. An A-algebra C is called locally étale if $C = B_n$ where B is étale over A and n is a prime ideal of B which contracts to m_A. We may similarly define locally unramified, replacing étale with unramified in the above definition.

It follows from [9, Theorem IV.17.4.1] that a homomorphism $A \to B$ of local rings (with respective maximal ideal m_A and m_B) is locally unramified if and only if B is a localization of a finitely presented A-algebra and $\Omega_{B|A} = 0$ and this is equivalent to B being a localization of a finitely presented A-algebra, $m_A B = m_B$ and B/m_B being a finite separable extension of A/m_A. Further, the condition of being locally unramified localizes (as follows from this characterization or as commented before [9, Proposition IV.17.3.8]). The homomorphism $A \to B$ is locally étale if and only if $A \to B$ is locally flat and locally nonramified, by Theorem 17.6.1 [9] (or [25, Theorem 2, page 55]).

Proposition 5.6. *Assume that $(L|K, v)$ is an extension of valued fields. Then the following are equivalent:*

1) $\mathcal{O}_L|\mathcal{O}_K$ is locally unramified.
2) $\mathcal{O}_L|\mathcal{O}_K$ is locally étale.
3) $L|K$ is a finite inertial extension.

Proof. It is proven in

Assume that $\mathcal{O}_L|\mathcal{O}_K$ is locally étale. Then $\mathcal{O}_L|\mathcal{O}_K$ is locally unramified by Theorem IV.17.6.1 [9] . Thus 2) implies 1).

Assume that $\mathcal{O}_L|\mathcal{O}_K$ is unramified. Since the property of being unramified localizes, $(L|K,v)$ is unramified, so that $L|K$ is a finite separable field extension. Let M be a finite Galois extension of K which contains L. Identify v with an extensions to L. Let B be the integral closure of $A=\mathcal{O}_K$ in L and C be the integral closure of A in M. Then there exist maximal ideals b in B and c in C such that $c\cap B=b$, $B_b=\mathcal{O}_L$ and $C_c=\mathcal{O}_M$. Since $B_b|A$ is locally unramified, L is contained in the inertia field of $M|K$ by 2) of [25, Chapter X, Theorem 1, page 103]. Thus 1) implies 3).

Now assume that 3) holds. Then L is contained in the inertia field of a finite Galois extension M of K which contains L. Let $(A,a)\subset(B,b)\subset(C,c)$ be as in the proof of 1)⇒3) (where $a=\mathcal{M}_K$). By assumption, we have that L is the fixed field $L=M^H$ where H is a subgroup of the inertia group of c over a. We then have that $B=C^H$ so that $\mathcal{O}_L|\mathcal{O}_K$ is locally étale by 1) of [25, Chapter X, Theorem 1, page 103]. Thus 3) implies 2) □

Proposition 5.7. *Assume that $(L|K,v)$ is a finite Galois extension. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\cong\Omega_{\mathcal{O}_L|\mathcal{O}_{K^{in}}}$$

where K^{in} is the inertia field of $(L|K,v)$.

Proof. This follows from Proposition 5.5 and the exact sequence of [19, Theorem 25.1]. □

5.1. **Henselization.**

We now recall some facts about henselization of fields and rings. A valued field (K,v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, all of its algebraic extensions are unibranched (cf. [7, Section 16]).

An extension (K^h,v^h) of a valued field (K,v) is called a **henselization** of (K,v) if (K^h,v^h) is henselian and for all henselian valued fields (L,ω) and all embeddings $\lambda:(K,v)\to(L,\omega)$, there exists a unique embedding $\tilde{\lambda}:(K^h,v^h)\to(L,\omega)$ which extends λ.

A henselization (K^h,v^h) of (K,v) can be constructed by choosing an extension v^s of v to a separable closure K^{sep} of K and letting K^h be the fixed field of the decomposition group

$$G^d(K^{sep}|K)=\{\sigma\in G(K^{sep}|K)\mid v^s\circ\sigma=v^s\}$$

of v^s, and defining v^h to be the restriction of v^s to K^h ([7, Theorem 17.11]).

Assume that A is a local ring and $g(X)\in A[X]$ is a polynomial. Let $\bar{g}(X)\in A/m_A[X]$ be the polynomial obtained by reducing the coefficients of $g(X)$ modulo m_A.

A local ring A is a henselian local ring if it has the following property: Let $f(X)\in A[X]$ be a monic polynomial of degree n. If $\alpha(X)$ and $\alpha'(X)$ are relatively prime monic polynomials in $A/m_A[X]$ of degrees r and $n-r$ respectively such that $\bar{f}(X)=\alpha(X)\alpha'(X)$, then there exist monic polynomials $g(X)$ and $g'(X)$ in $A[X]$ of degrees r and $n-r$ respectively such that $\bar{g}(X)=\alpha(X)$, $\bar{g}'(X)=\alpha'(X)$ and $f(X)=g(X)g'(X)$.

If A is a local ring, a local ring A^h which dominates A is called a henselization of A if any local homomorphism from A to a henselian local ring can be uniquely extended to A^h. A henselization always exists ([20, Theorem 43.5]). The construction is particularly nice when A is a normal local ring. Let K be the quotient field of A and Let K^{sep} be a separable closure of A. Let $\bar{A}$ be the integral closure of A in K^{sep} and let $\bar{m}$ be a maximal ideal of $\bar{A}$.

