On $\mathbb{R} - places$ and related topics

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Abstract

The frame of this survey is a formally real field K, which means that -1 is not a finite sum of squares of elements of K. It is well known from Artin-Schreier theory that such a field admits at least one total order compatible with the field structure.

After some background in Real Algebra, we introduce and study the space of $\mathbb{R}-places$. Thereafter, we present other mathematical notions, such as valuation fans, orderings of higher level and the real holomorphy ring. By use of these tools we obtain an outstanding result in Real Algebraic Geometry. Finally we provide some steps towards an abstract theory of $\mathbb{R} - places$.

1 Background in Real Algebra.

1.1 Preorderings, orderings.

In their Crelle paper (1927) Artin and Schreier introduced the notions of real fields and real-closed fields. These notions have since remained essentially unchanged. See for instance Moderne Algebra by Van der Waerden (1930), Lectures in Abstract Algebra by N. Jacobson (1964) and Algebra by S. Lang (1965). The notion of the positive cone associated to an order is due to J.P. Serre [S]. Basic references for classical theory of real fields are for instance [AS], [BCR], [R].

Definition 1 *A preordering T of K is a subset $T \subseteq K$, satisfying:*

$$T + T \subseteq T, \ T \cdot T \subseteq T, \ 0, 1 \in T, \ -1 \notin T$$
$$and \ T^* = T \backslash \{0\} \ is \ a \ subgroup \ of \ K^* = K \backslash \{0\}.$$

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Definition 2 *A preordering T is called a quadratic preordering if $K^2 \subseteq T$. If $K^{2n} \subseteq T$, T is said to be of level n. Preorderings with no level do exist.*

Zorn's lemma shows the existence of maximal quadratic preorderings; these are just the usual orderings, and are characterized by:

Definition 3 *A subset P of K is an ordering if:*

$$P + P \subseteq P, \ P \cdot P \subseteq P, \ P \cup -P = K, \ -1 \notin P.$$

From these properties one can deduce that $0, 1 \in P$, $P \cap -P = \{0\}$ and $\sum K^2 \subseteq P$. Here, and throughout the paper, $\sum K^{2n}$ denotes the set of all finite sums of $2n$-*th* powers.

Note also that a real field must have characteristic zero.

We can also call P a positive cone: to any such ordering P one can associate a binary relation $\leq_P$. This is a total order relation compatible with the field structure, defined as follows:

$$b - a \in P \Leftrightarrow a \leq_P b.$$

And P is the set of elements positive for the order relation $\leq_P$.

The set of orderings of a field K will be denoted by $\chi(K)$; it might also have been denoted by $Sper\,K$ to meet the usual notation in rings.

A very nice theorem from Artin-Schreier [AS] is:

Theorem 4 *Let K be a real field, $\sum K^2 = \underset{P_i \in \chi(K)}{\cap} P_i$.*

Example 5 *The field $\mathbb{R}$ admits only one ordering, and its set of positive elements is $\mathbb{R}^2$.*

Example 6 *The field $\mathbb{Q}(\sqrt[2]{2}) := \{a + b\sqrt[2]{2} \mid a, b \in \mathbb{Q}\}$ admits two orderings, one making $\sqrt[2]{2}$ positive and the other making $\sqrt[2]{2}$ negative.*

Example 7 *$\mathbb{R}((X))$, the power series field, admits also two orderings making X infinitesimal positive or negative.*

Example 8 *$\mathbb{R}(X)$ admits infinitely many orderings. For any $a \in \mathbb{R}$ one can define orderings $P_{a,+}$ and $P_{a,-}$ making $X - a$ infinitesimal positive or negative respectively.*

1.2   Real Valuations.

The main classic references on valuations are [K], [E], [R2]; see [EP] for a more modern treatment.

Definition 9 *A Krull valuation v on a field K is a surjective map*

$$v : K^* \twoheadrightarrow \Gamma$$

where Γ is a totally ordered abelian group (called the value group), such that
(1) $v(xy) = v(x) + v(y)$ for any x, y in K^*;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$, for any x, y in K^*, with $x + y$ in K^*.

The *valuation ring* of v is

$$A_v := \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

and its *maximal ideal* is

$$I_v := \{x \in K \mid x = 0 \text{ or } v(x) > 0\}.$$

$k_v := A_v/I_v$ is called the *residue field* of the valuation.
$U_v := A_v \setminus I_v$ denotes the *group of units*.

Definition 10 *A valuation v on a field K is said to be real if and only if the residue field k_v is formally real (meaning $-1 \notin \sum k_v^2$).*

A field admits real valuations if and only if it is formally real. Of course a formally real field admits real valuations, at least the trivial one.

The converse implication follows from the Baer-Krull theorem which ensures that if k_v admits an ordering, then K admits also at least one ordering.(see section 1.4).

We now recall the definition of a valuation ring and how one can associate a valuation to a given valuation ring.

Definition 11 *A subring A of a field K is a valuation ring if for any $x \in K$, either x or x^{-1} belongs to A.*

Definition 12 *The valuation associated to a valuation ring A of K, with maximal ideal I, is given by the canonical quotient map $v : K^* \to \Gamma$, where $\Gamma := K^*/(A \setminus I)$ is ordered by $v(x) \leq v(y) \Leftrightarrow yx^{-1} \in A$.*

Example 13 *Given an ordering P in a field K, the convex hull of $\mathbb{Q}$ in K is:*

$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \;\; r \pm x \in P\}.$$

$A(P)$ is a valuation ring in K with unique maximal ideal:

$$I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^* \;\; r \pm x \in P\}.$$

$A(P)$ is clearly a subring of K; it is a valuation ring because $b \notin A(P)$ implies $b^{-1} \in A(P)$: let $b \notin A(P)$, assume $b > 0$, since $b \notin A(P)$ we have in particular $1 < b$, therefore $0 < b^{-1} < 1$ which implies that $b^{-1} \in A(P)$ because $A(P)$ is convex in K with respect to P.

We shall see in 1.3 that the valuation associated to $A(P)$ is compatible with the ordering P and pushes down on the residue field an (archimedean) ordering, hence this valuation is real.

1.3   Compatibility of an ordering with a valuation.

For this part we can refer to [Be2] and [L]. Note that there is also a more recent book [EP].

Definition 14 *A quadratic preordering T in a field K is said to be fully compatible with a valuation v if and only if $1 + I_v \subset T$.*
In this case T induces on the residue field k_v a quadratic preordering $\overline{T}$.

In the case of an ordering P, we just say that P is *compatible* with v; then $\overline{P}$, induced by P on the residue field k_v, is an ordering of k_v.

Example 15 *The trivial valuation, sending every element of K to 0, is compatible with any ordering of K.*

Example 16 *The valuation v associated to an ordering P of K with valuation ring*
$$A(P) := \{x \in K \mid \exists r \in \mathbb{Q} \quad r \pm x \in P\}$$
is compatible with P.

Proof. $I(P) := \{x \in K \mid \forall r \in \mathbb{Q}^* \quad r \pm x \in P\}$ being the maximal ideal of $A(P)$ we have $1 + I(P) \subset P$. Hence the valuation is compatible with P. Then $\overline{P}$ induced by P on the residue field k_v is an archimedean ordering; in fact:

$\overline{P}$ is an ordering: clearly $\overline{P}$ is closed under addition and multiplication and $\overline{P} \cup -\overline{P} = k_v$. If -1 was in $\overline{P}$ we would have $-1 = \bar{a}$ for some $a \in P \cap A(P)$. Then $1 + a \in I(P)$, hence $-a \in 1 + I(P) \subset P$, so we would get $a = 0$ which is impossible.

This ordering $\overline{P}$ is archimedean: for any $x \in A(P)$ there exists some $r \in \mathbb{Z}$ such that $-r <_P x <_P r$, hence in the residue field we have $-r <_{\overline{P}} \bar{x} <_{\overline{P}} r$, and therefore $\overline{P}$ is an archimedean ordering of k_v.

Theorem 17 *Let P be an ordering of K, and v be a valuation on K; the following are equivalent:*
(1) $0 <_P a \leq_P b \Rightarrow v(a) \geq v(b)$ in Γ (the value group of v).
(2) The valuation ring A_v is convex in K with respect to P.
(3) The maximal ideal I_v of A_v is convex in K with respect to P.
(4) v is compatible with P (i.e. $1 + I_v \subset P$).

Proof. $(1) \Rightarrow (2)$ A_v convex in K means that if $x <_P y <_P z$, with $x, z \in A_v$ then $y \in A_v$, or equivalently $0 <_P a <_P b$ with $b \in A_v$ implies $a \in A_v$.

From (1) we deduce that $v(a) \geq v(b) \geq 0$ in Γ hence $a \in A_v$.

