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Valuations and Filtrations

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The classical theory of Gröbner bases, as developed by Bruno Buchberger, can be expanded to utilize objects more general than term orders. Each term order on the polynomial ring $k[\mathbf{x}]$ produces a filtration of $k[\mathbf{x}]$ and a valuation ring of the rational function field $k(\mathbf{x})$. The algorithms developed by Buchberger can be performed by using directly the induced valuation or filtration in place of the term order. There are many valuations and filtrations that are suitable for this general computational framework that are not derived from term orders, even after a change of variables. Here we study how to translate between properties of filtrations and properties in valuation theory, and give a characterization of which valuations and filtrations are derived from a term order after a change of variables. This characterization illuminates the properties of valuations and filtrations that are desirable for use in a generalized Gröbner basis theory.

1. Introduction

Bruno Buchberger's approach to the ideal membership problem led to the development of a reduction process that relied heavily upon term orders. Robbiano (1986) placed this theory in a larger context by examining graded structures and determining how term orders can be thought of as just a small part of the entire computational theory. Similarly, Sweedler (1986) developed a computational framework for working with ideals in polynomial rings and algebras by using a valuation-theoretic approach.

Let (V, m) be a valuation ring of the rational function field $k(\mathbf{x})$ such that $k \subset V$. If we wish to perform computations in an underlying polynomial ring $k[\mathbf{x}]$ of $k(\mathbf{x})$ in the setting of Sweedler (1986), then we require the following three conditions on V:

(i) $V \cap k[\mathbf{x}] = k$,
(ii) $k + m = V$, and
(iii) V is well-ordered in some sense with respect to $k[\mathbf{x}]$.

We address the first two properties in this paper, and discuss how they translate into the language of filtrations in Lemmas 4.7 and 4.8. One can begin with a term order on $k[\mathbf{x}]$ and define the associated valuation ring of $k(\mathbf{x})$ by

$$V = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{LT}(f) \leq \mathrm{LT}(g)\}.$$

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Properties (i) and (ii) can be demonstrated easily, and property (iii) follows from the well-orderedness of term orders.

Although the theory in Sweedler (1986) has been developed in terms of valuation rings, it is useful to translate the theory into results about valuations and filtrations. By translating properties about valuation rings into properties about valuations, we are able to simplify the language and proofs. It makes it easier to justify that our candidates for good valuation rings are well-ordered. In fact, one only has to check that the image of the polynomial ring under the valuation is a well-ordered subset of the value group. Although the computations are made easier with valuations rather than the intrinsic valuation rings, the theoretic framework is more closely related to classical Buchberger theory in the context of filtrations. Since we can think of Buchberger theory as a reduction process in terms of stepping through the pieces of a filtration or graded algebra, it makes sense to do much of the analysis in this area, and thus provides motivation for the second half of this paper.

We choose valuation theory as our first direction of exploration since the results are more easily stated and proved in this area than in the study of filtrations. It is worth mentioning that our definition of a valuation includes an ordering on the corresponding value group that is opposite to what is more commonly found in the literature. However, the notion of valuation given by Artin (1957) is essentially the same as ours except Artin includes 0 as a smallest element where we simply exclude it. Since our work is more closely linked to the theory of Gröbner bases and term orders than valuation theory we choose this order so that the results can be stated more simply.

As we saw, each term order on $k[\mathbf{x}]$ produces a valuation ring in $k(\mathbf{x})$, and hence, provides us also with the natural corresponding valuation. Conversely, we question whether we can characterize which valuation rings of $k(\mathbf{x})$ come from a term order on $k[\mathbf{x}]$, possibly with a change of variables. The reason for the *possible change of variables* is that we are looking for an intrinsic characterization involving ring-theoretic or module-theoretic conditions. The characterization should be independent of a specific parametrization of the polynomial ring. In other words, it should not be dependent upon a specific choice of variables for the polynomial ring. One would hope to find conditions whereby if a valuation or valuation ring satisfies those conditions, then it is possible to find a choice of variables for the polynomial ring whereby the filtration coincides with a term order valuation with regard to a term order on the chosen variables. We produce such a characterization in Theorem 2.6, which leads us to the discovery of valuations that do not come from a term order in suitable variables, yet are still well-behaved in that they are well-ordered and suitable for use in Sweedler (1986). In Mosteig (2000, 2002) and Mosteig and Sweedler (2001), we formulate such examples of valuations by using generalized power series.

Our second direction of exploration involves the study of nested filtrations. A nested filtration of a ring A is simply a collection of subsets of A that is totally ordered under inclusion. We study many properties of filtrations and demonstrate how to translate them into properties about valuations and valuation rings. In particular, we study filtrations with the following fundamental properties: full union (Section 3.1), intersects to zero (Section 3.1), one-dimensional graded components (Section 3.2), strong multiplicativity (Section 3.3), and non-negativity (Section 3.3). We say that a filtration is *regular* if it has all of the five listed properties above, and it turns out that regular filtrations are related to valuations that have properties (i) and (ii) above.

Graded algebras arise naturally as the associated graded algebras of an algebra with a filtration. Given a polynomial ring $k[\mathbf{x}]$ with a term order $\leq_\sigma$, there is a natural filtration we can define on $k[\mathbf{x}]$ (see Example 3.1). In this case, the associated graded algebra is isomorphic to $k[\mathbf{x}]$. Another way in which this can happen is similar to the manner in which an automorphism is used in our definition of valuations that come from a term order in suitable variables in Section 2.2. Namely, suppose that $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ is an algebra isomorphism. If $k[\mathbf{u}]$ has a filtration then the images, under ϕ, of the filtrands of $k[\mathbf{u}]$ form a filtration on $k[\mathbf{x}]$. On the level of associated graded algebras, ϕ induces an isomorphism between the associated graded algebra of $k[\mathbf{u}]$ and the associated graded algebra of $k[\mathbf{x}]$. Thus, by beginning with a term order filtration on $k[\mathbf{u}]$, and using ϕ to put a filtration on $k[\mathbf{x}]$, we see that the associated graded algebra of $k[\mathbf{x}]$ has one-dimensional graded components and is isomorphic $k[\mathbf{x}]$. Nevertheless the associated graded algebra of $k[\mathbf{x}]$ need not come from a term order filtration. Just as Theorem 2.6 characterizes which valuations come from a term order in suitable variables, we give a characterization of which filtrations come from a term order in suitable variables in Theorem 5.9.

Given a k-algebra A, we define a k-filtration to be a filtration in which the filtered pieces are k-subspaces of A. One of the conditions necessary for a filtration on $k[\mathbf{x}]$ to come from a term order in suitable variables is well-orderedness. In fact, well-orderedness is a required condition for most reduction algorithms, and so it suggests the important question of whether non-negative, strongly multiplicative k-filtrations on $k[\mathbf{x}]$ must necessarily be well-ordered. If this were so, it might allow the possibility of using all such orders in computer algebra systems. The importance of such filtrations is closely related to the importance of term orders themselves. Namely, they allow the formulation of an associated graded algebra, leading terms, a valuation, reduction, S-pairs and a generalization of Buchberger's test and algorithms as described in Sweedler (1986).

On the other hand, producing a non-negative, strongly multiplicative k-filtration on $k[\mathbf{x}]$ that is not well-ordered has some use because it cannot come from a term order filtration and so it would show that a filtration being a non-negative, strongly multiplicative k-filtration on $k[\mathbf{x}]$ does not imply that the filtration must be a term order filtration. As it turns out there are non-negative, strongly multiplicative k-filtrations on $k[\mathbf{x}]$ that are not well-ordered. The first such examples were given by Mosteig (2000). In addition, Mosteig (2000) demonstrates the existence of a regular well-ordered filtration on $k[x, y]$ for fields of both positive characteristic and characteristic zero. In characteristic zero, we yet have no example of a regular filtration that is not well-ordered, but we do have well-ordered filtrations (written in the language of valuations) that do not come from term orders in suitable variables (see Mosteig, 2000, 2002; Mosteig and Sweedler, 2001). This raises the open question of whether there exist regular filtrations on a polynomial ring of characteristic zero that are not well-ordered.

2. Valuations and Term Orders

We begin by reviewing the fundamentals of valuation theory, and then focus on a class of valuations on rational function fields that arise from term orders on underlying polynomial rings. Such valuations naturally give rise to the value monoid $\mathbb{N}^n$ of n-tuples of natural numbers. We then give criteria describing which valuations come from a term order in suitable variables.

2.1. Valuations

Whenever A is a ring, we use A^* to denote the multiplicative group of invertible elements of A and we use $A^\star$ to denote the non-zero elements of A.

By *monoid* we mean a set with an associative law of composition and a two-sided identity. Throughout this paper, we will assume that all monoids are commutative. We say $(M, \leq)$ is an *ordered monoid* if M is a monoid endowed with a total order $\leq$ such that for all $a, b, c \in M$,

$$a \leq b \Rightarrow a + c \leq b + c.$$

Given a total order $\leq$ on a monoid, we say that $a < b$, or that a is strictly less than b, whenever $a \leq b$ and $a \neq b$.

A *valuation ring* V of a field K is a subring such that for all $a \in K^*$, $a \in V$ or $a^{-1} \in V$. Given a valuation ring V of a field K, the corresponding *value group* is the multiplicative quotient group

$$G = K^*/V^*.$$

The natural quotient map

$$\nu : K^* \to K^*/V^*$$

is called the *valuation* associated to V where G is given the following total order: for $a, b \in K^*$,

$$\nu(a) \leq \nu(b) \qquad \text{iff } a/b \in V.$$

Since $-1 \in V$ it follows that $\nu(a) = \nu(-a)$ for all $a \in K^\star$. The order on G accommodates the following strong triangle inequality,

$$\nu(a + b) \leq \max(\nu(a), \nu(b)) \qquad \text{for } a, b \in K^* \text{ with } a + b \neq 0, \tag{1}$$

which can be strengthened further:

$$\nu(a + b) = \max(\nu(a), \nu(b)) \qquad \text{for } a, b \in K^* \text{ with } \nu(a) \neq \nu(b). \tag{2}$$

One may recover V from ν and the ordering on G. If id_G is the identity of G, the corresponding valuation ring is

$$V = \{0\} \cup \{a \in K^* \mid \nu(a) \leq \mathrm{id}_G\} \tag{3}$$

with unique maximal ideal

$$m = \{0\} \cup \{a \in K^* \mid \nu(a) < \mathrm{id}_G\}. \tag{4}$$

A useful consequence of these expressions for V and m is the observation that for $a, b \in K^*$,

$$\nu(a) \leq \nu(b) \qquad \text{iff } a = vb \text{ for some } v \in V; \tag{5}$$

$$\nu(a) < \nu(b) \qquad \text{iff } a = vb \text{ for some } v \in m. \tag{6}$$

Conversely, given a group homomorphism ν, from K^* to a totally ordered group G, that satisfies (1) and (2), we see that ν is a valuation with value group isomorphic to K^*/V^* where V is given by (3).

Definition. The *residue class field* of a valuation ν with valuation ring V is V/m. Given $v \in V, \pi(v)$ is called the image of v in the residue class field, where $\pi : V \to V/m$ is the canonical projection.

Lemma 2.1. *Let ν be a valuation on K with valuation ring V and let k be a subfield of K that lies in V. Assume that k maps isomorphically to the residue class field of ν, or equivalently, that $V = k \oplus m$ as Abelian groups. If a and b are non-zero elements* *of K such that $\nu(a) \leq \nu(b)$, then there exists a unique $\lambda \in k$ such that $a = \lambda b$ or $\nu(a - \lambda b) < \nu(b)$. Equivalently, there exists a unique $\lambda \in k$ such that $a - \lambda b = xb$ for some $x \in m$.*

Proof. Since $\nu(a) \leq \nu(b)$ we have by (5) that $a = vb$ for some $v \in V$. Since $V = k \oplus m$ there is a unique $\lambda \in k$ such that $v = \lambda + x$ for some $x \in m$. Thus $a - \lambda b = xb$. Clearly, $x = 0$ if and only if $a = \lambda b$. If $x \neq 0$, then by (6) we have $\nu(xb) < \nu(x)$. We see by (6) that if $\nu(a - \lambda b) < \nu(b)$, then $a - \lambda b = xb$ for some $x \in m$. □

Given a valuation on a field, we often focus on its restriction to subrings of the field, and so we make the following definition.

Definition. Suppose that ν is a valuation on K and A is a subring of K. The submonoid $\nu(A^\star)$ of the value group is called the *value monoid* of A.

Example 2.2. (Sweedler, 1986) Given a term order "$\leq_\sigma$" on $k[\mathbf{x}]$ in $k(\mathbf{x})$,

$$V_\sigma = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{LT}(f) \leq_\sigma \mathrm{LT}(g)\}$$

is a valuation ring with maximal ideal

$$m_\sigma = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{LT}(f) <_\sigma \mathrm{LT}(g)\}.$$

If ν denotes the associated valuation, we say that ν (and V) *comes from a term order.*

Given two non-zero polynomials $f, g \in k[\mathbf{x}], \mathrm{LT}(f) \leq_\sigma \mathrm{LT}(g)$ if and only if $f/g \in V_\sigma$, by the definition of V_σ. However, this is just the condition for the inequality defined on the value group and so

$$\mathrm{LT}(F) \leq_\sigma \mathrm{LT}(g) \Leftrightarrow \nu_\sigma(f) \leq \nu_\sigma(g). \tag{7}$$

Lemma 2.3. *Suppose ν is a valuation (with valuation ring V and maximal ideal m) on $k(\mathbf{x})$ that comes from a term order on $k[\mathbf{x}]$. Then*

(*i*) *$k \subset V$.*
(*ii*) *The residue class field of V is k.*
(*iii*) *The elements of the value monoid of $k[\mathbf{x}]$ are non-negative.*
(*iv*) *The value monoid of $k[\mathbf{x}]$ is isomorphic to $\mathbb{N}^n$.*

Proof.

(i) Since each $\lambda \in k$ can be written as $\lambda/1$, we have $k \subset V_\sigma$.
(ii) We see by (i) that k is embedded isomorphically into the residue class field via the canonical projection $\pi : V_\sigma \to V_\sigma/m_\sigma$. We need to show that the restriction of π to k is surjective. That is, given $f/g \in V_\sigma$, we wish to find $\lambda \in k$ such that $\pi(f/g) = \pi(\lambda)$.

Given $f/g \in V_\sigma$, we have $\mathrm{LT}(f) \leq_\sigma \mathrm{LT}(g)$. If $\mathrm{LT}(f) <_\sigma \mathrm{LT}(g)$, then $f/g \in m_\sigma$, and so $\pi(f/g) = \pi(0)$. If $\mathrm{LM}(f) = \mathrm{LM}(g)$, then $\mathrm{LT}(f) = \lambda \mathrm{LT}(g)$ for some $\lambda \in k^\star$, and so $\mathrm{LT}(f - \lambda g) <_\sigma \mathrm{LT}(g)$. Therefore, $\mathrm{LT}((f - \lambda g)/g) \in m_\sigma$, and so

$$\frac{f}{g} = \lambda + \frac{f - \lambda g}{g}$$

expresses f/g as the sum of λ and an element of m_σ, showing that $\pi(f/g) = \pi(\lambda)$.

(iii) Since $\mathrm{LT}(f) \geq_\sigma 1$ for all $f \in k[\mathbf{x}]^*$, we have $1/f \in V_\sigma$, and so by the order defined on the value group, $\nu(f) \geq \nu(1)$.

(iv) We define a map $\gamma : \nu(k[\mathbf{x}]^\star) \to \mathbb{N}^n$ and show that it is a monoid isomorphism. For each $f \in k[\mathbf{x}]^*$, we can uniquely write its leading term as $\mathrm{LT}(f) = \lambda \mathbf{x}^e$ for some $e \in \mathbb{N}^n$ and $\lambda \in k^*$. Define $\gamma(\nu(f))$ to be $\mathbf{e} \in \mathbb{N}^n$. The first matter is to show that γ is well-defined. Note that $\nu(f) = \nu(g)$ if and only if $f/g \in V_\sigma^*$. However, $f/g \in V_\sigma^\star$ whenever $\mathrm{LM}(f) = \mathrm{LM}(g)$. Thus if $\nu(f) = \nu(g)$, then f and g yield the same exponent $\mathbf{e} \in \mathbb{N}^n$, and so γ is well-defined. Since the exponent of the leading monomial of $1 \in k[\mathbf{x}]$ is $0 \in \mathbb{N}^n$, and the exponent of the leading monomial of a product of polynomials is the sum of the exponents of the leading monomials of each polynomial, γ is a monoid map. Now $\gamma(\nu(x_i)) = \mathbf{e}_i$ where $\mathbf{e}_i$ is the vector of length n consisting of all zeros except for a 1 in the ith position. Thus the image of γ contains a monoid generating set of $\mathbb{N}^n$, and hence γ is surjective. It remains to be shown that γ is injective. If $\gamma(\nu(f_1)) = \gamma(\nu(f_2))$, then $\mathrm{LM}(f_1) = \mathrm{LM}(f_2)$, and so $\nu(f_1) = \nu(f_2)$, Thus γ is an isomorphism of monoids. □

In Example 2.2, the valuation ring does not contain the original polynomial ring. However, by using a different construction we can guarantee that the polynomial ring is contained in the valuation ring. First, we define the trailing term trail(f) of a polynomial f as the smallest non-zero term (as given by the term order) that appears in the polynomial f. This gives another example of a valuation ring arising from a polynomial ring with a term order.

Example 2.4. Given a term order $\leq_\sigma$ on $k[\mathbf{x}]$ in $k(\mathbf{x})$, define

$$V_\sigma = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{trail}(f) \geq_\sigma \mathrm{trail}(g)\}.$$

Then V is a valuation ring with unique maximal ideal

$$m_\sigma = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{trail}(f) >_\sigma \mathrm{trail}(g)\}.$$

In this example, V contains the polynomial ring because $f = f/1$ and $\mathrm{trail}(f) \geq \mathrm{trail}(1) = 1$.

2.2. term orders in suitable variables

We wish to consider valuations and valuation rings that come from a term order on $k[\mathbf{x}]$ followed by an automorphism of $k[\mathbf{x}]$. This is the same as considering valuations that come from a term order on $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ with respect to a change of variables. We found that taking the direct approach, either considering $k[\mathbf{x}]$ together with an automorphism or considering $k[\mathbf{x}]$ together with a change of variables, leads to confusion. To avoid such confusion we introduce a second polynomial ring $k[\mathbf{u}] = k[u_1, \ldots, u_n]$ and a k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$.

Suppose that $k[\mathbf{u}]$ has a term order $\leq_\sigma$. As in Example 2.2, the order $\leq_\sigma$ gives rise to the valuation ring V_σ in $k(\mathbf{u})$. The k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ extends to an isomorphism $k(\mathbf{u}) \to k(\mathbf{x})$, which is also denoted by ϕ. The isomorphism $\phi : k(\mathbf{u}) \to k(\mathbf{x})$ maps V_σ (isomorphically) to $\phi(V_\sigma)$, a valuation ring in $k(\mathbf{x})$ with maximal ideal $\phi(m_\sigma)$. Since $V_\sigma = k \oplus m_\sigma$ and ϕ is the identity on k it follows that $\phi(V_\sigma) = k \oplus \phi(m_\sigma)$. In other words, $\phi(V_\sigma)$ has residue class field k. It is straightforward to check the following:

$$\begin{aligned}\phi(V_\sigma) &= \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{LT}(\phi^{-1}(f)) \leq_\sigma \mathrm{LT}(\phi^{-1}(g))\}\\ \phi(m_\sigma) &= \{0\} \cup \{f/g \mid f, g \in k[\mathbf{x}]^\star \text{ and } \mathrm{LT}(\phi^{-1}(f)) <_\sigma \mathrm{LT}(\phi^{-1}(g))\}.\end{aligned} \tag{8}$$

We introduce the notation $V_{\phi(\sigma)}$ and $m_{\phi(\sigma)}$ to denote $\phi(V_\sigma)$ and $\phi(m_\sigma)$, respectively. We use $\nu_{\phi(\sigma)}$ to denote the valuation on $k(\mathbf{x})$ arising from $V_{\phi(\sigma)}$, and we use ϕ^* to denote the restriction of ϕ to $k(\mathbf{u})^*$. Note that ϕ^* is a multiplicative group isomorphism from $k(\mathbf{u})^*$ to $k(\mathbf{x})^*$ and carries V_σ^* isomorphically to $V_{\phi(\sigma)}^*$. Hence ϕ^* induces an ordered group isomorphism of the value groups, $k(\mathbf{u})^*/V_\sigma^* \to k(\mathbf{x})^*/V_{\phi(\sigma)}^*$, which we denote by $\bar{\phi}^*$. We have the following commutative diagram, which is the key to proving Lemma 2.5.

$$\begin{array}{ccc} k(\mathbf{u})^* & \xrightarrow{\phi^*} & k(\mathbf{x})^* \\ \downarrow{\scriptstyle \nu_\sigma} & & \downarrow{\scriptstyle \nu_{\phi(\sigma)}} \\ k(\mathbf{u})^*/V_\sigma^* & \xrightarrow{\bar{\phi}^*} & k(\mathbf{x})^*/V_{\phi(\sigma)}^* \end{array} \tag{9}$$

Definition. Let ν be a valuation on $k(\mathbf{x})$ with valuation ring V. We say that ν or V comes from a term order *in suitable variables* on $k[\mathbf{x}]$ if there is a polynomial ring $k[\mathbf{u}]$ with term order $\leq_\sigma$ and k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ whereby $\nu = \nu_{\phi(\sigma)}$ or equivalently $V = V_{\phi(\sigma)}$. The isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ is called the *associated isomorphism.*

In the proof of Theorem 2.6, we are given a valuation ν and a term order $\leq_\sigma$, and we use them to construct an isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$. Note that we are forced to define ϕ in the given matter, and so it is uniquely defined, thus justifying the terminology "associated isomorphism".