Let H be the decomposition group

$$H=G^d(\bar{A}_{\bar{m}}|A)=\{\sigma\in G(K^{sep}|K)\mid \sigma(\bar{A}_{\bar{m}})=\bar{A}_{\bar{m}}\}.$$

Then A^h is the fixed ring of the action of H on $\bar{A}_{\bar{m}}$. We have

$$A^h=(\bar{A}\cap K^h)_{\bar{m}\cap(\bar{A}\cap K^H)}=\bar{A}_{\bar{m}}\cap K^h=(\tilde{A})_{\bar{m}\cap\tilde{A}}$$

where $\tilde{A}$ is the integral closure of A in K^h.

Lemma 5.8. *Assume that (K,v) is a valued field and (K^h,v^h) is a henselization of K. Then there is a natural isomorphism*

$$\mathcal{O}_{K^h}\cong\mathcal{O}_K^h.$$

Proof. Let v^s be an extension of v to K^{sep} and

$$H=\{\sigma\in\mathrm{Gal}(K^{sep}|K)\mid v^s\circ\sigma=v^s\},$$

so that K^h is the fixed field of H in K^{sep}. Let $\bar{V}$ be the integral closure of $\mathcal{O}_K$ in K^{sep}, and let $m=\bar{V}\cap\mathcal{M}_{K^{sep}}$, a maximal ideal in $\bar{V}$. Since K^{sep} is algebraic over K, we have that $\mathcal{O}_{K^{sep}}=\bar{V}_m$ by [34, Theorem 12, page 27]. Now, as is shown on the bottom of page 68 of [34], H is the decomposition group

$$H=G^d(\mathcal{O}_{K^{sep}}|\mathcal{O}_K)=\{\sigma\in G(K^{sep}|K)\mid\sigma(\mathcal{O}_{K^{sep}})=\mathcal{O}_{K^{sep}}\},$$

so that

$$\mathcal{O}_K^h=\bar{V}_m\cap K^h=\mathcal{O}_{K^{sep}}\cap K^h=\mathcal{O}_{K^h},$$

establishing the lemma.

□

Lemma 5.9. *Let K be a valued field and L be a field such that $K\subset L\subset K^h$. Then $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.*

Proof. The field K^{sep} is the directed union $K^{\mathrm{sep}}=\cup_i M_i$ of the finite Galois extensions M_i of K in K^{sep}. If M is a finite Galois extension of K in K^{sep}, then restriction induces a surjection of Galois groups $G(K^{\mathrm{sep}}|K)\rightarrow G(M|K)$, and an isomorphism $G(M|K)\cong G(K^{\mathrm{sep}}|K)/G(K^{\mathrm{sep}}|M)$. We have an isomorphism of profinite groups ([23, Example 1, page 271] or [18, Theorem VI.14.1, page 313])

$$G(K^{\mathrm{sep}}|K)\cong\lim_{\leftarrow}G(M_i|K).$$

Let $G^d(M|K)$ be the decomposition group of the valued field extension $M|K$, for M a Galois extension of K which is contained in K^{sep}. For M a finite Galois extension of K, restriction induces a homomorphism

$$G^d(K^{\text{sep}}|K) \to G^d(M|K). \tag{50}$$

Let $\sigma \in G^d(M|K)$. If N is a finite Galois extension of M contained in K^{sep}, then there exists $\bar{\sigma} \in G(N|K)$ such that $\bar{\sigma}|_M = \sigma$. Let A be the integral closure of $\mathcal{O}_M$ in N. There exists a maximal ideal p of A such that $A_p \cong \mathcal{O}_N$. Let $q = \bar{\sigma}(p)$, a maximal ideal of A. The group $G(N|M)$ acts transitively on the maximal ideals of A ([4, Lemma 21.8]) so there exists $\tau \in G(N|M)$ such that $\tau(q) = p$. Thus $\tau\bar{\sigma}(\mathcal{O}_N) = \mathcal{O}_N$ and $\tau\bar{\sigma}|_M = \sigma$ and so the homomorphism (50) is surjective with Kernel $G^d(K^{\text{sep}}|K) \cap G(K^{\text{sep}}|M)$. We have that

$$K^h = (K^{\text{sep}})^{G^d(K^{\text{sep}}|K)} = \cup M_i^{G^d(M_i|K)}.$$

Thus

$$L = L \cap (\cup_i M_i^{G^d(M_i|K)}) = \cup L_i$$

where $L_i = L \cap M_i^{G^d(M_i|K)}$. We have that $\Omega_{\mathcal{O}_{L_i}|\mathcal{O}_K} = 0$ for all i by Proposition 5.5. Thus

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \lim_{\rightarrow} \Omega_{\mathcal{O}_{L_i}|\mathcal{O}_K} = 0$$

by [6, Theorem 6.8]. □

Let K be a valued field. Fix an extension v^s of v to the separable closure K^{sep} of K. The field K^{sep} is henselian (for instance by the construction before Lemma 5.8); that is, the henselization $(K^{sep})^h = K^{sep}$ and $\mathcal{O}_{(K^{sep})^h} = \mathcal{O}_{K^{sep}}$.