$(2) \Rightarrow (3)$ Assume $0 <_P a <_P b$ with $b \in I_v$ then $0 <_P b^{-1} <_P a^{-1}$. Since $b^{-1} \notin A_v$ using (2) we deduce $a^{-1} \notin A_v$, hence $a \in I_v$, I_v being the ideal of non invertible elements of A_v.

$(3) \Rightarrow (4)$ Let $m \in I_v$, if $1 + m \notin P$ then $1 + m \in -P$, so $1 + m <_P 0$ hence $0 <_P 1 <_P -m$. Using the convexity of I_v in K for P, since $-m \in I_v$ too, this yields $1 \in I_v$ which is impossible.

(4) $\Rightarrow$ (1) Assume $0 <_P a \leq_P b$ and $v(a) < v(b)$ in Γ; then we deduce $0 < v(b) - v(a) = v(\frac{b}{a})$, hence $\frac{b}{a} \in I_v$, and also $-\frac{b}{a} \in I_v$ and $a \neq b$. From (4) we get $1 + (-\frac{b}{a}) \in P$, so $\frac{a-b}{a} >_P 0$, hence $a >_P b$ which is impossible.

Theorem 18 *Let $\mathcal{F}$ be the family of all valuation rings of K compatible with a given ordering P, then:*
(1) the valuation rings in $\mathcal{F}$ form a chain under inclusion;
(2) the smallest element of $\mathcal{F}$ is $A(P)$.

Proof. (1) Suppose $A, B \in \mathcal{F}$ and $A \nsubseteq B$, let $a \in A \backslash B$ and $a > 0$. We prove that $B \subset A$. Consider $0 < b \in B$, by the convexity of B in K we cannot have $0 < a \leq b$, so we must have $0 < b \leq a$. From the convexity of A in K, we deduce $b \in A$.

(2) Let $A \in \mathcal{F}$, A is convex in K and contains $\mathbb{Z}$, hence A contains $A(P)$ the convex hull of $\mathbb{Q}$ in K.

Note that any subring of K containing a valuation ring must itself be a valuation ring, hence $\mathcal{F}$ consists of all subrings of K containing $A(P)$. Remark also that $A \subset A'$ implies $I' \subset I$.

Definition 19 *The place associated to the valuation ring A of K, with valuation v on K, is an application $\lambda : K \to k_v \cup \{\infty\}$, where $\lambda \mid_A$ is the canonical surjection from A to k_v, and is an homomorphism for addition and multiplication extended to $k_v \cup \{\infty\}$ by $x + \infty = \infty$ and $x \cdot \infty = \infty$ (with $x \neq 0$).*

In fact if $a \in A$ then $\lambda(a) = \overline{a} = a + I$; and if $a \notin A$ then $\lambda(a) = \infty$.

1.4 The Baer-Krull theorem.

Original references are [Ba] and [K]. One can also refer to [L] and [BCR], this last one being the basis for the proof given below. There exists also a more general version in [BeBr].

Theorem 20 *Let A be a real valuation ring of K, and let v be the associated valuation. Let $\overline{P}$ be an ordering in the residue field k_v. Denote $\chi_{v,\overline{P}}$ the set of all orderings P_i in K inducing the given $\overline{P}$ in k_v. Then there is a bijection between $\chi_{v,\overline{P}}$ and $Hom(\Gamma, \mathbb{Z}/2)$ where Γ denotes the value group of v.*

The proof requires the following lemma.

Lemma 21 *Let K be a field and v be a real valuation on K; let γ be the positive cone of an ordering of k_v. Then there exists at least one ordering on K, with positive cone P, compatible with v (or with λ_v the place associated with v) such that $\overline{P} = \gamma$.*

Proof of lemma.
Let A be the valuation ring associated to v.
Let $T := \left\{ x \in K \mid \exists y \in K \ \exists z \in A \backslash I \quad \lambda_v(z) >_\gamma 0 \text{ and } x = y^2 z \right\}$, we first show that T is a proper quadratic preordering of K.

It is clear that if $x_1, x_2 \in T$, then $x_1 x_2 \in T$, and if $x \in T$ then $x^2 \in T$.

Now suppose that $-1 \in T$, then $\exists y \in K \ \exists z \in A \backslash I$ such that $\lambda_v(z) >_\gamma 0$ and $-1 = y^2 z$. Hence $z = -y^{-2}$, but $\lambda_v(-y^{-2}) \leq_\gamma 0$, so we cannot have $\lambda_v(z) >_\gamma 0$. Hence we get $-1 \notin T$.

To show that T is closed under addition, let $x_1, x_2 \in T$, so $x_1 = y_1^2 z_1$ and $x_2 = y_2^2 z_2$ with $z_1, z_2 \in A \backslash I$, $\lambda_v(z_1) >_\gamma 0$ and $\lambda_v(z_2) >_\gamma 0$. Then write $x_1 + x_2 = y_1^2 z_1 + y_2^2 z_2 = y_1^2 z_1 (1 + z_1^{-1} z_2 y_1^{-2} y_2^2)$. Assume that $y_2 y_1^{-1} \in A$, otherwise $y_1 y_2^{-1}$ is in A. Let $z = 1 + z_1^{-1} z_2 y_1^{-2} y_2^2$, $\lambda_v(z) = 1 + \lambda_v(z_1^{-1} z_2 y_1^{-2} y_2^2) = 1 + (\lambda_v(z_1))^{-1} \lambda_v(z_2)(\lambda_v(y_1^{-1} y_2))^2$, hence $z \in A \backslash I$ with $\lambda_v(z) >_\gamma 0$ and $x_1 + x_2 = (z_1 z) y_1^2$ is in T.

Now there exists P an ordering containing the proper preordering T. A is convex in K for P because from $1 + I \subset T$ we deduce $1 + I \subset P$. Suppose $x \in I$, $v(x) > 0$, hence $v(1 + x) = 0$ so $1 + x \in A \backslash I$ and $\lambda_v(1 + x) = 1 >_\gamma 0$. We can write $1 + x = (1 + x)1^2$ and deduce $1 + x \in T$.

We also have $\overline{P} = \gamma$ since $P \supset T$ and $\overline{T} = \gamma$. Indeed $\overline{T} \subset \gamma$ is clear. Let $\overline{z} \in \gamma$ and z be such that $\lambda_v(z) >_\gamma 0$, then $z \in A \backslash I$ and writing $z = z1^2$ we get $z \in T$.

Proof of Baer-Krull theorem.

From the lemma we know that there exists an ordering P, with A convex in K for P and $\overline{P} = \gamma$. Let Q be any element of $\chi(K)$ such that Q is compatible with v (or with λ_v) and $\overline{Q} = \gamma$.

Define the following mapping

$$\begin{aligned} \chi(K) &\rightarrow Hom(\Gamma, \mathbb{Z}/2) \\ Q &\rightarrow <P, Q> \end{aligned}$$

Here $<P, Q>$ is defined by $<P, Q>(v(x)) = 0$ if x has same sign for P and for Q, and $<P, Q>(v(x)) = 1$ otherwise.

We show that $<P, Q>$ is a well defined group homorphism from Γ to $\mathbb{Z}/2$.

It is clear that $x \longmapsto <P, Q>(v(x))$ is a group homomorphism from $K^* \rightarrow \mathbb{Z}/2$ with kernel containing $A \backslash I$: if $x \in A \backslash I$ then $\lambda_v(x) >_\gamma 0$ or $\lambda_v(x) <_\gamma 0$, so for any Q such that $\overline{Q} = \gamma$ we have $x >_Q 0$ or $x <_Q 0$. Hence having same sign for P and Q we get $<P, Q>(v(x)) = 0$. Hence $<P, Q>$ is a well defined group homomorphism from Γ to $\mathbb{Z}/2$.

The mapping $Q \mapsto <P, Q>$ is injective because $<P, Q>$ and P entirely determine Q (the sign of x for Q follows from knowing the sign of x for P and $<P, Q>(v(x))$).

We now have to show that the mapping $Q \mapsto <P, Q>$ is surjective.

Let $\varphi \in Hom(\Gamma, \mathbb{Z}/2)$. Define :

$Q := \{x \in K \mid x = 0$ or $(\varphi(v(x)) = 0$ if $x \in P)$ or $(\varphi(v(x)) = 1$ if $x \in -P)\}$.

We must prove that Q is the positive cone of an ordering. It is obvious that $Q \neq K$, $Q \cdot Q \subset Q$, $K^2 \subset Q$, and $Q \cup -Q = K$.

It remains to prove that $Q + Q \subset Q$. Let $x, y \in Q \backslash \{0\}$, assume $x^{-1} y \in A$ (otherwise $xy^{-1} \in A$), we distinguish two cases.