Lemma 2.5. *Suppose that ν comes from a term order in suitable variables on $k[\mathbf{x}]$. The value monoid of $k[\mathbf{x}]$ with respect to the valuation $\nu_{\phi(\sigma)}$ is isomorphic to $\mathbb{N}^n$ and consists solely of non-negative elements.*

Proof. Let $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ be the associated isomorphism. We shall show that $\bar{\phi}^*$ carries the value monoid of $k[\mathbf{u}]$ with respect to ν_σ isomorphically to the value monoid of $k[\mathbf{x}]$ with respect to $\nu_{\phi(\sigma)}$. Now, ν_σ is defined as the valuation of $k(\mathbf{u})^*$ that comes from the term order $\leq_\sigma$ on $k[\mathbf{u}]$. Thus by Lemma 2.3, the values monoid of $k[\mathbf{u}]$ with respect to the valuation ν_σ is isomorphic to $\mathbb{N}^n$ and consists solely of non-negative elements. Thus we need only demonstrate our original claim concerning $\bar{\phi}^*$.

Consider $k[\mathbf{u}]^\star \subset k(\mathbf{u})^*$ and follow its path under each composite in the commutative diagram in (9). For the first composite, $\phi^\star(k[\mathbf{u}]^\star) = k[\mathbf{x}]^\star$, and $\nu_{\phi(\sigma)}(k[\mathbf{x}]^\star)$ is the value monoid of $k[\mathbf{x}]$ with respect to the valuation $\nu_{\phi(\sigma)}$. For the second composite, $\nu_\sigma(k[\mathbf{u}]^\star)$ is the value monoid of $k[\mathbf{u}]$ with respect to ν_σ, and $\bar{\phi}^*(\nu_\sigma(k[\mathbf{u}]^\star))$ is an isomorphic copy of $\nu_\sigma(k[\mathbf{u}]^\star)$. Since the two composites are equal the lemma follows. $\square$

The following characterizes valuations on $k(\mathbf{x})$ that come from term orders in suitable variables.

Theorem 2.6. *Let ν be a valuation on $k(\mathbf{x})$ with valuation ring V and maximal ideal m. Then ν comes from a term order in suitable variables on $k[\mathbf{x}]$ if and only if it satisfies* *all of the following properties:*

(*i*) $k \subset V$.
(*ii*) *The residue class field of* V *is* k.
(*iii*) *The elements of the value monoid of* $k[\mathbf{x}]$ *are non-negative.*
(*iv*) *The value monoid of* $k[\mathbf{x}]$ *is isomorphic to* $\mathbb{N}^n$.

Proof. Suppose that ν comes from a term order in suitable variables on $k[\mathbf{x}]$ and let $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ be the associated isomorphism. By Lemma 2.5, parts (iii) and (iv) are satisfied. By Lemmas 2.1 and 2.3, $k \subset V_\sigma$ and $V_\sigma = k \oplus m_\sigma$. Since ϕ is a k-algebra isomorphism from $k(\mathbf{u})$ to $k(\mathbf{x})$ and it maps V_σ isomorphically to $V_{\phi(\sigma)}$, it follows that $k \subset V_{\phi(\sigma)}$ and $V_{\phi(\sigma)} = k \oplus m_{\phi(\sigma)}$. Hence (i) and (ii) are satisfied, concluding the proof of the *only if* direction.

Conversely, suppose that (i)–(iv) are satisfied. Since the value monoid is isomorphic to $\mathbb{N}^n$, we identify the two and assume the value monoid is precisely $\mathbb{N}^n$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{N}^n$ as a monoid where $\mathbf{e}_i$ is the vector of length n consisting of all zeros except for a 1 in the ith position. For each $\mathbf{e}_i$ choose $f_i \in k[\mathbf{x}]^*$ such that $\nu(f_i) = \mathbf{e}_i$. We will prove that $\{f_1, \ldots, f_n\}$ is an algebraically independent set that generates $k[\mathbf{x}]$ as an algebra. Assuming this has been established, we can define a k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ where $u_i \mapsto f_i$. By the choices we have made, $\nu(\phi(\mathbf{u}^\mathbf{e})) = \mathbf{e} \in \mathbb{N}^n$ for any monomial $\mathbf{u}^\mathbf{e} \in k[\mathbf{u}]$. Thus $\nu \circ \phi$ maps distinct monomials in $k[\mathbf{u}]$ to distinct values in $\mathbb{N}^n$. This allows us to define a term order on $k[\mathbf{u}]$ as follows: for monomials $\mathbf{u}^\mathbf{e}, \mathbf{u}^\mathbf{f} \in k[\mathbf{u}]$ define $\mathbf{u}^\mathbf{e} <_\sigma \mathbf{u}^\mathbf{f}$ if and only if $\nu(\phi(\mathbf{u}^\mathbf{e})) < \nu(\phi(\mathbf{u}^\mathbf{f}))$ in the ordered value group. All monomials of $k[\mathbf{u}]$ are greater than or equal to $\mathbf{u}^0 = 1$ since the value monoid is assumed to be non-negative. Moreover, compatibility of "$\leq_\sigma$" with multiplication of terms follows from ν being a homomorphism of multiplicative groups, and so "$\leq_\sigma$" is a term order. Since $\mathbf{u}^\mathbf{e} <_\sigma \mathbf{u}^\mathbf{f}$ if and only if $\nu(\phi(\mathbf{u}^\mathbf{e})) < \nu(\phi(\mathbf{u}^\mathbf{f}))$, it is easy to see by the strong triangle inequality (2) that for any $p \in k[\mathbf{u}]^*, \nu(\phi(p)) = \nu(\phi(\mathrm{LT}(p)))$. Therefore,

$$\begin{aligned}\nu(f) \leq \nu(g) &\Leftrightarrow \nu(\phi(\phi^{-1}(f))) \leq \nu(\phi(\phi^{-1}(g))) \\ &\Leftrightarrow \nu(\phi(\mathrm{LT}(\phi^{-1}(f)))) \leq \nu(\phi(\mathrm{LT}(\phi^{-1}(g)))) \\ &\Leftrightarrow \mathrm{LT}(\phi^{-1}(f)) \leq_\sigma \mathrm{LT}(\phi^{-1}(g)),\end{aligned}$$

and so by the top equation in (8),

$$\phi(V_\sigma) = \{0\} \cup \{f/g \mid f, g \in k[\mathbf{u}]^\star \text{ and } \nu(f) \leq \nu(g)\}.$$

However, this set can be rewritten as

$$\{0\} \cup \{h \in k(\mathbf{x})^* \mid \nu(h) \leq \mathrm{id}_G\},$$

and so it must be the valuation ring of ν. Hence ν comes from a term order in suitable variables on $k[\mathbf{x}]$.

It remains to be shown that $\{f_1, \ldots, f_n\}$ is an algebraically independent set that generates $k[\mathbf{x}]$ as an algebra. Note that $k[\mathbf{x}]$ has transcendence degree n over k and the set $\{f_1, \ldots, f_n\}$ only has n elements. Therefore, if $\{f_1, \ldots, f_n\}$ generates $k[\mathbf{x}]$, then it is an algebraically independent set.

Note that the order on the image-ring submonoid $\mathbb{N}^n$ induced by ν is a total order on $\mathbb{N}^n$ that is compatible with addition in $\mathbb{N}^n$. Moreover, all the elements of $\mathbb{N}^n$ are non-negative with respect to this order due to (iii), and so by Dickson's Lemma, the order induced by ν is a well-ordering on $\mathbb{N}^n$.

We now show that $k[\mathbf{x}] \subseteq k[f_1, \ldots, f_n]$. Suppose, for contradiction, there exists a $f \in k[\mathbf{x}]$ such that $f \notin k[f_1, \ldots, f_n]$. Among such elements, choose f so that $\nu(f) = (e_1, \ldots, e_n)$ is minimal with respect to the ordering on $\mathbb{N}^n$ induced by ν. Since $\nu(f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}) = (e_1, \ldots, e_n)$ it follows from Lemma 2.1 that for some $\lambda \in k$,

$$f = \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n} \text{ or } \nu(f - \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n}) < \nu(f).$$

If $f = \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n}$, then $f \in k[f_1, \ldots, f_n]$, a contradiction. If $f \neq \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n}$ and $\nu(f - \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n}) < \nu(f)$, then by the minimality of $\nu(f)$ it follows that $f - \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n} \in k[f_1, \ldots, f_n]$. But then $f = (f - \lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n}) + (\lambda f_1^{e_1} f_2^{e_2} \ldots f_n^{e_n})$, thus showing that f is the sum of two elements of $k[f_1, \ldots, f_n]$, and hence itself lies in $k[f_1, \ldots, f_n]$. This is, once again, a contradiction, and so $k[\mathbf{x}] = k[f_1, \ldots, f_n]$. □

This leads us to the following corollary as suggested by Dexter Kozen.

Corollary 2.7. *If ν be a valuation on $k(x_1, \ldots, x_n)$ with valuation ring V, then ν comes from a term order on $k[\mathbf{x}]$ if and only if it satisfies:*

(*i*) *$k \subset V$.*
(*ii*) *The residue class field of V is k.*
(*iii*) *The elements of the value monoid of $k[\mathbf{x}]$ are non-negative.*
(*iv*) *The value monoid of $k[\mathbf{x}]$ is isomorphic to $\mathbb{N}^n$.*
(*v*) *If $\mathbf{x}^\mathbf{m}$ and $\mathbf{x}^\mathbf{n}$ are distinct monomials, then $\nu(\mathbf{x}^\mathbf{m}) \neq \nu(\mathbf{x}^\mathbf{n})$.*

Proof. If ν comes directly from a term order, then property (v) certainly holds, and (i)–(iv) hold due to Theorem 2.6.

Conversely, suppose (i)–(v) hold. Given distinct terms $\mathbf{x}^\mathbf{m}, \mathbf{x}^\mathbf{n}$, we have that $\nu(\mathbf{x}^\mathbf{m}) \neq \nu(\mathbf{x}^\mathbf{n})$, and so ν defines a total order on the set of monomials. By Theorem 2.6, we know there exists an automorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ and a term order on $k[\mathbf{u}]$ such that $\nu(\mathbf{x}^\mathbf{m}) < \nu(\mathbf{x}^\mathbf{n})$ whenever $\mathrm{LT}(\phi^{-1}(\mathbf{x}^\mathbf{m})) < \mathrm{LT}(\phi^{-1}(\mathbf{x}^\mathbf{n}))$. Using this we can show that all monomials in $k[\mathbf{x}]$ are non-negative and that the order is compatible with multiplication, and so ν comes from a term order. □

3. Filtrations

In this section we present the material that is fundamental to working with nested filtrations on an arbitrary ring A. Throughout this section and the following sections, A is assumed to be a commutative ring containing a field k. It may be helpful for the reader to consider the conditions and results in this section for the particular case where A is a polynomial ring. In fact, we will see in Example 3.1 a filtration on a polynomial ring that arises from a term order. This fundamental example raised many of the questions that motivated much of the present research.

We discuss filtrations and many of the desirable properties needed for the computational framework discussed in the valuation-theoretic context in Sweedler (1986). Each filtration provides a quasi-order and equivalence relation on the original ring. In the case the ring A is a k-algebra, this quasi-order will provide us with a method of constructing an associated k-vector space. If we require more properties of our filtration such as multiplicativity, then this k-vector space can be made into a k-algebra.

3.1. NESTED FILTRATIONS

In the general definition of an *ascending* filtration on a set A, one has $\mathcal{F}$, a set of subsets of A, where for $S, T \in \mathcal{F}$ there exists $U \in \mathcal{F}$ with $S \subset U \supset T$. A **descending** filtration $\mathcal{F}$ has the property such that for $S, T \in \mathcal{F}$, there exists $U \in \mathcal{F}$ with $S \supset U \subset T$. Because this paper is concerned with the interaction between term orders, valuations and filtrations, the relevant filtrations are nested filtrations. Thus we only consider nested filtrations, and we frequently say "filtration" when we mean "nested filtration".

DEFINITION. Let $\mathcal{F}$ be a subset of the power set of A. That is, $\mathcal{F}$ is a set whose elements are subsets of A. We call $\mathcal{F}$ a *nested* filtration on A if for any $S, T \in \mathcal{F}$, either $S \subset T$ or $S \supset T$. In this case, the elements of $\mathcal{F}$ are called the *filtered pieces* or *filtrands* of A.

DEFINITION. A filtration $\mathcal{F}$ has *full union* if

$$A = \bigcup_{\{S \in \mathcal{F} \mid S \neq A\}} S.$$

Note that we do not say that $\mathcal{F}$ has full union if $A = \cup_{S \in \mathcal{F}} S$ but rather we exclude A from the union. This is because given a filtration $\mathcal{F}$, if $A \notin \mathcal{F}$, then one may add A to $\mathcal{F}$ to form $\mathcal{F}' = \{A\} \cup \mathcal{F}$. Moreover, if $A \in \mathcal{F}$, then one may exclude A from $\mathcal{F}$ by forming $\mathcal{F}' = \mathcal{F} \setminus \{A\}$. Either way, $\mathcal{F}'$ and $\mathcal{F}$ have the same *essential* properties, as seen in Proposition 3.4. Therefore, the presence or absence of A in $\mathcal{F}$ should not be a factor in definitions, properties, results, etc. In fact, as the reader proceeds through this section, numerous instances will be seen where the presence or absence of A in $\mathcal{F}$ is irrelevant. This is typically achieved by conditioning or indexing over the set $\mathcal{F} \setminus \{A\} = \{S \in \mathcal{F} \mid S \neq A\}$.

DEFINITION. A filtration $\mathcal{F}$ *intersects to zero* if

$$\{0\} = \bigcap_{S \in \mathcal{F}} S.$$

As will become evident from examples, it is common for $\{0\}$ to be the unique minimal element of $\mathcal{F}$. In such situations, $\mathcal{F}$ obviously intersects to zero.

Here now is the filtration on a polynomial ring arising from a term order. This filtration exhibits the properties we look for in other filtrations. Many of the results in this paper arise from considering possible generations of term order filtrations.

EXAMPLE 3.1. Let $k[\mathbf{x}]$ be a polynomial ring with term order "$\leq_\sigma$". We denote the collection of monomials in $k[\mathbf{x}]$ by $\mathbb{M}[\mathbf{x}]$. We define a *term order filtration* $\mathcal{F}_\sigma$ on $k[\mathbf{x}]$ as follows. For each $\mathbf{e} \in \mathbb{N}^n$ define the filtrand

$$k[\mathbf{x}]_{\leq \sigma^e} = \mathrm{Span}_k(\{\mathbf{x^u} \in \mathbb{M}[\mathbf{x}] \mid \mathbf{x^u} \leq_\sigma \mathbf{x^e}\})$$

where Span_k represents the k-subspace of $k[\mathbf{x}]$ spanned by the indicated monomials. Let $\mathcal{F}_\sigma$ consist of $\{0\}$ together with all of the subspaces of $k[\mathbf{x}]$ of the form $k[\mathbf{x}]_{\leq \sigma^e}$ for $\mathbf{e} \in \mathbb{N}^n$. We also define

$$k[\mathbf{x}]_{< \sigma^e} = \mathrm{Span}_k(\{\mathbf{x^u} \in \mathbb{M}[\mathbf{x}] \mid \mathbf{x^u} <_\sigma \mathbf{x^e}\})$$

and

$$k[\mathbf{x}]_{\sim \sigma^{\mathbf{e}}} = \{f \in k[\mathbf{x}]^\star \mid \mathrm{LT}(f) = \mathbf{x^e}\}.$$

We see that $\mathcal{F}_\sigma$ is totally ordered because $\{0\}$ lies in each $k[\mathbf{x}]_{\leq_\sigma \mathbf{e}}$, and for distinct monomials $\mathbf{x}^\mathbf{u}, \mathbf{x}^\mathbf{v}$, either $\mathbf{x}^\mathbf{u} <_\sigma \mathbf{x}^\mathbf{v}$ or $\mathbf{x}^\mathbf{v} <_\sigma \mathbf{x}^\mathbf{u}$, in which case, respectively, $k[\mathbf{x}]_{\leq_\sigma \mathbf{u}} \subset k[\mathbf{x}]_{\leq_\sigma \mathbf{v}}$ or $k[\mathbf{x}]_{\leq_\sigma \mathbf{v}} \subset k[\mathbf{x}]_{\leq_\sigma \mathbf{u}}$. Thus $\mathcal{F}_\sigma$ is a filtration. Now, $\mathcal{F}_\sigma$ has full-union because 0 lies in all elements of $\mathcal{F}_\sigma$ and for all $f \in k[\mathbf{x}]^\star$ such that $\mathrm{LM}(f) = \mathbf{x}^\mathbf{e}$, we have $f \in k[\mathbf{x}]_{\leq_\sigma \mathbf{e}}$. Also, $\mathcal{F}_\sigma$ intersects to zero because $\{0\}$ is its unique minimal element.

Note that the set $\mathcal{F}_\sigma \setminus \{\{0\}\}$ is indexed by $\mathbb{N}^n$ since each $k[\mathbf{x}]_{\leq_\sigma \mathbf{e}}$ is distinct for distinct values of $\mathbf{e} \in \mathbb{N}^n$. This is easily seen because for distinct monomials $\mathbf{x}^\mathbf{u}, \mathbf{x}^\mathbf{v}$ with $\mathbf{x}^\mathbf{u} <_\sigma \mathbf{x}^\mathbf{v}$ we have that $\mathbf{x}^\mathbf{v} \in k[\mathbf{x}]_{\leq_\sigma v} \setminus k[\mathbf{x}]_{\leq_\sigma u}$.

The above filtration satisfies and motivates some properties that we will soon define. For example, a term order filtration $\mathcal{F}_\sigma$ is a k-filtration since all of the elements of $\mathcal{F}_\sigma$ are k-subspaces of $k[\mathbf{x}]$. It will also be seen that a term order filtration has one-dimensional graded components and is strongly multiplicative. Because term order filtrations provide motivation for many of the definitions and results in this paper, we show how these definitions and results specifically apply to such filtrations.

3.2. QUASI-ORDERS AND ASSOCIATED GRADED VECTOR SPACES

A quasi-order is a binary relation that is reflexive, transitive and total. In other words, a quasi-order is like a total order except a quasi-order lacks the property "$a \leq b$ and $b \leq a \Rightarrow a = b$". To describe the correspondence between valuations and filtrations, a natural first step is to observe that a natural quasi-order and an equivalence relation on the ring A can be constructed using a filtration. In the case A is a k-algebra, these constructions allow us to form an associated graded k-vector space.

DEFINITION. A filtration $\mathcal{F}$ on A yields a quasi-order on A called the *filtration order*. Given $a, b \in A$, we write $a \leq_\mathcal{F} b$ if for each $S \in \mathcal{F}$ where $b \in S$, it follows that $a \in S$. We write $a <_\mathcal{F} b$ if $a \leq_\mathcal{F} b$ but $b \not\leq_\mathcal{F} a$. Equivalently, $a <_\mathcal{F} b$ if $a \leq_\mathcal{F} b$ and there exists $S \in \mathcal{F}$ where $a \in S$ but $b \notin S$.

In Section 5, we will see that for a term order filtration on $k[x]$ we can describe the quasi-ordering by means of leading monomials. Quasi-orders are transitive, and so they yield an equivalence relation.

DEFINITION. A filtration $\mathcal{F}$ on A yields an equivalence relation on A called the *filtration equivalence*. Given $a, b \in A$, we write $a \sim_\mathcal{F} b$ if $a \leq_\mathcal{F} b$ and $b \leq_\mathcal{F} a$.

Using the filtration order and filtration equivalence, we define the following subsets of A. Given $a \in A$, we define

$$A_{\leq_\mathcal{F} a} = \{b \in A \mid b \leq_\mathcal{F} a\} \tag{10}$$

$$A_{<_\mathcal{F} a} = \{b \in A \mid b <_\mathcal{F} a\} \tag{11}$$

$$A_{\sim_\mathcal{F} a} = \{b \in A \mid b \sim_\mathcal{F} a\}. \tag{12}$$

Note that $A_{\leq_\mathcal{F} a}$ and $A_{<_\mathcal{F} a}$ may or may not be filtrands of A. Generally $A_{\sim_\mathcal{F} a}$ will not be a filtrand of A. Section 5 exhibits (10)–(12) for the specific case in which A is a polynomial ring.

Here are some fundamental properties of filtrations.

LEMMA 3.2. *Let $\mathcal{F}$ be a filtration on A and let $a, b \in A$. Then*

(*i*) *The quasi-order is total in the sense that either* $a <_{\mathcal{F}} b$ *or* $b \leq_{\mathcal{F}} a$.
(*ii*) $A_{\sim_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} a} \setminus A_{<_{\mathcal{F}} a}$.
(*iii*) $A_{\leq_{\mathcal{F}} a} = \cap_{\{S\in\mathcal{F}|a\in S\}} S$. *If* $\{S \in \mathcal{F} \mid a \in S\} = \emptyset$, *we set* $\bigcap_{\{S\in\mathcal{F}|a\in S\}} S = A$.
(*iv*) $A_{<_{\mathcal{F}} a} = \cup_{\{S\in\mathcal{F}|a\notin S\}} S$. *If* $\{S \in \mathcal{F} \mid a \notin S\} = \emptyset$, *we set* $\bigcup_{\{S\in\mathcal{F}|a\notin S\}} S = A$.
(*v*) $A_{\sim_{\mathcal{F}} a} = \cap_{\{S\in\mathcal{F}|a\in S\}} S \setminus \cup_{\{S\in\mathcal{F}|a\notin S\}} S$.
(*vi*) *If* $a \sim_{\mathcal{F}} b$ *then* $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}, A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$ *and* $A_{\sim_{\mathcal{F}} a} = A_{\sim_{\mathcal{F}} b}$.
(*vii*) *If* $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}$ *or* $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$ *or* $A_{\sim_{\mathcal{F}} a} \subset A_{\sim_{\mathcal{F}} b}$ *or* $A_{\sim_{\mathcal{F}} a} \supset A_{\sim_{\mathcal{F}} b}$ *then* $a \sim_{\mathcal{F}} b$.

PROOF.

