On generalized series fields and exponential-logarithmic series fields with derivations.*

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Abstract

We survey some important properties of fields of generalized series and of exponential-logarithmic series, with particular emphasis on their possible differential structure, based on a joint work of the author with S. Kuhlmann [KM12b, KM11].

1 Introduction on generalized series fields

Definition 1.1 (Generalized series and their natural valuation) Let k be a field and Γ be a totally ordered Abelian group. A **generalized series** with coefficients in k and exponents in Γ is a map $a : \Gamma \to k$, denoted by :

$$a = \sum_{\alpha \in \Gamma} a_\alpha t^\alpha$$

with its support $\mathrm{Supp}\ a := \{\alpha \in \Gamma \mid a_\alpha \neq 0\}$ that is well-ordered.

We denote by $k((\Gamma))$ the set of generalized series, which is actually a **field** when endowed with the componentwise sum and (the straightforward generalization of) the convolution product.

The **canonical valuation** on $k((\Gamma))$ is:

$$\begin{array}{rccc} v: & k((\Gamma)) & \to & \Gamma \cup \{\infty\} \\ & a & \mapsto & \min(\mathrm{Supp}\ a) \\ & 0 & \mapsto & \infty \end{array}$$

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Since Γ is an ordered group, one can define on it the **Archimedean equivalence relation** :

$$\forall \alpha, \beta \in \Gamma,\ \alpha \sim_+ \beta \Leftrightarrow \exists n,\ n\alpha \leq \beta \text{ and } n\beta \leq \alpha$$

and the corresponding set Φ of **Archimedean equivalence classes** $\phi := [\gamma]$ for $\gamma \in \Gamma$. *Phi* inherits naturally a total order $\leq$ from the one of Γ:

$$\phi_1 = [\gamma_1] \leq \phi_2 = [\gamma_2] \leq \infty := [0] \Leftrightarrow 0 \leq \gamma_2 \leq \gamma_1.$$

The order type of Φ is called the **rank** of Γ, and as well the **rank** of $k((\Gamma))$.

Generalized series are ubiquitous objects in mathematics. They generalize the classical field of Laurent series ($\Gamma = \mathbb{Z}$), and Puiseux series form a subfield of $k((\Gamma))$ for $\Gamma = \mathbb{Q}$.

A few years after the pioneering work of Veronese and Levi-Civita on infinite and infinitesimal numbers [Ver91, LC94, LC98], H. Hahn introduced in [Hah07] the generalized series in the case where $k = \mathbb{R}$, along the proof of his embedding theorem for (totally) ordered Abelian groups:

Theorem 1.2 (Hahn's embedding) *Any ordered Abelian group is order isomorphic to a subgroup of the Hahn group over the divisible hull of its skeleton (see Section 2).*

To quote Hahn himself (p. 614-615, translated by ourselves):
" *We want to show that to the sizes in an arbitrary non archimedean system of orders of magnitude, in unambiguous way symbols can be assigned of the form*

$$\sum a_\gamma e_\gamma$$

where a_γ means a real number, e_γ are symbols for a ranking ("units") and summation applies to well-ordered amounts."
For a survey on this topic, we recommand [Ehr95].

Almost at the same time, G.H. Hardy published his monograph enhancing P. Du Bois-Reymond's work on *orders of infinity* [Har10]. The central question that this work adresses is: even though orders of growth of real functions have no bound (by Du Bois-Reymond's results), can we find asymptotic scales to describe them ? With the field of *logarithmic-exponential functions*, Hardy provides an important example of such a generalized asymptotic scale. It is what is called now a *Hardy field* (see [Bou76, Section V, Appendix] and below).

In his seminal article [Kru32], W. Krull provides a common framework for these works, and many others: *abstract valuation theory*. In [Kru32, Section 13], fields of generalized series are key examples of **maximal valued fields** - i.e. fields which admit no proper immediate extension - with any given value group and residue field. Note that Krull proves in the same article that *any* valued field possesses an immediate maximal extension, and puts the following crucial question (see the end of Section 13):
"*Under which conditions a maximal extension of a given valued field may be viewed as a generalized series field?*"

The answer for valued fields having same characteristic as its residue field is given by Kaplansky in [Kap42a, Kap42b] almost ten years after. More precisely:

Theorem 1.3 (Kaplansky's embedding) *In the equal characteristic case, every valued field is analytically isomorphic to a subfield of a suitable generalized series field*

"Suitable" means that in certain cases, one might need *factor sets* to express the multiplication rule for monomials of the generalized series. But in the context we are interested in (real closed or algebraically closed zero characteristic residue field), these are not needed.

Note that there is also a result for the mixed characteristic case, that we leave aside since we are mostly interested in the zero characteristic for differential structures. Just keep in mind that generalized series are not less represented in number theory (see e.g. [Sch37, Poo93, Ked01b, Ked01a, MR93, Ber00])

For the same reason, we will not detail the various results for what are called now *Mal'cev-Neumann series* [Neu49, Mal48] (when k is only supposed to be a ring, and Γ is not necessarily commutative) (see e.g. [Rib95, Rib97]).

As for the classical formal power series, generalized series have great algebraic properties [Rib92]. E.g., they are always Henselian valued fields. If Γ is divisible and k is real closed, respectively algebraically closed, then so is $k((\Gamma))$. As wrote S. Maclane [Mac39]: *generalized power series are universal as valued fields*.

In topological terms, a valued field (K, v) is an *ultrametric* space. In this context, generalized series fields are **spherically complete**: the intersection of any decreasing sequence of balls is nonempty. This holds if and only if every *pseudo-Cauchy sequence* in K has a pseudo limit in K, which means that the field K is maximally valued. In a spherically complete space, many of the classical results of functional analysis hold, as the Hahn-Banach, Banach-Steinhaus and Open Mapping Theorems [Sch02]. In particular, one has a *Banach Fixed Point Theorem* [PCR93, Kuh11].

At the interplay between model theory and geometry sits the *order minimal* or *o-minimal* geometry, a generalization of *semialgebraic* and *subanalytic* geometry [vdD98]. There, generalized series provide non standard models for various theories, like certain o-minimal *expansions* of the field of real numbers: by the power functions corresponding to a subfield of the reals [Mil94]; by the global real exponential function possibly with restricted analytic functions [vdDMM97, Kuh00, vdH06]; by solutions of Pfaffian differential equations [Spe99]. Moreover, generalized series are themselves used to build other o-minimal structures, by expanding the field of reals by *multisummable series* [vdDS00], by *convergent generalized series* [vdDS98].