Proposition 5.10. *Let (K, v) be a valued field. Then $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \cong \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}}$.*

Proof. We may embed K^h into K^{sep} (by the construction before Lemma 5.8) giving a tower of valued field extensions $K \subset K^h \subset K^{sep}$. By [19, Theorem 25.1], we have an exact sequence $\Omega_{\mathcal{O}_{K^h}|\mathcal{O}_K} \otimes_{\mathcal{O}_{K^h}} \mathcal{O}_{K^{sep}} \to \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \to \Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}} \to 0$. The proposition now follows from Lemma 5.9. □

Lemma 5.11. *Assume that $(L|K, v)$ is a finite separable extension of valued fields. Then*

$$\Omega_{\mathcal{O}_L^h|\mathcal{O}_K^h} \cong (\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{L^h}\ . \tag{51}$$

In particular, by Lemma 5.3, we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} = 0$.

Proof. We have that

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \cong \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_K} \tag{52}$$

by Lemma 5.9 and the exact sequence of [19, Theorem 25.1]. By [25, Theorem 1, page 87], there exist étale extensions $A_i|\mathcal{O}_L$ and maximal ideals m_i of A_i such that $\mathcal{O}_{L^h} = \lim_{\rightarrow}(A_i)_{m_i}$. We have the exact sequences

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} A_i \xrightarrow{\alpha} \Omega_{A_i|\mathcal{O}_K} \to \Omega_{A_i|\mathcal{O}_L} \to 0$$

of Theorem 25.1 [19]. Since $A_i|\mathcal{O}_L$ is étale, we have that this map is formally étale ([9, Definition IV.17.3.1]) and is thus formally unramified and formally smooth ([9, Definition IV.17.1.1]). Thus $\Omega_{A_i|\mathcal{O}_L} = 0$ by [9, Proposition IV.17.2.1] and α is injective by [9, Proposition IV.17.2.3]. Thus by this calculation and [6, Proposition 16.9],

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}(A_i)_{m_i}\cong(\Omega_{A_i|\mathcal{O}_K})\otimes_{A_i}(A_i)_{m_i}\cong\Omega_{(A_i)_{m_i}|\mathcal{O}_K}. \tag{53}$$

By Theorem 16.8 [6] and equations (52) and (53),

$$\begin{aligned}\Omega_{\mathcal{O}_L^h|\mathcal{O}_K^h} &\cong \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_K}\cong\lim_{\rightarrow}[(\Omega_{(A_i)_{m_i}|\mathcal{O}_K})\otimes_{(A_i)_{m_i}}\mathcal{O}_{L^h}]\\ &\cong \lim_{\rightarrow}[(\Omega_{\mathcal{O}_L|\mathcal{O}_K})\otimes_{\mathcal{O}_L}\mathcal{O}_{L^h}]\cong(\Omega_{\mathcal{O}_L|\mathcal{O}_K})\otimes_{\mathcal{O}_L}\mathcal{O}_{L^h}.\end{aligned} \tag{54}$$

□

5.2. **Proofs of Theorems 5.1 and 5.2.**

We first prove Theorem 5.1.

The natural sequence of $\mathcal{O}_M$-modules

$$0\rightarrow\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}\mathcal{O}_M\rightarrow\Omega_{\mathcal{O}_M|\mathcal{O}_K}\rightarrow\Omega_{\mathcal{O}_M|\mathcal{O}_L}\rightarrow 0 \tag{55}$$

computed from the extensions of rings $\mathcal{O}_K\subset\mathcal{O}_L\subset\mathcal{O}_M$ is right exact (but the first map might not be injective) by [19, Theorem 25.1]. Tensor this sequence with $\mathcal{O}_M^h$ over $\mathcal{O}_M$ to get a right exact sequence of $\mathcal{O}_M^h$-modules

$$0\rightarrow(\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}\mathcal{O}_M)\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\rightarrow\Omega_{\mathcal{O}_M|\mathcal{O}_K}\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\rightarrow\Omega_{\mathcal{O}_M|\mathcal{O}_L}\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\rightarrow 0. \tag{56}$$

By (51), we have isomorphisms

$$\Omega_{\mathcal{O}_M|\mathcal{O}_L}\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\cong\Omega_{\mathcal{O}_M^h|\mathcal{O}_L^h},\ \Omega_{\mathcal{O}_M|\mathcal{O}_K}\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\cong\Omega_{\mathcal{O}_M^h|\mathcal{O}_K^h}$$

and

$$\begin{aligned}&(\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}\mathcal{O}_M)\otimes_{\mathcal{O}_M}\mathcal{O}_M^h\cong\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}\mathcal{O}_M^h\\ &\cong(\Omega_{\mathcal{O}_L|\mathcal{O}_K}\otimes_{\mathcal{O}_L}\mathcal{O}_L^h)\otimes_{\mathcal{O}_L^h}\mathcal{O}_M^h\cong\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_L^h}\mathcal{O}_M^h.\end{aligned}$$

Thus (56) is the right exact sequence

$$0\rightarrow\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_{L^h}}\mathcal{O}_{M^h}\rightarrow\Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}}\rightarrow\Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{L^h}}\rightarrow 0 \tag{57}$$

of [19, Theorem 25.1]. Since $\mathcal{O}_M^h$ is a faithfully flat $\mathcal{O}_M$-module, we have that (55) is exact if and only if (57) is exact.