(i) If $b \not\leq_{\mathcal{F}} a$, then by definition there exists $S \in \mathcal{F}$ where $a \in S$ but $b \notin S$. Suppose that $T \in \mathcal{F}$ with $b \in T$. Since we are working filtrations that are nested, either $S \subset T$ or $S \supset T$. It cannot happen that $S \supset T$ because $b \notin S$ but $b \in T$. Thus, $S \subset T$ and $a \in T$. We have shown that for any $T \in \mathcal{F}$ with $b \in T$, T must also contain a. Thus $a \leq_{\mathcal{F}} b$. Therefore the conditions for $a <_{\mathcal{F}} b$ are satisfied.
(ii) Clearly, $A_{\sim_{\mathcal{F}} a} \subset A_{\leq_{\mathcal{F}} a} \setminus A_{<_{\mathcal{F}} a}$. On the other hand if $b \notin A_{<_{\mathcal{F}} a}$ then $a \leq_{\mathcal{F}} b$ by part (i). Thus if $b \in A_{\leq_{\mathcal{F}} a} \setminus A_{<_{\mathcal{F}} a}$, it follows that $b \sim_{\mathcal{F}} a$. Hence $A_{\leq_{\mathcal{F}} a} \setminus A_{<_{\mathcal{F}} a} \subset A_{\sim_{\mathcal{F}} a}$.
(iii) We have $b \in \cap_{\{S\in\mathcal{F}|a\in S\}} S$ if and only if $b \in S$ for every $S \in \mathcal{F}$ such that $a \in S$. This is just the condition for $b <_{\mathcal{F}} a$; i.e. $b \in A_{<_{\mathcal{F}} a}$.
(iv) We have $b \in \cup_{\{S\in\mathcal{F}|a\notin S\}} S$ if and only if there exists $S \in \mathcal{F}$ such that $a \notin S$ and $b \in S$. This is just the condition for $b \in A < \mathcal{F}a$; i.e. $b \in A_{<_{\mathcal{F}} a}$.
(v) This follows from (ii)–(iv).
(vi) If $a \sim_{\mathcal{F}} b$ then $\{S \in \mathcal{F} \mid a \in S\} = \{S \in \mathcal{F} \mid b \in s\}$ and $\{S \in \mathcal{F} \mid a \notin S\} = \{S \in \mathcal{F} \mid b \notin S\}$. By part (iii), $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}$, and by part (iv), $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$. Putting this together in conjunction with (ii), we get $A_{\sim_{\mathcal{F}} a} = A_{\sim_{\mathcal{F}} b}$.
(vii) If $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}$ then $b \in A_{\leq_{\mathcal{F}} a}$ and $a \in A_{\leq_{\mathcal{F}} b}$. Thus $a \sim_{\mathcal{F}} b$. If $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$ then $b \notin A_{<_{\mathcal{F}} a}$ and $a \notin A_{<_{\mathcal{F}} b}$. The order is total, and so $b \not<_{\mathcal{F}} a$ implies that $b \geq_{\mathcal{F}} a$. Similarly $a \not<_{\mathcal{F}} b$ implies that $a \geq_{\mathcal{F}} b$. Thus $a \sim_{\mathcal{F}} b$. If $A_{\sim_{\mathcal{F}} a} \subset A_{\sim_{\mathcal{F}} b}$ then $a \in A_{\sim_{\mathcal{F}} b}$ since $a \in A_{\sim_{\mathcal{F}} a}$. Thus $a \sim_{\mathcal{F}} b$. The result follows similarly for $A_{\sim_{\mathcal{F}} a} \supset A_{\sim_{\mathcal{F}} b}$. □

Lemma 3.2 has the following corollary whose proof is left to the reader.

COROLLARY 3.3. *Let* $\mathcal{F}$ *be a filtration on* A *and let* $a \in A$.

(*i*) $A_{\leq_{\mathcal{F}} a} = A \Leftrightarrow a \notin \cup_{\{S\in\mathcal{F}|S\neq A\}} S$.
(*ii*) $A_{<_{\mathcal{F}} a} = \emptyset \Leftrightarrow a \in \cap_{S\in\mathcal{F}} S$.

The following proposition shows that regarding the quasi-order arising from $\mathcal{F}$, it makes no difference whether or not $A \in \mathcal{F}$.

PROPOSITION 3.4. *If* $\mathcal{F}$ *is filtration on* A, *then* $\mathcal{F} \cup \{A\}$ *and* $\mathcal{F} \setminus \{A\}$ *are filtrations on* A. *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be filtrations on* A *where* $\mathcal{F}' = \mathcal{F} \uplus \{A\}$ (*where* $\uplus$ *stands for disjoint union*). *For* $a \in A$, *we have the following:*

$$A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}'} a}, \quad A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}'} a}, \quad A_{\sim_{\mathcal{F}} a} = A_{\sim_{\mathcal{F}'} a}.$$

PROOF. Since a filtration is simply a nested collection of subsets of A, the assertions about $\mathcal{F} \cup \{A\}$ and $\mathcal{F} \setminus \{A\}$ being filtrations are immediate.

If $a, b \in A$ with $a \leq_{\mathcal{F}} b$, then for each $S \in \mathcal{F}$ with $b \in S$, S must contain a. For $T \in \mathcal{F}'$ with $T = A$, both $a, b \in T$. For $T \in \mathcal{F}'$ with $T \neq A$, it follows that $T \in \mathcal{F}$ and hence if $b \in T$, then T contains a. Thus $a \leq_{\mathcal{F}'} b$. On the other hand, if $a \leq_{\mathcal{F}'} b$, then for each $T \in \mathcal{F}'$ with $b \in T, T$ must contain a. Since $\mathcal{F} \subset \mathcal{F}'$ it follows that for each $T \in \mathcal{F}$ with $b \in T, T$ must contain a. Thus $a \leq_{\mathcal{F}} b$. This proves $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}'} a}$.

Next, suppose that $\mathcal{F}$ and $\mathcal{F}'$ be filtrations on A where $\mathcal{F}' = \mathcal{F} \uplus \{A\}$. If $a, b \in A$ with $a <_{\mathcal{F}} b$, then there exists $S \in \mathcal{F}$ with $a \in S$ and $b \notin S$. But $S \in \mathcal{F}'$ since $\mathcal{F} \subset \mathcal{F}'$. Thus $a <_{\mathcal{F}'} b$. On the other hand if $a <_{\mathcal{F}'} b$, then there exists $T \in \mathcal{F}'$ with $a \in T$ and $b \notin T$. Since $b \notin T$ it follows that $T \neq A$ and thus $T \in \mathcal{F}$. Hence, $a <_{\mathcal{F}} b$. This proves $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}'} a}$.

It immediately follows from $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}'} a}$ and $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}'} a}$ that $A_{\sim_{\mathcal{F}} a} = A_{\sim_{\mathcal{F}'} a}$ because $A_{\sim_{\mathcal{F}} a}$ is the complement to $A_{<_{\mathcal{F}} a}$ in $A_{\leq_{\mathcal{F}} a}$ and $A_{\sim_{\mathcal{F}'} a}$ is the complement to $A_{<_{\mathcal{F}'} a}$ in $A_{\leq_{\mathcal{F}'} a}$. $\square$

The significance of the preceding proposition is that the quasi-order arising from a filtration is not affected by the presence or absence of A in the filtration. Thus a filtration cannot be uniquely determined by the quasi-order it induces. We would always change the filtration by adding or removing A to obtain a different filtration which induces the same quasi-order. In fact, A is not the only possible addition to a filtration $\mathcal{F}$ which will not change the quasi-order arising from $\mathcal{F}$. It is not difficult to show that if $a \in A$ and $A_{<_{\mathcal{F}} a} \notin \mathcal{F}$ then one may augment $\mathcal{F}$ to include $A_{<_{\mathcal{F}} a}$ without changing the partial order determined by $\mathcal{F}$. Such a possible change is illustrated by the following example.

Example 3.5. If $k[x, y]$ has the pure lex term order with $y <_{lex} x$, then $k[y] = k[x, y]_{<_{lex} x}$ is not one of the term order filtrands. Thus one can augment the term order filtration to include $k[y] = k[x, y]_{<_{lex} x}$ without changing the quasi-order determined by the term order filtration.

Since $\sim_{\mathcal{F}}$ is an equivalence relation, it may be used to partition A into equivalence classes of the form $A_{\sim_{\mathcal{F}} a}$.

Definition. Let $\mathcal{F}$ be a filtration on A. We denote by $A/\!/\mathcal{F}$ the set of "$\sim_{\mathcal{F}}$" equivalence classes. We denote by $A \wr \mathcal{F}$ the set of subsets of A of the form $A_{\leq_{\mathcal{F}} a}$. The map $\varsigma : A \to A \wr \mathcal{F}$ is defined by $\varsigma(a) = A_{\leq_{\mathcal{F}} a}$ for $a \in A$.

We shall primarily use $A \wr \mathcal{F}$ in place of $A/\!/\mathcal{F}$. In fact, we often identify $A/\!/\mathcal{F}$ and $A \wr \mathcal{F}$ since Proposition 3.6 (i) shows that

$$\begin{aligned} A/\!/\mathcal{F} &\to A \wr \mathcal{F} \\ A_{\sim_{\mathcal{F}} a} &\mapsto A_{\leq_{\mathcal{F}} a} = \varsigma(a) \end{aligned} \tag{13}$$

is a well-defined bijective map.

Let us briefly investigate the set $A \wr \mathcal{F}$. Given $a, b \in A$ such that $a \leq_{\mathcal{F}} b$, it follows that $\varsigma(a) = A_{\leq_{\mathcal{F}} a} \subset A_{\leq_{\mathcal{F}} b} = \varsigma(b)$. Thus $A \wr \mathcal{F}$ is a set of nested subsets of A and $A \wr \mathcal{F}$ has a total order with respect to inclusion; that is, for $S, T \in A \wr \mathcal{F}$, we have $S \leq T$ if and only if $S \subset T$. In Section 5, we consider $A \wr \mathcal{F}$ as a filtration on A. We show that $A \wr \mathcal{F}$ is in some sense a *normalization* of $\mathcal{F}$. Section 5 also presents a description of $A \wr \mathcal{F}$ and related considerations for the specific case where A is a polynomial ring and $\mathcal{F}$ comes from a term order.

Proposition 3.6. *Let $\mathcal{F}$ be a filtration on A and let $a, b \in A$. Consider the map $\varsigma : A \to A \wr \mathcal{F}$ defined in* (13).

(i) $\varsigma(a) = \varsigma(b)$ *if and only if* $a \sim_{\mathcal{F}} b$.
(ii) $\varsigma(a) < \varsigma(b)$ *if and only if* $a <_{\mathcal{F}} b$.
(iii) *If* $A \neq \cup_{\{S \in \mathcal{F} | S \neq A\}} S$, *then* $A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$ *is a "*$\sim_{\mathcal{F}}$*" equivalence class of* A.
(iv) *If* $a \in A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$, *then* $\varsigma(a)$ *is the unique maximal element of* $A \wr \mathcal{F}$.
(v) *If* $A \wr \mathcal{F}$ *has any maximal element, then* $A \neq \cup_{\{S \in \mathcal{F} | S \neq A\}} S$.
(vi) *If* $\emptyset \neq \cap_{S \in \mathcal{F}} S$, *then* $\cap_{S \in \mathcal{F}} S$ *is a "*$\sim_{\mathcal{F}}$*" equivalence class of* A.
(vii) *If* $a \in \cap_{S \in \mathcal{F}} S$, *then* $\varsigma(a)$ *is the unique minimal element of* $A \wr \mathcal{F}$.
(viii) *If* $A \wr \mathcal{F}$ *has any minimal element, then* $\emptyset \neq \cap_{S \in \mathcal{F}} S$.

Proof.

(i) Now, $a \sim_{\mathcal{F}} b$ if and only if $a \leq_{\mathcal{F}} b$ and $b \leq_{\mathcal{F}} a$. If $a \leq_{\mathcal{F}} b$, then $\varsigma(a) = A_{\leq_{\mathcal{F}} a} \subset A_{\leq_{\mathcal{F}} b} = \varsigma(b)$. In terms of the order on $A \wr \mathcal{F}$ this means that $\varsigma(a) \leq \varsigma(b)$. Similarly, $b \leq_{\mathcal{F}} a$ implies that $\varsigma(b) \leq \varsigma(a)$. Thus $\varsigma(a) = \varsigma(b)$. Conversely, if $\varsigma(a) = \varsigma(b)$, then $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}$. Thus $a \in A_{\leq_{\mathcal{F}} b}$ and $b \in A_{\leq_{\mathcal{F}} a}$; i.e. $a \leq_{\mathcal{F}} b$ and $b \leq_{\mathcal{F}} a$.
(ii) Here $\varsigma(a) < \varsigma(b)$ if and only if $A_{\leq_{\mathcal{F}} a} \subsetneq A_{\leq_{\mathcal{F}} b}$ if and only if ($a \leq_{\mathcal{F}} b$ and $b \not\leq_{\mathcal{F}} a$) if and only if $a <_{\mathcal{F}} b$.
(iii) Suppose that $A \neq \cup_{\{S \in \mathcal{F} | S \neq A\}} S$. Let a, b be elements in $A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$. By Corollary 3.3 (i), $A_{\leq_{\mathcal{F}} a} = A$, and thus $b \leq_{\mathcal{F}} a$. Similarly, $a \leq_{\mathcal{F}} b$ and so $a \sim_{\mathcal{F}} b$. Thus $A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$ is contained in a "$\sim_{\mathcal{F}}$" equivalence class of A. Now suppose $a \sim_{\mathcal{F}} b$ and $a \in A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$. By Lemma 3.2 (vi), $A_{\leq_{\mathcal{F}} a} = A_{\leq_{\mathcal{F}} b}$. By Corollary 3.3 (i), $A_{\leq_{\mathcal{F}} a} = A$, and so $A = A_{\leq_{\mathcal{F}} b}$. Another application of Corollary 3.3 (i) yields $b \in A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$.
(iv) As pointed out in the proof of part (iii), if $a \in A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$, then $A = A_{\leq_{\mathcal{F}} a} = \varsigma(a)$. Certainly A is the unique maximal subset of A, and so it is the unique maximal element of $A \wr \mathcal{F}$.
(v) Suppose that $A \wr \mathcal{F}$ has a maximal element. This means that there exists $a \in A$ with $A_{\leq \mathcal{F} a}$ maximal along sets of the form $A_{\leq_{\mathcal{F}} b}$. Since a filtration is nested, for $b \in A$, we have $b \in A_{\leq_{\mathcal{F}} b} \subset A_{\leq_{\mathcal{F}} a}$ and hence $A_{\leq_{\mathcal{F}} a} = A$. By Corollary 3.3 (i), $a \in A \setminus \cup_{\{S \in \mathcal{F} | S \neq A\}} S$. In particular, $A \neq \cup_{\{S \in \mathcal{F} | S \neq A\}} S$.
(vi) Suppose that $\emptyset \neq \cap_{\{S \in \mathcal{F}\}} S$. Let $a \in \cap_{S \in \mathcal{F}} S$. By Corollary 3.3 (ii), $A_{<_{\mathcal{F}} a} = \emptyset$. Thus for $b \in A$ we have $b \not<_{\mathcal{F}} a$ and so by Lemma 3.2 (i), $a \leq_{\mathcal{F}} b$. Similarly, if $b \in \cap_{S \in \mathcal{F}} S$, then $b \leq_{\mathcal{F}} a$. Thus if $a, b \in \cap_{S \in \mathcal{F}} S$, it follows that a $\sim_{\mathcal{F}} b$, and so $\cap_{S \in \mathcal{F}} S$ is contained in a "$\sim_{\mathcal{F}}$" equivalence class of A. Now suppose $a \sim_{\mathcal{F}} b$ and $a \in \cap_{S \in \mathcal{F}} S$. By Corollary 3.3 (ii), $A_{\leq_{\mathcal{F}} a} = \emptyset$. By Lemma 3.2 (vi), $A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$, and so $A_{<_{\mathcal{F}} b} = \emptyset$. Thus by Corollary 3.3 (ii), $b \in \cap_{S \in \mathcal{F}} S$.
(vii) As pointed out in the proof of part (vi), if $a \in \cap_{S \in \mathcal{F}} S$ then $a \leq_{\mathcal{F}} b$ for any $b \in A$. Thus $A_{\leq_{\mathcal{F}} a} \subset A_{\leq_{\mathcal{F}} b}$ for all $b \in A$ and we have shown that $A_{\leq_{\mathcal{F}} a}$ is minimal among subsets of A of the form $A_{\leq_{\mathcal{F}} b}$. Equivalently, $\varsigma(a)$ is the unique minimal element of $A \wr \mathcal{F}$.
(viii) Suppose that $A \wr \mathcal{F}$ has a minimal element. This means that there exists $a \in A$ with $A_{\leq_{\mathcal{F}} a}$ minimal among sets of the form $A_{\leq_{\mathcal{F}} b}$. Thus $a \leq_{\mathcal{F}} b$ for all $b \in A$ and so $A_{<_{\mathcal{F}} a} = \emptyset$. By Corollary 3.3 (ii), it follows that $a \in \cap_{S \in \mathcal{F}} S$. In particular, $\cap_{S \in \mathcal{F}} S \neq \emptyset$. □

Note that Proposition 3.6 (iii–v) shows that a filtration $\mathcal{F}$ has full union if and only if $A \wr \mathcal{F}$ does not have a maximal element. Moreover, Proposition 3.6 (vi–viii) shows that a filtration $\mathcal{F}$ intersects to zero if and only if $\varsigma(0)$ is a minimal element of $A \wr \mathcal{F}$ and $\varsigma(0) <_{\mathcal{F}} \varsigma(a)$ for $a \in A^{\star}$.

Definition. A filtration $\mathcal{F}$ on A is a *k-filtration* on A if each $S \in \mathcal{F}$ is a k-subspace of A.

Lemma 3.7. *Let $\mathcal{F}$ be a k-filtration on A.*

(*i*) *The set $A_{\leq_{\mathcal{F}} a}$ is a k-subspace of A for $a \in A$.*
(*ii*) *The set $A_{<_{\mathcal{F}} a}$ is a k-subspace of A for $a \in A \setminus \cup_{S\in\mathcal{F}} S$.*
(*iii*) $\varsigma(0)$ *is the unique minimal element of $A \wr \mathcal{F}$.*

Proof.

(i) By Lemma 3.2 (iii), it follows that if $\mathcal{F}$ is a k-filtration on A, then $A_{\leq_{\mathcal{F}} a}$ is a k-subspace of A. This is because an intersection of subspaces is again a subspace or in the degenerate case $A_{\leq_{\mathcal{F}} a} = A$.
(ii) It follows from Lemma 3.2 (iv) that $A_{<_{\mathcal{F}} a}$ is a k-subspace of A when it is non-empty because a (non-empty) union of nested subspaces is again a subspace. The union is non-empty by Corollary 3.3 (ii).
(iii) Since the elements of $\mathcal{F}$ are subspaces, $0 \in \cap_{S\in\mathcal{F}} S$. By Proposition 3.6 (vi), $\cap_{S\in\mathcal{F}} S$ is a "$\sim_{\mathcal{F}}$" equivalence class of A. Thus $\varsigma(0) = A_{\leq_{\mathcal{F}} 0} = \cap_{S\in\mathcal{F}} S$. By Proposition 3.6 (vii), $\varsigma(0) = \cap_{S\in\mathcal{F}} S$ is the unique minimal element of $A \wr \mathcal{F}$. □

The associated graded vector space of k-filtration has graded components of the form $A_{\leq_{\mathcal{F}} a}/A_{<_{\mathcal{F}} a}$ for $a \notin \cap_{S\in\mathcal{F}} S$. However, we must be careful to avoid duplicates; i.e. we must avoid using elements of A that are "$\sim_{\mathcal{F}}$" equivalent. This is done by parameterizing over $A \wr \mathcal{F}$.

Definition. Let $\mathcal{F}$ be a k-filtration on A. The notation $(A \wr \mathcal{F})^*$ denotes $(A \wr \mathcal{F}) \setminus \{\varsigma(0)\}$. Given $C \in (A \wr \mathcal{F})^*$ with $C = \varsigma(c)$, $c \in A$, define the *(graded) component*

$$\mathrm{gr}_C A = A_{\leq_{\mathcal{F}} c}/A_{<_{\mathcal{F}} c}.$$

Define the *associated graded k-vector space* as

$$\mathrm{gr} A = \bigoplus_{C \in (A\wr\mathcal{F})^*} \mathrm{gr}_C A.$$

An element of $\mathrm{gr}A$ is called *homogeneous* if it is an element of a single graded component.

Note that $\mathrm{gr}_C A$ is well-defined because if $c' \in A$ is another element with $\varsigma(c') = C$, then $A_{\leq_{\mathcal{F}} c} = A_{\leq_{\mathcal{F}} c'}$ and $A_{<_{\mathcal{F}} c} = A_{<_{\mathcal{F}} c'}$.

Definition. A k-filtration $\mathcal{F}$ of A has *finite-dimensional graded components* if for each $C \in (A \wr \mathcal{F})^*$ the k-vector space $\mathrm{gr}_C A$ is finite-dimensional. The filtration has *one-dimensional graded components* if for each $C \in (A \wr \mathcal{F})^{\star}$ the k-vector space $\mathrm{gr}_C A$ is one-dimensional.

Suppose that $A \neq \cup_{\{S\in\mathcal{F}|S\neq A\}}S$. By Proposition 3.6 (iv), each element of $A \setminus \cup_{\{S\in\mathcal{F}|S\neq A\}}S$ corresponds to the largest element of $(A \wr \mathcal{F})^*$ under the map ς defined in (13). The corresponding graded component is $A/\cup_{\{S\in\mathcal{F}|S\neq A\}} S$. If the filtration has finite-dimensional graded components, then $A/\cup_{\{S\in\mathcal{F}|S\neq A\}} S$ is finite-dimensional, and if the filtration has one-dimensional graded components then $A/\cup_{\{S\in\mathcal{F}|S\neq A\}} S$ is one-dimensional.

LEMMA 3.8. *Let $\mathcal{F}$ be a k-filtration on A with one-dimensional graded components. Assume that $\mathcal{F}$ intersects to zero. Let $a \in A \setminus \{0\}$, and let $\bar{a}$ denote the image of a in $A_{\leq_{\mathcal{F}} a}/A_{<_{\mathcal{F}} a} = \mathrm{gr}_{\varsigma(a)}A$, the $\varsigma(a)$-component of* $\mathrm{gr}A$.