O-minimality tells us that the definable subsets of the structure have finitely many connected components, justifying the *tameness* of such geometry in the sense of [Gro97]. Other important tameness results about differential equations may be put in parallel: the proof of the Dulac conjecture (finiteness of the number of limit cycles of a planar polynomial vector field: Hilbert 16th problem, part 2) using *transseries* [É92, vdH06]; the desingularization result for 3-dimensional real analytic vector fields in [CMR05] using rank 1 generalized series.

In o-minimal geometry as for differential equations, another important algebraic object is omnipresent: the **Hardy fields**, i.e. fields of germs at $+\infty$ of real functions closed under differentiation. As the germs in such a field are non oscillating, they carry a total order and therefore the corresponding natural valuation, so they are *differential valued fields* [Ros83b] and therefore key tools for the study of asymptotics. For examples, take $\mathbb{R}(x)$ or $\mathbb{R}(x^{\mathbb{R}}, \exp(x)^{\mathbb{R}})$ or the above cited Hardy's log-exp functions. Also, for any o-minimal structure over $\mathbb{R}$, the germs at $+\infty$ of unary definable functions form a Hardy field. As another important example, in the context of an isolated singularity of a real analytic vector field, take the field of meromorphic functions evaluated on a germ of *non oscillating* integral curve.

Hardy fields may also be generated by adjoining to a given Hardy field certain solutions of differential equations [Ros83a, Ros95, Kuh11, PCR04]. In [AvdD02] the authors have introduced the notion of *H-fields*, an axiomatized version of Hardy fields. Note that by [Bos86] there exist also infinite rank Hardy fields that are not *exponentially bounded*, i.e. for which any element is bounded by some iterate of exp. For a nice survey on these topics, see [AvdD05a].

How far can one push the connection between Hardy fields and generalized series fields ? For instance, we showed in [Mat11] that one can solve differential equations over finite rank generalized series similarly as over finite rank Hardy fields. In [vdDMM01, Corollary 3.12], the authors prove that the infinite rank Hardy field of the above cited o-minimal structure $\mathbb{R}_{an,\exp}$ embeds as ordered differential field into the field of *logarithmic-exponential series*. Recently, J. van der Hoeven introduced in [vdH09] the notion of a *transserial Hardy field*, namely a Hardy field which is differentially order isomorphic to a subfield of the field of transseries. There he proves [vdH09, Theorem 9] [AvdDvdH12, Theorem 1.1] that : *the field of transseries that are differentially algebraic over $\mathbb{R}$ is isomorphic as ordered differential field to some Hardy field.* Even more recently [1], M. Aschenbrenner, L. van den Dries and J. van der Hoeven state the following result for H-fields - so in particular for Hardy fields - [AvdDvdH12, Theorem 4.1]: *every real closed H-field has an immediate maximal H-field extension.* By the Kaplansky's embedding Theorem 1.3, this implies that:

Theorem 1.4 *Any H-field is analytically and differentially isomorphic to a subfield of a suitable field of generalized series endowed with* some *derivation.*

The authors assert that the derivation on such differential maximal immediate extension needs not be unique (see the proof of their Proposition 5.4), whereas the extension is unique as a valued field by Kaplansky's work. Now follows an important question: what kind of derivation can we expect for the generalized series field? More precisely: can we expect the derivation on generalized series to be the one that we *all want*? For generalized series, one may not only require the derivation to mimic the valuative properties of the derivation in a Hardy field. One may expect the derivation to behave the same way as the derivation for the classical formal power series, e.g. that it commutes with infinite sums. This is what we call a **series derivation of Hardy type** (see below and Definition 2.3). Together with S. Kuhlmann, we propose the following conjecture, problem and questions:

[1]The author thanks Matthias Aschenbrenner for having indicated him this preprint

Differential Kaplansky embedding conjecture for Hardy fields. *Any Hardy field is analytically and differentially isomorphic to a subfield of a suitable generalized series field endowed with a* series derivation of Hardy type.

Differential Kaplansky embedding problem. *Describe which differential valued fields are analytically and differentially isomorphic to a subfield of a suitable generalized series field endowed with a* series derivation of Hardy type.

Question 1. Is there uniqueness of the differential maximal extension in the case of the field of transseries ?

Question 2. Can we obtain the same result as in Theorem 1.4 for valued field endowed with a differential valuation in the sense of Rosenlicht ?

In the following sections, the author will survey the results in [KM12b] describing how to endow generalized series fields with such **series derivations of Hardy type**. Thus, such differential generalized series fields are H-fields. This has already been done in [AvdD05b, Section 11] in the particular case of a value group Γ divisible with a property called (*) (i.e. admitting a valuation basis in [Kuh00]) and carrying an *asymptotic couple* in the sense of [Ros81]. In [KM11] we continued our study in the case of *exponential-logarithmic series fields* in the sense of [Kuh00] (see Section 5), to provide a large family of *exponential H-fields*.

2 What kind of derivations for generalized series ?

We follow the ideas in [KM12b, KM11], but with *additive notations* as for the classical Krull valuations.

By a **derivation** on a field K, we mean a map $d : K \to K$ which is linear and verifies the Leibniz rule : $d(ab) = d(a)b + ad(b)$. The derivation for the usual power series verifies two further key properties :

- it *commutes with infinite sums* $d\left(\sum_n a_n x^n\right) = \sum_n a_n d(x^n)$;
- it *generalizes the Leibniz rule to real powers* $d(x^\alpha) = \alpha x^{\alpha-1} d(x) = \alpha x^\alpha d(x)/x$, with in particular $d(1) = 0$.

How can we generalize these properties to the case of our fields $k((\Gamma))$? If we impose that d is **strongly linear**, i.e. $d\left(\sum_\alpha a_\alpha t^\alpha\right) = \sum_\alpha a_\alpha d(t^\alpha)$, two questions arise:

1. how do we define $d(t^\alpha)$, $\alpha \in \Gamma$, where Γ is an arbitrary ordered Abelian group ?
2. how do we ensure that $\sum_\alpha a_\alpha d(t^\alpha)$ is itself well-defined and in $k((\Gamma))$?