By assumption, $L|K$ and $M|L$ are towers of Galois extensions

$$K=K_0\subset K_1\subset\cdots\subset K_r=L \text{ and } L=L_0\subset L_1\subset\cdots\subset L_s=M$$

so

$$K^h=K_0^h\subset K_1^h\subset\cdots\subset K_r^h=L^h \text{ and } L^h=L_0^h\subset L_1^h\subset\cdots\subset L_s^h=M^h$$

are towers of Galois extensions. Since each $K_{i+1}^h|K_i^h$ is unibranched, there exist factorizations

$$K_i^h\subset U_i^1\subset U_i^2\subset\cdots\subset U_i^{m_i}=K_{i+1}^h$$

where U_i^1 is the inertia field of $K_{i+1}^h|K_i^h$ and $U_i^{j+1}|U_i^j$ is Galois of prime degree. These extensions are all necessarily unibranched, so $U_i^1|K_i^h$ satisfies 2) of Theorem 5.4 and $U_i^{j+1}|U_i^j$ satisfies 1) of Theorem 5.4 for $1\le j$. Similarly, we have factorizations

$$L_i^h\subset V_i^1\subset V_i^2\subset\cdots\subset V_i^{n_i}=L_{i+1}^h$$

where $V_i^1|L_i^h$ satisfies 2) of Theorem 5.4 and $V_i^{j+1}|V_i^j$ satisfies 1) of Theorem 5.4 for $1 \le j$. By Theorem 5.4, we have exact sequences

$$
\begin{array}{c}
0 \to \Omega_{\mathcal{O}_{U_0^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_0^1}} \mathcal{O}_{U_0^2} \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{U_0^1}} \to 0\\
0 \to \Omega_{\mathcal{O}_{U_0^2}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_0^2}} \mathcal{O}_{U_0^3} \to \Omega_{\mathcal{O}_{U_0^3}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_0^3}|\mathcal{O}_{U_0^2}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{K_1^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{K_1^h}} \mathcal{O}_{U_1^1} \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K_1^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{U_1^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{U_1^1}} \mathcal{O}_{U_1^2} \to \Omega_{\mathcal{O}_{U_1^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{U_1^2}|\mathcal{O}_{U_1^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{V_0^1} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{L^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_0^1}} \mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_{V_0^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{L_1^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L_1^h}} \mathcal{O}_{V_1^1} \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{L_1^h}} \to 0\\
0 \to \Omega_{\mathcal{O}_{V_1^1}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_1^1}} \mathcal{O}_{V_1^2} \to \Omega_{\mathcal{O}_{V_1^2}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{V_1^2}|\mathcal{O}_{V_1^1}} \to 0\\
\vdots\\
0 \to \Omega_{\mathcal{O}_{V_{s-1}^{n_s-1}}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{V_{s-1}^{n_s-1}}} \mathcal{O}_{M^h} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{V_{s-1}^{n_s-1}}} \to 0.
\end{array}
$$

In particular, differentiation defines an injection of $\mathcal{O}_{V_0^1}$-modules

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_{L^h}}\mathcal{O}_{V_0^1} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_K^h}.$$

Since $\mathcal{O}_{V_0^2}$ is a flat $\mathcal{O}_{V_0^1}$-module, we have injections

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_{L^h}}\mathcal{O}_{V_0^2} \cong (\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_{L^h}}\mathcal{O}_{V_0^1})\otimes_{\mathcal{O}_{V_0^1}}\mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^1}|\mathcal{O}_{K^h}}\otimes_{\mathcal{O}_{V_0^1}}\mathcal{O}_{V_0^2} \to \Omega_{\mathcal{O}_{V_0^2}|\mathcal{O}_K^h}$$

and continuing, we obtain that differentiation gives an injection of $\mathcal{O}_{M^h}$-modules

$$\Omega_{\mathcal{O}_{L^h}|\mathcal{O}_{K^h}} \otimes_{\mathcal{O}_{L^h}} \mathcal{O}_{M^h} \to \Omega_{\mathcal{O}_{M^h}|\mathcal{O}_{K^h}}$$

so that (57) is short exact and thus (55) is short exact.

Since $\mathcal{O}_M$ is a faithfully flat $\mathcal{O}_L$, module, we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_M = 0$ if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$, and so $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ if and only if $\Omega_{\mathcal{O}_M|\mathcal{O}_L} = 0$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

We now prove Theorem 5.2. We first prove Statement 1). By [6, Theorem 16.8], we have an isomorphism of $\mathcal{O}_{K^{sep}}$-modules

$$\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} \cong \lim_{\to}[(\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{K^{sep}}].$$

where the limit is over finite Galois subextensions $L|K$ of K^{sep}.

If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions of K^{sep}, then it follows immediately from the above formula that $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0$.