(*i*) *The singleton set $\{\bar{a}\}$ is a basis for* $\mathrm{gr}_{\varsigma(a)}A$.
(*ii*) *All components of* $\mathrm{gr}A$ *have a basis of this form.*
(*iii*) *Let $S \subset A$ such that $\varsigma(S)$ is a system of representatives of $(A \wr \mathcal{F})^\star$; i.e. under the surjective map $\varsigma : A \to (A \wr \mathcal{F})^\star$, S maps bijectively to $(A \wr \mathcal{F})^\star$. The subset $\{\bar{s}\}_{s\in S}$ of* $\mathrm{gr}A$ *is a basis for* $\mathrm{gr}A$.
(*iv*) *If $b \leq_{\mathcal{F}a}$, then there is a unique $\lambda \in k$ such that $b - \lambda a = 0$ or $\varsigma(b - \lambda a) < \varsigma(a)$ in the order on $(A \wr \mathcal{F})^\star$ given by the filtration.*
(*v*) *If $b \in A_{\leq_{\mathcal{F}} a}$, then there is a unique $\lambda \in k$ where $b - \lambda a \in A_{<_{\mathcal{F}} a}$.*

PROOF. (i)–(iii) Since the graded components are assumed to be one-dimensional, any non-zero element is a basis. Such an element is given by $\bar{a}$ since $a \notin A_{<_{\mathcal{F}} a}$. Since A is the direct sum of the components, the union of the bases of the components is a basis for A.

(iv) We leave it to the reader to prove the equivalence of (iv) and (v).

(v) Let $[b]$ denote the image of b under the quotient map $A_{\leq_{\mathcal{F}} a} \to A_{\leq_{\mathcal{F}} a}/A_{<_{\mathcal{F}} a}$. Since $A_{\leq_{\mathcal{F}} a}/A_{<_{\mathcal{F}} a}$ is one-dimensional with basis $\bar{a}$ there is a unique element $\lambda \in k$ with $[b] = \lambda\bar{a}$. This is the unique $\lambda \in k$ where $b - \lambda a \in A_{<_{\mathcal{F}} a}$. □

3.3. MULTIPLICATIVITY OF FILTRATIONS

We now study multiplicative properties of filtrations and their ramifications. In particular, we connect multiplicativity with cancellativity (Lemma 3.9) and eventually give a sufficient condition for the notions of weak and strong multiplicativity to coincide (Proposition 3.10). Finally, if we are given a strongly multiplicative k-filtration $\mathcal{F}$ (on A) with one-dimensional components, and if $\mathcal{F}$ has the extra condition of non-negativity, then $k^\star$ consists precisely of the invertible elements of A (Lemma 3.11).

DEFINITION. A filtration is *weakly multiplicative* if for all $a, b, c \in A$,

$$a \leq_{\mathcal{F}} b \Rightarrow ac \leq_{\mathcal{F}} bc.$$

A weakly multiplicative filtration is called *strongly multiplicative* if for $c \neq 0$,

$$a <_{\mathcal{F}} b \Rightarrow ac <_{\mathcal{F}} bc.$$

Strong multiplicativity implies the cancellative property given in Lemma 3.9 (iv). The converse can also be shown: if $\mathcal{F}$ is weakly multiplicative and has the cancellative property then it is strongly multiplicative.

The power series ring over k in one variable, $k[[x]]$, is an example of an algebra with a strongly multiplicative k-filtration $\mathcal{F}$ where $A \neq \cup_{\{S\in\mathcal{F}|S\neq A\}}S$. For each positive integer

i let $\langle x^i \rangle$ be the ideal generated by x^i. Let $\mathcal{F}$ consist of all of these ideals. The filtration has one-dimensional graded components. If we had started with the power series ring in several variables and let $\mathcal{F}$ consist of the powers of the maximal ideal then it would still be the case that the filtration is strongly multiplicative and $A \neq \cup_{\{S \in \mathcal{F} | S \neq A\}} S$. However, in this case the filtration would not have one-dimensional graded components, but rather would only have finite-dimensional graded components. In these power series examples, $\varsigma(1_A)$ is the unique maximal element of $(A \wr \mathcal{F})^\star$. This coincides with Proposition 3.12 (iii).

Here are some fundamental multiplicative properties of weakly and strongly multiplicative filtrations, which we state without proof. We will exploit these properties in the next section to obtain valuations from filtrations.

Lemma 3.9. *Let $\mathcal{F}$ be a weakly multiplicative filtration on A and let $a, b, c, d \in A$.*

(*i*) *If $a \sim_{\mathcal{F}} b$, then $ac \sim_{\mathcal{F}} bc$.*
(*ii*) *If $ac <_{\mathcal{F}} bc$, then $a <_{\mathcal{F}} b$.*

Now assume that the filtration is strongly multiplicative.

(*iii*) *If $c \neq 0$ and $ac \leq_{\mathcal{F}} bc$, then $a \leq_{\mathcal{F}} b$.*
(*iv*) *If $c \neq 0$ and $ac \sim_{\mathcal{F}} bc$, then $a \sim_{\mathcal{F}} b$.*
(*v*) *If $ad \leq_{\mathcal{F}} bc, a >_{\mathcal{F}} b$, and c or d is non-zero, then $d <_{\mathcal{F}} c$.*

It is possible for a weakly multiplicative k-filtration on an integral domain to fail to be strongly multiplicative. Let A be a k-algebra with a k-filtration that consists of just three subsets of $A : \mathcal{F} = \{\{0\}, \{k\}, \{A\}\}$. The filtration ordering is directly described by $0 <_{\mathcal{F}} \lambda <_{\mathcal{F}} a$ for $\lambda \in k^*$, $a \in A \setminus k$. Thus if $ab \notin k$ for $a, b \in A \setminus k$, the filtration is weakly multiplicative. It is not strongly multiplicative if $A \setminus k \neq \emptyset$. Thus if A is a polynomial ring in one or more variables, this gives a weakly multiplicative but not strongly multiplicative k-filtration on an integral domain. The next result shows that adding additional hypotheses forces the notions of weak and strong multiplicativity to coincide.

Proposition 3.10. *Let $\mathcal{F}$ be a weakly multiplicative k-filtration on A that intersects to zero.*

(*i*) *$A_{<_{\mathcal{F}} 1_A}$ contains all zero-divisors of A.*
(*ii*) *If $\mathcal{F}$ has one-dimensional graded components and A is an integral domain, then $\mathcal{F}$ is strongly multiplicative.*

Proof.

(i) Suppose that $a \in A$ is a zero-divisor with $1_A \leq_{\mathcal{F}} a$. Since a is a zero-divisor there is a non-zero element $b \in A$ with $ab = 0$. Multiplying both sides of $1_A \leq_{\mathcal{F}} a$ by b and applying weak multiplicativity yields $b = 1_A b \leq_{\mathcal{F}} ab = 0$. Thus $b \leq_{\mathcal{F}} 0$. By Proposition 3.6 (vii) and Lemma 3.7 (iii), $\varsigma(0)$ is the unique minimal element in $A \wr \mathcal{F}$ and $\varsigma(0)$ consists of $\cap_{S \in \mathcal{F}} S$. By the "intersects to zero" hypothesis it follows that $\cap_{S \in \mathcal{F}} S = \{0\}$ and thus $b \leq_{\mathcal{F}} 0$ implies that $b = 0$, a contradiction.

(ii) Suppose that $a, b, c \in A$ with $a <_{\mathcal{F}} b$ and $c \neq 0$. We must show that $ac <_{\mathcal{F}} bc$. Since $a <_{\mathcal{F}} b$ it follows that $b \neq 0$, and hence $bc \neq 0$ since A is an integral domain. By weak multiplicativity, it follows that $ac \leq_{\mathcal{F}} bc$. By Lemma 3.8 (iv), there exists $\lambda \in k$ where $ac - \lambda bc <_{\mathcal{F}} bc$. We shall show that $\lambda = 0$, which implies that $ac <_{\mathcal{F}} bc$ as desired. Suppose that $\lambda \neq 0$. Then the element $a - \lambda b$ cannot lie in the k-subspace $A_{\leq_{\mathcal{F}} a}$. This is because $a \in A_{\leq_{\mathcal{F}} a}$, and if $\lambda \neq 0$ it would follow that $b \in A_{\leq_{\mathcal{F}} a}$, thus contradicting $a <_{\mathcal{F}} b$. Thus $a <_{\mathcal{F}} a - \lambda b$ and so $a \in A_{\leq_{\mathcal{F}} a-\lambda b}$. Since $A_{\leq_{\mathcal{F}} a-\lambda b}$ is a k-subspace containing $a - \lambda b$ it follows that $b \in A_{\leq_{\mathcal{F}} a-\lambda b}$; i.e. $b \leq_{\mathcal{F}} a - \lambda b$. By weak multiplicativity this implies that $bc \leq_{\mathcal{F}} (a - \lambda b)c$, contradicting $ac - \lambda bc <_{\mathcal{F}} bc$. □

By Proposition 3.10 all zero divisors of A lie in $A_{<_{\mathcal{F}} 1_A}$. Thus if $\varsigma(1_A)$ is the smallest element of $(A \wr \mathcal{F})^{\star}$, it follow that 0 is the only zero divisor in A and A is an integral domain.

Definition. We will call a k-filtration on A *non-negative* if it intersects to zero and $\varsigma(1_A)$ is the smallest element of $(A \wr \mathcal{F})^{*}$.

If $k[\mathbf{x}]$ has term order "$\leq_\sigma$", the term order filtration $\mathcal{F}_\sigma$ is non-negative. This immediately follows from the definition of a term order.

Lemma 3.11. *Let $\mathcal{F}$ be a non-negative k-filtration on A with one-dimensional graded components. Let a be an element of A.*

(*i*) *If $a \in A \setminus k$, then $a >_{\mathcal{F}} 1_A$.*
(*ii*) *If $a \in k^{*}$, then $a \sim_{\mathcal{F}} 1_A$.*
Now assume that the filtration is strongly multiplicative.
(*iii*) *If a is invertible, then $a \in k$.*
(*iv*) *k is the unique maximal subfield of A.*

Proof.

(i) Suppose that a does not lie in k. If $a \leq_{\mathcal{F}} 1_A$, then by Lemma 3.8 (v) there exists $\lambda \in k$ with $a - \lambda 1_A \in A_{<_{\mathcal{F}} 1_A}$. By the non-negativity assumption this implies that $a - \lambda 1_A = 0$, contradicting $a \notin k$.
(ii) On one hand, $k \subset A_{\leq_{\mathcal{F}} 1_A}$, and on the other hand $\varsigma(1_A)$ is the smallest element of $(A \wr \mathcal{F})^{\star}$.
(iii) If $a \notin k$, then $a >_{\mathcal{F}} 1_A$ by part (i). We replace d by a^{-1} and both b and c by 1_A in Lemma 3.9 (v). Since $a >_{\mathcal{F}} b$ and $c \neq 0$, it follows that $d <_{\mathcal{F}} c$; that is, $a^{-1} <_{\mathcal{F}} 1_A$. Since $a^{-1} \neq 0$, this contradicts the minimality of 1_A in $(A \wr \mathcal{F})^{\star}$.
(iv) This is clear because by part (ii), k contains all the invertible elements of A. □

3.4. the associated graded algebra

We show how strong multiplicativity implies that the associated graded vector space is an algebra, and then discuss properties of this associated graded algebra.

Proposition 3.12. *Suppose that $\mathcal{F}$ is a weakly multiplicative k-filtration on A.*

(*i*) *$A \wr \mathcal{F}$ is a multiplicative monoid. The identity is $\varsigma(1_A)$ and the product is defined as follows. For $C, D \in A \wr \mathcal{F}$ the product CD is defined as $\varsigma(cd)$ where $c, d \in A$ with $\varsigma(c) = C$ and $\varsigma(d) = D$. For $C, D, E \in A \wr \mathcal{F}$, if $C \leq D$ then $CE \leq DE$.*

Now assume that $\mathcal{F}$ is strongly multiplicative and $1_A \notin \cap_{S\in\mathcal{F}} S$. (Note that this second condition always holds when $\mathcal{F}$ intersects to zero.)

(*ii*) *$(A \wr \mathcal{F})^\star$ is a submonoid of $A \wr \mathcal{F}$. For $C, D, E \in (A \wr \mathcal{F})^\star$, if $C < D$ then $CE < DE$. $(A \wr \mathcal{F})^\star$ is cancellative in that $CE = DE$ implies $C = D$.*

(*iii*) *If $A \neq \cup_{\{S\in\mathcal{F}|S\neq A\}} S$, then $\varsigma(1_A)$ is the unique maximal element of $(A \wr \mathcal{F})^\star$.*

(*iv*) *The graded k-vector space $\mathrm{gr}A$ is a graded algebra with respect to the monoid $(A\wr\mathcal{F})^\star$ with product defined as follows. For $C, D \in (A \wr \mathcal{F})^\star$, with $C = \varsigma(c)$ and $D = \varsigma(d)$ for $c, d \in A$, let $x \in \mathrm{gr}_C A$ and $y \in \mathrm{gr}_D A$. In other words x is a coset in $A_{\leq_\mathcal{F} c}/A_{<_\mathcal{F} c}$ and y is a coset in $A_{\leq_\mathcal{F} d}/A_{<_\mathcal{F} d}$. If either x or y is zero the product xy is (defined to be) zero. If x and y are non-zero, let $a \in A_{\leq_\mathcal{F} c}$ be a coset representative for x and let $b \in A_{\leq_\mathcal{F} d}$ be a coset representative for y. Then xy is the element of $\mathrm{gr}_{CD}A$ represented by the coset of ab in $A_{\leq_\mathcal{F} cd}/A_{<_\mathcal{F} cd}$. This definition is extended to all elements of grA by distributivity.*

Proof.

(i) Let $c, c', d, d' \in A$ with $\varsigma(c) = \varsigma(c') = C$ and $\varsigma(d) = \varsigma(d') = D$. The elements c, c', d, d' are all non-zero because $C, D \in (A \wr \mathcal{F})^\star$. Since $c \sim_\mathcal{F} c'$ it follows that $c \leq_\mathcal{F} c'$, and by weak multiplicativity, $cd \leq_\mathcal{F} c'd$. By symmetry $c'd \leq_\mathcal{F} cd$ and hence $cd \sim_\mathcal{F} c'd$. Similarly, $c'd \sim_\mathcal{F} c'd'$ and so $cd \sim_\mathcal{F} c'd'$. This shows that the product on $(A \wr \mathcal{F})^\star$ is well defined. Associativity follows from associativity of the product on A. The fact that $\varsigma(1_A)$ is the unit of $(A \wr \mathcal{F})^*$ follows from the fact that 1_A is the unit of A.

Suppose that $C, D, E \in A \wr \mathcal{F}$ and $c, d, e \in A$ with $\varsigma(c) = C$, $\varsigma(d) = D$ and $\varsigma(e) = E$. Assume that $C \leq D$ or equivalently $c \leq_\mathcal{F} d$. If $e \neq 0$, then by weak multiplicativity, $ce \leq de$. Hence, $CE \leq DE$. If $e = 0$, then $ce = 0 = de$ and $CE = \varsigma(0) = DE$. So again $CE \leq DE$.

(ii) Since $1_A \notin \cap_{S\in\mathcal{F}} S$ it follows that $\varsigma(1_A) \in (A \wr \mathcal{F})^\star$. Thus to conclude that $(A \wr \mathcal{F})^\star$ is a submonoid of $A \wr \mathcal{F}$ it remains to show that $(A \wr \mathcal{F})^\star$ is closed under products. Suppose that $C, D \in (A\wr\mathcal{F})^\star$ with $c, d \in A$ where $\varsigma(c) = C$ and $\varsigma(d) = D$. We must show that $CD \in (A\wr\mathcal{F})^\star$. Since $C \in (A\wr\mathcal{F})^\star$ it follows that $\varsigma(0) \neq C$ and so $0 <_\mathcal{F} c$. Since $D \in (A\wr\mathcal{F})^\star$ it follows that $\varsigma(0) \neq D$ and so $0 \neq d$. By strong multiplicativity, $0 = 0 \cdot d <_\mathcal{F} c \cdot d$, and so $\varsigma(0) < \varsigma(cd) = CD \in A \wr \mathcal{F}$. In particular $\varsigma(0) \neq CD$ and $CD \in (A \wr \mathcal{F})^\star$. This concludes the proof that $(A \wr \mathcal{F})^\star$ is a submonoid of $A \wr \mathcal{F}$.

Suppose that $C, D, E \in (A \wr \mathcal{F})^\star$, and $c, d, e \in A$ such that $\varsigma(c) = C, \varsigma(d) = D$, and $\varsigma(e) = E$. Assume that $C < D$ or equivalently $c <_\mathcal{F} d$. Now $e \neq 0$ because $E \in (A\wr\mathcal{F})^\star$. By strong multiplicativity $ce <_\mathcal{F} de$. Hence, $CE < DE$. To show that $(A\wr\mathcal{F})^\star$ is cancellative we must show that $C = D$ if $CE = DE$. The order on $A\wr\mathcal{F}$ is a total order and so $C < D$, $C = D$ or $C > D$. If $C < D$ or $C > D$ we have just shown that $CE < DE$ or $CE > DE$. Thus $CE = DE$ implies that $C = E$ and $(A \wr \mathcal{F})^\star$ is cancellative.

(iii) If $A \neq \cup_{\{S\in\mathcal{F}|S\neq A\}} S$, then $(A\wr\mathcal{F})^\star$ has a unique maximal element by Proposition 3.6 (vi, vii). Let D denote this unique maximal element. If $D \neq \varsigma(1_A)$, then $D > \varsigma(1_A)$

by the maximality of D. In this case multiply both sides of $D > \varsigma(1_A)$ by D and apply part (i) to conclude that $D\varsigma(1_A) = D$ and part (ii) to conclude that $D^2 > D_\varsigma(1_A)$. But $D^2 > D$ contradicts D being the unique maximal element of $(A \wr \mathcal{F})^\star$, and this contradiction proves that $D = \varsigma(1_A)$. Hence, $\varsigma(1_A)$ is the unique maximal element of $(A \wr \mathcal{F})^\star$.

(iv) First we show that the product on gr$A$ is well-defined. Suppose that $a' \in A_{\leq_\mathcal{F} c}$ is another coset representative for x and $b' \in A_{\leq_\mathcal{F} d}$ is another coset representative for y. Since x is non-zero both a and a' lie in $A_{\leq_\mathcal{F} c} \setminus A_{<_\mathcal{F} c}$. Thus $a \sim_\mathcal{F} c \sim_\mathcal{F} a'$ and similarly $b \sim_\mathcal{F} d \sim_\mathcal{F} b'$.

From part (i) it follows that $ab \sim_\mathcal{F} cd \sim_\mathcal{F} a'b'$. Thus both ab and $a'b'$ lie in $A_{\leq_\mathcal{F} cd} \setminus A_{<_\mathcal{F} cd}$ and we may ask if they represent the same coset in $A_{\leq_\mathcal{F} cd}/A_{<_\mathcal{F} cd}$. Since a and a' represent the same coset in $A_{\leq_\mathcal{F} c}/A_{<_\mathcal{F} c}$ it follows that $a - a' \in A_{<_\mathcal{F} c}$, and so $a - a' <_\mathcal{F} c$. Now, b is non-zero since y is non-zero, and so by strong multiplicativity $(a - a')b <_\mathcal{F} cb$. Hence, ab and ab' represent the same coset in $A_{\leq_\mathcal{F} cd}/A_{<_\mathcal{F} cd}$. Similarly ab' and $a'b'$ represent the same coset in $A_{\leq_\mathcal{F} cd}/A_{<_\mathcal{F} cd}$, showing that the product on gr$A$ is well-defined. Associativity and the fact that the coset of 1_A in $\mathrm{gr}_{\varsigma(1_A)}$ is the unit of gr_A follow from the respective properties of A. □

Given a non-zero element $u \in \mathrm{gr}A, u$ is the sum of a finite number of non-zero homogeneous elements. Since the order on $(A \wr \mathcal{F})^\star$ is a total order, u has a largest non-zero homogeneous component, which is denoted by $\mathrm{LT}(u)$.

Let us check back with $k[\mathbf{x}]$ equipped with a term order "$\leq_\sigma$" and the term order filtration $\mathcal{F}_\sigma$. Using (25) we see that the associated graded algebra is indexed over $\mathbb{N}^n$. For $\mathbf{e} \in \mathbb{N}^n$, the coset of $\mathbf{x}^\mathbf{e}$ is a basis of the $\mathbf{e}$-component of the associated graded algebra. Mapping this coset to $\mathbf{x}^\mathbf{e}$ in $k[\mathbf{x}]$ sets up an isomorphism between the associated graded algebra and $k[\mathbf{x}]$. Under this isomorphism, components $\mathrm{LT}(u)$ of gr$A$ correspond to leading terms with respect to a term order.

PROPOSITION 3.13. *Suppose that $\mathcal{F}$ is a strongly multiplicative k-filtration on A where $1_A \notin \cap_{S\in\mathcal{F}} S$.*

(*i*) *For $u, v \in \mathrm{gr}A$, $\mathrm{LT}(uv) = \mathrm{LT}(u)\mathrm{LT}(v)$.*
(*ii*) gr$A$ *is an integral domain.*
(*iii*) *If $a, b \in A$ with $ab = 0$, then a or b lies in $\cup_{S\in\mathcal{F}} S$.*
(*iv*) *If the filtration intersects to zero, then A is an integral domain.*

PROOF.

(i) The result holds trivially if $u = 0$ or $v = 0$. Assume that $u \neq 0 \neq v$. Then u is the sum of a finite number of non-zero homogeneous elements. Thus there is a finite set $\mathcal{C}_u \subset (A \wr \mathcal{F})^\star$ such that for each $C \in \mathcal{C}_u$ there is a non-zero homogeneous element $u_C \in \mathrm{gr}_C A$ where $u = \sum_{C\in\mathcal{C}_u} u_C$. Let C' be the largest element of $\mathcal{C}_u$ so that $u_{C'} = \mathrm{LT}(u)$. Similarly there is a finite set $\mathcal{D}_v \subset (A \wr \mathcal{F})^\star$, and for each $D \in \mathcal{D}_v$ there is a non-zero homogeneous element $v_D \in \mathrm{gr}_D A$ where $v = \sum_{D\in\mathcal{D}_v} v_D$. Let D' be the largest element of $\mathcal{D}_v$ so that $v_{D'} = \mathrm{LT}(v)$. The product $uv = \sum_{C\in\mathcal{C}_u, D\in\mathcal{D}_v} u_C v_D$. Since $C' > C$ for $C \in \mathcal{C}_u \setminus \{C'\}$ and $D' > D$ for $D \in \mathcal{D}_v \setminus \{D'\}$, it follows from Proposition 3.12 (ii) that $C'D' > CD$ if $C \neq C'$ or $D \neq D'$. This proves that

$u_C v_D = \mathrm{LT}(uv)$ once we show that $u_C v_D \neq 0$. That is, we have reduced the problem to showing that the product of non-zero homogeneous elements is non-zero.