Case 1. If $x^{-1} y \in I$, $v(x^{-1} y) > 0$, then $v(1 + x^{-1} y) = 0$, $1 + x^{-1} y \in A \backslash I$ and $1 + x^{-1} y \in P$ because $1 + I \subset P$. Hence $1 + x^{-1} y \in (A \backslash I) \cap P$ which implies

that $1 + x^{-1}y \in Q$ since $(A \backslash I) \cap P \subset Q$. Writing $x + y = x(1 + x^{-1}y)$ we get $x + y \in Q$ as a product of two elements of Q.

Case 2. If $x^{-1}y \notin I$ then $x^{-1}y \in A \backslash I$ and $v(x^{-1}y) = 0$ implying in turn $\varphi(v(x^{-1}y)) = 0$. Since $x^{-1}y \in Q$ we deduce from the definition of Q that $x^{-1}y \in P$. Thus $1 + x^{-1}y \in P$, but also $1 + x^{-1}y \in A$ and since $\lambda_v(1 + x^{-1}y) = \lambda_v(1) + \lambda_v(x^{-1}y)$ we get $\lambda_v(1 + x^{-1}y) > 0$ and $1 + x^{-1}y \notin I$. Finally $1 + x^{-1}y \in (A \backslash I) \cap P$, hence belongs to Q. Again writing $x + y = x(1 + x^{-1}y)$ we get $x + y \in Q$ as a product of two elements of Q.

Verify now that A is Q-convex : let $m \in I$, $v(m) > 0$ hence $v(1 + m) = 0$, $\lambda_v(1 + m) = 1 >_\gamma 0$, and $1 + m \in P$. $v(1 + m) = 0$ and $1 + m \in P$ imply in turn that $1 + m \in Q$.

Also $\overline{Q} = \gamma$ is obvious from $\overline{P} = \gamma$ and definition of Q.

As a consequence of the Baer-Krull theorem, if $\Gamma/2\Gamma$ has, as vector space over $\mathbb{Z}/2$, a basis of n classes, then $\chi_{v,\overline{P}}$ has 2^n elements P_i. Hence the lifting of $\overline{P}$ to K is unique if and only if Γ is $2-$divisible.

2   On $\mathbb{R}$-places.

2.1 $\mathbb{R}$-place associated to an ordering.

For a complete presentation of these notions one can refer to [L], or in a more geometrical setting to [Schü1], [Schü2] and [Schü3].

Let K be a real field and P be an ordering on K. Let v denote the valuation associated to the valuation ring $A(P)$. From previous results we know that $(k_v, \overline{P})$ can be uniquely embedded in $(\mathbb{R}, \mathbb{R}^2)$ since $\overline{P}$ is archimedean. Denote this embedding by i and let π be the canonical mapping from K into $k_v \cup \{\infty\}$ (where if $a \notin A(P)$, then $\pi(a) = \infty$).

Definition 22 *The $\mathbb{R}$-place associated to P is $\lambda_P : K \to \mathbb{R} \cup \{\infty\}$ defined by the following commutative diagram:*

$$
\begin{array}{ccc}
K & \xrightarrow{\lambda_P} & \mathbb{R} \cup \{\infty\} \\
\pi \searrow & & \nearrow i \\
& k_v \cup \{\infty\} &
\end{array}
$$

Explicitly $\lambda_P(a) = \infty$ when $a \notin A(P)$, and $\lambda_P(a) = \inf\{r \in \mathbb{Q} \mid a \leq_P r\} = \sup\{r \in \mathbb{Q} \mid r \leq_P a\}$ if $a \in A(P)$. In fact it is known that any $\mathbb{R}$-place arises in this way from some ordering P (see [L], 9.1).

2.2 The space of $\mathbb{R}$-places.

The space of $\mathbb{R}$-places of a field K is the set $M(K) = \{\lambda_P \mid P \in \chi(K)\}$, where $\chi(K)$ denotes the space of orderings of K. $M(K)$ is equipped with the coarsest topology making continuous the evaluation mappings defined for every $a \in K$ by:

$$e_a : M(K) \longrightarrow \mathbb{R} \cup \{\infty\}$$

$$\lambda_P \mapsto \lambda_P(a)$$

Recall that the usual topology on $\chi(K)$ is the Harrison topology generated by the open-closed Harrison sets:

$$\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}.$$

With this topology $\chi(K)$ is a compact totally disconnected space. Craven has shown in [C] that every compact totally disconnected space is homeomorphic to the space of orderings $\chi(K)$ of some field K.

Now consider the mapping Λ defined by:

$$\Lambda : \chi(K) \longrightarrow M(K)$$

$$P \mapsto \lambda_P$$

With the previous topologies on $\chi(K)$ and $M(K)$ the mapping Λ is a continuous, surjective and closed mapping.

$M(K)$ equipped with the above topology is a compact Hausdorff space. Remark that this topology on $M(K)$ is also the quotient topology inherited from the above topology on $\chi(K)$.

2.3 $\mathbb{R}$-places and the Real Holomorphy Ring.

We now provide some facts on the real holomorphy ring which has heavy links with orderings and $\mathbb{R}$-places.

Definition 23 *The real holomorphy ring, denoted $H(K)$, is the intersection of all real valuation rings of K.*

From the results in part 1 we obtain $H(K) = \underset{P \in \chi(K)}{\cap} A(P)$.

We also have:

$$H(K) = A(\sum K^2) = \{a \in K \mid \exists n \in \mathbb{N}, n \geq 1 \mid n \pm a \in \sum K^2\}.$$

$H(K)$ is a Prüfer ring with quotient field K (see [L], p.85). Recall that a Prüfer ring is a ring $R \subset K$ such that, for any prime ideal $\mathfrak{p}$ in R, the localization $R_{\mathfrak{p}}$ is a valuation ring in K.

In the sequel we denote the real spectrum of the real holomorphy ring of K by:

$$Sper(H(K)) = \{\alpha = (\mathfrak{p},\overline{\alpha}), \ \mathfrak{p} \in Sper(H(K)), \ \overline{\alpha} \ ordering \ of \ quot(H(K)/\mathfrak{p})\}.$$

Relations between $\chi(K)$, $M(K)$ and $H(K)$ are given by the next theorem.

Theorem 24 *(Becker-Gondard, [BG2]). The following diagram is commutative:*

$$
\begin{array}{ccc}
\chi(K) & \overset{sper\ i}{\Longrightarrow} & MinSperH(K) \\
\downarrow \Lambda & & \downarrow sp \\
M(K) \overset{res}{\Rightarrow} Hom(H(K),\mathbb{R}) & \overset{j}{\to} & MaxSperH(K)
\end{array}
$$

where the horizontal mappings are homeomorphisms, and the vertical ones continuous surjective mappings (see definitions below).

Hence $\chi(K)$ the space of orderings of K is homeomorphic to $MinSperH(K)$, and the space $M(K)$ of $\mathbb{R}$-places on K is homeomorphic to $MaxSperH(K)$.

The mappings in the above diagram are defined as follows:

$\Lambda : \chi(K) \longrightarrow M(K)$ is given by $P \mapsto \lambda_P$ (see 2.2).

$sper\ i : \chi(K) \longrightarrow MinSperH(K)$ is given by $P \mapsto P \cap H(K)$.

$sp : MinSperH(K) \longrightarrow MaxSperH(K)$ is given by $\alpha \longmapsto \alpha^{\max}$, where $\alpha^{\max}$ is the unique maximal specialization of α.

$res : M(K) \longrightarrow Hom(H(K),\mathbb{R})$ is given by $\lambda \mapsto \lambda_{|H(K)}$.

$j : Hom(H(K),\mathbb{R}) \longrightarrow MaxSperH(K)$ is given by $\varphi \mapsto \alpha_\varphi$, where $\alpha_\varphi = \varphi^{-1}(\ \mathbb{R}^2)$ or, using the notation for the real spectrum, $\alpha_\varphi = (\ker \varphi, \overline{\alpha})$ with $\overline{\alpha} = \mathbb{R}^2 \cap quot(\varphi(H(K))$.

All the spaces in the diagram are compact and the topologies of $M(K)$ and $MaxSperH(K)$ are the quotient topologies inherited through Λ and sp.

3 Fans (level 1 case).

In this section we mainly follow the notations and proofs of [L].

3.1 Quadratic preorderings.

The compatibility of a quadratic preordering with a valuation can be of two types. Given T a proper quadratic preordering in a real field K, v a valuation on K is *compatible* with T if it is compatible with *some* ordering P containing T. v is called *fully compatible* with T if it is compatible with *every* ordering P containing T. Below we give alternative characterizations.

Definition 25 *Given T a proper quadratic preordering in a real field K, and v a valuation on K with unique maximal ideal I_v in the associated valuation ring A_v:*

    (1) v is fully compatible with T if and only if $1 + I_v \subset T$.
    (2) v is compatible with T if and only if $(1 + I_v) \cap -T = \emptyset$.
    (3) v is compatible with T if and only if $\overline{T}$ is a preordering in the residue field k_v.