To answer question 1, we will apply the above cited Hahn's embedding theorem (1.2). To any totally ordered (non trivial) Abelian group $(\Gamma, \leq)$, one can associate its **skeleton** $[\Phi, (A_\phi)_{\phi\in\Phi}]$ as follows. As before Φ denotes the ordered set of all Archimedean equivalence classes. The map $v_\Gamma : \Gamma \to \Phi \cup \{\infty\}$ defined by $v_\Gamma(\alpha) := [\alpha] = \phi$ is the corresponding **natural valuation** on Γ (as ordered group). To any value $\phi \in \Phi$ corresponds two subgroups of Γ: $C_\phi := \{\alpha \in \Gamma \mid v_\Gamma(\alpha) \geq \phi\}$ and $D_\phi := \{\alpha \in \Gamma \mid v_\Gamma(\alpha) > \phi\}$, with the following properties:

- $D_\phi \subsetneq C_\phi$;
- $(0) \subset \cdots D_\phi \subsetneq C_\phi \cdots \subset \Gamma$ forms the chain of *isolated subgroups* of Γ;
- $A_\phi := C_\phi / D_\phi$ is an Archimedean group, and therefore is (order isomorphic to) some subgroup of $\mathbb{R}$ (Hölder's theorem).

The groups A_ϕ, $\phi \in \Phi$, are usually called the **ribs** and Φ the **spine** of Γ.

Given a skeleton $[\Phi, (A_\phi)_{\phi\in\Phi}]$, one can always build the corresponding **Hahn group** $\prod_{\phi\in\Phi} A_\phi$, i.e. the subgroup of the group product of the A_ϕ's of elements with well-ordered support ordered lexicographically. We can give now a more precise version of Theorem 1.2 (see e.g. [Fuc63], [Gla99, Theorem 4.C] or [Kuh00, Theorem 0.26]).

Theorem 2.1 (Hahn's Embedding) $(\Gamma, \leq)$ *embeds as an ordered group in the Hahn group* $\prod_{\phi\in\Phi} \overline{A}_\phi$ *where* $\overline{A}_\phi$ *is the divisible closure of* A_ϕ *in* $\mathbb{R}$ *(i.e. the rational vector subspace of* $\mathbb{R}$ *generated by* A_ϕ*).*

Note that the ordering of an ordered group extends uniquely to its divisible closure. *From now on, we consider* Γ *as a subgroup of a given Hahn group* $\prod_{\phi\in\Phi} \overline{A}_\phi$*, which is itself a subgroup of* $\prod_{\phi\in\Phi} \mathbb{R}$. So, any element $\alpha \in \Gamma$ is written $\alpha = \sum_\phi a_\phi \mathbb{1}_\phi$, and $v_\Gamma(\alpha) = \min(\operatorname{supp} \alpha)$. Returning to the generalized series field $k((\Gamma))$, we set $t^\alpha := \prod_\phi t_\phi^{\alpha_\phi}$ where t_ϕ denotes for simplicity $t^{\mathbb{1}_\phi}$. The multiplication rule is componentwise, and note that we will also speak of the support $\operatorname{supp} t^\alpha$ of such formal product: $\operatorname{supp} t^\alpha := \operatorname{supp} \alpha := \{\phi \in \Phi \mid \alpha_\phi \neq 0\}$ which is well-ordered in Φ.

Now we define the derivative of t^α, by imposing that it verifies a **strong Leibniz rule**, i.e. $d(t^\alpha) = t^\alpha \sum_\phi \alpha_\phi \frac{d(\phi)}{\phi}$. Subsequently, three new questions arise:

3. how do we define $d(\phi)$ for $\phi \in \Phi$, with Φ being an arbitrary ordered set ?
4. how do we make sense of α_ϕ as coefficient in the series ?
5. how do we ensure that $\sum_\phi \alpha_\phi \frac{d(\phi)}{\phi}$ is itself well-defined and in $k((\Gamma))$?

To answer question 3, one can set $d(\phi) \in k((\Gamma))$. In other words, one can pick *a priori* any map $d : \Phi \to k((\Gamma))\backslash\{0\}$ extended by $d(1) = 0$.

To answer question 4, we need to impose that $\alpha_\phi \in k$ for any $\alpha \in \Gamma$ and $\phi \in \operatorname{Supp} \alpha \subset \Phi$. In other words, k has to contain the union of the groups $\overline{A}_\phi$ in $\mathbb{R}$. For simplicity, one can take $k = \mathbb{R}$ (as we did in [KM12b, KM11]) or $k = \mathbb{C}$.

Example 2.2 In the finite rank case, say $r \in \mathbb{N}$, the two preceding answers also solve questions 2 and 5. Indeed, write any element $\alpha \in \Gamma$ as $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r}$, and for any $a \in k((\Gamma))$:

$$\begin{array}{rcl} d(a) & = & d\left(\sum_\alpha a_\alpha t^\alpha\right) \\ & = & \sum_\alpha a_\alpha d(t^\alpha) \\ & = & \sum_\alpha a_\alpha t^\alpha \left(\alpha_1 \frac{d(t_1)}{t_1} + \cdots + \alpha_r \frac{d(t_r)}{t_r}\right) \\ & = & \frac{d(t_1)}{t_1} \sum_\alpha (a_\alpha \alpha_1) t^\alpha + \cdots + \frac{d(t_r)}{t_r} \sum_\alpha (a_\alpha \alpha_r) t^\alpha \end{array}$$

So $d(a)$ is well-defined whatever value in $k((\Gamma))$ we choose for the $\frac{d(t_i)}{t_i}$'s.

It remains to answer questions 2 and 5 in the general case.

Definition 2.3 *We call* ***series derivation*** *on $k((\Gamma))$ any derivation on $k((\Gamma))$ that is strongly linear (solving question 2) and that verifies a strong Leibniz rule (solving question 5).*

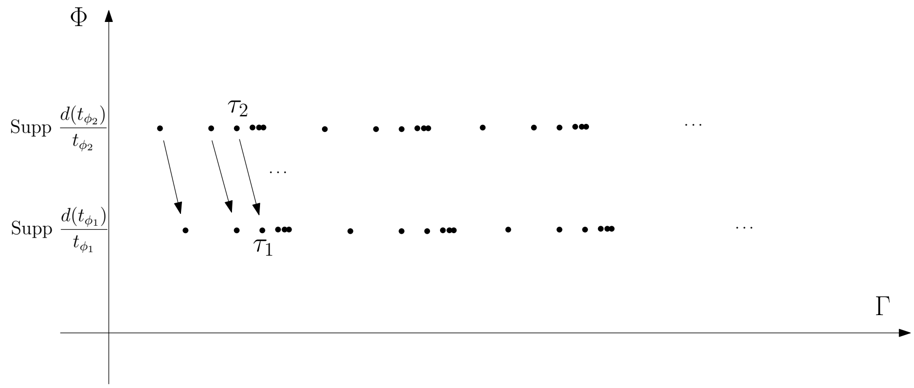
Thus the problem consists in finding a criterion on the map $d : \Phi \to k((\Gamma))$ so that the formulas in questions 2 and 5 involve always **summable families** of series. A family of series is said to be summable if the union of their support is well-ordered and if to any value in this union corresponds only finitely many series. This is done in [KM12b, Theorem 3.7], using Ramsey's theory type arguments [Ros82, Exercise 7.5, p. 112]:

Theorem 2.4 *A map $d : \Phi \to k((\Gamma))\backslash\{0\}$ extends to a series derivation on $k((\Gamma))$ if and only both of the following conditions* hold*:*

(C1) *for any strictly increasing sequence $(\phi_n)_{n\in\mathbb{N}} \subset \Phi$ and any sequence $(\tau_n)_{n\in\mathbb{N}} \subset \Gamma$ with for any n, $\tau_n \in \mathrm{Supp}\, \frac{d(t_{\phi_n})}{t_{\phi_n}}$, then $(\tau_n)_{n\in\mathbb{N}}$ cannot be decreasing (i.e. there is $N \in \mathbb{N}$ such that $\tau_{N+1} > \tau_N$).*

(C2) *For any strictly decreasing sequences $(\phi_n)_{n\in\mathbb{N}} \subset \Phi$ and $(\tau_n)_{n\in\mathbb{N}} \subset \Gamma$ such that for any n $\tau_n \in \mathrm{Supp}\, \frac{d(t_{\phi_n})}{t_{\phi_n}}$, there is $N \in \mathbb{N}$ such that $v_\Gamma(\tau_{N+1} - \tau_N) > \phi_{N+1}$.*

More precisely, (C1) gives an answer to the question 5. Roughly speaking, it says that for "almost all" pairs of elements $\phi_1 < \phi_2$ of Φ and "almost all" two *pairwise corresponding* elements $\tau_1 \in \mathrm{Supp}\, \frac{d(t_{\phi_1})}{t_{\phi_1}}$ and $\tau_2 \in \mathrm{Supp}\, \frac{d(t_{\phi_2})}{t_{\phi_2}}$, we must have $\tau_1 > \tau_2$.



To the question 2, the answer is (C2). It says that for "almost all" elements $\phi_1 > \phi_2$ of Φ and "almost all" elements $\tau^{(1)} \in \text{Supp}\ \dfrac{d(t_{\phi_1})}{t_{\phi_1}}$ and $\tau^{(2)} \in \text{Supp}\ \dfrac{d(t_{\phi_2})}{t_{\phi_2}}$ with $\tau^{(1)} > \tau^{(2)}$, their principal parts up to the $\mathbb{1}_{\phi_2}$-component must be equal.

Example 2.5 We illustrate (2.4) by the following easy example. Consider the Hardy field of germs at $+\infty$ of real functions $\mathbb{H} = \mathbb{R}(\log(x)^{\mathbb{R}}, x^{\mathbb{R}}, \exp(x)^{\mathbb{R}})$, which can be viewed also as a subfield of a field $k((\Gamma))$ of generalized series of rank 3, with e.g. $t_1 = \exp(-x)$, $t_2 = 1/x$ and $t_3 = 1/\log(x)$ and $\Gamma = \mathbb{R} \times_{\text{lex}} \mathbb{R} \times_{\text{lex}} \mathbb{R}$. Note that $t_1 < t_2 < t_3 < 1$ for the ordering in $\mathbb{H}$ as well as $v_\Gamma(t_1) = \phi_1 < v_\Gamma(t_2) = \phi_2 < v_\Gamma(t_3) = \phi_3$ in Φ, whereas $(0,0,0) < v(t_3) = (0,0,1) < v(t_2) = (0,1,0) < v(t_1) = (1,0,0)$ in Γ as well as $[t_3] \subset [t_2] \subset [t_1]$ as Archimedean equivalence classes in Γ. We compute in $\mathbb{H}$:

$$\begin{cases} \dfrac{d(t_1)}{t_1} = -1 & \text{with support } \{(0,0,0)\} \\ \dfrac{d(t_2)}{t_2} = \dfrac{1}{x} = -t_2 & \text{with support } \{(0,1,0)\} \\ \dfrac{d(t_3)}{t_3} = \dfrac{-1}{x\log(x)} = -t_2 t_3 & \text{with support } \{(0,1,1)\} \end{cases}$$

Consider for instance t_2 and t_3. One has that $v\left(\dfrac{d(t_3)}{t_3}\right) = (0,1,0) < (0,1,1) = v\left(\dfrac{d(t_2)}{t_2}\right)$ and their difference equals $(0,0,1)$. $v_\Gamma(0,0,1) = \phi_3$ which is indeed bigger than ϕ_2 in Φ.

Our criterion in (2.4) permits to build families of series derivations for a large class of generalized series fields [KM12b, Section 5]. Moreover we will be able to obtain in certain cases series derivations of Hardy type, as will be shown in (3.3).

3 On Hardy type series derivations

Definition 3.1 *Let (K, v, d) be a differential valued field. Denote by $\mathcal{O}_v$ the valuation ring and $\mathfrak{m}_v$ its maximal ideal. The derivation $d : K \to K$ is said to be a* **Hardy type derivation** *if :*

(HD1) $\mathcal{O}_v = \ker d + \mathfrak{m}_v$;

(HD2) *d verifies **l'Hospital's rule**:* $\forall a, b \in K \backslash \{0\}$ *with* $v(a) \neq 0$ *and* $v(b) \neq 0$,

$$v(a) \leq v(b) \Leftrightarrow v(d(a)) \leq v(d(b)).$$

(HD3) *the **logarithmic derivation is compatible with the valuation**:* $\forall a, b \in K$

$$|v(a)| > |v(b)| > 0 \Rightarrow v\left(\frac{d(a)}{a}\right) \leq v\left(\frac{d(b)}{b}\right)$$

$$\text{with:} \quad v\left(\frac{d(a)}{a}\right) = v\left(\frac{d(b)}{b}\right) \Leftrightarrow v(a),\ v(b) \text{ are Archimedian equivalent.}$$

Axioms (HD1) and (HD2) are those which define a *differential valuation* in the sense of M. Rosenlicht ([Ros80, Definition p. 303]). Axiom (HD3) corresponds to the Principle (*) in [Ros81, p. 992]. This principle is itself an abstract version of properties obtained in [Ros83b, Propositions 3 and 4] and [Ros80, Principle (*) p. 314] in the context of Hardy fields.