Suppose that x and y as the proof of Proposition 3.12 (iv) are non-zero. As shown in the proof of part (iv), $ab \in A_{\leq_{\mathcal{F}} cd} \backslash_{<_{\mathcal{F}} cd}$. Thus the coset of ab in $A_{\leq_{\mathcal{F}} cd}/A_{<_{\mathcal{F}} cd}$ is non-zero. This shows that the product of non-zero homogeneous elements of gr$A$ is not zero.

(ii) In the proof of (i), we showed that if $u \neq 0 \neq v$, then $\mathrm{LT}(uv) = u_C v_D \neq 0$ and hence $uv \neq 0$.

(iii) Let us show that $ab \neq 0$ if $a, b \in A \setminus \cap_{S\in\mathcal{F}} S$. Since $a, b \in A \setminus \cap_{S\in\mathcal{F}} S$ it follows that both $\varsigma(a)$ and $\varsigma(b)$ lie in $(A \wr \mathcal{F})^\star$. Let $\bar{a}$ denote the coset of a in $A_{\leq_{\mathcal{F}} a}/A_{<_{\mathcal{F}} a}$ and let $\bar{b}$ denote the coset of b in $A_{\leq_{\mathcal{F}} b}/A_{<_{\mathcal{F}} b}$. Both $\bar{a}$ and $\bar{b}$ are non-zero homogeneous elements in gr$A$. By part (ii), $\bar{a}\bar{b} \neq 0$. By definition, $\bar{a}\bar{b}$ is the coset of ab in $A_{\leq_{\mathcal{F}} ab}/A_{<_{\mathcal{F}} ab}$. Thus $ab \neq 0$.

(iv) If the filtration intersects to zero, then $\cap_{S\in\mathcal{F}} S = \{0\}$. This together with the result in part (iii) implies that $ab \neq 0$ for $a, b \in A^\star$. Thus A is an integral domain. $\square$

4. Valuations, Valuation Rings, and Filtration

In Section 3, we saw that a nested filtration on a ring A induces a quasi-order on A. In this section we shall see that if A is an integral domain, the order on A naturally extends to a quasi-order on the field of fractions K of A and gives rise to a valuation ring in K, and conversely. This will enable us to study and give examples of filtrations by considering the associated valuation rings and valuations instead. After extending the filtration quasi-order on A to its quotient field, we translate various properties about filtrations into properties about valuation rings and valuations. We also discuss the normalization of a filtration, and then combine the results of this section to create a bijection between special classes of valuation rings and filtrations.

4.1. THE CORRESPONDENCES

We begin by simply extending the filtration order on a domain to a quasi-order on its field of fractions. The following simple lemma is stated without proof.

LEMMA 4.1. *Let A be an integral domain with field of fractions K and let $\mathcal{F}$ be a strongly multiplicative filtration on A. For $a, b, c, d \in A$, define a binary relation on K by*

$$\frac{a}{b} \leq_{\mathcal{F}} \frac{c}{d} \qquad \text{if and only if } ad \leq_{\mathcal{F}} bc.$$

This puts a quasi-order on K that extends the quasi-order "$\leq_{\mathcal{F}}$" of A.

With respect to the quasi-order in the lemma above, we can define an equivalence relation on the field of fractions K by defining

$$\frac{a}{b} \sim_{\mathcal{F}} \frac{c}{d} \qquad \text{if and only if } \frac{a}{b} \leq_{\mathcal{F}} \frac{c}{b} \text{ and } \frac{c}{d} \leq_{\mathcal{F}} \frac{a}{b}$$

and

$$\frac{a}{b} <_{\mathcal{F}} \frac{c}{d} \qquad \text{if and only if } \frac{a}{b} \leq_{\mathcal{F}} \frac{c}{d} \text{ and } \frac{c}{d} \not\leq_{\mathcal{F}} \frac{a}{b}.$$

Using the filtration order and filtration equivalence, we extend some definitions from A to its field of fractions K as follows. Given $a \in K$, we set

$$\begin{aligned} K_{\leq_{\mathcal{F}} a} &= \{b \in K \mid b \leq_{\mathcal{F}} a\} \\ K_{<_{\mathcal{F}} a} &= \{b \in K \mid b <_{\mathcal{F}} a\} \\ K_{\sim_{\mathcal{F}} a} &= \{b \in K \mid b \sim_{\mathcal{F}} a\}. \end{aligned}$$

The following lemma can be proved in a straightforward manner.

LEMMA 4.2. *Let A be an integral domain with field of fractions K and let $\mathcal{F}$ be a strongly multiplicative filtration on A.*

- (*i*) $\frac{a}{b} <_{\mathcal{F}} \frac{c}{d}$ *if and only if* $ad <_{\mathcal{F}} bc$.
- (*ii*) $\frac{a}{b} \sim_{\mathcal{F}} \frac{c}{d}$ *if and only if* $ad \sim_{\mathcal{F}} bc$.
- (*iii*) *Assuming that* $\frac{a}{b} \neq 0 \neq \frac{c}{d}$.

$$\frac{a}{b} \leq_{\mathcal{F}} \frac{c}{d} \Leftrightarrow \frac{b}{a} \geq_{\mathcal{F}} \frac{d}{c}, \quad \frac{a}{b} <_{\mathcal{F}} \frac{c}{d} \Leftrightarrow \frac{b}{a} >_{\mathcal{F}} \frac{d}{c}, \quad \frac{a}{b} \sim_{\mathcal{F}} \frac{c}{d} \Leftrightarrow \frac{b}{a} \sim_{\mathcal{F}} \frac{d}{c}.$$

- (*iv*) *The quasi-order on K is total and is strongly multiplicative.*
- (*v*) *If $\mathcal{F}$ is a k-filtration, then $K_{\leq_{\mathcal{F}} \frac{a}{b}}$ is a k-subspace of K. If $0 \neq \frac{a}{b}$ then $K_{<_{\mathcal{F}} \frac{a}{b}}$ is a k-subspace of K.*

THEOREM 4.3. *Let A be an integral domain with field of fractions K and let $\mathcal{F}$ be a strongly multiplicative k-filtration on A.*

- (*i*) *The set $K_{\leq_{\mathcal{F}} 1_K}$ is a k-subalgebra of K and is a valuation ring of K. It is called the valuation ring associated to $\mathcal{F}$. The set $K_{<_{\mathcal{F}} 1_K}$ is the maximal ideal of $K_{\leq_{\mathcal{F}} 1_K}$ and $K_{\sim_{\mathcal{F}} 1_K}$ is the set of invertible elements of $K_{\leq_{\mathcal{F}} 1_K}$.*
- (*ii*) *The order "$\leq_{\mathcal{F}}$" on K or A may be recovered from either the valuation or the valuation ring. For $u, v \in K$ with $v \neq 0$ and the valuation ν of the valuation ring $K_{\leq_{\mathcal{F}} 1_K}$,*

$$u \leq_{\mathcal{F}} v \textit{ if and only if } \frac{u}{v} \in K_{\leq_{\mathcal{F}} 1_K} \textit{ if and only if } \nu(u) \leq \nu(v).$$

- (*iii*) *$K_{\leq_{\mathcal{F}} 1_K}$ has residue class field k if and only if $\mathcal{F}$ has one-dimensional graded components.*
- (*iv*) *The value monoid of A is non-negative if and only if $\mathcal{F}$ is a non-negative filtration.*

PROOF.

- (i) By Lemma 4.2 (v), $K_{\leq_{\mathcal{F}} 1_K}$ is k-subspace, and so it is closed under addition. To prove that it is a k-algebra, we need only demonstrate that it is closed under multiplication. In fact, we eventually prove that it is a valuation ring. Clearly, $1_K \in K_{\leq_{\mathcal{F}} 1_K}$. By Lemma 4.2 (iii), if $\frac{a}{b} >_{\mathcal{F}} 1_K$, then $\frac{b}{a} <_{\mathcal{F}} 1_K$, thus showing that if $u \in K \setminus K_{\leq_{\mathcal{F}} 1_K}$, then $u^{-1} \in K_{\leq_{\mathcal{F}} 1_K}$. Thus it remains to be demonstrated that $K_{\leq_{\mathcal{F}} 1_K}$ is closed under products to show that it is a valuation ring. This is clear because if $u \leq_{\mathcal{F}} 1_K$ and $v \leq_{\mathcal{F}} 1_K$ then by strong (or weak) multiplicativity,

$$u \leq_{\mathcal{F}} 1_K \Rightarrow uv \leq_{\mathcal{F}} 1_K v = v \leq_{\mathcal{F}} 1_K.$$

We now show that $K_{\sim_{\mathcal{F}} 1_K}$ is the set of invertible elements of $K_{\leq_{\mathcal{F}} 1_K}$. Suppose u is an invertible element of $K_{\leq_{\mathcal{F}} 1_K}$. Then $u^{-1} \leq_{\mathcal{F}} 1_K$, and so by strong (or weak) multiplicativity, we can multiply both sides of this inequality by u to get $1_K \leq_{\mathcal{F}} u$. However, since $u \in K_{\leq_{\mathcal{F}} 1_K}$, we have $u \leq_{\mathcal{F}} 1_K$, and so $u \sim_{\mathcal{F}} 1_K$. Conversely, suppose $u \in K_{\sim_{\mathcal{F}} 1_K}$; i.e. $u \sim_{\mathcal{F}} 1_K$. Then by Lemma 4.2 (iii), $u^{-1} \sim_{\mathcal{F}} 1_K$, and so $u^{-1} \in K_{\leq_{\mathcal{F}} 1_K}$. Thus $K_{\sim_{\mathcal{F}} 1_K}$ is the set of invertible elements of $K_{\leq_{\mathcal{F}} 1_K}$. This also shows that $K_{\leq_{\mathcal{F}} 1_K} \setminus K_{\sim_{\mathcal{F}} 1_K} = K_{<_{\mathcal{F}} 1_K}$ is the set on non-invertible elements of $K_{\leq_{\mathcal{F}} 1_K}$ and hence is the maximal ideal of $K_{\leq_{\mathcal{F}} 1_K}$.

(ii) Let $a, b, c, d \in K$ with $u = \frac{a}{b}$ and $v = \frac{c}{d}$. Then $\frac{u}{v} = \frac{ad}{bc}$ and we have

$$\frac{u}{v} \in K_{\leq_{\mathcal{F}} 1_K} \Leftrightarrow \frac{ad}{bc} \leq_{\mathcal{F}} 1_K \Leftrightarrow ad \leq_{\mathcal{F}} bc \Leftrightarrow \frac{a}{b} \leq_{\mathcal{F}} \frac{c}{d} \Leftrightarrow u \leq_{\mathcal{F}} v.$$

Since $K_{\leq_{\mathcal{F}} 1_K}$ is the valuation ring, we have

$$\nu(u) \leq \nu(v) \Leftrightarrow \nu\left(\frac{u}{v}\right) \leq \mathrm{id}_G \Leftrightarrow \frac{u}{v} \in K_{\leq_{\mathcal{F}} 1_K}$$

where id_G denotes the identity of the value group. Note that this recovers the "$\leq_{\mathcal{F}}$" order on A as well as on K since the order on K extends that of A.

(iii) Now, $K_{<_{\mathcal{F}} 1_K}$ is the maximal ideal of $K_{\leq_{\mathcal{F}} 1_K}$ and $k \subset K_{\leq_{\mathcal{F}} 1_K}$. Thus the residue class field is k if and only if $K_{\leq_{\mathcal{F}} 1_K} = k \oplus K_{<_{\mathcal{F}} 1_K}$. Since $k \cap K_{<_{\mathcal{F}} 1_K} = \{0\}$, the residue class field is k if and only if $K_{\leq_{\mathcal{F}} 1_K} = k + K_{<_{\mathcal{F}} 1_K}$. Now suppose that $a/b \in K_{\leq_{\mathcal{F}} 1_K} \setminus K_{<_{\mathcal{F}} 1_K}$ with $a, b \in A^\star$. Since $a/b \sim_{\mathcal{F}} 1_K$, it follows that $a \sim_{\mathcal{F}} b$.

If $\mathcal{F}$ has one-dimensional graded components, there exists $c \in A_{\leq_{\mathcal{F}} a}$ where $A_{\leq_{\mathcal{F}} a} = kc \oplus A_{<_{\mathcal{F}} a}$. Thus there are $\lambda, \gamma \in k^*$ and $a', b' \in A_{<_{\mathcal{F}} a} = A_{<_{\mathcal{F}} b}$ with $a = \lambda c + a'$ and $b = \gamma c + b'$. Hence, $a - \frac{\lambda}{\gamma} b = a' - \frac{\lambda}{\gamma} b'$. Moreover,

$$\frac{a}{b} = \frac{\lambda}{\gamma} + \frac{a - \frac{\lambda}{\gamma} b}{b} = \frac{\lambda}{\gamma} + \frac{a' - \frac{\lambda}{\gamma} b'}{b}. \tag{14}$$

Since $a' - \frac{\lambda}{\gamma} b' \in A_{<_{\mathcal{F}} b}$, it follows that

$$\frac{a' - \frac{\lambda}{\gamma} b'}{b} \in K_{<_{\mathcal{F}} 1_K},$$

and so (14) shows that $a/b \in k + K_{<_{\mathcal{F}} 1_K}$.

Conversely, suppose that $K_{\leq_{\mathcal{F}} 1_K}$ has residue class field k. To demonstrate that $\mathcal{F}$ has one-dimensional graded components we show that for any $b \in A^\star$ the vector space $A_{\leq_{\mathcal{F}} b}/A_{<_{\mathcal{F}} b}$ is one-dimensional. In fact, we show that the image of b is basis. Suppose that $a \in A_{\leq_{\mathcal{F}} b}$. By Lemma 2.1 there exists $\lambda \in k$ with $a - \lambda b = 0$ or $\nu(a - \lambda b) < \nu(b)$. If $a - \lambda b = 0$, then $a - \lambda b \in A_{<_{\mathcal{F}} b}$. If $\nu(a - \lambda b) < \nu(b)$, then $a - \lambda b \in A_{<_{\mathcal{F}} b}$ because in part (ii) we showed that the "$\leq_{\mathcal{F}}$" order agrees with the order coming from ν and the value group.

(iv) We must prove that $\mathrm{id}_G \leq \nu(a)$ for all $a \in A^\star$ if and only if $1_K \leq_{\mathcal{F}} a$ for all $a \in A^\star$. But this is immediate from (b) since the "$\leq_{\mathcal{F}}$" order agrees with the order coming from ν and the value group. □

Note that

$$\nu(a) \leq \nu(b) \Leftrightarrow \frac{a}{b} \in K_{\leq_{\mathcal{F}} 1_K} \qquad \text{and} \qquad a \leq_{\mathcal{F}} b \Leftrightarrow \frac{a}{b} \in K_{\leq_{\mathcal{F}} 1_K}.$$

The first inequality follows by the order defined on the value group, and the second follows by Theorem 4.3 (ii). Thus the order on the value group coincides with the order given by $\mathcal{F}$. Since the valuation was obtained from the filtration, it is not happenstance that the two orders coincide.

We describe how Theorem 4.3 applies to term order filtrations. In this case, we have $k[\mathbf{x}]$ with term order "$\leq_\sigma$" and term order filtration $\mathcal{F}_\sigma$. Let K denote $k(\mathbf{x})$, the field of fractions of $k[\mathbf{x}]$. Then the valuation ring $K_{\leq_{\mathcal{F}_\sigma} 1_K}$ in Theorem 4.3 (i) coincides with the valuation ring previously constructed from a term order in Example 2.2.

Theorem 4.4. *Let A be an integral domain with subfield k and field of fractions K. Let ν be a valuation (with value group G) on K where $\nu(\lambda) = \mathrm{id}_G$ for $\lambda \in K^*$. Let $M = \nu(A^\star)$ be the value monoid of A. For each $m \in M$, we associate a subset of A:*

$$A_{\leq_\nu m} = \{0\} \cup \{a \in A^\star \mid \nu(a) \leq_m\}.$$

Let $\mathcal{F}_\nu$ be the set of subsets of A consisting of $\{0\}$ and the $A_{\leq_\nu m}$'s.

(*i*) *$\mathcal{F}_\nu \setminus \{\{0\}\}$ is indexed by M. That is, $A_{\leq_\nu n} \neq \{0\}$ for $n \in M$ and $A_{\leq_\nu m} \neq A_{\leq_\nu n}$ for $m, n \in M$ with $m \neq n$. If $m, n \in M$ with $m < n$, then $A_{\leq_\nu m} \subset A_{\leq_\nu n}$.*
(*ii*) *$\mathcal{F}_\nu$ is a k-filtration on A.*
(*iii*) *$\mathcal{F}_\nu$ has full union and intersects to zero.*
(*iv*) *In terms of the $\mathcal{F}_\nu$ filtration order "$\leq_{\mathcal{F}_\nu}$" and the subsets $A_{\leq_{\mathcal{F}_\nu} a}$,*

$$A_{\leq_{\mathcal{F}_\nu} 0} = \{0\} \text{ and } A_{\leq_{\mathcal{F}} g} = A_{\leq_\nu m} \text{ when } a \neq 0 \text{ and } m = \nu(a).$$

(*v*) *For $a, b \in A^\star$,*

$$a \leq_{\mathcal{F}_\nu} b \Leftrightarrow \nu(a) \leq \nu(b), \quad a <_{\mathcal{F}_\nu} b \Leftrightarrow \nu(a) < \nu(b), \quad a \sim_{\mathcal{F}_\nu} b \Leftrightarrow \nu(a) = \nu(b).$$

(*vi*) *The filtration $\mathcal{F}_\nu$ is strongly multiplicative and the valuation coming from the filtration $\mathcal{F}_\nu$ given in Theorem 4.3 (i) is equivalent to the original valuation ν.*
(*vii*) *The filtration $\mathcal{F}_\nu$ has one-dimensional graded components if and only if the residue class field of ν is k.*
(*viii*) *The filtration $\mathcal{F}_\nu$ is non-negative if and only if $\nu(1_A)$ is the smallest element of M.*

Proof.

(i) Suppose that $m, n \in M$ with $m \neq n$. Since the order on the value group is total, either $m < n$ or $n < m$ by the order defined on the value group. Suppose that $m < n$. Then $\nu(a) \leq m < n$ for any $a \in A_{\leq_\nu m} \setminus \{0\}$ and so $a \in A_{\leq_\nu n}$, thus showing that $A_{\leq_\nu m} \subset A_{\leq_\nu n}$. Since M is the value monoid of A, there exists $b \in A^\star$ with $\nu(b) = n$. Then $b \in A_{\leq \nu n}$, but $b \notin A_{\leq_\nu m}$ since $\nu(b) = n \not\leq m$. Thus $A_{\leq_\nu m} \neq A_{\leq_\nu n}$. We just showed that for $n \in M$ there exists $0 \neq b \in A_{\leq_\nu n}$, thus proving that $A_{\leq_\nu n} \neq \{0\}$.
(ii) Clearly, the element $\{0\} \in \mathcal{F}_\nu$ is a subset of every other element of $\mathcal{F}_\nu$. Moreover, $\{0\}$ is also a k-subspace of A. Since the order on the value group is total and we showed in part (i) that $A_{\leq_\nu m} \subset A_{\leq_\nu n}$ if $m < n$, it follows that $\mathcal{F}_\nu$ is nested. Thus $\mathcal{F}_\nu$ is a filtration. Let us show that each $A_{\leq_\nu m}$ is a k-subspace of A. Now, $0 \in A_{\leq_\nu m}$ by the definition of $A_{\leq_\nu m}$. For $a \in A_{\leq_\nu m} \setminus \{0\}$ and $\lambda \in k^*$, $\nu(\lambda a) = \nu(\lambda)\nu(a) = \mathrm{id}_G \nu(a) = \nu(a) \leq m$. Thus $\lambda a \in A_{\leq_\nu m}$. Finally, if $a, a' \in A_{\leq_\nu m} \setminus \{0\}$ with $a + a' \neq 0$, then by the triangle inequality, $\nu(a + a') \leq \max(\nu(a), \nu(a')) \leq m$. Hence $a + a' \in A_{\leq_\nu m}$, and so $A_{\leq_\nu m}$ is k-subspace of A.

(iii) Let a be any element of A. If $a = 0$, then $a \in A_{\leq_\nu m}$ for all $m \in M$. For $a \neq 0$ let $m = \nu(a) \in M$. Then $a \in A_{\leq_\nu m}$. This shows that the union of the $A_{\leq_\nu m}$'s is all of A, that is, $\mathcal{F}_\nu$ has full union. Moreover, $\mathcal{F}_\nu$ intersects to zero because $\{0\} \in \mathcal{F}_\nu$.

(iv) Now, $\{0\}$ is the smallest element of $\mathcal{F}_\nu$. This directly implies that $A_{\leq_\mathcal{F} 0} = \{0\}$. Next suppose that $a \in A^\star$ and let us verify that $A_{\leq_\mathcal{F} g} = A_{\leq_\nu m}$ for $m = \nu(a)$. Suppose that $b \in A_{\leq_\nu m}$ so that $b = 0$ or $\nu(b) \leq \nu(a)$. If $b = 0$, then b lies in all elements of $\mathcal{F}_\nu$ containing a because 0 lies in all elements of $\mathcal{F}_\nu$. Suppose that $b \neq 0$ and $S \in \mathcal{F}_\nu$ with $a \in S$. By part (i), $S = A_{\leq_\nu n}$ for a unique $n \in M$. Since $a \in A_{\leq_\nu n}$ it follows that $\nu(a) \leq n$. Since $b \in A_{\leq_\nu m}$ it follows that $\nu(b) \leq m = \nu(a) \leq n$, and so $b \in A_{\leq_\nu n} = S$. Thus b lies in all elements of $\mathcal{F}_\nu$ containing a. We have shown that $b \leq_{\mathcal{F}_\nu} a$, and so $b \in A_{\leq_\mathcal{F} g}$. This proves $A_{\leq_\nu m} \subset A_{\leq_{\mathcal{F}_\nu} a}$.