    We set $\chi_{/T} := \{P \text{ ordering} \mid P \supset T\}$.
    A way of building fully compatible preorderings is to use the "wedge product" introduced in 1978 by Becker in [Be1] and Becker, and Brocker in [BeBr].

Definition 26 *Let K be a real field, let A be a valuation ring in K, and $\pi : A \longrightarrow k_v$ be the projection map. Let T be a preordering of K and let S be a preordering of k_v such that $S \supset \overline{T}$. The wedge product is defined by $T \wedge S := T \cdot \pi^{-1}(S \backslash \{0\})$.*

    We refer the reader to Lam's book ([L], p.21) to verify that $T \wedge S$ is a preordering in K, fully compatible with v, and such that residually $\overline{T \wedge S} = S$.
    There is also an alternative definition for the wedge product:
    $T \wedge S = \cap \left\{ \text{orderings } P \mid P \supset T \text{ and } \overline{P} \in \chi_{/S} \right\}$

3.2 Fans of level 1.

In the context of preorderings fans were first presented by Becker and Köpping in [BK].

Definition 27 *Let K be a real field and let T be a proper quadratic preordering in K. T is a fan if and only if for any $S \supset T$, such that $-1 \notin S$ and such that $S^* = S \backslash \{0\}$ is a subgroup of K^* satisfying $[K^* : S^*] = 2$, S is an ordering in K.*

    Note that if T is a fan any preordering containing T is again a fan. There is an alternative useful characterization of a fan given in [L] (p.40), with proof of equivalence:

Proposition 28 *A preordering T is a fan if and only if for any $a \in K^* \backslash -T$ we have $T + aT \subset T \cup aT$. Such an element a is said T-rigid.*

First examples of fans are the trivial fans : these are orderings P and intersection of two orderings $P_1 \cap P_2$.

Further examples will be given later when dealing with orderings of higher level.

Another example is the pullback $\widehat{S}$ of a trivial fan S in k_v. Namely $\widehat{S} = K^2 \wedge S = K^2 \cdot \pi^{-1}(S \setminus \{0\})$ is a fan in K. In fact Bröcker's trivialization theorem given later in 3.3 says that all fans arise in this way.

Fans are well behaved for compatibility with real valuations.

Theorem 29 *Let K be a real field, v a valuation on K, and T a preordering in K. Then the followings hold:*
(a) If v is compatible with T, T is a fan implies that $\overline{T}$ is a fan in k_v;
(b) If v is fully compatible with T, T is a fan if and only if $\overline{T}$ is a fan.

Proof.

(a) We use proposition 28 charactrizing a fan. Let $b \in A \setminus I$ such that $\overline{b} \notin -\overline{T}$ we shall show that $\overline{b}$ is $\overline{T}$-*rigid*. T being a fan let $t_1 + t_2 b \in T + bT \subset T \cup bT$ hence there exist t_3 or t_4 such that $t_1 + t_2 b = t_3$ or $t_1 + t_2 b = t_4 b$. Going down to k_v we get $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_3}$ or $\overline{t_1} + \overline{t_2}\overline{b} = \overline{t_4}\overline{b}$ hence $\overline{t_1} + \overline{t_2}\overline{b} \in \overline{T} \cup \overline{b}\overline{T}$, and $\overline{T}$ is a fan.

(b) We use the definition of a fan. Assume v is fully compatible with T and $\overline{T}$ is a fan we have to prove that T is a fan. Let $W \supset T$ be such that $-1 \notin W$, $W^* = W \setminus \{0\}$ is a subgroup of K^* and $[K^* : W^*] = 2$, we have to prove that W is an ordering. We first show that $\overline{W}$ is an ordering. If $-1 = \overline{w}$ for some $w \in W \cap A$, then $-1 = w + m$ for some $m \in I$, so $-w = 1 + m \in 1 + I \subset T \subset W$ hence $-1 \in W$ which is impossible. Since $\overline{T}$ is a fan and $\overline{W^*}$ a subgroup of k_v^* such that $\left[k_v^* : \overline{W^*}\right] = 2$, $\overline{W}$ is an ordering. Form the wedge product $W \wedge \overline{W} = W \cdot \pi^{-1}(\overline{W} \setminus \{0\}) = W \cdot (1 + I) \subset W \cdot T \subset W$, since from [L] (p.22) $W \cdot \pi^{-1}(\overline{W} \setminus \{0\}) = W \cdot (1 + I)$); then $W \wedge \overline{W} \subset W$ holds, hence $W = W \wedge \overline{W}$ is an ordering.

3.3 Trivialization of fans.

A remarkable result is Bröcker's theorem on trivialization of fans ([Brö]).

Theorem 30 *Let K be a real field and $T \subset K$ be a fan. Then there exists a valuation v, fully compatible with T, such that the pushdown $\overline{T}$ in the residue field k_v is a trivial fan.*

The theorem follows from propositions 31 and 32 below. We use the proof given by Lam ([L], p. 94).

Proposition 31 *Let T be a non-trivial fan in the field K. Then there exists a non-trivial valuation v on K, fully compatible with T.*

The proof of proposition 31 requires three lemmas.

Lemma 1. *Let G be an ordered group (written additively), and H be a subgroup of G. If H does not contain a non-trivial convex subgroup of G, then for any positive element $h \in H$ there exists $g \in G \backslash H$ such that $0 < g < h$.*

Proof of lemma 1. Let $C := \{g \in G \mid \exists n \in \mathbb{N} \quad -nh \leq g \leq nh\}$. C is the convex hull of the subgroup of G generated by h, hence a convex subgroup. Assume there does not exist an element g as in the statement, then for any $g \in G$, $0 \leq g \leq h$ implies $g \in H$. By easy induction on n it follows that for any $n \in \mathbb{N}$, $-nh \leq g \leq nh$ implies $g \in H$. Hence $\{0\} \neq C \subseteq H$, contradicting the assumption that H does not contain a non-trivial convex subgroup of G.

Lemma 2. *Let T be a fan in the field K. Let v_1 be a valuation on K with value group Γ_1; if $v_1(T^*)$ does not contain a non-trivial convex subgroup of Γ_1, then v_1 is fully compatible with T.*

Proof of lemma 2. We claim that the condition:
"for every m in the unique maximal ideal $\mathcal{M}_1$, and for every $t \in U_1 \cap T$, a unit belonging to T, $t + m \in T$ implies that $1 + \mathcal{M}_1 \subset T$"
entails that v_1 is fully compatible with T.

We distinguish two cases:
Case 1. Assume $v_1(m) \notin v_1(T^*)$.
In this case $(T \cdot m) \cap U_1 = \emptyset$; so in particular $m \notin -T$, since $v_1(m) > 0$. Since T is a fan, $t + m \in T + T \cdot m = T \cup T \cdot m$. We have to show that $t + m \in T$. Clearly $t + m \in U_1$ because $v_1(t + m) = 0$ since $v_1(t) = 0$ and $v_1(m) > 0$. Since $(T \cdot m) \cap U_1 = \emptyset$ we get $t + m \notin T \cdot m$ hence $t + m \in T$.
Case 2. Assume $v_1(m) \in v_1(T^*)$.
Apply lemma 1 to $H := v_1(T^*)$. Since $v_1(m)$ is a positive element of H there exists x such that $v_1(x) \notin H$ and $0 < v_1(x) < v_1(m)$. Now let $t + m = t' + m'$ where $t' := t + x$ and $m' = m - x$. From $x \in \mathcal{M}_1$ we get $t' \in U_1$, and since $v_1(m') \notin v_1(T^*)$, case 1 gives $t' \in T$. Finally from $v_1(x) < v_1(m)$ we get $v_1(m') = v_1(m - x) = \min\{v_1(m), v_1(x)\} = v_1(x) \notin v_1(T^*)$. Thus using again case 1, we get $t' + m' \in T$, and hence $t + m \in T$.

Lemma 3. *Let $T \subset K$ be a non trivial fan and $P \in \chi_{/T}$. Let $v_P : K^* \longrightarrow \Gamma$ be the canonical valuation associated with P; then $v_P(T^*) \neq \Gamma$. In particular v_P is not the trivial valuation so every ordering in $\chi_{/T}$ is non archimedean.*

For the proof of this last lemma we refer to Lam [L], corollary 12-11 of lemma 12-10 p. 95.

Proof of proposition 31. Given a non trivial fan $T \subset K$, fix $v_0 : K^* \twoheadrightarrow \Gamma_0$ such that $v_0(T^*) \neq \Gamma_0$ (for instance, take $P \in \chi_{/T}$ and let v_0 be the valuation v_P associated with $A(P)$). Now consider the convex subgroups of Γ_0 contained in $v_0(T^*)$; they form a chain under inclusion. The union of them Δ is the largest convex subgroup contained in $v_0(T^*)$. By quotienting we can coarsen the valuation v_0 into a valuation $v_1 : K^* \twoheadrightarrow \Gamma_1 := \Gamma_0/\Delta$. Then $v_1(T^*)$ cannot

contain a non-trivial convex subgroup of Γ_1. Hence, by lemma 2, v_1 is fully compatible with T. Since $[\Gamma_1 : v_1(T^*)] = [\Gamma_0 : v_0(T^*)] > 1$, v_1 is a non trivial valuation.

