

REDUCTION OF LOCAL UNIFORMIZATION TO THE CASE OF RANK ONE VALUATIONS FOR RINGS WITH ZERO DIVISORS

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ABSTRACT. This is a continuation of a previous paper by the same authors. In the former paper, it was proved that in order to obtain local uniformization for valuations centered on local domains, it is enough to prove it for rank one valuations. In this paper, we extend this result to the case of valuations centered on rings which are not necessarily integral domains and may even contain nilpotents.

1. INTRODUCTION

For an algebraic variety X over a field k , the problem of resolution of singularities is whether there exists a proper birational morphism $X' \rightarrow X$ such that X' is regular. The problem of local uniformization can be seen as the local version of resolution of singularities for an algebraic variety. For a valuation ν of $k(X)$ having a center on X , the local uniformization problem asks whether there exists a proper birational morphism $X' \rightarrow X$ such that the center of ν on X' is regular. This problem was introduced by Zariski in the 1940's as an important step to prove resolution of singularities. Zariski's approach consists in proving first that every valuation having a center on the given algebraic variety admits local uniformization. Then one has to glue these local solutions to obtain a global resolution of all singularities.

Zariski succeeded in proving local uniformization for valuations centered on algebraic varieties over a field of characteristic zero (see [11]). He used this to prove resolution of singularities for algebraic surfaces and threefolds over a field of characteristic zero (see [12]). Abhyankar proved (see [1]) that local uniformization can be obtained for valuations centered on algebraic surfaces in any characteristic and used this fact to prove resolution of singularities for surfaces (see [2] and [4]). He also proved local uniformization and resolution of singularities for threefolds over fields of characteristic other than 2, 3 and 5 (see [3]). Very recently, Cossart and Piltant proved resolution of singularities (and, in particular, local uniformization) for threefolds over any field of positive characteristic, as well as in the arithmetic case (see [5] and [6]). They proved it using the approach of Zariski. However, the

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problem of local uniformization remains open for valuations centered on algebraic varieties of dimension greater than three over fields of positive characteristic.

Since local uniformization is a local problem, we can work with local rings instead of algebraic varieties. A valuation ν centered on a local integral domain R is said to admit local uniformization if there exists a local local ring $R^{(1)}$ dominated by \mathcal{O}_ν and dominating R such that $R^{(1)}$ is regular. Let \mathcal{N} be the category of all noetherian local domains and $\mathcal{M} \subseteq \mathcal{N}$ be a subcategory of \mathcal{N} which is closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. We want to know for which subcategories \mathcal{M} with these properties, all valuations centered on objects of \mathcal{M} admit local uniformization. In Section 7.8 of [7], Grothendieck proved that any category of schemes, closed under passing to subschemes and finite radical extensions, in which resolution of singularities holds, is a subcategory of quasi-excellent schemes (it is known that the category of quasi-excellent schemes is closed under all the operations mentioned above). He conjectured (see Remark 7.9.6 of [7]) that resolution of singularities holds in this most general possible context: that of quasi-excellent schemes. Translated into our local situation, this conjecture says that the subcategory of \mathcal{N} which optimizes local uniformization is the category of all quasi-excellent local rings. This subcategory has the properties above. For a discussion on quasi-excellent and excellent local rings see Section 7.8 of [7]. However, this conjecture is widely open.

In most of the successful cases, including those mentioned above, local uniformization was first proved for rank one valuations. Then the general case was reduced to this a priori weaker one. In [9], we prove that this reduction works under very general assumptions. Namely, we consider a subcategory \mathcal{M} of the category of all noetherian local integral domains, closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. The main result of [9] is that if every rank one valuation centered on an object of \mathcal{M} admits local uniformization, then all the valuations centered on objects of \mathcal{M} admit local uniformization. The main goal of this paper is to extend this result to rings which are not necessarily integral domains and, in particular, may contain nilpotent elements. The importance of non-integral and non-reduced schemes in modern algebraic geometry is well known. Even if one were only interested in reduced schemes to start with, one is led to consider non-reduced ones as they are produced by natural constructions, for example, in deformation theory. Therefore, it appears desirable to study the problem of local uniformization for such schemes and, in particular, to extend our earlier results on reducing the problem to the rank one case to this more general context.

If R is not reduced we cannot expect, in general, to make $R^{(1)}$ be regular by blowings up. The natural extension to this case is to require $(R^{(1)})_{\text{red}}$ to be regular and $I_{(1)}^n/I_{(1)}^{n+1}$ to be an $(R^{(1)})_{\text{red}}$ -free module for every $n \in \mathbb{N}$ (here $I_{(1)}$ denotes the nilradical of $R^{(1)}$). For more precise definitions see Section 2. Let \mathcal{N} be the category of all noetherian local rings and $\mathcal{M} \subseteq \mathcal{N}$ be a subcategory of \mathcal{N} which

is closed under taking homomorphic images and localizing any finitely generated birational extension at a prime ideal. Our main result is the following:

Theorem 1.1. *Assume that for every noetherian local ring R in $\text{Ob}(\mathcal{M})$, every rank one valuation centered on R admits local uniformization. Then all the valuations centered on objects of \mathcal{M} admit local uniformization.*