For the opposite inclusion suppose that $b \in A_{\leq_\mathcal{F} g}$. Then b lies in all elements of $\mathcal{F}_\nu$ containing a. In particular, $b \in A_{\leq_\nu m}$. This proves $A_{\leq_{\mathcal{F}_\nu} a} \subset A_{\leq_\nu m}$.

(v) By part (iv), we have

$$a \leq_{\mathcal{F}_\nu} b \Leftrightarrow a \in A_{\leq_{\mathcal{F}_\nu} b} = A_{\leq_\nu \nu(b)} \Leftrightarrow \nu(a) \leq \nu(b).$$

By definition, $a <_\mathcal{F} b$ is equivalent to $a \leq_\mathcal{F} b$ and $b \nleq_\mathcal{F} a$. But by what we have just shown, this is equivalent to $\nu(a) \leq \nu(b)$ and $\nu(b) \nleq \nu(a)$. This is equivalent to $\nu(a) < \nu(b)$. The verification that $a \sim_\mathcal{F} b \Leftrightarrow \nu(a) = \nu(b)$ is similar and left to the reader.

(vi) Let $a, b, c \in A$ with $a <_\mathcal{F} b$ and $c \neq 0$. If $a = 0$, then certainly $ac <_\mathcal{F} bc$ and so we may assume that $a \neq 0$. By part (v), it follows that $\nu(a) < \nu(b)$. Since the value group is an ordered group, $\nu(a)\nu(c) < \nu(b)\nu(c)$. Since ν is multiplicative we have $\nu(ac) < \nu(bc)$. Using part (v) again it follows that $ac <_\mathcal{F} bc$. Similarly, we can show that $a <_\mathcal{F} b$ implies $ac \leq_\mathcal{F} bc$, and so the filtration is strongly multiplicative. By definition, the valuation ring of the filtration is $K_{\leq_\mathcal{F} \nu} 1_K$. Suppose that $a, b \in A$ with $a/b \in K_{\leq_{\mathcal{F}_\nu} 1_K}$. By the definition (in Lemma 4.1) of the extension of "$\leq_\mathcal{F}$" to K, it follows that $a \leq_\mathcal{F} b$. By part (v), this gives $\nu(a) \leq \nu(b)$ and hence a/b lies in the valuation ring of ν. The argument is reversible, thus showing that $K_{\leq_{\mathcal{F}_\nu} 1_K}$ is equal to the valuation ring of ν. Hence ν is equivalent to the valuation coming from $\mathcal{F}_\nu$ since they have the same valuation ring.

(vii) Let V be the valuation ring of ν and let m be its maximal ideal. Note that $k \subset V$ since $\nu(\lambda) = \mathrm{id}_G$ for $\lambda \in k^*$. Thus m and V are k-subspaces of K. Suppose that $\mathcal{F}_\nu$ has one-dimensional graded components. Let us show that k is a complement to m in V. Let $v \in V$ and write v as a/b where $a, b \in A$ with $b \neq 0$. Then by Lemma 3.8, there is a unique $\lambda \in k$ where $a - \lambda b \in A_{<_\mathcal{F} b}$. We have

$$\frac{a}{b} = \lambda + \frac{a - \lambda b}{b}. \tag{15}$$

Since $a - \lambda b \in A_{<_\mathcal{F} b}$ it follows that $(a - \lambda b)/b$ lies in the maximal ideal of the valuation ring and we have shown that k is a complement to the maximal ideal. Thus if $\mathcal{F}_\nu$ has one-dimensional graded components, then the residue class field is k.

Conversely if the residue class field is k, then k is a complement to the maximal ideal of V. Let us show that for any $a \in A^\star$, the coset of a is a k-basis for $A_{\leq_\mathcal{F} a}/A_{<_\mathcal{F} a}$. This would establish that $\mathcal{F}_\nu$ has one-dimensional graded components. Suppose that $b \in A_{\leq_\mathcal{F} a}$. Since $b \leq_{\mathcal{F}_\nu} a$ it follows that $\nu(b) \leq \nu(a)$ and by (5), $b = va$ for some $v \in V$. Since k is a complement to m, we have $v = \lambda + x$ for some $\lambda \in k$ and $x \in m$. Thus $b = \lambda a + xa$. Since both b and λa lie in A it follows that $xa \in A$.

By (6) it follows that $\nu(xa) < \nu(a)$, and so $xa <_{\mathcal{F}_\nu} a$. Thus $xa \in A_{\leq_{\mathcal{F}} a}$, and b is congruent to λa modulo $A_{\leq_{\mathcal{F}} a}$.

(viii) By part (v), the filtration order coincides with the order induced by the valuation ν, and so by Theorem 4.3 (iv), the filtration is non-negative if and only if the value monoid M is non-negative. Thus $\mathcal{F}_\nu$ is non-negative if and only if there does not exist $\nu(a) \in M$ such that $\nu(a) < \mathrm{id}_G$. Since $\nu(1_A) = \mathrm{id}_G$, this is equivalent to the condition that $\nu(1_A)$ is the smallest element of M. □

4.2. NORMALIZED FILTRATIONS

Our next task is the matter of starting with a filtration and obtaining a normalized filtration which induces the same quasi-order. The important restriction here is that if $\mathcal{F}$ and $\mathcal{G}$ are two filtrations on A which induce the same quasi-order then they should have the same normalization. Also, the normalization of a normalization should be the first normalization. Earlier we defined $A \wr \mathcal{F}$ as the set of subsets of A of the form $A_{\leq_{\mathcal{F}} a}$. We shall show that $A \wr \mathcal{F}$ is a filtration on A and fulfills the required conditions for being the normalization of $\mathcal{F}$. We show that a filtration which arises from a valuation (as in Theorem 4.4) is normalized.

PROPOSITION 4.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be filtrations on A.*

(*i*) *$A \wr \mathcal{F}$ is a filtration on A.*

(*ii*) *The quasi-order "$\leq_{A\wr\mathcal{F}}$" on A is the same as "$\leq_{\mathcal{F}}$". That is, for $a \in A$:*

$$A_{\leq_{\mathcal{F}} a} = A_{\leq_{A\wr\mathcal{F}} a}, \quad A_{<_{\mathcal{F}} a} = A_{<_{A\wr\mathcal{F}} a}, \quad A_{\sim_{\mathcal{F}} a} = A_{\sim_{A\wr\mathcal{F}} a}.$$

(*iii*) *If $\mathcal{F}$ and $\mathcal{G}$ induce the same quasi-order on A then $A \wr \mathcal{F} = A \wr \mathcal{G}$.*
$A \wr \mathcal{F}$ is considered to be the normalization *of $\mathcal{F}$ and $\mathcal{F}$ is called* normalized *if $\mathcal{F} = A \wr \mathcal{F}$.*

(*iv*) *Since $A\wr\mathcal{F}$ is a filtration on A, one can form $A\wr(A\wr\mathcal{F})$. Passing to the normalization is a "unipotent" process in that*

$$A \wr \mathcal{F} = A \wr (A \wr \mathcal{F}).$$

PROOF.

(i) By Lemma 3.2 (i) it follows that for $a, b \in A$ either $a \in A_{<_{\mathcal{F}} b}$ or $b \in A_{\leq_{\mathcal{F}} a}$. In the former case $A_{\leq_{\mathcal{F}} a} \subset A_{<_{\mathcal{F}} b} \subset A_{\leq_{\mathcal{F}} b}$ and in the latter $A_{\leq_{\mathcal{F}} b} \subset A_{\leq_{\mathcal{F}} a}$. Hence, $A \wr \mathcal{F}$ is nested and so is a filtration on A.

(ii) Let us show that for $a \in A$,

$$A_{\leq_{\mathcal{F}} a} = \cap_{\{b \in A | a \in A_{\leq_{\mathcal{F}} b}\}} A_{\leq_{\mathcal{F}} b}. \tag{16}$$

Since $a \in A_{\leq_{\mathcal{F}} a}$ it follows that $a \in \{b \in A \mid a \in A_{\leq_{\mathcal{F}} b}\}$. Thus $A_{\leq_{\mathcal{F}} a}$ is the unique minimal set among those being intersected on the right-hand side of (16). Hence the intersection is as claimed. The sets on the right-hand side of (16) are the sets in the $A\wr\mathcal{F}$ filtration which contain a. Thus by Lemma 3.2 (iii) applied to the $A\wr\mathcal{F}$ filtration we have that for $a \in A$,

$$A_{\leq_{A\wr\mathcal{F}} a} = \cap_{\{T \in A\wr\mathcal{F} | a \in T\}} T = \cap_{\{b \in A | a \in A_{\leq_{\mathcal{F}} b}\}} A_{\leq_{\mathcal{F}} b} = A_{\leq_{\mathcal{F}} a}. \tag{17}$$

Next let us show that for $a \in A$,

$$A_{<_{\mathcal{F}}a} = \cap_{\{b \in A | a \notin A_{\leq_{\mathcal{F}}b}\}} A_{\leq_{\mathcal{F}}b}. \tag{18}$$

Since the quasi-order "$\leq_{\mathcal{F}}$" on A is total, if $a \notin A_{\leq_{\mathcal{F}}b}$ then $b \in A_{<_{\mathcal{F}}a}$ and $A_{\leq_{\mathcal{F}}b} \subset A_{<_{\mathcal{F}}a}$. Thus each set among those being unioned on the right-hand side of (18) lies in $A_{<_{\mathcal{F}}a}$ and so $A_{<_{\mathcal{F}}a} \supset \cup_{\{b \in A | a \notin A_{\leq_{\mathcal{F}}b}\}} A_{\leq_{\mathcal{F}}b}$. For the opposite inclusion, if $b \in A_{<\mathcal{F}a}$ then $a \notin A_{\leq_{\mathcal{F}}b}$ and so $A_{\leq_{\mathcal{F}}b}$ occurs among the sets being unioned on the right-hand side of (18). Since $b \in A_{\leq_{\mathcal{F}}b}$ this shows that $A_{<_{\mathcal{F}}a} \subset \cup_{\{b \in A | a \notin A_{\leq_{\mathcal{F}}b}\}} A_{\leq_{\mathcal{F}}b}$, and thus establishes (18). The sets on the right-hand side of (18) are the sets in the $A \wr \mathcal{F}$ filtration which do not contain a. Thus by Lemma 3.2 (iv) applied to the $A \wr \mathcal{F}$ filtration we have that for $a \in A$,

$$A_{<_{A \wr \mathcal{F}}a} = \cup_{\{T \in A \wr \mathcal{F} | a \notin T\}} T = \cup_{\{b \in A | a \notin A_{\leq_{\mathcal{F}}b}\}} A_{\leq_{\mathcal{F}}b} = A_{<_{\mathcal{F}}a}. \tag{19}$$

Now, (17) and (19) show that "$\leq_{A \wr \mathcal{F}}$" is the same quasi-order on A as "$\leq_{\mathcal{F}}$". It immediately follows from $A_{\leq_{\mathcal{F}}a} = A_{\leq_{A \wr \mathcal{F}}a}$ and $A_{<_{\mathcal{F}}a} = A_{<_{A \wr \mathcal{F}}a}$ that $A_{\sim_{\mathcal{F}}a} = A_{\sim_{A \wr \mathcal{F}}a}$, because $A_{\sim_{\mathcal{F}}a}$ is the complement to $A_{<_{\mathcal{F}}a}$ in $A_{\leq_{\mathcal{F}}a}$ and $A_{\sim_{A \wr \mathcal{F}}a}$ is the complement to $A_{<_{A \wr \mathcal{F}}a}$ in $A_{\leq_{A \wr \mathcal{F}}a}$.

Note that it was necessary to verify both (17) and (19) because "$\mathcal{F}$" and "$A \wr \mathcal{F}$" are quasi-orders and not orders.

(iii) $A \wr \mathcal{F}$ is the set of subsets of A of the form $A_{\leq_{\mathcal{F}}a}$ and $A \wr \mathcal{G}$ is the set of subsets of A of the form $A_{\leq_{\mathcal{G}}a}$. By assumption $\mathcal{F}$ and $\mathcal{G}$ induce the same quasi-order on A. Thus $A_{\leq_{\mathcal{F}}a} = A_{<_{\mathcal{G}}a}$ for $a \in A$. Hence, $A \wr \mathcal{F} = A \wr \mathcal{G}$.

(iv) By part (ii), $A \wr \mathcal{F}$ induces the same quasi-order on A as does $\mathcal{F}$. Thus (iv) follows from (iii) with $\mathcal{G} = A \wr \mathcal{F}$. □

Theorem 4.4 defines a filtration $\mathcal{F}_\nu$ in terms of a valuation ν. Let us show that $\mathcal{F}_\nu$ is a normalized filtration.

Proposition 4.6. *The filtration $\mathcal{F}_\nu$ defined in Theorem 4.4 is a normalized filtration.*

Proof. Suppose that $a \in A$. Theorem 4.4 (iv), if $a = 0$ then $A_{<_{\mathcal{F}_\nu}0} = \{0\}$, and if $a \neq 0$ then $A_{\leq_{\mathcal{F}_\nu}a} = A_{\leq_\nu \nu(a)}$. Note that $\mathcal{F}_\nu$ is defined to be precisely these subsets of A. Thus $\mathcal{F}_\nu = A \wr \mathcal{F}_\nu$, proving that $\mathcal{F}_\nu$ is a normalized filtration. □

4.3. regular filtrations and complementary valuation rings

According to Sweedler (1986), there are two properties of valuation rings of primary interest in the context of computing analogs of Gröbner bases via valuation theory. The first property is well-orderedness, and it is discussed in the introduction. The second property of importance is described in the following definition.

Definition. Let A be an integral domain with field of fractions K, and let k be a field contained in A. Let V be a valuation ring in K and let m denote the maximal ideal of V. We say V is a *complementary valuation ring* to A in K if $A \cap m = \{0\}$ and $V = (A \cap V) + m$. If, in addition, $k \subset V$ and $A \cap V = k$, then we say that V is a *k-complementary valuation ring* to A in K.

According to Sweedler (1986), V is a complementary valuation ring to $k[\mathbf{x}]$ in its field of fractions in Example 2.2.

Definition. We call a k-filtration $\mathcal{F}$ on a ring A *regular* if it has full union (Section 3.1), intersects to zero (Section 3.1), has one-dimensional graded components (Section 3.2), is strongly multiplicative (Section 3.3), and is non-negative (Section 3.3).

Lemma 4.7. *Each k-complementary valuation ring V to A gives rise to regular, normalized filtration on A.*

Proof. Let V be a k-complementary valuation ring to A in K and let ν be the associated valuation. Let $\mathcal{F}_\nu$ be the associated filtration, as defined in Theorem 4.4. By Theorem 4.4, $\mathcal{F}_\nu$ is a k-filtration with full union and intersects to zero. Since V is a k-complementary valuation ring to A in $K, V = (A \cap V) + m$ and $A \cap V = k$. Thus $V/m \cong A \cap V = k$, and so by Theorem 4.4 (vi), the filtration has one-dimensional graded components. By Theorem 4.4 (vi), the filtration is strongly multiplicative. We need now only justify that $\mathcal{F}_\nu$ is non-negative. Suppose $a \in A$ and $a <_{\mathcal{F}_\nu} 1_K$. So $\nu(a) <_\nu (1_K)$, and so $a \in m$. Since $A \cap m = \{0\}$, we have $a = 0$. Thus $\mathcal{F}_\nu$ is non-negative. The filtration $\mathcal{F}_\nu$ is normalized by Proposition 4.6. □

Lemma 4.8. *Each regular filtration on A gives rise to a k-complementary valuation ring V to A.*

Proof. From Theorem 4.3, we get a valuation ring $V = \{a \in K \mid a \leq_{\mathcal{F}_\nu} 1_K\}$ with maximal ideal $m = \{a \in K \mid a <_{\mathcal{F}_\nu} 1_K\}$. We need to prove that V is a k-complementary valuation ring to A in K. First, we prove that $A \cap V = k$.

Since $\mathcal{F}$ is a k-filtration, we have that for all $\lambda \in k, \lambda \sim_{\mathcal{F}_\nu} 1_K$. Therefore, $\lambda \in V$, and so $k \subseteq V$. Since $k \subseteq A$, we have $k \subseteq A \cap V$.

Conversely, suppose a is a non-zero element of $A \cap V$. Therefore, $a \leq_{\mathcal{F}_\nu} 1_K$. Since $\mathcal{F}$ is one-dimensional, there exists $\lambda \in k$ such that $a - \lambda <_{\mathcal{F}_\nu} 1_K$. Since $\mathcal{F}$ is non-negative, $a = \lambda$. Thus $A \cap V \subseteq k$, and so we have equality.

We now need to show that $V = (A \cap V) + m$. The reverse containment is clear; we need only justify the forward containment. Let v be a non-zero element of V. Therefore, $v \leq_{\mathcal{F}_\nu} 1_K$. Since V sits insides the field of fractions of A, there exist non-zero $a, b \in A$ such that $v = a/b$. In this case, $a \leq_{\mathcal{F}_\nu} b$. There exists $\lambda \in k$ such that $a - \lambda b <_{\mathcal{F}_\nu} b$, and so by dividing by b (using strong multiplicativity), we get $v - \lambda <_{\mathcal{F}_\nu} 1_K$. Therefore, $v - \lambda \in m$. Since $v = \lambda + (v - \lambda)$ and $\lambda \in k = A \cap V$, we have $v \in (A \cap V) + m$. □

For a k-complementary valuation ring V to A, let $\alpha(V)$ be the regular, normalized filtration on A defined as in Lemma 4.7. For a regular filtration $\mathcal{F}$ on A, let $\beta(\mathcal{F})$ be the k-complementary valuation ring V to A defined as in Lemma 4.8. This gives the following correspondence.

$$\left\{\begin{matrix} k\text{-complementary} \\ \text{valuation rings to } A \end{matrix}\right\} \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} \left\{\begin{matrix} \text{regular, normalized} \\ \text{filtrations on } A \end{matrix}\right\}. \tag{20}$$

Proposition 4.9. *The operations α and β are inverses to one another, and hence give a bijective correspondence.*

Proof. The fact that $\beta \circ \alpha$ is the identity map was shown in Theorem 4.4 (vi).

We now show that $\alpha \circ \beta$ is the identity map. Given a filtration $\mathcal{F}$, Theorem 4.3 produces the valuation $K_{\le_{\mathcal{F}} 1_K}$. Then Theorem 4.4 produces a filtration whose filtrands are of the form $A_{\le_\nu m}$, where $m = \nu(a)$. Thus by Theorem 4.3 (ii), $u \in A_{\le_\nu m} \Leftrightarrow \nu(u) \le \nu(a) \Leftrightarrow u \le_{\mathcal{F}} a \Leftrightarrow u \in A_{\le a}$. Since $\mathcal{F}$ is normalized, all of its filtrands are of the form $A_{\le a}$, and so the filtrands of $\mathcal{F}$ and $[\alpha \circ \beta](\mathcal{F})$ are identical. □

5. Filtrations and Term Orders

In this section, we discuss filtrations on polynomial rings that come from term orders in suitable variables, and give an intrinsic characterization of such filtrations. We begin with basic definitions, and then we specialize some of the results from Section 3 to polynomial rings with term order filtrations. To complete our characterization, we make a few intermediate developments concerning general filtrations. In particular, we show how to place a natural monoid structure on the normalization of a filtration in case $\mathcal{F}$ is weakly multiplicative.

5.1. term order filtrations

In preparation of the characterization of which filtrations on the polynomial ring arise from term orders in suitable variables, we need to carefully describe the properties of filtrations arising from term orders. Let us begin by specifically describing the sets (10)–(12) for the case where $A = k[\mathbf{x}]$ and the filtration is $\mathcal{F}_\sigma$ for a term order "$\le_\sigma$" on $k[\mathbf{x}]$. It is left to the reader to verify the following equalities:

$$k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} 0} = \{0\}, \quad k[\mathbf{x}]_{<_{\mathcal{F}_\sigma} 0} = \emptyset, \quad k[\mathbf{x}]_{\sim_{\mathcal{F}_\sigma} 0} = \{0\}. \tag{21}$$

The sets $k[\mathbf{x}]_{\le_\sigma \mathbf{e}}$, $k[\mathbf{x}]_{<_\sigma \mathbf{e}}$ and $k[\mathbf{x}]_{\sim_\sigma \mathbf{e}}$ are defined in Example 3.1. Recall that $k[\mathbf{x}]_{\le_\sigma \mathbf{e}}$ is defined as $\mathrm{Span}_k(\{\mathbf{x}^\mathbf{u} \in \mathbb{M}[\mathbf{x}] \mid \mathbf{x}^\mathbf{u} \le_\sigma \mathbf{x}^\mathbf{e}\})$, and $k[\mathbf{x}]_{<_\sigma \mathbf{e}}$ and $k[\mathbf{x}]_{\sim_\sigma \mathbf{e}}$ are defined similarly. If $\mathrm{LM}(g) = \mathbf{x}^\mathbf{e}$, then

$$k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} g} = k[\mathbf{x}]_{\le_\sigma \mathbf{e}}, \quad k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} g} = k[\mathbf{x}]_{<_\sigma \mathbf{e}}, \quad k[\mathbf{x}]_{\sim_{\mathcal{F}_\sigma} g} = k[\mathbf{x}]_{\sim_\sigma \mathbf{e}}. \tag{22}$$

Recall that $\mathcal{F}_\sigma$ consists of $\{0\}$ together with all of the subspaces of $k[\mathbf{x}]$ of the form $k[\mathbf{x}]_{\le_\sigma \mathbf{e}}$ for $\mathbf{e} \in \mathbb{N}^n$. Thus $\mathcal{F}_\sigma$ consists of the sets $k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} f}$ for $f \in k[\mathbf{x}]$. Since $k[\mathbf{x}] \wr \mathcal{F}_\sigma$ is defined as the set of subsets of A of the form $A_{\le_{\mathcal{F}_\sigma} a}$, we have shown that

$$\mathcal{F}_\sigma = k[\mathbf{x}] \wr \mathcal{F}_\sigma. \tag{23}$$

As is immediate from the definition, for a term order filtration the sets in $\mathcal{F}_\sigma$ are k-subspaces of $k[\mathbf{x}]$.