Proposition 32 *For any preordering T in a field K, the followings are equivalent:*

(1) T is a fan in K.

(2) There exists a valuation v_1 on K, fully compatible with T, such that, with respect to v_1, T pushes down to a trivial fan in the residue field, hence $\left[\overline{K}^* : \overline{T}^*\right] \leq 4$.

Proof of proposition 32.

$(2) \Rightarrow (1)$ Trivially if v_1 exists, is fully compatible with T, and pushes down to a trivial fan $\overline{T}$, then T is a fan.

$(1) \Rightarrow (2)$ From the previous proposition we know that there exists a valuation v fully compatible with T, hence $\overline{T}$ is a fan in the residue field k_v.

If $\left[k_v^* : \overline{T}^*\right] \geq 8$, then $\overline{T}$ would be a non-trivial fan, and applying lemma 3 to $\overline{T}$ in k_v we would get a non-trivial valuation on k_v fully compatible with $\overline{T}$. But from proposition 12-3 in [L], k_v has no non-trivial valuation fully compatible with $\overline{T}$. Then just take $v_1 = v$.

For the geometric point of view on fans we refer to [AR] and [ABR].

4 Valuation fans and examples.

From now on preorderings are NOT supposed to be quadratic.

Let us recall the definition of a general preordering. A preordering T in a field K is a subset $T \subseteq K$, satifying:

$T + T \subseteq T$, $T \cdot T \subseteq T$, $0, 1 \in T$, $-1 \notin T$, $T^* = T \backslash \{0\}$ is a subgroup of K^*.

4.1 Valuation fans (of any level).

Definition 33 *(Jacob, [J1]). Let K be a field; a valuation fan in K is a preordering T such that there exists v a real valuation on K, v fully compatible with T (meaning $1 + I_v \subset T$), and v induces an archimedean ordering on the residue field k_v.*

More precisely, a preordering T in K is a valuation fan if and only if $A(T) = \{x \in K \mid \exists r \in \mathbb{Q} \; r \pm x \in T\}$ is a valuation ring with associated valuation v fully compatible with T, and $\overline{T}$ in k_v is an (archimedean) ordering.

There is an alternative characterization for valuation fans, sometimes useful in model theory:

Proposition 34 *(Jacob, [J2]). A preordering T in a field K is a valuation fan if and only if for any $x \notin \pm T$ we have either $1 \pm x \in T$ or $1 \pm x^{-1} \in T$.*

Example 35 *Usual orderings P are valuation fans (of level 1, i.e. $\sum K^2 \subset P$).*

It is I think important for real algebraic geometry to understand minimal valuation fans of level 1. They are defined as valuation fans not properly containing any valuation fan which is a quadratic preordering. Of course such a minimal valuation fan T_0 pushes down an archimedean ordering in the residue field of K for the valuation associated to the valuation ring given by: $A(T_0) = \{x \in K \mid \exists r \in \mathbb{Q} \; r \pm x \in T_0\}$.

But a better way to understand these minimal valuation fans, in relation with $\mathbb{R}$-places, is:

Example 36 *Let λ be a $\mathbb{R} - place$ on a field K, let $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$, then $T = \cap P_i$ is a valuation fan and it is a minimal valuation fan of level 1.*

4.2 Orderings of higher level.

Further examples of valuation fans are provided by Becker's orderings of higher level.

Definition 37 *(Becker, [Be1]). Let K be a commutative formally real field, $P \subset K$ is an ordering of level n if: $\sum K^{2n} \subset P$, $P + P \subset P$, $P.P \subset P$.*

Hence P^* is a subgroup of K^*. When $K^*/P^* \simeq \mathbb{Z}/2n\mathbb{Z}$, then the ordering is called of exact level n.

A very interesting paper on sums of d-th powers in rings with some relation to orderings of higher level is [Jo].

The orderings of level 1 are the usual total orderings.

Example 38 *If $K = \mathbb{R}((X))$, there exist two usual orderings:*

$$P_+ = K^2 \cup XK^2 \text{ and } P_- = K^2 \cup -XK^2$$

And for every prime p there exist two orderings of exact level p:

$$P_{p,+} = K^{2p} \cup X^p K^{2p} \text{ and } P_{p,-} = K^{2p} \cup -X^p K^{2p}.$$

All these orderings are associated to the unique $\mathbb{R}$-place of $\mathbb{R}((X))$, and for the associated valuation they all induce the same archimedean ordering in the residue field.

These higher level orderings have important links with sums of powers; we refer the reader to [Be4] and just mention the following important theorems:

Theorem 39 *(Becker, [Be1]). Let K be a real field, then:*
$\sum K^{2n} = \cap \{P_i \mid P_i \text{ ordering of level dividing } n\}$.

Theorem 40 *(Becker, [Be1]). Let K be a field, and let p be a prime. The followings are equivalent:*
(1) $\sum K^2 \neq \sum K^{2p}$.
(2) K admits an ordering of exact level p.

In the case where the level is a power of 2, Becker's results yield ([Be1]):

Theorem 41 *In a field K the followings are equivalent:*
(1) $\forall a \in K \quad a^2 \in \sum K^4$;
(2) Every real valuation on K has a 2-divisible value group.
(3) K does not admit any ordering of exact level 2.

On the side of $\mathbb{R}$-places we obtain as a corollary that $\lambda_P = \lambda_Q$ if and only if P and Q are two usual orderings beginning a 2-primary chain of higher level orderings. Such a chain has been defined by Harman in [H] (see later definition 45).

Hence the mapping $\Lambda : \chi(K) \longrightarrow M(K)$ is a bijection if and only if K does not admit any ordering of exact level 2.

4.3   Another approach with signatures.

Usual orderings can be recast in terms of signatures. A signature is a group morphism, $\sigma : K^* \longrightarrow \{\pm 1\}$, with additivily closed kernel; then $P = ker\sigma \cup \{0\}$ is an ordering of K.

The notion of a signature has a higher level analog:

Definition 42   *(Becker, [Be3]). A signature of level n on a field K is a morphism of abelian groups:*
$$\sigma : K^* \to \mu_{2n}$$
such that the kernel is additively closed, where μ_{2n} denotes the group of $2n$-th roots of 1.

Clearly if σ is a signature of level n, then $P = ker\sigma \cup \{0\}$ is an ordering of higher level with exact level dividing n.

There exists also a much more general notion of signature involving valuation fans:

Definition 43 *(Schwartz, [S2]). A generalized signature in a field K is a morphism of abelian groups, $\sigma : K^* \to G$, such that the kernel is a valuation fan.*

5   Algebraic closure of a field equipped with a valuation fan.

Several notions of a closure, under algebraic extensions, of a field equipped with either higher level orderings or higher level signatures, either valuation fans or generalized signatures, have been introduced and studied in the literature.

Since $\mathbb{R}$-places are closely related with some valuation fans of level 1, one can also consider a notion of closure under algebraic extensions of a field equipped with a $\mathbb{R}$-place λ, by considering the closure of a field equipped with the valuation fan $T = \cap \{P_i \mid P_i \in \Lambda^{-1}(\lambda)\}$ where $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$. Such closures might be important in Real Algebraic Geometry.

All these notions of closure can be unified in one theory, the theory of Henselian Residually Real-Closed fields (HRRC fields).

In this section we present without any proof the main features of this theory, from an algebraic point of view.

Definition 44 *(Becker, Berr, Gondard, [BBG]). A field K is henselian residually real-closed (HRRC) if and only if it admits an henselian valuation v with real-closed residue field k_v.*

Recall that a valuation v on a field K, with valuation ring A_v, is *henselian* if it satisfies Hensel's lemma : *"For any monic polynomial $f \in A_v[X]$, if $\overline{f}$ has a simple root $\beta \in k_v$, then f has a root $b \in A_v$ such that $\overline{b} = \beta$ "*.

The henselian residually real-closed fields have been variously named in the literature: they are called real henselian fields in Brown [Br], [Br2], fields real-closed with respect to a signature in Schwartz [S2] and almost real-closed fields in Delon-Farre [DF].

5.1 Examples of HRRC fields.

The basic examples of henselian residually real-closed fields arise in a classical way as follows (see [Fu]): given R a real-closed field, and Γ a totally ordered abelian group, let $R((\Gamma)) = \{\sum_{\gamma} a_\gamma t^\gamma \mid \gamma \in \Gamma, a_\gamma \in R\}$ be the set of generalized power series with support well ordered, where $support \sum_{\gamma} a_\gamma t^\gamma = \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$.