Assume that $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0$ and $L|K$ is a finite Galois subextension of K^{sep}. If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$, then there exists a finite Galois extension N of K such that N contains L and the image of $(\Omega_{\mathcal{O}_L|\mathcal{O}_K}) \otimes_{\mathcal{O}_L} \mathcal{O}_{K^{\mathrm{sep}}}$ in $(\Omega_{\mathcal{O}_N|\mathcal{O}_K}) \otimes_{\mathcal{O}_N} \mathcal{O}_{K^{\mathrm{sep}}}$ is zero. Since $\mathcal{O}_{K^{\mathrm{sep}}}$ is a faithfully flat $\mathcal{O}_N$-module (by Lemma 5.3) we have that the image of $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_N$ in $\Omega_{\mathcal{O}_N|\mathcal{O}_K}$ is zero, so that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \otimes_{\mathcal{O}_L} \mathcal{O}_N = 0$ by Theorem 5.1. Thus $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ since $\mathcal{O}_N$ is a faithfully flat $\mathcal{O}_L$-module.

We now prove Statement 2). We have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \Omega_{\mathcal{O}_L|\mathcal{O}_{K^{in}}}$ by Proposition 5.7. For $0 \le i \le \ell-1$, $\Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_0}} = 0$ if and only if $\Omega_{\mathcal{O}_{K_i}|\mathcal{O}_{K_0}} \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}} = 0$ since $\mathcal{O}_{K_{i+1}}$ is a faithfully flat $\mathcal{O}_{K_i}$-module by Lemma 5.3. Statement 2) now follows from Theorem 5.4 by induction on i in equation (41). □

6. Proof of Theorems 1.2 and 1.3

Take a valued field (K,v) and extend v to the separable closure K^{sep} of K. Recall that we call (K,v) a deeply ramified field if it satisfies (DRvg) and (DRvr).

Throughout we assume that $\operatorname{char} Kv = p > 0$. If $\operatorname{char} K = 0$, then we set $K' := K(\zeta_p)$ with ζ_p a primitive p-th root of unity and extend v to K'. If $\operatorname{char} K = p$, then we set $K' := K$. The next proposition will show that in our proof of Theorems 1.2 we can assume that $K = K'$.

Proposition 6.1. *1) If $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$, then $\Omega_{\mathcal{O}_L|\mathcal{O}_{K'}} = 0$ holds for every Galois extension $(L|K',v)$ of degree p.*
2) If (K',v) is a deeply ramified field, then so is (K,v).

Proof. 1): Assume that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$. By part 1) of Theorem 5.2 this implies that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for every finite Galois extension $(L|K,v)$. If $K = K'$, then there is nothing more to show; so we assume that $\operatorname{char} K = p > 0$ and $K \ne K'$. We note that $(K'|K,v)$ is a Galois extension. Every Galois extension $L|K'$ of degree p is a Kummer extension, and moreover, $L|K$ is also a Galois extension, so we have $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. We infer from Theorem 5.1 that $\Omega_{\mathcal{O}_L|\mathcal{O}_{K'}} = 0$. Therefore, every Galois extension $(L|K',v)$ of degree p satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_{K'}} = 0$.
2): This follows from [15, Theorem 1.8]. □

We split Theorem 1.2 into the following two theorems, which we will prove separately. In view of Proposition 6.1 it suffices to prove them under the assumption that K contains a primitive p-th root of unity if $\operatorname{char} K > 0$.

Theorem 6.2. *If $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$, then (K,v) is deeply ramified.*

Theorem 6.3. *If (K,v) is deeply ramified, then $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$.*

One of the implications of Theorem 1.3 will be proved in Proposition 6.5, and the other in Proposition 6.6.

6.1. **Proof of Theorem 6.2.**

We will need some preparations. If the valued field (K,v) is of characteristic 0 with residue characteristic $p > 0$, then we decompose $v = v_0 \circ v_p \circ \overline{v}$, where v_0 is the finest coarsening of v that has residue characteristic 0, v_p is a rank 1 valuation on Kv_0, and $\overline{v}$ is the valuation induced by v on the residue field of v_p (which is of characteristic $p > 0$). The valuations v_0 and $\overline{v}$ may be trivial. For simplicity, we will write v_0v_p for $v_0 \circ v_p$ and $v_p\overline{v}$ for $v_p \circ \overline{v}$.

In this decomposition, the valuation v_p is at the center, so we define $\operatorname{crf}(K,v) := (Kv_0)v_p$ as one may call it the "central residue field". Further, we denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp, that is, $(vK)_{vp} = \mathcal{C}_{vK}(vp)$. We note that $(vK)_{vp}$ is equal to the value group $v_p\overline{v}\,(Kv_0)$.

Now take any valued field (K,v) of residue characteristic $p > 0$.

We will use the following observation:

Proposition 6.4. *If $(vK)_{vp}$ is p-divisible, Kv is perfect and all Galois extensions $(L|K,v)$ of prime degree p with nontrivial defect satisfy $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$, then (K,v) satisfies (DRvr).*

Proof. We will show that the assumptions imply that $\mathrm{crf}\,(K,v)$ is perfect. Then the assertion follows from [15, Proposition 4.13] since by [5, Theorem 1.4], all Galois extensions $(L|K,v)$ of prime degree p with nontrivial defect that satisfy $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ have independent defect in the sense of [15, 5].