The question we want to address here is:

How can we endow $k((\Gamma))$ *with a Hardy type series derivation ?*

As we showed in [KM12b, Theorem 4.3 and Corollary 4.4], the answer depends only on the valuations $v(d(t_\phi)/t_\phi)$, say θ_ϕ, $\phi \in \Phi$:

Theorem 3.2 *A series derivation d on* $k((\Gamma))$ *verifies* (HD2) *and* (HD3) *if and only if the following condition holds:*

(H3) $\forall \phi_1, \phi_2 \in \Phi$, $\phi_1 < \phi_2 \ \Rightarrow\ \theta_{\phi_1} > \theta_{\phi_2}$ *and* $v_\Gamma\left(\theta_{\phi_1} - \theta_{\phi_2}\right) > \phi_1$.

In this case (HD1) holds with $k = \ker d$.

As an illustration, consider the example in (2.5).

In [AvdD02] is developed the notion of *H-field*, which generalizes the one of Hardy field. An **H-field** is an ordered field (denote by v the natural valuation on it) endowed with a derivation $d : K \to K$ such that:

(HF1) $\mathcal{O}_v = \ker d + \mathfrak{m}_v$;

(HF2) for any $f \in K_{>0}$, $v(f) < 0 \Rightarrow d(f) > 0$.

Note that (HD1) is (HF1). Therefore, if we suppose that k is ordered and consequently so is $k((\Gamma))$ lexicographically, then:

$$k((\Gamma)) \text{ is an H-field if and only if for any } \phi \in \Phi,\ \frac{d(t_\phi)}{t_\phi} < 0.$$

Indeed, for any series $a > 0$ with $v(a) < 0$, denote $a = a_\alpha t^\alpha + \cdots$ where $v(a) = \alpha$ and $a_\alpha > 0$. Denote $v_\Gamma(\alpha) = \phi$ and the coefficient of $\mathbb{1}_\phi$ in α by α_0. Note that $\alpha_0 < 0$ since $v(a) = \alpha < 0$. By the strong linearity, the strong Leibniz rule and (H3), we have:

$$d(a) = a_\alpha d(t^\alpha) + d(\cdots) = a_\alpha t^\alpha \left(\alpha_0 \frac{d(t_\phi)}{t_\phi} + \cdots \right) + d(\cdots) \tag{1}$$

So $d(a)$ has same sign as $-\dfrac{d(t_\phi)}{t_\phi}$.

Example 3.3 We use (2.4) and (3.2) to build general examples of series derivations of Hardy type on $k((\Gamma))$. If we restrict our attention to the value $\theta_\phi = v(d(t_\phi)/t_\phi)$, then (H3) tells us that the map $\phi \mapsto \theta_\phi$ has to be a *section* of Φ in Γ. It says also that for any $\phi_1 < \phi_2$, the principal parts of θ_{ϕ_1} and θ_{ϕ_2} up to the component $\mathbb{1}_{\phi_1}$ have to be identical. This can be achieved easily in the following cases. We leave the verifications to the reader.

- If Φ admits an order-preserving map $\sigma : \Phi \to \Phi$ which is a **right-shift**, i.e. such that $\sigma(\phi) > \phi$, then for any ϕ pick $\theta_\phi \in \Gamma_{>0}$ such that $v_\Gamma(\theta_\phi) = \sigma(\phi)$ and set for example $\dfrac{d(t_\phi)}{t_\phi} = c_\phi t^{\theta_\phi}$ for arbitrary $c_\phi \in k$. This includes for example the cases where Φ carries the structure of an ordered group, or where Φ is a limit ordinal.

- If Φ is the concatenation of *any* family of sets as in the preceding example, then consider the union of the correspondingly defined maps and derivatives for each of these subsets of Φ.

- If Φ has a biggest element ϕ_0 and carries a right-shift $\sigma : \Phi \setminus \{\phi_0\} \to \Phi$ then we can set the θ_ϕ's and the logarithmic derivatives as before, just completing the definition for ϕ_0 by setting $\theta_{\phi_0} := 1$. This includes for example the cases where Φ is a successor ordinal or where Φ is isomorphic to an interval of $\mathbb{R}$ with greatest element.

We treat now the case of subsets of $\mathbb{R}$. Note that this includes any *countable* well-ordered, reverse well-ordered or more generally scattered set (they all embed into $\mathbb{Q}$). It includes also the case of some tricky *dense* subsets of $\mathbb{R}$ as the Dushnik-Miller example (a dense subset of $\mathbb{R}$ that admits no non trivial self-embedding). We invite the reader to read through [Ros82] for the various references and classical results.

Proposition 3.4 *Let Φ be isomorphic to a subset S of $\mathbb{R}$. Suppose that:*

1. *if Φ has a greatest element ϕ_0, then $S \subset A_{\phi_0}$;*

2. *if Φ has no greatest element, then there is a strictly increasing sequence $(\phi_n)_{n\in\mathbb{N}}$ partitioning Φ (with possibly $\phi_0 \mapsto -\infty$) such that for all n either $(\phi_n, \phi_{n+1}] \simeq S_n \subset A_{\phi_{n+1}}$ (case a.) or $(\phi_n, \phi_{n+1}] \simeq S_n \subset A_{\phi_{n+2}}$ (case b.);*

Then $k((\Gamma))$ carries a series derivation of Hardy type.