In $R((\Gamma))$ one can define:

- <u>Multiplication</u> by: $t^\gamma t^\delta = t^{\gamma+\delta}$;

- <u>Addition</u> by: $\sum_{\gamma} a_\gamma t^\gamma + \sum_{\delta} b_\delta t^\delta = \sum_{\alpha} (a_\alpha + b_\alpha) t^\alpha$;
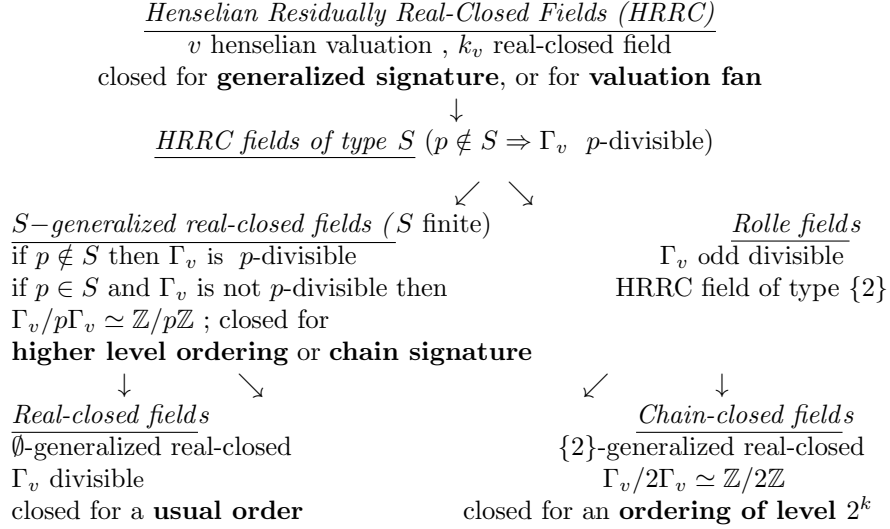
- <u>Order</u> by: $\sum_{\gamma} a_\gamma t^\gamma >_K 0 \Leftrightarrow a_m >_R 0$, where $m = \min(support \sum_{\gamma} a_\gamma t^\gamma)$;

- <u>Valuation</u> by: $v : R((\Gamma)) \to \Gamma$ and $v(\sum_{\gamma} a_\gamma t^\gamma) = m = \min(support \sum_{\gamma} a_\gamma t^\gamma)$.

It is well-known that $R((\Gamma))$ is a field, admitting v as a henselian valuation with real-closed residue field R and value group Γ; hence $R((\Gamma))$ is an HRRC field.

5.2 Subtheories of the theory of HRRC fields.

Let v be a real valuation on a field K, k_v its residue field, Γ_v its value group, and let S be a set of primes. Relations between various subtheories of the theory of HRRC fields are described by the following diagram where arrows indicate subtheories.

$$\underline{\textit{Henselian Residually Real-Closed Fields (HRRC)}}$$
$$v \text{ henselian valuation }, \ k_v \text{ real-closed field}$$
closed for **generalized signature**, or for **valuation fan**

$$\downarrow$$

$$\underline{\textit{HRRC fields of type S}} \ (p \notin S \Rightarrow \Gamma_v \ \ p\text{-divisible})$$

$$\swarrow \quad \searrow$$

$\underline{\textit{S$-$generalized real-closed fields (S} } \text{ finite)} \qquad\qquad \underline{\textit{Rolle fields}}$
if $p \notin S$ then Γ_v is p-divisible $\qquad\qquad\qquad \Gamma_v$ odd divisible
if $p \in S$ and Γ_v is not p-divisible then $\qquad$ HRRC field of type $\{2\}$
$\Gamma_v/p\Gamma_v \simeq \mathbb{Z}/p\mathbb{Z}$; closed for
higher level ordering or **chain signature**

$$\downarrow \qquad \searrow \qquad\qquad\qquad \swarrow \qquad \downarrow$$

$\underline{\textit{Real-closed fields}} \qquad\qquad\qquad\qquad \underline{\textit{Chain-closed fields}}$
$\emptyset$-generalized real-closed $\qquad\qquad\qquad \{2\}$-generalized real-closed
Γ_v divisible $\qquad\qquad\qquad\qquad\qquad \Gamma_v/2\Gamma_v \simeq \mathbb{Z}/2\mathbb{Z}$
closed for a **usual order** $\qquad\qquad$ closed for an **ordering of level** 2^k

In the diagram above, most of the theories correspond to some notion of closure, under algebraic extensions, of a field equipped with some object. With an *ordering* (real-closed field), with an *ordering of exact level a power of* 2 (chain-closed field), with an *ordering of exact level a power of p where p is prime* ($\{p\}$-real-closed fields), with an *ordering of exact level n* (S-generalized real-closed fields of exact type S ($p \in S \Leftrightarrow \Gamma_v$ not divisible, and for all $p \in S$, $p \mid n$), or with a *valuation fan* (henselian residually real-closed field).

5.3 On the question of the uniqueness of closure.

For a field equipped with a usual ordering it is well known that the real closure is unique up to K-isomorphism.

Even for chain-closed fields this is not true anymore. In order to recover the uniqueness of the closure, up to K-isomorphism, one needs to consider a closure for a whole chain of orderings with levels powers of 2 in the sense of Harman:

Definition 45 *(Harman, [H]). A 2-primary chain of orderings in a field K is:*

$$(P_n)_{n\in\mathbb{N}} = (P_0, P_1, ..., P_n, ...)$$

P_0 being a usual ordering and P_n an ordering of level 2^{n-1}, such that

$$P_n \cup -P_n = (P_0 \cap P_{n-1}) \cup -(P_0 \cap P_{n-1}).$$

Theorem 46 *(Harman, [H]). A field K equipped with a 2-primary chain of orderings admits a closure under algebraic extensions unique up to K-isomorphism. The closure is called a chain-closed field and it is equal to the intersection of two real-closures of K for P_0 and P_1.*

For generalized real-closed fields, in order to recover the uniqueness up to K-isomorphism, Niels Schwartz has introduced the notion of chain signature.

Definition 47 *(Schwartz, [S1]). A chain signature on a field K is a homomorphism:*
$$\varphi : K^* \to \{1, -1\} \times \widehat{\mathbb{Z}}$$
such that $\ker\varphi$ is a valuation fan, where $\widehat{\mathbb{Z}} = \prod \widehat{\mathbb{Z}}_p$ and $\widehat{\mathbb{Z}}_p$ denotes the additive group of p-adic integers.

One can recover orderings of higher level by taking:
$P_n(\varphi) = \varphi^{-1}(1 \times n\widehat{\mathbb{Z}}) \cup \{0\}$.

Theorem 48 *(Schwartz, [S1]). A field K equipped with a chain signature φ admits a closure under algebraic extensions unique up to K-isomorphism. This closure is a HHRC field.*

In the more general situation of a field equipped with a valuation fan we can also ensure the uniqueness of the closure by considering a field equipped, not only with a single valuation fan, but with a whole chain of valuation fans.
From Brown's work we can derive the following:

Theorem 49 *(Brown, [Br]). Let R and R' be two HRRC fields, algebraic extensions of a field K, then the followings are equivalent:*
(1) R and R' are K-isomophic.
(2) $R^{2n} \cap K = R'^{2n} \cap K$ for all $n \in \mathbb{N}$.

In fact these $T_n = R^{2n} \cap K$ are valuation fans, which form a chain of valuation fans $(T_n)_{n \in \mathbb{N}}$ as defined below; this chain is said to be induced on K by R.

Definition 50 *(Becker-Berr-Gondard, [BBG]). A chain of valuation fans in a field K is defined as $(T_n)_{n \in \mathbb{N}}$ such that:*
(1) $K^{2n} \subset T_n$;
(2) $T_{n.m} \subset T_n$;
(3) $(T_n)^m \subset T_{n.m}$;
(4) $T_n^*/T_{n.m}^* \subset T_1^*/T_{n.m}^*$ is the subgroup of elements of exponent m.

With this notion we have been able to obtain the following theorem:

Theorem 51 *(Becker-Berr-Gondard, [BBG]). Any field K, equipped with a chain of valuation fans $(T_n)_{n \in \mathbb{N}}$, admits a closure under algebraic extensions R, unique up to K-isomorphism. Then R is a HRRC field, and R induces on K a chain of valuation fans $(T_n)_{n \in \mathbb{N}}$ (i.e. $T_n = R^{2n} \cap K$ for all n).*

5.4 Properties of HHRC fields.

Henselian residually real-closed fields have a lot of nice properties ; we list, again without any proof, some of them below. Main reference is [BBG].