In the equal characteristic case, $\mathrm{crf}\,(K,v)=Kv$ and there is nothing to show. So we assume that (K,v) has mixed characteristic. Take any nonzero element of $\mathrm{crf}\,(K,v)$; it can be written as bv_0v_p with $b\in K$. Consider the extension $K(\eta)|K$ with $\eta^p=b$. We have that ηv_0v_p is a p-th root of bv_0v_p in $\mathrm{crf}\,(K(\eta),v)$.

Suppose that bv_0v_p does not have a p-th root in $\mathrm{crf}\,(K,v)$, so $K(\eta)|K$ is a Kummer extension of degree p. Then $(K(\eta)v_0v_p|Kv_0v_p$ is purely inseparable of degree p. It follows that $v_0v_pK(\eta)=v_0v_pK$ and that $(K(\eta)|K,v_0v_p)$ and $(K(\eta)v_0v_p|Kv_0v_p,\bar{v})$ are unibranched. Consequently, $(K(\eta)|K,v)$ is unibranched. Further, as $(vK)_{vp}$ and thus also $\overline{v}(Kv_0v_p)$ is p-divisible, we have $\overline{v}(K(\eta)v_0v_p)=\overline{v}(Kv_0v_p)$ and therefore, $vK(\eta)=vK$. Moreover, $K(\eta)v=K(\eta)v_0v_p\bar{v}$ is a purely inseparable extension of $Kv=Kv_0v_p\bar{v}$ and since Kv is perfect, we find that $K(\eta)v=Kv$. Thus $(K(\eta)|K,v)$ is an extension with nontrivial defect. Since (K,v) is an independent defect field, the defect must be independent. Hence by [5, part 2) of Theorem 1.4],

$$v(b-K^p)\;=\;\frac{p}{p-1}vp-\{\alpha\in pvK\mid \alpha>H\}$$

for some convex subgroup H of vK that does not contain vp, so also does not contain $\frac{p}{p-1}vp$. It follows that there is some $a\in K$ such that $v(b-a^p)>vp$, whence $(b-a^p)v_0v_p=0$. This shows that $(av_0v_p)^p=bv_0v_p$, so that bv_0v_p has a p-th root in $\mathrm{crf}\,(K,v)$, which contradicts our assumption.

We have now proved that $\mathrm{crf}\,(K,v)$ is perfect, as desired. □

Now we are ready to prove one part of Theorem 1.3:

Proposition 6.5. *If $K=K'$ and if $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ for all unibranched Galois extensions $(L|K,v)$ of prime degree p, then (K,v) is a deeply ramified field.*

Proof. By Proposition 6.4 it suffices to show that $(vK)_{vp}$ is p-divisible, and Kv is perfect.

We first deal with the equal characteristic case. In this case, $(vK)_{vp}=vK$. Suppose that vK is not p-divisible and take some $a\in K$ such that $va\notin pvK$. We may assume that $va<0$. Take $\vartheta\in K^{sep}$ such that $\vartheta^p-\vartheta=a$. Then $pv\vartheta=va$ and $(K(\vartheta)|K,v)$ is an Artin-Schreier extension with $\mathrm{e}\,(K(\vartheta)|K,v)=p$. Hence by Theorem 4.4, $\Omega_{\mathcal{O}_{K(\vartheta)}|\mathcal{O}_K}\neq 0$, contradiction.

Suppose that Kv is not perfect, and take $b\in\mathcal{O}_K^\times$ such that bv does not have a p-th root in Kv. Take $c\in K$ such that $0>vc\in vK$ and $\vartheta\in K^{sep}$ such that $\vartheta^p-\vartheta=c^pb$. Then $(K(\vartheta)|K,v)$ is an Artin-Schreier extension with $K(\vartheta)v=Kv(bv^{1/p})$. Hence by Theorem 4.3, $\Omega_{\mathcal{O}_{K(\vartheta)}|\mathcal{O}_K}\neq 0$, which again is a contradiction. We have shown that in the equal characteristic case, (K,v) is deeply ramified.

Now we deal with the mixed characteristic case. If we are able to show that (K,v) satisfies (DRvg), $(vK)_{vp}$ is p-divisible and Kv is perfect, then we can apply

Proposition 6.4 to conclude that (K,v) also satisfies (DRvr), showing that (K,v) is deeply ramified.

Suppose that there is an archimedean component of vK which is discrete. Pick $\alpha\in vK$ such that $\alpha<0$ and $\alpha+\mathcal{C}_{vK}^+(\alpha)$ is the largest negative element in $\mathcal{A}_{vK}(\alpha)$. Take $\eta\in K^{sep}$ such that $\eta^p\in K$ with $v\eta^p=\alpha$. Then $v\eta+\mathcal{C}_{vL}^+(v\eta)$ is the largest negative element in $\mathcal{A}_{vL}(v\eta)$, not contained in $\mathcal{C}_{vK}(v\eta^p)$, and $(K(\eta)|K,v)$ is a Kummer extension with $\mathrm{e}\,(K(\eta)|K,v)=p$. There is no $c\in K$ such that $vc\eta\in\mathcal{C}_{vL}^+(v\eta)$ since otherwise, $v\eta+\mathcal{C}_{vL}^+(v\eta)=-vc+\mathcal{C}_{vL}^+(v\eta)$ so that $v\eta\in\mathcal{C}_{vK}(v\eta^p)$. Case (i) of Theorem 4.6 applies, and we obtain from it that $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K}\neq 0$, contradiction.