Next let us consider the structure of $k[\mathbf{x}] \wr \mathcal{F}_\sigma$. Since $\mathcal{F}_\sigma$ intersects to zero, $\{0\}$ is a "$\sim_{\mathcal{F}_\sigma}$" equivalence class and

$$\varsigma(0) = k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} 0} = k[\mathbf{x}]_{\sim_{\mathcal{F}_\sigma} 0} = \{0\}$$

is the unique minimal element of $k[\mathbf{x}] \wr \mathcal{F}_\sigma$. Let us denote this element of $k[\mathbf{x}] \wr \mathcal{F}_\sigma$ by "$-\infty$". As observed in (22), $k[\mathbf{x}]_{\sim_{\mathcal{F}_\sigma} g} = k[\mathbf{x}]_{\sim_\sigma \mathbf{e}}$ and as observed below, $\mathcal{F}_\sigma \setminus \{\{0\}\}$ is indexed by $\mathbb{N}^n$. This gives us the following bijective correspondence:

$$\{-\infty\} \uplus \mathbb{N}^n \longleftrightarrow k[\mathbf{x}] \wr \mathcal{F}_\sigma. \tag{24}$$

Under this correspondence $-\infty$ corresponds to $k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} 0}$ and $\mathbf{e} \in \mathbb{N}^n$ corresponds to $k[\mathbf{x}]_{\le_{\mathcal{F}_\sigma} \mathbf{x}^\mathbf{e}}$.

The order on $\{-\infty\} \uplus \mathbb{N}^n$ corresponding to the order on $k[\mathbf{x}] \wr \mathcal{F}_\sigma$ is given as follows. We say that $-\infty$ is the unique minimal element and for $\mathbf{e}, \mathbf{f} \in \mathbb{N}^n, \mathbf{e} \leq \mathbf{f}$ if and only if $\mathbf{x}^\mathbf{e} \leq_\sigma \mathbf{x}^\mathbf{f}$. In other words, this order agrees with the term order "$\leq_\sigma$".

For $k[\mathbf{x}]$ with term order "$\leq_\sigma$" and term order filtration $\mathcal{F}_\sigma$, (24) gives the following correspondence:

$$\mathbb{N}^n \longleftrightarrow (k[\mathbf{x}] \wr \mathcal{F}_\sigma)^\star. \tag{25}$$

It follows from (22) that for $\mathbf{e} \in \mathbb{N}^n$, the one-dimensional k space spanned by the monomial $\mathbf{x}^\mathbf{e}$ is a vector space complement to $k[\mathbf{x}]_{<_{\mathcal{F}_\sigma} g}$ in $k[\mathbf{x}]_{\leq_{\mathcal{F}_\sigma} g}$. Thus $k[\mathbf{x}]_{\leq_{\mathcal{F}_\sigma} g}/k[\mathbf{x}]_{<_{\mathcal{F}_\sigma} g}$ is one-dimensional and the filtration $\mathcal{F}_\sigma$ has one-dimensional graded components.

Let $k[\mathbf{x}]$ be a polynomial ring with term order "$\leq_\sigma$". A term order has the following compatibility properties.

$$\mathbf{x}^\mathbf{e} \leq \mathbf{x}^\mathbf{f} \Rightarrow \mathbf{x}^\mathbf{e}\mathbf{x}^\mathbf{g} \leq \mathbf{x}^\mathbf{f}\mathbf{x}^\mathbf{g},$$
$$\mathbf{x}^\mathbf{e} < \mathbf{x}^\mathbf{f} \Rightarrow \mathbf{x}^\mathbf{e}\mathbf{x}^\mathbf{g} < \mathbf{x}^\mathbf{f}\mathbf{x}^\mathbf{g},$$

which must be inherited by the term order filtration, as stated in the next result.

PROPOSITION 5.1. *Every term order filtration is strongly multiplicative.*

Using the notion of normalization, we can rephrase (23) as follows.

PROPOSITION 5.2. *Every term order filtration is a normalized filtration.*

5.2. THE ASSOCIATED MONOID

Now we present the natural monoid structure on $A \wr \mathcal{F}$ when $\mathcal{F}$ is a weakly multiplicative filtration and the natural submonoid structure on $(A \wr \mathcal{F})^*$ when $\mathcal{F}$ is a strongly multiplicative filtration. To begin, assume that $\mathcal{F}$ is a weakly multiplicative filtration on A. By Lemma 3.9 (i), if $a, b, c, d \in A$ with $a \sim_\mathcal{F} b$ and $c \sim_\mathcal{F} d$ and $ac \sim_\mathcal{F} bc \sim_\mathcal{F} bd$, hence there is a well-defined product on $A//\mathcal{F}$, the set of "$\sim_\mathcal{F}$" equivalence classes (3.6), where

$$A_{\sim_\mathcal{F} a} * A_{\sim_\mathcal{F} c} = A_{\sim_\mathcal{F} ac}. \tag{26}$$

Because the multiplicative structure of A is associative and has unit 1_A we have the following proposition.

PROPOSITION 5.3. *The product on $A//\mathcal{F}$ given by* (26) *is associative and has unit $A_{\sim_\mathcal{F} 1_A}$.*

In (13) we identify $A//\mathcal{F}$ with $A \wr \mathcal{F}$. Therefore where (26) gives the product on $A//\mathcal{F}$, Proposition 3.12 gives the corresponding product on $A \wr \mathcal{F}$ via

$$A_{\leq_\mathcal{F} a} A_{\leq_\mathcal{F} c} = A_{\leq_\mathcal{F} ac}.$$

PROPOSITION 5.4. *Let $\mathcal{F}$ be a weakly multiplicative filtration on A.*

(*i*) *Using the monoid structure on $A \wr \mathcal{F}$ given by Proposition* 3.12 *and the monoid structure on $A//\mathcal{F}$ given by* (26), *the map $A//\mathcal{F} \to A \wr \mathcal{F}$ given by* (13) *is a monoid isomorphism.*

(*ii*) *Ignoring the additive structure of A and simply viewing A as a multiplicative monoid, the map $A \to A//\mathcal{F}, a \to A_{\sim_{\mathcal{F}} c}$, and the map $\varsigma\colon A \to A \wr \mathcal{F}$ are monoid maps.*

Proof.

(i) The map $A//\mathcal{F} \to A \wr \mathcal{F}$ sends $A_{\sim_{\mathcal{F}} a}$ to $A_{\leq_{\mathcal{F}} a}$ and has already been shown to be bijective. The description of the monoid structure (26) on $A//\mathcal{F}$ and the description in Proposition 3.12 of the monoid structure on $A \wr \mathcal{F}$ show that $A//\mathcal{F} \to A \wr \mathcal{F}$ is a monoid map.

(ii) Multiplicativity of both maps simply comes down to $A_{\sim_{\mathcal{F}} a} * A_{\sim_{\mathcal{F}} c} = A_{\sim_{\mathcal{F}} ac}$ in $A//\mathcal{F}$ and $\varsigma(a)\varsigma(c) = \varsigma(ac)$ in $A \wr \mathcal{F}$, for $a, c \in A$. Each map carries 1_A to the identity of their respective ranges because $A_{\sim_{\mathcal{F}} 1_A}$ is the identity of $A//\mathcal{F}$ and $\varsigma(1_A) = A_{\leq_{\mathcal{F}} 1_A}$ is the identity of $A \wr \mathcal{F}$. □

Section 4.1 shows that a strongly multiplicative filtration on an integral domain A gives rise to a valuation ν on K, the field of fractions of A, and vice versa. When A is viewed as a multiplicative monoid as in Proposition 5.4 (ii), $A^\star$ is a submonoid. If ν is a valuation on K, then the restriction

$$\nu\,|_{A^\star}\colon A^\star \to \nu(K^\star) \tag{27}$$

is a monoid map whose image $\nu(A^\star)$ is the value monoid of A. It is natural to ask how the monoid map (27) relates to the monoid maps in (iv) above. Our next goal is to show that $(A \wr \mathcal{F})^\star$ is naturally isomorphic to the value monoid of A with respect to the valuation induced by $\mathcal{F}$.

Proposition 5.5. *Let A be an integral domain with field of fractions K. Suppose that $\mathcal{F}$ is a strongly multiplicative k-filtration on A and $K_{\leq_{\mathcal{F}} 1_K}$ is the valuation ring in K arising from $\mathcal{F}$ as given in Theorem 4.3 (i). Let ν be the valuation coming from $K_{\leq_{\mathcal{F}} 1_K}$ and let $\nu(A^\star)$ be the value monoid of A. There are bijective, monoid, order preserving maps $\nu(A^\star) \to (A \wr \mathcal{F})^\star$ and $(A \wr \mathcal{F})^\star \to \nu(A^\star)$ defined as follows. For $a \in A^\star$:*

$$\nu(A^\star) \to (A \wr \mathcal{F})^\star, \quad \nu(a) \mapsto \varsigma(a) \text{ and } (A \wr \mathcal{F})^\star \to \nu(A^\star), \quad \varsigma(a) \mapsto \nu(a).$$

These maps are inverse ordered monoid isomorphisms to one another.

Proof. First let us show that the proposed maps $\nu(A^\star) \to (A \wr \mathcal{F})^\star$ and $(A \wr \mathcal{F})^\star \to \nu(A^\star)$ are well-defined. For $a, b \in A^\star$, it follows from Theorem 4.3 (ii) that

$$\nu(a) \leq \nu(b) \Leftrightarrow a \leq_{\mathcal{F}} b \Leftrightarrow \varsigma(a) \leq \varsigma(b),$$

and hence

$$\nu(a) = \nu(b) \Leftrightarrow a \sim_{\mathcal{F}} b \Leftrightarrow \varsigma(a) = \varsigma(b).$$

This shows that each of the proposed maps is independent of the coset representative used to define that map, and hence the proposed maps are well-defined. It is clear that they are inverses of one another, and hence they are bijective. This also shows that the maps are order preserving.

It remains to show that the maps $\nu(A^\star) \to (A \wr \mathcal{F})^\star$ and $(A \wr \mathcal{F})^\star \to \nu(A^\star)$ are monoid maps. This follows from the fact that the surjective maps

$$\varsigma\,|_{A^\star}\colon A^\star \to (A \wr \mathcal{F})^\star \text{ and } \nu\,|_{A^\star}\colon A^* \to \nu(A^\star)$$

are monoid maps. □

We next describe the monoid structure on $k[\mathbf{x}] \wr \mathcal{F}_\sigma$ and $(k[\mathbf{x}] \wr \mathcal{F}_\sigma)^\star$ where $\mathcal{F}_\sigma$ is the filtration defined in Example 3.1 (for a term order "$\leq_\sigma$" on $k[\mathbf{x}]$). For this purpose we refer to the description of $k[\mathbf{x}] \wr \mathcal{F}_\sigma$ and $(k[\mathbf{x}] \wr \mathcal{F}_\sigma)^\star$ in (24) and (25).

Choose 0 as a coset representative of $\varsigma(0) = k[\mathbf{x}]_{\leq_{\mathcal{F}_\sigma} 0}$, which is the element denoted "$-\infty$" in (24). For $\mathbf{e} \in \mathbb{N}^n$ choose $\mathbf{x}^\mathbf{e}$ as a coset representative of $\varsigma(\mathbf{x}^\mathbf{e}) = k[\mathbf{x}]_{\leq_{\mathcal{F}_\sigma} \mathbf{x}^\mathbf{e}}$ which is the element denoted "$\mathbf{e}$" in (24). Using Proposition 5.4 and the fact that $0 \cdot 0 = 0, 0 \cdot \mathbf{x}^\mathbf{e} = 0$ and $\mathbf{x}^\mathbf{e} \cdot \mathbf{x}^\mathbf{f} = \mathbf{x}^{\mathbf{e}+\mathbf{f}}$, it follows that the correspondence (24) is a monoid isomorphism between the additive monoid $\{-\infty\} \uplus \mathbb{N}^n$ and the multiplicative monoid $k[\mathbf{x}] \wr \mathcal{F}_\sigma$. Moreover, the correspondence (25) is a monoid isomorphism between the additive monoid $\mathbb{N}^n$ and the multiplicative monoid $(k[\mathbf{x}] \wr \mathcal{F}_\sigma)^\star$.

5.3. Characterizing term order filtrations

We are finally prepared to discuss filtrations that come from term orders in suitable variables and give a characterization. We begin with a few preliminary results.

Definition. Suppose that $\mathcal{F}$ is a filtration on A and $a, c \in A$ where $a <_\mathcal{F} \mathcal{C}$. We say that *$a$ lies just$_\mathcal{F}$ below c* and that *c lies just$_\mathcal{F}$ above a* if the conditions of Lemma 5.6 are satisfied. Condition (i) is the motivation for the terminology "just$_\mathcal{F}$ below" and "just$_\mathcal{F}$ above".

Lemma 5.6. *Suppose that $\mathcal{F}$ is a filtration on A and $a, c \in A$ where $a <_\mathcal{F} c$. The following conditions are equivalent.*

(i) If $b \in A$ with $a \leq_\mathcal{F} b \leq_\mathcal{F} c$, then either $a \sim_\mathcal{F} b$ or $b \sim_\mathcal{F} c$.
(ii) If $T \in \mathcal{F}$ with $a \in T$ but $c \notin T$, then $T = A_{\leq_\mathcal{F} a}$.
(iii) $A_{\leq_\mathcal{F} a} = A_{<_\mathcal{F} c}$.
(iv) $A_{\leq_\mathcal{F} c}$ is the smallest element of $A \wr \mathcal{G}$ properly containing $A_{\leq_\mathcal{F} a}$.

Proof.

(i) $\Rightarrow$ (iv) We have $A_{\leq_\mathcal{F} a} \subset A_{\leq_\mathcal{F} c}$ since $a <_\mathcal{F} c$. Suppose that $b \in A$ where $A_{\leq_\mathcal{F} a} \subset A_{\leq_\mathcal{F} b} \subset A_{\leq_\mathcal{F} c}$. Then $a \leq_\mathcal{F} b \leq_\mathcal{F} c$ and so either $a \sim_\mathcal{F} b$ or $b \sim_\mathcal{F} c$. If $a \sim_\mathcal{F} b$, then $A_{\leq_\mathcal{F} a} = A_{\leq_\mathcal{F} b}$, and if $b \sim_\mathcal{F} c$, then $A_{\leq_\mathcal{F} b} = A_{\leq_\mathcal{F} c}$. Hence $A_{\leq_\mathcal{F} c}$ is the smallest element of $A \wr \mathcal{G}$ properly containing $A_{\leq_\mathcal{F} a}$.

(iv) $\Rightarrow$ (iii) We have $A_{\leq_\mathcal{F} a} \subset A_{<_\mathcal{F} c}$ since $a <_\mathcal{F} c$. Let us show the opposite inclusion. If $b \in A_{<_\mathcal{F} c}$, then $b <_\mathcal{F} c$, and so $A_{\leq_\mathcal{F} b} \subsetneq A_{\leq_\mathcal{F} c}$. By the minimality of $A_{\leq_\mathcal{F} c}$, it cannot be the case that $A_{\leq_\mathcal{F} a} \subsetneq A_{\leq_\mathcal{F} b}$. By the fact that $A \wr \mathcal{G}$ is totally ordered with respect to inclusion it follows that $A_{\leq_\mathcal{F} b} \subset A_{\leq_\mathcal{F} a}$ and so $b \in A_{\leq_\mathcal{F} a}$. Thus we have shown that $A_{<_\mathcal{F} c} \subset A_{\leq_\mathcal{F} a}$. This proves $A_{<_\mathcal{F} a} = A_{<_\mathcal{F} c}$ and finishes (iv) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) Let us show that $A_{\leq_\mathcal{F} a} \subset T \subset A_{<_\mathcal{F} c}$. Since we are assuming $A_{\leq_\mathcal{F} a} = A_{<_\mathcal{F} c}$, this will show that $T = A_{\leq_\mathcal{F} a}$. Suppose that $a \in T$ and $b \in A_{\leq_\mathcal{F} a}$. Then $b \in S$ for all $S \in \mathcal{F}$ with $a \in S$. In particular, $b \in T$ and so $A_{\leq_\mathcal{F} a} \subset T$. Suppose that $c \notin T$ and $b \in T$. Then T is an element of $\mathcal{F}$ containing b and excluding c. Thus $b <_\mathcal{F} c$, and so $T \subset A_{<_\mathcal{F} c}$.

(ii) $\Rightarrow$ (i) Since $b \leq_\mathcal{F} c$, if $b \nsim_\mathcal{F} c$, then $b <_\mathcal{F} c$. Hence there exists $T \in \mathcal{F}$ with $b \in T$ and $c \notin T$. Since $a \leq_\mathcal{F} b$, it follows that $a \in T$ and so (ii) implies that $T = A_{\leq_\mathcal{F} a}$. since $b \in T$ this shows that $b \leq_\mathcal{F} a$. Hence $a \sim_\mathcal{F} b$. $\square$

Definition. Suppose that $U \subset A$ and $a \in A$. We say that a is a *maximal*$_{\mathcal{F}}$ element of U if $a \in U$ and $u \leq_{\mathcal{F}} a$ for all elements $u \in U$.

Proposition 5.7. *Suppose that $\mathcal{F}$ is a filtration on $A, a \in A$ and $S \in \mathcal{F}$.*

(*i*) *If a lies just$_{\mathcal{F}}$ below some element of A, then $A_{\leq_{\mathcal{F}} a}$ is an element of $\mathcal{F}$.*
(*ii*) *If $\{\zeta \in A \wr \mathcal{F} \mid \zeta > \varsigma(a)\}$ has a minimal element, then $A_{\leq_{\mathcal{F}} a} \neq A$ and $A_{\leq_{\mathcal{F}} a}$ is an element of $\mathcal{F}$.*
(*iii*) *The element $a \in A$ is a maximal$_{\mathcal{F}}$ element of $A_{\leq_{\mathcal{F}} a}$.*
(*iv*) *If a is a maximal$_{\mathcal{F}}$ element of S, then $S = A_{\leq_{\mathcal{F}} a}$.*

Proof.

(i) Suppose that $c \in A$ where c lies just$_{\mathcal{F}}$ above a. Then $a <_{\mathcal{F}} c$ and so there exists $T \in \mathcal{F}$ with $a \in T$ but $c \notin T$. Hence, by Lemma 5.6 (ii), $T = A_{\leq_{\mathcal{F}} a}$.
(ii) Suppose that $\{\zeta \in A \wr \mathcal{F} \mid \zeta > \varsigma(a)\}$ has a minimal element $\varsigma(c)$ for $c \in A$. Then $a <_{\mathcal{F}} c$ and so $A_{\leq_{\mathcal{F}} a} \subsetneq A_{\leq_{\mathcal{F}} c}$. In particular, $A_{\leq_{\mathcal{F}} a} \neq A$.
Let us show that a lies just$_{\mathcal{F}}$ below c. Suppose that $a \leq_{\mathcal{F}} b \leq_{\mathcal{F}} c$ so that $\varsigma(a) \leq \varsigma(b) \leq \varsigma(c)$. If $a <_{\mathcal{F}} b$, then $\varsigma(b) \in \{\zeta \in A \wr \mathcal{F} \mid \zeta > \varsigma(a)\}$. By the minimality of $\varsigma(c)$ it follows that $\varsigma(c) \leq \varsigma(b)$ and so $b \sim_{\mathcal{F}} c$, showing that a lies just$_{\mathcal{F}}$ below c. It now follows from part (i) that $A_{\leq_{\mathcal{F}} a}$ is an element of $\mathcal{F}$.
(iii) Certainly, $a \in A_{\leq_{\mathcal{F}} a}$ and by the definition of $A_{\leq_{\mathcal{F}} a}$ it follows that $u \leq_{\mathcal{F}} a$ for all $u \in A_{\leq_{\mathcal{F}} a}$.
(iv) Suppose that a is a maximal$_{\mathcal{F}}$ element of S. First we show that $A_{\leq_{\mathcal{F}} a} \subset S$. Suppose that $b \in A_{\leq_{\mathcal{F}} a}$. By the definition of "$\leq_{\mathcal{F}}$", $b \in S'$ for all $S' \in \mathcal{F}$ where $a \in S'$. Thus $b \in S$ and we have shown that $A_{\leq_{\mathcal{F}} a} \subset S$. To verify the opposite containment, suppose that $s \in S$. By the maximality$_{\mathcal{F}}$ of a, it follows that $s \leq_{\mathcal{F}} a$. Thus $s \in A_{\leq_{\mathcal{F}} a}$ and so $S \subset A_{\leq_{\mathcal{F}} a}$. Therefore, a is a maximal$_{\mathcal{F}}$ element of S, and so $S = A_{\leq_{\mathcal{F}} a}$. □

Definition. A k-filtration is *well-ordered* if $A \wr \mathcal{F}$ is well-ordered where its elements are compared via set-inclusion.

Note that in the case of a k-filtration, $\varsigma(0)$ is always the unique minimal element of $A \wr \mathcal{F}$. Hence $A \wr \mathcal{F}$ is well-ordered if and only if $(A \wr \mathcal{F}) \setminus \{\varsigma(0)\} = (A \wr \mathcal{F})^{\star}$ is well-ordered. The fact that term order filtrations are well-ordered comes from (24).

Corollary 5.8. *Let $\mathcal{G}$ be a well-ordered filtration on A where $A \wr \mathcal{G}$ does not have a maximal element. Then $\mathcal{G} = A \wr \mathcal{G}$ if and only if each $W \in \mathcal{G}$ has a maximal$_{\mathcal{G}}$ element.*

Proof.

($\Rightarrow$): Now, $A \wr \mathcal{G}$ consists of sets of the form $A_{\leq_{\mathcal{G}} a}$ for $a \in A$ and by Proposition 5.7 (iii), a is a maximal$_{\mathcal{G}}$ element of $A_{\leq_{\mathcal{G}} a}$. Hence if $\mathcal{G} = A \wr \mathcal{G}$, then every $W \in \mathcal{G}$ has a maximal$_{\mathcal{G}}$ element.
($\Leftarrow$): We show that $\mathcal{G} \subset A \wr \mathcal{G}$ and $A \wr \mathcal{G} \subset \mathcal{G}$. Suppose that $W \in \mathcal{G}$. By hypothesis, if w is a maximal$_{\mathcal{G}}$ element of W, then $W = A_{\leq_{\mathcal{G}} w}$ by Proposition 5.7 (iv). Hence, $W = \varsigma(a) \in A \wr \mathcal{G}$ and $\mathcal{G} \subset A \wr \mathcal{G}$.