Proof. Case 1. There is an isomorphism $s : \Phi \to S \subset A_{\phi_0}$. For any $\phi \in \Phi$ pick some element $\theta_\phi = s(\phi)\mathbb{1}_{\phi_0}$ of Γ and set for example $\dfrac{d(t_\phi)}{t_\phi} = c_\phi t_{\phi_0}^{\theta_\phi}$ where $c_\phi \in k$. Note that $v_\Gamma(\theta_{\phi_0}) = \phi_0$.
Case 2.a. There is an isomorphism $s_n : (\phi_n, \phi_{n+1}] \to S_n \subset A_{\phi_{n+1}}$. For any $\phi \in (\phi_n, \phi_{n+1}]$ pick some element $\alpha_\phi = s_n(\phi)\mathbb{1}_{\phi_{n+1}} + \cdots$ of Γ θ_ϕ. Then set $\theta_\phi := s_1(\phi_1)\mathbb{1}_{\phi_1} + \cdots + s_n(\phi_n) + \alpha_\phi$ (the principal part is needed to comply the second part of (H3)) and for example $\dfrac{d(t_\phi)}{t_\phi} = c_\phi t^{\theta_\phi} = t_{\phi_1}^{s_1(\phi_1)} \cdots t_{\phi_n}^{s_n(\phi_n)} t_{\phi_{n+1}}^{s_{n+1}(\phi_{n+1})} \cdots$ where $c_\phi \in k$.
Case 2.b. There is an isomorphism $s_n : (\phi_n, \phi_{n+1}] \to S_n \subset A_{\phi_{n+2}}$. For any $\phi \in (\phi_n, \phi_{n+1}]$ pick some element $\alpha_\phi = s_n(\phi)\mathbb{1}_{\phi_{n+2}} + \cdots$ of Γ θ_ϕ. Then set $\theta_\phi := \alpha_\phi$. □

For instance, these hypothesis 1. or 2. are fulfilled whenever the corresponding A_{ϕ_n} are isomorphic to $\mathbb{R}$. More generally:

Corollary 3.5 *Let Φ is isomorphic to the concatenation of a family of subsets of $\mathbb{R}$ over any set which admits a right-shift endomorphism. Suppose that $A_\phi \simeq \mathbb{R}$ for any $\phi \in \Phi$. Then $k((\Gamma))$ carries a series derivation of Hardy type.*

Indeed, the construction in case 2.b. of (3.4) applies. Note that if Φ is isomorphic to the concatenation of a family of subsets of $\mathbb{R}$ over *any* limit ordinal, we can also apply the construction 2.a.

Some of these abstract examples may be illustrated by or enhanced using germs in some Hardy fields in the spirit of (2.5). For instance, take the following Hardy fields:

- $\mathbb{R}(\exp(x)^{\mathbb{R}}, \exp_2(x)^{\mathbb{R}}, \ldots, \exp_n(x)^{\mathbb{R}}, \ldots)$
- $\mathbb{R}(x^{\mathbb{R}}, \exp(x^\alpha)^{\mathbb{R}}; \alpha \in \mathbb{R}_{>0})$
- $\mathbb{R}(x^{\mathbb{R}}, \log(x)^{\mathbb{R}}, \ldots, \log_n(x)^{\mathbb{R}}, \ldots)$
- $\mathbb{R}(\ldots, \exp_n(x)^{\mathbb{R}}, \ldots, \exp_2(x)^{\mathbb{R}}, \exp(x)^{\mathbb{R}}, x^{\mathbb{R}}, \log(x)^{\mathbb{R}}, \ldots, \log_n(x)^{\mathbb{R}}, \ldots)$

We let the reader verify what kind of Hardy type series derivation may model these cases.

The preceding examples give partial solution to the following natural problem:

Describe the generalized series fields which can carry a Hardy type series derivation.

We believe that a complete answer can be derived from (3.2). This would also help to characterize which groups may belong to an asymptotic couple in the sense of [Ros81, AvdD05b].

4 On asymptotic integration, integration and logarithms.

Definition 4.1 *Let (K, v, d) be a differential valued field, and $a \in K$. The element a is said to admit an* ***asymptotic integral*** *b if there exists $b \in K \setminus \{0\}$ such that $v(d(b) - a) > v(a)$. The element a is said to admit an* ***integral*** *b if there exists $b \in K \setminus \{0\}$ such that $d(b) = a$.*

For a differential valued field, the existence of a valuation permits to deal with approximate solutions of equations, for instance the basic differential equation corresponding to integration:

$$d(y) = a \tag{2}$$

In the context of Hardy fields, the problem of computing asymptotic integrals has been solved by M. Rosenlicht [Ros83b, Proposition 2 and Theorem 1]. In [KM12b] we observed that his proof applies to the more general context of valued fields with Hardy type derivations:

Theorem 4.2 *[Rosenlicht] Let (K, v, d) be a differential valued field with d of Hardy type. Let $a \in K\backslash\{0\}$, then a admits an asymptotic integral if and only if*

$$v(a) \neq l.u.b. \left\{ v\left(\frac{d(b)}{b}\right);\ b \in K\backslash\{0\},\ v(b) \neq 0 \right\}$$

Moreover, for any such a, there exists $u_0 \in K\backslash\{0\}$ with $v(u_0) \neq 0$ such that for any $u \in K\backslash\{0\}$, $|v(u_0)| > |v(u)| > 0$, then $a.\dfrac{au/d(u)}{d(au/d(u))}$ is an asymptotic integral of a.

Note that in our context of a generalized series field $k((\Gamma))$ endowed with a series derivation of Hardy type, by the computation in (1), one has for any $b \in K\backslash\{0\}$ with $v(b) \neq 0$ that:

$$v\left(\frac{d(b)}{b}\right) = v\left(\frac{d(t^\beta)}{t^\beta}\right) = v\left(\frac{d(t_\phi)}{t_\phi}\right) = \theta_\phi$$

where $\beta = v(b)$ and $\phi = v_\Gamma(\beta)$. Therefore in the hypothesis of (4.2) we can replace the set $\left\{ v\left(\frac{d(b)}{b}\right);\ b \in k((\Gamma)) \setminus \{0\},\ v(b) \neq 0 \right\}$ by the set $\left\{\theta_\phi;\ \phi \in \Phi\right\}$.

We can derive from (4.2) in our context explicit formulas for the computation of a specific asymptotic integral: the unique one which is a monomial $c_\gamma t^\gamma$ [KM12b, Corollary 6.3] or [KM11, Section 4.1]:

Corollary 4.3 *Let $\alpha \in \Gamma$ with $\alpha \neq \tilde{\theta}$, $\alpha = \alpha_0 \mathbb{1}_{\phi_0} + \cdots$. There exists a uniquely determined $\psi_\alpha \in \Phi$ which satisfies $\alpha - \theta_{\psi_\alpha} = \gamma_0 \mathbb{1}_{\psi_\alpha} + \cdots$.*
*Set $\dfrac{d(t_{\psi_\alpha})}{t_{\psi_\alpha}} = c_{\psi_\alpha} t^{\theta_{\psi_\alpha}} + \cdots$. Consequently, any series $a = a_\alpha t^\alpha + \cdots$ admits as **monomial asymptotic integral**:*

$$\text{a.i.}(a) = \frac{a_\alpha}{\gamma_0 c_{\psi_\alpha}} t^{\alpha - \theta_{\psi_\alpha}}$$

A natural question now is:

What is the relation between asymptotic integration and integration ?