Suppose that $(vK)_{vp}$ is not p-divisible and take some $a\in K$ such that $va\in(vK)_{vp}\setminus p(vK)_{vp}$. We may assume that $va<0$. Take $\eta\in K^{sep}$ such that $\eta^p=a$. Then $pv\eta=va$ and $(K(\eta)|K,v)$ is a Kummer extension with $\mathrm{e}\,(K(\eta)|K,v)=p$. Case (i) of Theorem 4.6 applies, and we have that $vI\cap\mathcal{C}_{vL}(vp)\neq\emptyset$. Hence $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K}\neq 0$, contradiction.

Suppose that Kv is not perfect, and take $b\in\mathcal{O}_K^\times$ such that bv does not have a p-th root in Kv. Take $\eta\in K^{sep}$ such that $\eta^p=b$. Then $(K(\eta)|K,v)$ is a Kummer extension with $K(\eta)v=Kv(bv^{1/p})$. Hence by Theorem 4.5, $\Omega_{\mathcal{O}_{K(\eta)}|\mathcal{O}_K}\neq 0$, which is again a contradiction. This finishes the proof that (K,v) is deeply ramified. □

Now Theorem 6.2 follows from Proposition 6.5 in conjunction with part 1) of Proposition 6.1.

6.2. **Proof of Theorem 6.3.**

We first observe:

Proposition 6.6. *Take a deeply ramified field (K,v). Then every unibranched extension $(L|K,v)$ which is an Artin-Schreier extension or a Kummer extension of prime degree satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$. In particular, if $K=K'$, then every unibranched Galois extension $(L|K,v)$ of prime degree satisfies $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.*

Proof. Take a deeply ramified field (K,v) with $\mathrm{char}\,Kv=p$ and an extension $(L|K,v)$ as in the assumption of the proposition. In view of Theorem 1.4, we only have to deal with the case of defectless extensions.

Assume that $\mathrm{char}\,K=p$ and $(L|K,v)$ is an Artin-Schreier extension of degree p. We have that vK is p-divisible and Kv is perfect by [15, Lemma 4.2]. Thus, the case of $\mathrm{e}\,(L|K)=p$ cannot appear and we must have that $\mathrm{f}\,(L|K)=p$ with the extension $Lv|Kv$ separable. Hence, $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ by Theorem 4.2.

Assume that $(L|K,v)$ is a Kummer extension of degree q with $\mathrm{f}\,(L|K)=q$. Again, $Lv|Kv$ is separable, so $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$ by Theorem 4.2.

Finally, assume that $(L|K,v)$ is a Kummer extension of degree q with $\mathrm{e}\,(L|K)=q$. We apply Theorem 4.6. If $q\neq p$, then $vq=0$ and case i) holds with $vI\cap\mathcal{C}_{vL}(vq)=\emptyset$. If $q=p$, then necessarily $\mathrm{char}\,K=0$. By [15, part (1) of Lemma 4.3], $(vK)_{vp}$ is p-divisible. If case ii) of Theorem 4.6 would hold, then using (8), $0<v(\eta-1)<v(1-\zeta_p)=\frac{vp}{p-1}$ with $v(\eta-1)\notin vK$, whence $v(\eta-1)\in(vL)_{vp}$ and $((vL)_{vp}:(vK)_{vp})=p$. As this contradicts the fact that $(vK)_{vp}$ is p-divisible, case ii) cannot appear and moreover, $vI\cap\mathcal{C}_{vL}(vp)=\emptyset$ since $\mathcal{C}_{vL}(vp)=(vL)_{vp}=(vK)_{vp}$.

As (K,v) satisfies (DRvg), no archimedean component of vK is discrete, hence we obtain from case i) of Theorem 4.6 that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}=0$.

This proves the first assertion of our proposition. The second follows from the first because if $K = K'$, then every Galois extension of prime degree is an Artin-Schreier or a Kummer extension. $\square$

Take any deeply ramified field (K,v). By [15, Corollary 1.7 (2)], also the henselization $(K,v)^h$ of (K,v) inside of (K^{sep},v) is a deeply ramified field. By Proposition 5.10 it suffices to prove that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_{K^h}} = 0$. We may therefore assume from the start that (K,v) is henselian.