Next, let us show that $\varsigma(a) = A_{\leq_{\mathcal{G}} a} \in \mathcal{G}$ for $a \in A$. This will prove that $A \wr \mathcal{G} \subset \mathcal{G}$. By hypothesis, $A_{\leq_{\mathcal{G}} a}$ is not a maximal element of $A \wr \mathcal{G}$ and so there are elements of $A \wr \mathcal{G}$ properly containing $A_{\leq_{\mathcal{G}} a}$. By the hypothesis of $A \wr \mathcal{G}$ being well-ordered there exists $c \in A$ where $A_{\leq_{\mathcal{G}} c}$ is the smallest element of $A \wr \mathcal{G}$ properly containing $A_{\leq_{\mathcal{G}} a}$. By Lemma 5.6 (iv), it follows that a lies just$_{\mathcal{G}}$ below c. By Proposition 5.7 this implies that $A_{\leq_{\mathcal{G}} a} \in \mathcal{G}$ for $a \in A$. □

In Section 2.2 we defined what it means for a valuation or valuation ring to come from a term order in suitable variables. In the same spirit we shall define what it means for a filtration to come from a term order in suitable variables.

Suppose that $k[\mathbf{u}]$ is a polynomial ring with term order "$\leq_\sigma$" and k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$. The term order "$\leq_\sigma$" gives rise to the filtration $\mathcal{F}_\sigma$ on $k[\mathbf{u}]$. The elements of $\mathcal{F}_\sigma$ are subspaces of $k[\mathbf{u}]$ and the k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ carries these subspaces to subspaces of $k[\mathbf{x}]$, giving the filtration on $k[\mathbf{x}]$. We denote this filtration on $k[\mathbf{x}]$ by $\phi(\mathcal{F}_\sigma)$.

Definition. Suppose that $k[\mathbf{u}]$ is a polynomial ring with term order $\leq_\sigma$. Let $\mathcal{F}_\sigma$ denote the term order filtration on $k[\mathbf{u}]$. Suppose that $k[\mathbf{x}]$ is a polynomial ring and ϕ is a k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$. The filtration on $k[\mathbf{x}]$ consisting of the subsets of $k[\mathbf{x}]$ that are the images under ϕ of the sets in the filtration $\mathcal{F}_\sigma$ on $k[\mathbf{u}]$ is denoted "$\phi(\mathcal{F}_\sigma)$". More precisely,

$$\phi(\mathcal{F}_\sigma) = \{\phi(S) \mid S \in \mathcal{F}_\sigma\}.$$

Since ϕ is an isomorphism the filtration $\phi(\mathcal{F}_\sigma)$ shares the properties of $\mathcal{F}_\sigma$ such as being strongly multiplicative, intersecting to zero, and having one-dimensional graded components. A filtration $\mathcal{G}$ on $k[\mathbf{x}]$ is said to come from a term order *in suitable variables* on $k[\mathbf{x}]$ if there is a polynomial ring $k[\mathbf{u}]$ with term order $\leq_\sigma$ and k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ whereby $\mathcal{G}$ equals $\phi(\mathcal{F}_\sigma)$. The isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ is called the associated isomorphism.

As in the case of valuations coming from a term order in suitable variables, the associated isomorphism is uniquely determined. We note in the proof of Theorem 5.9 that there is only one way to define the isomorphism ϕ.

By using the valuation associated to a filtration, one may characterize when a filtration induces and some quasi-order as filtration that comes from a term order in suitable variables. This would proceed by starting with the filtration, using Theorem 4.3 to obtain a valuation ring, and then using Theorem 2.6 to characterize when the valuation ring comes from a term order in suitable variables. Rather than pursue this course, here is a more intrinsic, more direct characterization of when a quasi-order and a filtration come from a term order in suitable variables.

Theorem 5.9. *Let $\mathcal{G}$ be a filtration on $k[\mathbf{x}]$.*

A. $k[\mathbf{x}] \wr \mathcal{G}$ *comes from a term order filtration in suitable variables if conditions (i)–(iv) are satisfied.*

B. $\mathcal{G}$ *comes from a term order filtration in suitable variables if and only if (i)–(v) are satisfied.*

(*i*) $\mathcal{G}$ *is a strongly multiplicative k-filtration that intersects to zero.*

(*ii*) *$\mathcal{G}$ has one-dimensional graded components.*
(*iii*) *$\varsigma(1_{k[\mathbf{x}]})$ is the smallest element of $(k[\mathbf{x}] \wr \mathcal{G})^\star$.*
(*iv*) *$(k[\mathbf{x}] \wr \mathcal{G})^\star$ and $\mathbb{N}^n$ are isomorphic as monoids.*
(*v*) *Each $W \in \mathcal{G}$ has a maximal$_{\mathcal{G}}$ element.*

Proof. Suppose that conditions (i)–(iv) are satisfied. Since $(k[\mathbf{x}] \wr \mathcal{G})^\star$ is isomorphic to $\mathbb{N}^n$, choose and fix one such isomorphism as an identification. In this way consider $(k[\mathbf{x}] \wr \mathcal{G})^\star$ to be $\mathbb{N}^n$. By Proposition 3.12(ii), the order on $(k[\mathbf{x}] \wr \mathcal{G})^\star$ is strongly multiplicative. By hypothesis $\varsigma(1_{k[\mathbf{x}]})$ is the smallest element of $(k[\mathbf{x}] \wr \mathcal{G})^\star$. A term order is defined as a total order (with additional properties) either on the monomials in $k[\mathbf{x}]$, or equivalently, on the set $\mathbb{N}^n$. Since we are identifying $(k[\mathbf{x}] \wr \mathcal{G})^\star$ with $\mathbb{N}^n$, we note that the order on $(k[\mathbf{x}] \wr \mathcal{G})^\star$ is a term order, and by Dickson's lemma (Becker and Weispfenning, 1993), there are no infinite descending sequences in $(k[\mathbf{x}] \wr \mathcal{G})^\star$. Hence every non-empty subset has a minimal element and $(k[\mathbf{x}] \wr \mathcal{G})^\star$ is well-ordered. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{N}^n$ where $\mathbf{e}_i$ is the vector of length n consisting of all zeros except for a 1 in the ith position. For each $\mathbf{e}_i$, choose $f_i \in k[\mathbf{x}]$ such that $\varsigma(f_i) = \mathbf{e}_i$. Let us show that $\{f_1, \ldots, f_n\}$ is an algebraically independent set that generates $k[\mathbf{x}]$ as an algebra.

Since $k[\mathbf{x}]$ has transcendence degree n over k and the set $\{f_1, \ldots, f_n\}$ only has n elements, if $\{f_1, \ldots, f_n\}$ generates $k[\mathbf{x}]$ as an algebra, then it is an algebraically independent set. Suppose that $k[f_1, \ldots, f_n]$, the subalgebra of $k[\mathbf{x}]$ generated by $f_1, \ldots, f_n$, is a proper subalgebra of $k[\mathbf{x}]$. Among the elements in $k[\mathbf{x}] \setminus k[f_1, \ldots, f_n]$ choose f where $\varsigma(f)$ is minimal. This is possible because $(k[\mathbf{x}] \wr \mathcal{G})^\star$ is well-ordered. Suppose $\varsigma(f) = (e_1, \ldots, e_n)$. Since $\varsigma(f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n})$ also equals $(e_1, \ldots, e_n)$ it follows that $f \sim_{\mathcal{F}} f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}$. By the hypothesis concerning one-dimensional graded components, one may apply Lemma 3.8 (iv) to conclude that there exists $\lambda \in k$ with

$$f - \lambda f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n} = 0$$

or

$$\varsigma(f - \lambda f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}) < \varsigma(f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}) = (e_1, \ldots, e_n).$$

It cannot be that $f - \lambda f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n} = 0$ since $f \in k[\mathbf{x}] \setminus k[f_1, \ldots, f_n]$. It cannot be that $\varsigma(f - \lambda f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n}) < (e_1, \ldots, e_n)$ since $f - \lambda f_1^{e_1} f_2^{e_2} \cdots f_n^{e_n} \in k[\mathbf{x}] \setminus k[f_1, \ldots, f_n]$ and this would contradict the minimality of $\varsigma(f)$. Hence, there cannot exist $f \in k[\mathbf{x}] \setminus k[f_1, \ldots, f_n]$, and so $k[\mathbf{x}] = k[f_1, \ldots, f_n]$.

Let us define a k-algebra isomorphism $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ where $u_i \mapsto f_i$ or equivalently $\phi(g(u_1, \ldots, u_n)) = g(f_1, \ldots, f_n)$. The map ϕ is an algebra isomorphism because $\{f_1, \ldots, f_n\}$ generates $k[\mathbf{x}]$ is an algebra and is an algebraically independent set.

For $(b_1, \ldots, b_n) = \mathbf{b} \in \mathbb{N}^n$, consider

$$\varsigma(\phi(\mathbf{u}^{\mathbf{b}})) = \varsigma(f_1^{b_1} f_2^{b_2} \cdots f_n^{b_n}) = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n = (b_1, b_2, \ldots, b_n) = \mathbf{b}.$$

Since the order on $\mathbb{N}^n$ is a term order, we use ϕ to pull the order on $\mathbb{N}^n$ back to a term order on $k[\mathbf{u}]$. Let us denote this term order on $k[\mathbf{u}]$ by "$<_\rho$" which is defined as follows. For $\mathbf{b}, \mathbf{c} \in \mathbb{N}^n$ set $\mathbf{u}^{\mathbf{b}} <_\rho \mathbf{u}^{\mathbf{c}}$ if and only if $\mathbf{b} < \mathbf{c}$ in the order on $\mathbb{N}^n$. Now ϕ carries the $\mathcal{F}_\rho$ filtration over to $k[\mathbf{x}]$ to give the filtration $\phi(\mathcal{F}_\rho)$ on $k[\mathbf{x}]$ as given by the definition of a filtration coming from a term order in suitable variables. Here ϕ is the associated isomorphism. We shall show that $\phi(\mathcal{F}_\rho)$ equals $k[\mathbf{x}] \wr \mathcal{G}$. This will prove that $k[\mathbf{x}] \wr \mathcal{G}$ comes from a term order filtration in suitable variables.

We verify that $\phi(\mathcal{F}_\rho)$ equals $k[\mathbf{x}] \wr \mathcal{G}$ by the usual strategy of showing that $\phi(\mathcal{F}_\rho) \supset k[\mathbf{x}] \wr \mathcal{G}$ and $\phi(\mathcal{F}_\rho) \subset k[\mathbf{x}] \wr \mathcal{G}$. Thus we must justify the following two statements:

(a) $\{0\} \in \phi(\mathcal{F}_\rho)$ and for $f \in k[\mathbf{x}]^\star$ there exists $\mathbf{c} \in \mathbb{N}^n$ where $k[\mathbf{x}]_{\le_\mathcal{G} f} = \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$.

(b) $\phi(\{0\}) \in k[\mathbf{x}] \wr \mathcal{G}$ and if $\mathbf{c} \in \mathbb{N}^n$, then there exists $f \in k[\mathbf{x}]$ where $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) = k[\mathbf{x}]_{\le_\mathcal{G} f}$.

(a) We have $\{0\} \in \phi(\mathcal{F}_\rho)$ because $\{0\} = \phi(\{0\})$ for $\{0\} \in \mathcal{F}_\rho$. Next suppose that $f \in k[\mathbf{x}]^\star$. We shall show that $k[\mathbf{x}]_{\le_\mathcal{G} f} = \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$ where $c \in \mathbb{N}^n$ with $\varsigma(f) = \mathbf{c}$. If $\mathbf{b} \in \mathbb{N}^n w$ here $\mathbf{u}^b \le_\rho \mathbf{u}^\mathbf{c}$, then by the definition of the term order in $k[\mathbf{u}]$ it follows that $\varsigma(\phi(\mathbf{u}^\mathbf{b})) = \mathbf{b} \le \mathbf{c} = \varsigma(f)$. Hence by Proposition 3.6 (iv) it follows that $\phi(\mathbf{u}^\mathbf{b}) \le_\mathcal{G} f$. Since $k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$ is spanned by $\{\mathbf{u}^\mathbf{b}\}_{\mathbf{u}^\mathbf{b} \le_\rho \mathbf{u}^\mathbf{c}}$, it follows that $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) \subset k[\mathbf{x}]_{\le_\mathcal{G} f}$. For the opposite inclusion, suppose that $k[\mathbf{x}]_{\le_\mathcal{G} f} \not\subset \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$. So there exists $v \in k[\mathbf{x}]_{\le_\mathcal{G} f}$ where $v \notin \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$. Since $\mathbb{N}^n$ is well-ordered, among the elements $v \in k[\mathbf{x}]_{\le_\mathcal{G} f}$ where $v \notin \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$ choose such a v where $\varsigma(v)$ is minimal. Set $\mathbf{b} = \varsigma(v)$. Then $\mathbf{b} = \varsigma(v) \le \varsigma(f) = \mathbf{c}$. Hence $\mathbf{u}^\mathbf{b} \in k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$ and $\phi(\mathbf{u}^\mathbf{b}) \in k[\mathbf{x}]_{\le_\mathcal{G} f}$. Since $\varsigma(v) = \mathbf{b} = \varsigma(\phi(\mathbf{u}^\mathbf{b}))$ it follows from Lemma 3.8 (iv) that there exists $\lambda \in k$ with $v - \lambda\phi(\mathbf{u}^\mathbf{b}) = 0$ or $\varsigma(v - \lambda\phi(\mathbf{u}^\mathbf{b})) < \mathbf{b}$. But $v - \lambda\phi(\mathbf{u}^\mathbf{b})$ cannot equal zero because $v \notin \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$. If $\varsigma(v - \lambda\phi(\mathbf{u}^\mathbf{b})) < \mathbf{b}$, then by the minimality of $\mathbf{b}$ it follows that $v - \lambda\phi(\mathbf{u}^\mathbf{b}) \in \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$. This would again imply that $v \in \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$, another contradiction. Hence it follows that $k[\mathbf{x}]_{\le_\mathcal{G} f} \subset \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$. Hence $k[\mathbf{x}]_{\le_\mathcal{G} f} = \phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}})$, thus finishing (a).

(b) First we show that $\phi(\{0\}) \in k[\mathbf{x}] \wr \mathcal{G}$. Since $\phi(\{0\}) = \{0\}$ we must show that $\{0\} \in k[\mathbf{x}] \wr \mathcal{G}$. Suppose not. Then $\cap_{S \in k[\mathbf{x}] \wr \mathcal{G}} S = \cap_{\{0\} \ne S \in k[\mathbf{x}] \wr \mathcal{G}} S$. By hypothesis (iii), $\varsigma(1_{k[\mathbf{x}]})$ is the smallest element of $(k[\mathbf{x}] \wr \mathcal{G})^\star$. Thus $\cap_{\{0\} \ne S \in k[\mathbf{x}] \wr \mathcal{G}} S = k[\mathbf{x}]_{\le_\mathcal{G} 1_{k[\mathbf{x}]}}$. This would contradict hypothesis (i), which states that $\mathcal{G}$ intersects to zero. Hence, $\phi(\{0\}) = \{0\} \in k[\mathbf{x}] \wr \mathcal{G}$.

For the second part of (**b**), suppose that $\mathbf{c} \in \mathbb{N}^n$. We must show $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) \in k[\mathbf{x}] \wr \mathcal{G}$, i.e. that there exists $f \in k[\mathbf{x}]$ where $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) = k[\mathbf{x}]_{\le_\mathcal{G} f}$. In fact we show that f can be chosen as $\phi(\mathbf{u}^\mathbf{c})$, i.e. that $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) = k[\mathbf{x}]_{\le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})}$.

Suppose that $\mathbf{b} \in \mathbb{N}^n$ where $\mathbf{u}^\mathbf{b} \le_\rho \mathbf{u}^\mathbf{c}$. Then, as above, by the definition of the term order on $k[\mathbf{u}]$ it follows that $\varsigma(\phi(\mathbf{u}^\mathbf{b})) = \mathbf{b} \le \mathbf{c} = \varsigma(\phi(\mathbf{u}^\mathbf{c}))$. Hence, $\phi(\mathbf{u}^\mathbf{b}) \le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})$. Since $k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$ is spanned by $\{\mathbf{u}^\mathbf{b}\}_{\mathbf{u}^\mathbf{b} \le_\rho \mathbf{u}^\mathbf{c}}$ it follows that $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) \subset k[\mathbf{x}]_{\le_\mathcal{G} (\mathbf{u}^\mathbf{c})}$. To show that $\phi(k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}) \supset k[\mathbf{x}]_{\le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})}$ we must show that if $g \in k[\mathbf{x}]$ with $g \le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})$, then $g = \phi(h)$ for $h \in k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$. Since we have already shown that ϕ is an isomorphism, there is a unique element $h \in k[\mathbf{u}]$ with $g = \phi(h)$. It remains to show that $h \in k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$. Let $\mathrm{LM}(h) = \mathbf{u}^\mathbf{b}$. By the construction of the term order on $k[\mathbf{u}]$ it follows that $\phi(\mathbf{u}^\mathbf{b}) \sim_\mathcal{G} \phi(h) = g$. Since $g \le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})$ it follows that $\phi(\mathbf{u}^\mathbf{b}) \le_\mathcal{G} \phi(\mathbf{u}^\mathbf{c})$. Applying ς to both sides gives

$$\mathbf{b} = \varsigma(\phi(\mathbf{u}^\mathbf{b})) \le_\mathcal{G} \varsigma(\phi(\mathbf{u}^\mathbf{c})) = \mathbf{c}.$$

By the definition of the term order filtration on $k[\mathbf{u}]$ this implies that $\mathbf{u}^\mathbf{b} \in k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$ and since $\mathbf{u}^\mathbf{b}$ is the lead monomial of h this implies $h \in k[\mathbf{u}]_{\le_\rho \mathbf{u}^\mathbf{c}}$.

We have completed the opposite inclusion establishing (**b**) and that $\phi(\mathcal{F}_\rho)$ equals $k[\mathbf{x}] \wr \mathcal{G}$. Hence, $k[\mathbf{x}] \wr \mathcal{G}$ comes from a term order filtration in suitable variables. The next part of the proof is to show that if the hypothesis (v) is also satisfied, then $\phi(\mathcal{F}_\rho)$ equals $\mathcal{G}$. This is easy to establish because we have established that $\phi(\mathcal{F}_\rho)$ equals $k[\mathbf{x}] \wr \mathcal{G}$ and with hypothesis (v) we can apply Corollary 5.8 to conclude that $k[\mathbf{x}] \wr \mathcal{G}$ equals $\mathcal{G}$. Hence, $\phi(\mathcal{F}_\rho)$ equals $\mathcal{G}$ and $\mathcal{G}$ comes from a term order filtration in suitable variables.

It remains to show that if $k[\mathbf{x}] \wr \mathcal{G}$ comes from a term order filtration in suitable variables, then (i)–(iv) are satisfied and if $\mathcal{G}$ comes from a term order filtration in suitable variables, then (i)–(v) are satisfied. Suppose that $\mathcal{G}$ comes from a term order filtration in suitable variables. To be more specific, suppose that $k[\mathbf{u}]$ is a polynomial ring with term order $\le_\sigma$.

Let $\mathcal{F}_\sigma$ denote the term order filtration on $k[\mathbf{u}]$ and let $\phi : k[\mathbf{u}] \to k[\mathbf{x}]$ be the k-algebra isomorphism whereby $\mathcal{G}$ equals $\phi(\mathcal{F}_\sigma)$.

By Example 3.1, a term order filtration is a k-filtration. As observed in the remarks after (25), a term order filtration has one-dimensional graded components. Since $\mathbf{0}$ is the smallest element of $\mathbb{N}^n$ with respect to a term order, $\varsigma(1_{k[\mathbf{u}]})$ is the smallest element of $(k[\mathbf{u}] \wr \mathcal{F}_\sigma)^\star$. By Proposition 5.1, a term order filtration is strongly multiplicative. As shown in the remarks following Proposition 5.5, $(k[\mathbf{u}] \wr \mathcal{F}_\sigma)^\star$ is isomorphic to $\mathbb{N}^n$. A term order filtration intersects to zero because $\{0\}$ is one of the filtrands. Finally, in a term order filtration all of the filtrands are of the form $k[\mathbf{u}]_{\leq_\sigma h}$ for $h \in k[\mathbf{u}]$ (which has a maximal element, namely h). Since ϕ is an isomorphism, $\mathcal{G}$ has all these properties and so $\mathcal{G}$ satisfies (i)–(v). □

Example 3.5 shows that hypothesis (v) in Theorem 5.9 is needed. In the example, a pure lex term order filtration is augmented to include $k[x, y]_{<_{lex} x} = k[y]$ without changing the quasi-order determined by the term order filtration. Call this new filtration $\mathcal{G}'$. Since the quasi-order is unchanged, hypotheses (i)–(iv) still hold for $\mathcal{G}'$. However, the new filtrand $k[y] = k[x, y]_{<_{lex} x} \in \mathcal{G}'$, does not have a maximal$_{\mathcal{G}'}$ element and hence $\mathcal{G}'$ cannot come from a term order filtration in suitable variables. To see that $k[x, y]_{<_{lex} x}$ does not have a maximal$_{\mathcal{G}'}$ element, note that $1 <_{lex} y <_{lex} y^2 <_{lex} y^3 <_{lex} \cdots$, and for any polynomial $a \in k[y] = k[x, y]_{<_{lex} x}$ there is a positive integer m where $a <_{lex} y^m$. Hence, $k[x, y]_{<_{lex} x}$ cannot have a maximal$_{\mathcal{G}'}$ element. (It is appropriate to use "$<_{lex}$" interchangeably with "$<_{\mathcal{G}'}$" since the quasi-order determined by $\mathcal{G}'$ is the same as the quasi-order determined by $\mathcal{G}$, the "$<_{lex}$" quasi-order.)

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