The answer is given in [Kuh11, Theorem 55] for spherically complete differential valued fields:

Asymptotic integration immplies integration.

Applying this result to our context we obtain that:

Corollary 4.4 *Let $k((\Gamma))$ be endowed with a series derivation of Hardy type d. Set $\tilde{\theta} = l.u.b.\left\{\theta^{(\phi)} \; ; \; \phi \in \Phi\right\}$. Then any series $a \in k((\Gamma))$ with $v(a) > \tilde{\theta}$ admits an integral in $k((\Gamma))$. Moreover $k((\Gamma))$ is closed under integration if and only if $\tilde{\theta} \notin \Gamma$.*

A particular case of the problem of integration that we want to solve is the one for the *logarithm*:

$$d(y) = \frac{d(a)}{a} \Leftrightarrow y = \log|a| + c \tag{3}$$

This is the subject of the section 4.2 in [KM11]. There, such solutions log to (3) are called **pre-logarithms** since they might not be surjective as a real logarithm is. Of course, our study is rooted in the studies of logarithmic and exponential maps in the non-archimedean context, e.g. [All62, Kuh00, Res93, vdDMM97, vdH06].

By [All62], in order to define a pre-logarithm on $k((\Gamma))$, we suppose *from now on* that the coefficient field k carries a logarithm log, e.g. $k = \mathbb{R}$. So in particular k is *ordered*, and therefore so is $k((\Gamma))$ lexicographically. Moreover, the **logarithm on 1-units** is naturally defined as the following isomorphism of ordered groups:

$$\begin{array}{cccc} \log \;: & 1 + k((\Gamma_{>0})) & \rightarrow & k((\Gamma_{>0})) \\ & 1+\epsilon & \mapsto & \displaystyle\sum_{n\geq 1}(-1)^{n-1}\frac{\epsilon^n}{n} \end{array} \tag{4}$$

Since any generalized series may be written $a = a_\alpha t^\alpha(1+\epsilon)$, it remains only to define a pre-logarithm on the monic monomials t^α. In particular, we are interested in pre-logarithms verifying the **growth axiom scheme** as a real logarithm does:

(GA) $\forall \alpha \in \Gamma_{<0},\; v(\log(t^\alpha)) > \alpha.$

In the non differential case, this problem is discussed extensively in [Kuh00]. The perspective we adopt in the differential case is to consider pre-logarithms that verify equation (3). Moreover, in the context of generalized series fields, we consider pre-logarithms that are **series morphisms**, i.e. such that:

(L) $\forall \alpha = \sum_{\phi\in\Phi} \alpha_\phi \mathbb{1}_\phi \in \Gamma,\;\; \log(t^\alpha) = \sum_{\phi\in\Phi}\alpha_\phi \log(t_\phi).$

We prove in [KM11, Theorem 4.10] that:

Theorem 4.5 *Let $k((\Gamma))$ be endowed with a series derivation of Hardy type d. Set $\tilde{\theta} = l.u.b.\left\{\theta^{(\phi)} \; ; \; \phi \in \Phi\right\}$. There exists a unique pre-logarithm* log *on $k((\Gamma))$ which is a series morphism if and only if the following two conditions hold:*

1. $\tilde{\theta} \notin \bigcup_{\phi\in\Phi} Supp\, \frac{d(t_\phi)}{t_\phi}$;
2. $\forall \phi \in \Phi,\, \forall \tau_\phi \in Supp\, \frac{d(t_\phi)}{t_\phi},\; v\left(\text{a.i.}(\tau_\phi)\right) < 0.$

Moreover, this pre-logarithm verifies (GA).

In [KM11, Corollary 4.13], we obtain explicit formulas for this pre-logarithm.

Generally speaking, the principle is the same as before: to get the good property on $k((\Gamma))$ -here that any $\frac{d(a)}{a}$ has an integral - it suffices to have it for the $\frac{d(t_\phi)}{t_\phi}$'s. Once again, the key property that we use in the proof is the spherical completeness of $k((\Gamma))$, more precisely the existence of a *fixed point principle* as in [Kuh11, PCR93].

As a conclusion to this section, note that one can derive from the preceding results methods of construction of pre-logarithms and classes of examples in the same spirit as (3.3). Conversely, one may also try to deduce from a given pre-logarithmic structure a corresponding series derivation of Hardy type: see [KM11, Section 5]

Example 4.6 Suppose that Φ carries a right-shift endomorphism $\sigma : \Phi \to \Phi$, then the map defined by:

$$\log_\sigma(t^{\sum_{\phi\in\Phi}\gamma_\phi \mathbb{1}_\phi}) = \log_\sigma(\prod_{\phi\in\Phi} t_\phi^{\gamma_\phi}) = \sum_{\phi\in\Phi} \gamma_\phi t^{-1}_{\sigma(\phi)}$$

induces a pre-logarithm on $k((\Gamma))$. Moreover if σ is surjective, denote the **convex orbit** of any ϕ by $C_\phi = \{\psi \in \Phi \mid \exists k \in \mathbb{N},\ \sigma^k(\phi) \preccurlyeq \psi \preccurlyeq \sigma^{-k}(\phi)\}$. Then the pre-logarithm $\log_\sigma$ corresponds to a series derivation of Hardy type such that:

For any $\phi_1 < \phi_2 \in \Phi$, the principal part up to C_{ϕ_1} of $\theta_{\phi_1} - \theta_{\phi_2}$ is equal to the one of $\sum_{j=1}^{\infty} \mathbb{1}_{\sigma^j(\phi_2)} - \mathbb{1}_{\sigma^j(\phi_1)}$, with in particular $\theta_{\sigma^k(\phi)} = \theta_\phi - \sum_{j=1}^{k} \mathbb{1}_{\sigma^j(\phi)}$ for any $k \in \mathbb{N}$.

5 Exponential-logarithmic series fields

The last but not least differential equation we are interested in is the one for the *exponential*:

$$d(y) = d(a)y \Leftrightarrow y = c\exp(a) \tag{5}$$

According to [KKS97], there is no hope for defining an exponential and logarithmic structure on a generalized series field. Nevertheless there are several similar ways to obtain non Archimedian exponential-logarithmic fields seen as *subfields* of generalized series fields [vdDMM97, vdH06, Kuh00]. See also [KT] for a comparison between LE-series and EL-series. The original idea is due to [Dah84, DG87] and may be seen as an abstraction of Hardy's construction of log-exp functions [Har10]. In each case, it consists in starting with some initial field of generalized series, and then taking the closure under towering logarithmic and/or exponential extensions.