Let K be an HRRC field then:

(1) K is a real field;

(2) Every algebraic extension of K is a radical extension;

(3) K has no real extension of degree $p \in \mathbb{P} \backslash S$.
Note that whenever $2 \in S$, one can replace (3) by (3') "K has no extension of degree $p \in \mathbb{P} \backslash S$";

(4) $\forall n \in \mathbb{N}$, K is n-pythagorean : $K^{2n} + K^{2n} = K^{2n}$;

(5) K is *hereditarily pythagorean*, i.e., every algebraic extension is again a pythagorean field;

(6) $\forall n \in \mathbb{N}$, K^{2n} is a *fan* (refer to definion 27, or to characteriztion 28 for such preorderings);

(7) $\forall n \in \mathbb{N}$, K^{2n} is a *valuation fan*, i.e. it is a preordering such that:
$\forall x \notin \pm K^{2n}$ either $1 \pm x \in K^{2n}$ or $1 \pm x^{-1} \in K^{2n}$;

(8) All real valuations on K are henselian;

(9) The set of real valuation rings in K is totally ordered by inclusion;

(10) The *smallest* real valuation ring in K is:

$$A(K^2) = A(K^{2n}) = H(K)$$

where $A(T) = \{x \in K \mid \exists n \in \mathbb{N} \ \ n \pm x \in T\}$, T being a valuation fan, and where $H(K)$ is the real holomorphy ring (i.e. the intersection of all real valuation rings);

(11) K admits a unique $\mathbb{R}$-place which can be defined using the valuation ring $A(K^2)$ and the associated valuation (see definition 22);

(12) Jacob's ring $J(\underset{n \in \mathbb{N}}{\cap} K^{2n})$ is the *biggest* valuation ring with real-closed residue field. This ring is defined as follows. If T is a valuation fan, the ring $J(T)$ is equal to $J_1(T) \cup J_2(T)$ where:

$$\begin{aligned} J_1(T) &= \{x \in K \mid x \notin \pm T \text{ et } 1 + x \in T\} \\ &\text{and} \\ J_2(T) &= \{x \in K \mid x \in \pm T \text{ et } xJ_1(T) \subset J_1(T)\}). \end{aligned}$$

These fields have been studied from a model theoretic point of view; the preceding theories are all elementary theories, with nice first order axiomatizations (see [BBG], [De1], [De2], [Di], [G1] and [G3]).

The significance of Jacob's ring for the model theory of these fields appears in [J2], and also later with the transfer theorem obtained by Delon and Farre [DF].

6 $\mathbb{R}$-places in Real Algebraic Geometry.

6.1 Separation of connected components in $M(K)$.

Higher level orderings provide a tool to separate connected components in the space of $\mathbb{R}$-places $M(K)$.

Theorem 52 *(Becker-Gondard, [BG2]). Let K be a real field. Two $\mathbb{R}$-places λ_P and λ_Q, associated to usual orderings P and Q, are in two distinct connected components of $M(K)$ if and only if:*

$$\exists b \in K^* \ (b \in P \cap -Q \ and \ \beta^2 \in \sum K^4).$$

Proof. This criterion is obtained using higher level orderings, more precisely orderings of exact level 2.

Recall $\mathcal{H}(a) = \{P \in \chi(K) \mid a \in P\}$, $\chi(K) = \mathcal{H}(a) \cup \mathcal{H}(-a)$ and for $a \neq 0$ $\mathcal{H}(a) \cap \mathcal{H}(-a) = \varnothing$, but $\Lambda(\mathcal{H}(a)) \cap \Lambda(\mathcal{H}(-a))$ may be non empty.

Nevetherless, if there exist $b \notin \sum K^2$ with $b^2 \in \sum K^4$, then there does not exist $P \in \mathcal{H}(b)$ and $Q \in \mathcal{H}(-b)$ such that $\lambda_P = \lambda_Q$.

Otherwise $b \notin (P \cap Q) \cup -(P \cap Q)$ and $\lambda_P = \lambda_Q$ imply, as said before, that there exists an ordering of level 2, P_2, such that:

$$P_2 \cup -P_2 = (P \cap Q) \cup -(P \cap Q)$$

with $b \notin P_2 \cup -P_2$, hence $b^2 \notin P_2$, so $b^2 \notin \sum K^4 = \cap P_{2,i}$, where the $P_{2,i}$ run over the set of all orderings with level dividing 2.

Assume that λ_P et λ_Q (with $P \neq Q$) are in the same connected component C of $M(K)$, and that there exists $b \in K^*$such that $b \in P \cap -Q$ with $b^2 \in \sum K^4$. Λ being closed $C \cap \Lambda(\mathcal{H}(b))$, and $C \cap \Lambda(\mathcal{H}(-b))$ form a partition of C into two non empty closed sets, impossible.

Conversely:

If λ_P et λ_Q are in C and C', two distinct connected components of $M(K)$, $M(K)$ being a compact Hausdorff space there exists an open-closed set U such that $U \supset C$ and $U^c = (M(K) \backslash U) \supset C'$.

Let $X = \Lambda^{-1}(U)$ and $Y = \Lambda^{-1}(U^c)$. X and Y form a partition of $\chi(K)$. Λ being surjective we get : $\Lambda^{-1}(\Lambda(\Lambda^{-1}(U))) = \Lambda^{-1}(U)$ so $\Lambda^{-1}(\Lambda(X)) = X$, and similarly $\Lambda^{-1}(\Lambda(Y)) = Y$.

The following lemma from Harman ensures then the existence of b such that $X = \mathcal{H}(b)$ and $Y = \mathcal{H}(-b)$ with $b^2 \in \sum K^4$, hence we have $b \in P \cap -Q$ with $b^2 \in \sum K^4$.

Harman's Lemma ([H]). If $\chi(K) = \chi_1 \cup \chi_2$, where χ_1 and χ_2 are disjoint open-closed sets such that $\Lambda^{-1}(\Lambda(\chi_1)) = \chi_1$ and $\Lambda^{-1}(\Lambda(\chi_2)) = \chi_2$, then there exist a such that $\chi_1 = \mathcal{H}(a)$ and $\chi_2 = \mathcal{H}(-a)$.

6.2 Number of connected components of a smooth real projective variety.

Using $\mathbb{R}$-places, the real holomorphy ring and a result of Bröcker on the fibers of central points, we have been able to obtain the following theorem:

Theorem 53 *(Becker-Gondard, [BG2]). Let Y be a smooth projective variety on $\mathbb{R}$, with function field $K = \mathbb{R}(Y)$. Suppose $Y(\mathbb{R}) \neq \emptyset$, then $\mid \pi_0(Y(\mathbb{R})) \mid$, the number of connected components of $Y(\mathbb{R})$, is given by:*

$$\mid \pi_0(Y(\mathbb{R})) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2].$$

This result is in the spirit of Harnack's result giving as upper bound, $g + 1$, for the number of connected components of a smooth projective curve $V(\mathbb{R})$ where g is the genius of V; but here we obtain a formula with equality and for any dimension.

The theorem also shows clearly the known fact that the number of connected components is a birational invariant among the smooth varieties with given function field.

The first proof (1992) of this result is given in [BG2]. The theorem is a corollary of the two lemmas below which make use of the connected components of the space of $\mathbb{R}$-places $M(K)$.

Two new proofs have been found in 2003-2004 by Jean-Louis Colliot-Thélène (see [CT]), who derived the result from one of his previous papers, and by Claus Scheiderer (see [Sche]), who used geometric arguments avoiding $\mathbb{R}$-places, higher level orderings and real holomorphy ring.

Lemma 54 *Let Y be a smooth projective variety on $\mathbb{R}$, with function field $K = \mathbb{R}(Y)$. Suppose $Y(\mathbb{R}) \neq \emptyset$, and let $\mid \pi_0(Y(\mathbb{R})) \mid$ denote the number of connected components of $Y(\mathbb{R})$. Then holds:*

$$\mid \pi_0(Y(\mathbb{R})) \mid = \mid \pi_0(M(K) \mid .$$

In lemma 54 and below, $\mid \pi_0(M(K) \mid$ denotes the number of connected components of $M(K)$ equipped with the topology defined in 2.2.

Lemma 55 *For any real field K:*

$$\mid \pi_0(M(K)) \mid = 1 + \log_2[(K^{*2} \cap \sum K^4) : (\sum K^{*2})^2].$$

Sketch of proof of the lemma 54:

We use the center map $c : M(K) \to Y(\mathbb{R})$, defined by $x = c(\lambda) = c(V_\lambda)$ the unique point (since Y is projective) whose local ring $\mathfrak{o}_x$ is dominated by V_λ, the valuation ring associated to the $\mathbb{R}$-place λ.

- In this case it is known [e.g. [BCR], Prop. 7.6.2 (ii), p. 133] that c is surjective, the central points being the closure of regular points. And one can prove that c is continuous.