Part 1) of Theorem 5.2 shows that in order to prove that $\Omega_{\mathcal{O}_{K^{sep}}|\mathcal{O}_K} = 0$ it suffices to prove that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all finite Galois subextensions $(L|K,v)$ of $(K^{sep}|K,v)$. Proposition 4.1 shows that after enlarging $(L|K,v)$ to a finite Galois extension $(M|K,v)$ if necessary, there is a tower of field extensions

$$K \subset M_0 \subset M_1 \subset \cdots \subset M_m = M$$

where M_0 is the inertia field of $(M|K,v)$ and each extension $M_{i+1}|M_i$ is a Kummer extension of prime degree, or an Artin-Schreier extension if the extension is of degree $p = \operatorname{char} K$. By part 2) of Theorem 5.2, to prove that $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$ it suffices to prove that $\Omega_{\mathcal{O}_{M_{i+1}}|\mathcal{O}_{M_i}} = 0$ for $0 \le i \le m-1$. By Theorem 1.5, (M_i,v) is a deeply ramified field for each i, hence $\Omega_{\mathcal{O}_{M_{i+1}}|\mathcal{O}_{M_i}} = 0$ by Proposition 6.6. We have shown that $\Omega_{\mathcal{O}_M|\mathcal{O}_K} = 0$.

Since $M|K$ is a Galois extension, so is $M|L$. Hence we can apply Theorem 5.1 to conclude that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$. This completes our proof of Theorem 1.2.

References

[1] Atiyah, M.F. – MacDonald, I.G.: *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969

[2] Blaszczok, A. – Kuhlmann, F.-V.: *On maximal immediate extensions of valued fields*, Mathematische Nachrichten **290** (2017), 7–18

[3] Coates, J.H. – Greenberg, R.: *Kummer theory for abelian varieties over local fields*, Invent. Math. **124** (1996), 129–174

[4] Cutkosky, S.D.: *Introduction to Algebraic Geometry*, Graduate Studies in Mathematics, 188. American Mathematical Society, Providence, R.I., 2018

[5] Cutkosky, S.D. – Kuhlmann, F.-V. – Rzepka, A.: *Characterizations of Galois extensions with independent defect* (2023), submitted; arXiv:2305.10022

[6] Eisenbud, D.: *Commutative Algebra with a view toward Algebraic Geometry*, Springer-Verlag, New York, 1995

[7] Endler, O.: *Valuation theory*, Springer-Verlag, Berlin, 1972

[8] Gabber, O. – Ramero, L.: *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003

[9] Grothendieck, A. – Dieudonné. J.: *Éléments de Géométrie Algébrique, IV Étude locale des schémas et des morphismes de schémas, quatriéme partie*, Pub. Math. IHES 32 (1967)

[10] Jahnke, F. – Kartas, K.: *Beyond the Fontaine-Wintenberger theorem*, arXiv:2304.05881

[11] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

[12] Kunz, E.: *Kähler differentials*, Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1986

[13] Kuhlmann, F.-V.: *A classification of Artin–Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[14] Kuhlmann, F.-V.: *Approximation of elements in henselizations*, manuscripta math. **136** (2011), 461–474

[15] Kuhlmann, F.-V. – Rzepka, A.: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738
[16] Kuhlmann, F.-V. – Rzepka, A.: *Galois extensions of prime degree of valued fields*, in preparation
[17] Kuhlmann, F.-V. – Vlahu, I.: *The relative approximation degree*, Math. Z. **276** (2014), 203–235
[18] Lang, S.: *Algebra*, revised 3rd. ed., Springer-Verlag, New York, 2002
[19] Matsumura, H.: *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge UK, 1989
[20] Nagata, M.: *Local Rings*, Interscience Publishers, New York, London, 1962
[21] Nart, E. – Novacoski, J.H.: *The defect formula*, Adv. Math. **428** (2023), Paper No. 109153, 44 pp
[22] Nart, E. – Novacoski, J.H.: *Minimal limit key polynomials* (2023), arXiv:2311.13558
[23] Neukirch, N.: *Algebraic Number Theory*, Berlin Springer Verlag, Heidelberg, 1999.
[24] Novacoski, J.H. – Spivakovsky, M.: *Kähler differentials, pure extensions and minimal key polynomials* (2023), submitted; arXiv:2311.14322
[25] Raynaud, M.: *Anneaux Locaux Henséliens*, Lecture Notes in Mathematics **169**, Springer-Verlag, Berlin Heidelberg New York, 1970
[26] Rotman, J.: *An Introduction to Homological Algebra*, Pure and Applied Mathematics, 85. Academic Press, Inc., New York-London, 1979
[27] Rotman, J.: *An introduction to homological algebra*, Second edition. Universitext. Springer, New York, 2009
[28] Scholze, P.: *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313
[29] Silva de Souza, C.H. – Novacoski, J.H. – Spivakovsky, M.: *Graded rings associated to valuations and direct limits*, Journal of Pure and Applied Algebra **227** (2023), no. 6, Paper No. 106965
[30] Tang, H.T.: *Gauss' lemma*, Proc. Amer. Math. Soc. **35** (1972), 372–376
[31] Thatte, V.: *Ramification theory for Artin-Schreier extensions of valuation rings*, J. Algebra **456** (2016), 355–389
[32] Thatte, V.: *Ramification theory for degree p extensions of valuation rings in mixed characteristic (0,p)*, J. Algebra **507** (2018), 225–248
[33] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. I, D. Van Nostrand, Princeton N.J., 1958
[34] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin, 1960

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
Email address: `cutkoskys@missouri.edu`

Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland
Email address: `fvk@usz.edu.pl`