The fields of LE-series and of grid-based transseries are naturally equipped with a series derivation of Hardy type. It is also possible to equip certain fields of well-ordered transseries with such a derivation [Sch01]. Our aim here is to show that it is also possible to do that for EL-series fields.

Here we get started directly with a generalized series field $k((\Gamma))$ endowed with a pre-logarithm strong morphism log and a Hardy type series derivation d as in Section

4. The construction we present here is the construction of the exponential-logarithmic closure of $k((\Gamma))$ as in [Kuh00]. It consists in iterating the following exponential extension procedure.

By definition, the pre-logarithm defines an embedding of ordered groups $\log : t^\Gamma \hookrightarrow k((\Gamma_{<0}))$. Keep in mind that this map *cannot* be surjective [KKS97]. In other words, the exponential of the elements of $k((\Gamma_{<0})) \setminus \log(\Gamma)$ are missing in the left-hand side. Set the following multiplicative copy of the ordered additive group $\Gamma^\sharp := k((\Gamma_{<0}))$:

$$t^{\Gamma^\sharp} := \{t^a \mid a \in k((\Gamma_{<0})),\ \text{with } t^{\log(t^\alpha))} := t^\alpha,\ \alpha \in \Gamma\}$$

By construction $t^\Gamma \subset t^{\Gamma^\sharp}$, so the generalized series field $k((\Gamma^\sharp))$ extends $k((\Gamma))$. Define also a pre-logarithm by:

$$\begin{array}{rccc} \log^\sharp : & t^{\Gamma^\sharp} & \hookrightarrow & k((\Gamma^\sharp_{<0})) \\ & t^a & \mapsto & a \end{array}$$

$\log^\sharp$ extends log, and $k((\Gamma_{<0}))$ embeds in $k((\Gamma^\sharp_{<0}))$. Call $\left(k((\Gamma^\sharp)), \log^\sharp\right)$ the **exponential extension** of $(k((\Gamma)), \log)$.

By iterating this exponential extension procedure, we can define the n'th exponential extension $\left(k((\Gamma^{\sharp n})), \log^{\sharp n}\right)$, obtaining thus an inductive system of pre-logarithmic fields. The inductive limit is defined as the **exponential-logarithmic series (EL-series) field** (induced by $(k((\Gamma)), \log)$ in the sense of [Kuh00], say $\left(k((\Gamma))^{\mathrm{EL}}, \log^{\mathrm{EL}}\right)$.

Theorem 5.1 (Kuhlmann) *For any generalized series field $k((\Gamma))$ endowed with a pre-logarithm* log*, the couple $\left(k((\Gamma))^{\mathrm{EL}}, \log^{\mathrm{EL}}\right)$ is such that:*

$$\log^{\mathrm{EL}} : \left(k((\Gamma))^{\mathrm{EL}}_{>0}, ., \leq\right) \to \left(k((\Gamma))^{\mathrm{EL}}, +, \leq\right)$$

is an isomorphism of ordered groups, verifying the growth axiom scheme (GA). Therefore it admits a well-defined exponential map $\exp^{\mathrm{EL}}$ *as inverse map, and $k((\Gamma))^{\mathrm{EL}}$ is a non-Archimedian exponential-logarithmic field.*

Note that $k((\Gamma))^{\mathrm{EL}}$ is a strict subfield of the generalized series field $k((\Gamma^{\mathrm{EL}}))$ where $\Gamma^{\mathrm{EL}} := \bigcup_{n\in\mathbb{N}} \Gamma^{\sharp n}$. For example, the series $t_\phi^{-1} + t^{-t_\phi} + t^{-t^{t_\phi}} + \cdots$ belongs to $k((\Gamma^{\mathrm{EL}}))\setminus k((\Gamma))^{\mathrm{EL}}$. Nevertheless, Γ^{EL} is the value group of $k((\Gamma))^{\mathrm{EL}}$ and $k((\Gamma^{\mathrm{EL}}))$, and k is their residue field, so that $k((\Gamma))^{\mathrm{EL}} \subset k((\Gamma^{\mathrm{EL}}))$ is an immediate extension.

Considering $k((\Gamma))$ endowed with a Hardy type series derivation d, we are interested in extending d to $k((\Gamma))^{\mathrm{EL}}$. In [KM11, Theorem 6.2], we showed that this is always feasible:

Theorem 5.2 *The series derivation of Hardy type d on $k((\Gamma))$ extends to a series derivation of Hardy type on $k((\Gamma))^{\mathrm{EL}}$, and this extension is uniquely determined.*

Of course it is understood that the derivations and exponential and logarithmic maps we consider verify the corresponding differential equations (3) and (5): such $k((\Gamma))^{\mathrm{EL}}$ is a *non-archimedian differential exponential-logarithmic field*.

Moreover $k((\Gamma))^{\mathrm{EL}}$ inherits the properties of $k((\Gamma))$ concerning asymptotic integration and integration. Denote as before $\tilde{\theta} = l.u.b.\left\{\theta^{(\phi)}\ ;\ \phi \in \Phi\right\}$ where Φ is the rank of the *initial* valued group Γ. By [KM11, Theorem 7.1 and Corollary 7.2], we have that:

Theorem 5.3 *A series $a \in k((\Gamma))^{\mathrm{EL}}$ admits an asymptotic integral if and only if $a \not\asymp \tilde{\theta}$. The EL-series field $k((\Gamma))^{\mathrm{EL}}$ is closed under integration if and only if $\tilde{\theta} \notin \Gamma^{\mathrm{EL}}$.*

To conclude this section and the article, recall from our introduction that generalized series fields are universal domains for valued fields. In the characteristic zero case, with the EL-series construction, they provide also universal domains for ordered exponential fields. This is brilliantly illustrated by the construction of *surreal numbers* [Con01, Gon86, All87]. See also [All85] for a specific emphasis on the generalized series structure and set theoretic topics. These non Archimedian numbers form a *class* which can be endowed at the same time with the structure of a generalized series field and an exponential-logarithmic field. With S. Kuhlmann, we conjecture that:

Differential EL-series conjecture for surreal numbers. The class of surreal numbers carries the structure of a differential EL-series field, and consequently is a universal domain for exp-log differential fields.

Toward this question, we describe in a recent work [KM12a] the exp-log equivalence classes of surreal numbers.

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