- Bröcker proved, in an unpublished manuscript, that the fiber of a central point has a finite number of connected components, and that if x is a regular point then the fiber is connected.

Now we just have to use the following topological lemma: *"If a mapping between two compact spaces X and Y is continuous and surjective, and if each fiber is connected, then it induces a bijection between $\pi_0(X)$ and $\pi_0(Y)$, the sets of connected components of X and Y"*.

Sketch of proof of lemma 55:

It is known from [B2] that $| \pi_0(M(K)) | = \log_2[E : E^+]$, where E is the group of units of the real holomorphy ring $H(K)$, and $E^+ = E \cap \sum K^2$.

Then we prove that the quotient group $(K^{*2} \cap \sum K^4)/(\sum K^{*2})^2$ is isomorphic to $E/(E^+ \cup -E^+)$.

7 Towards abstract $\mathbb{R}$-places.

The space of orderings of a field, studied in relation with quadratic forms and real valuations, have been the origin of the theory of abstract spaces of orderings (1979-80) and of Marshall's problem:

"Is every abstract space of orderings the space of orderings of some field ?"

In [M] it is proved that one can always associate to an abstract space of orderings a "*P*-structure" (partition of the space of orderings into subspaces which are fans, and such that any fan intersects only one or two classes). Such a *P*-structure is a candidate to be analogous to the space of $\mathbb{R}$-places in the field case. But it appeared that not any *P*-structure is a Hausdorff space, hence we have to improve this notion to fit with the space of $\mathbb{R}$-places in the field case.

7.1 Abstract spaces of orderings (level 1 case).

Abstract space of orderings have been introduced using signatures by Marshall in [M]:

Definition 56 *An abstract space of orderings is (X, G), where G is a group of exponent 2 (hence abelian), -1 a distinguished element of G, and X a subset of $Hom(G, \{1, -1\})$ such that:*

(1) X is a closed subset of $Hom(G, \{1, -1\})$;

(2) $\forall \sigma \in X \quad \sigma(-1) = -1$;

(3) $\bigcap_{\sigma \in X} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);

(4) For any f and g quadratic forms over G:

$$D_X(f \oplus g) = \cup\{D_X \langle x, y \rangle \mid x \in D_X(f),\ y \in D_X(g)\}.$$

In the above definition $D_X(f)$ denotes the set $\{a \in G$ represented by $f\}$, i.e. there exists g such that $f \equiv_X \langle a \rangle \oplus g$ where $f \equiv_X h$ if and only if f and h have same dimension, and have for any $\sigma \in X$ same signature.

On the side of fans, seen as sets of signatures on a field, a four elements fan of level 1 is characterized by:

$$\sigma_0 \sigma_1 \sigma_2 \sigma_3 = 1$$

and it corresponds to the fan seen as a preordering: $T = \overset{3}{\underset{i=0}{\cap}} \ker \sigma_i \cup \{0\}$.

In the abstract situation a abstract fans have been defined by Marshall.

Definition 57 *An abstract fan is an abstract space of orderings (X, G) such that $X = \{\sigma \in Hom(G, \{1, -1\}) \mid \sigma(-1) = -1\}$.*
It is also characterized by: if $\sigma_0, \sigma_1, \sigma_2 \in X$ then the product $\sigma_0 \sigma_1 \sigma_2 \in X$.

What was expected to correspond to the space of $\mathbb{R}$-places of the field case in the context of abstract spaces of orderings is called a P-structure and has been defined as follows by Marshall in [M3].

Definition 58 *A P-structure is an equivalence relation on a space of orderings (X, G) such that the canonical mapping $\Lambda : X \to M$, where M is the set of equivalence classes, satisfies:*

(1) Each fiber is a fan;

(2) If $\sigma_0 \sigma_1 \sigma_2 \sigma_3 = 1$ then $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ has a non empty intersection with at most two fibers.

Marshall has proved that every abstract space of orderings has a P-structure, generally not unique. But unlikely the case of the space of $\mathbb{R}$-places in a field, this P-structure M equipped with the quotient topology, is not always Hausdorff.

7.2 Abstract spaces of signatures (higher level)

In the higher level case, one can also define abstract spaces of signatures (similar to 4.3 in the field case).

Definition 59 *An abstract space of signatures of of level 2^n is (X, G), G abelian group of exponent 2^n, $X \subset Hom(G, \mu_{2^n})$ such that:*

(0) $\forall \sigma \in X$, $\forall k \in \mathbb{N}$ with k odd, $\sigma^k \in X$;

(1) X is a closed subset of $Hom(G, \mu_{2^n})$;

(2) $\forall \sigma \in X$ $\sigma(-1) = -1$ (-1 distinguished element of μ_{2^n});

(3) $\underset{\sigma \in X}{\cap} \ker \sigma = 1$ (where $\ker \sigma = \{a \in G \mid \sigma(a) = 1\}$);

(4) For any f and g forms over G

$$D_X(f \oplus g) = \cup\{D_X \langle x, y \rangle \mid x \in D_X(f), \ y \in D_X(g)\}.$$

In fields, the space of $\mathbb{R}$-places is known as soon as one knows the usual orderings and the orderings of level 2. Using this idea in the abstract situation we have been able to obtain a theorem which can be seen as the first case of a P-structure which looks like an abstract space of $\mathbb{R}$-places.

Theorem 60 *(Gondard-Marshall, [GM]). Let (X, G) be a subspace of a space of signatures (X', G') with 2-power exponent.*

For $\sigma_0, \sigma_1 \in X$, define $\sigma_0 \sim \sigma_1$ if $\sigma_0 \sigma_1 = \tau^2 \in X'^2$.

Then the followings are equivalent:

(1) If $\sigma_0 \sigma_1 \sigma_2 \sigma_3 = 1$, then either σ_0 is in relation by $\sim$ with exactly one of the $\sigma_1, \sigma_2, \sigma_3$, or σ_0 is in relation by $\sim$ with everyone of the $\sigma_1, \sigma_2, \sigma_3$.

(2) $\sim$ defines a P-structure on X.

Moreover in this case the induced P-structure defined on X by $\sim$ has a Hausdorff topology.

The key idea for proving the theorem is that in the field case, studied by Harman in [H], for any P_2, ordering of level 2, holds for some orderings P_0, P_1:

$a^2 \in P_2 \iff a \in P_2 \cup -P_2 = (P_0 \cap P_1) \cup (-(P_0 \cap P_1))$.

Hence on the side of abstract signatures we get $\tau(a^2) = \tau(a)^2 = \sigma_0(a)\sigma_1(a)$.

7.3 Open problems.

1 - Study in the field case the space of level 1 valuation fans $\mathcal{VF}(K)$, and its relation with $Sper H(K)$. The motivation is that $\chi(K)$, isomorphic to $\min Sper H(K)$, consists of valuation fans P_i, and that to a $\mathbb{R}$-place λ in $M(K)$, which is isomorphic to $\max Sper H(K)$, can be associated a valuation fan of level 1: $T_\lambda = \cap \{P_i \mid P_i \in \Lambda^{-1}(\lambda)\}$ where $\Lambda^{-1}(\lambda) = \{P_i \mid \lambda_{P_i} = \lambda\}$. Or work on the same question dealing with signatures.

2 - Characterize the topological spaces which are realizable as spaces of $\mathbb{R}$-places. Partial results in that direction have been recently obtained in [EO], [KK], [KMO] and [MMO].

It will be useful to study for a field K the space of connected components of the space of $\mathbb{R}$-places of K, $\pi_0(M(K))$. This might be some kind of space of orderings. Another question in this area is: in which cases are the connected components of $M(K)$ homeomorphic?

Also it is known from Schülting's results (see [Schü1]) that $M(K(X))$ have the same number of connected components as $M(K)$. Does $M(K((X)))$ have the same number of connected components as $M(K)$? (Conjecture is yes).

3 - Construct a finer theory for abstract spaces of orderings taking into account the $\mathbb{R}$-places. For example, $\mathbb{Q}(2^{\frac{1}{2}})$ and $\mathbb{R}((X))$ have isomorphic spaces of orderings, but the first one has two $\mathbb{R}$-places and no ordering of level 2, and the second one has only one $\mathbb{R}$-place but has a 2-primary chain of higher level orderings.

4 - Try to define a notion of abstract space of valuation fans, and write a theory of abstract $\mathbb{R}$-places. Both are linked because of the minimal valuation fans of level 1 defined from a $\mathbb{R}$-place λ by $T_\lambda = \cap \left\{ P_i \mid P_i \in \Lambda^{-1}(\lambda) \right\}$, where $\Lambda^{-1}(\lambda) = \{ P_i \mid \lambda_{P_i} = \lambda \}$. Then use abstract $\mathbb{R}$-places to solve Marshall's problem of realizability of abstract spaces of orderings.

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