The proof of Theorem 1.1 consists of three main steps. The first step is to prove that for every local ring R and every valuation ν centered on R , there exists a local blowing up (see Definition 2.2) $R \rightarrow R^{(1)}$ such that $R^{(1)}$ has only one associated prime ideal. Then we consider a decomposition $\nu = \nu_1 \circ \nu_2$ of ν such that $\text{rk}(\nu_1) < \text{rk}(\nu)$ and $\text{rk}(\nu_2) < \text{rk}(\nu)$. Using induction, we can assume that both ν_1 and ν_2 admit local uniformization. The second main step consists in using this to prove that there exists a local blowing up $R^{(1)} \rightarrow R^{(2)}$ such that $(R^{(2)})_{\text{red}}$ is regular. The third and final step is to prove that there exists a further local blowing up $R^{(2)} \rightarrow R^{(3)}$ such that $(R^{(3)})_{\text{red}}$ is regular and $I_{(3)}^n/I_{(3)}^{n+1}$ is an $(R^{(3)})_{\text{red}}$ -free module for every $n \in \mathbb{N}$ (here $I_{(3)}$ denotes the nilradical of $R^{(3)}$).

This paper is divided as follows. In Section 2 we present the basic definitions and results that will be used in the sequel. Sections 3, 4 and 5 are dedicated to prove the results related to the first, second and third steps, respectively. In the last section we present a proof of our main theorem.

2. PRELIMINARIES

Let R be a noetherian commutative ring with unity and Γ an ordered abelian group. Set $\Gamma_\infty := \Gamma \cup \{\infty\}$ and extend the addition and order from Γ to Γ_∞ as usual.

Definition 2.1. A valuation ν on R is a mapping $\nu : R \rightarrow \Gamma_\infty$ with the following properties:

- (V1) $\nu(ab) = \nu(a) + \nu(b)$ for every $a, b \in R$;
- (V2) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for every $a, b \in R$;
- (V3) $\nu(1) = 0$ and $\nu(0) = \infty$;
- (V4) The **support** of ν , which is defined by $\text{supp}(\nu) := \{a \in R \mid \nu(a) = \infty\}$, is a minimal prime ideal of R .

Take a multiplicative system S of R such that $\text{supp}(\nu) \subseteq R \setminus S$. Then the extension (which we call again ν) of ν to R_S given by $\nu(a/s) := \nu(a) - \nu(s)$ is again a valuation. Indeed, the three first axioms are easily checked. The minimality of $\text{supp}(\nu)$ as a prime ideal of R_S follows from the fact that the prime ideals of R_S are in a bijective correspondence to the prime ideals of R contained in $R \setminus S$. From now on, we will freely make such extensions of ν to R_S without mentioning it explicitly.

A valuation ν on R is said to have a center if $\nu(a) \geq 0$ for every $a \in R$. In this case, the **center** of ν on R is defined by $\mathfrak{C}_\nu(R) := \{a \in R \mid \nu(a) > 0\}$. Moreover, if R is a local ring with unique maximal ideal \mathfrak{m} (in which case we say “the local ring

$(R, \mathfrak{m})^n$), then a valuation ν on R is said to be **centered** at R if $\nu(a) \geq 0$ for every $a \in R$ and $\nu(a) > 0$ for every $a \in \mathfrak{m}$. We observe that if ν is a valuation having a center on R , then ν is centered on $R_{\mathfrak{C}_\nu(R)}$. The **value group of ν** , denoted by νR , is defined as the subgroup of Γ generated by $\{\nu(a) \mid a \in R\}$. The **rank of ν** is the number of proper convex subgroups of νR .

For an element $b \in R \setminus \text{supp}(\nu)$ we consider the canonical map $\Phi : R \rightarrow R_b$ given by $\Phi(a) = a/1$. Let

$$J(b) := \ker \Phi = \bigcup_{i=1}^{\infty} \text{ann}_R(b^i).$$

We have a natural embedding $R/J(b) \subseteq R_b$. Take $a_1, \dots, a_r \in R$ such that

$$\nu(a_i) \geq \nu(b) \quad \text{for each } i, 1 \leq i \leq r.$$

Consider the subring $R' := R/J(b)[a_1/b, \dots, a_r/b]$ of R_b . Then the restriction of ν to R' has a center $\mathfrak{C}_\nu(R')$ in R' . We set $R^{(1)} := R'_{\mathfrak{C}_\nu(R')}$.

Definition 2.2. The canonical map $R \rightarrow R^{(1)}$ will be called the **local blowing up of R** with respect to ν along the ideal (b, a_1, \dots, a_r) . For a valuation μ having a center on R we will say that $R \rightarrow R^{(1)}$ is **μ -compatible** if $b \notin \mathfrak{C}_\mu(R)$ and $a_i \in \mathfrak{C}_\mu(R)$ for every $i, 1 \leq i \leq r$.

Lemma 2.3. *The composition of finitely many local blowings up is again a local blowing up. Moreover, if each of these local blowings up is μ -compatible, then their composition is again μ -compatible.*

Proof. It is enough to prove that for two local blowings up $\pi : R \rightarrow R^{(1)}$ and $\pi' : R^{(1)} \rightarrow R^{(2)}$ with respect to ν , there exists a local blowing up $R \rightarrow R^{(3)}$ with respect to ν such that $R^{(3)} \simeq R^{(2)}$. We write

$$R^{(1)} = R'_{\mathfrak{C}_\nu(R')} \text{ for } R' = R/J(b)[a_1/b, \dots, a_r/b]$$

for some $a_1, \dots, a_r, b \in R$ and

$$R^{(2)} = R''_{\mathfrak{C}_\nu(R''^{(1)})} \text{ for } R''^{(1)} = R^{(1)}/J(\beta)[\alpha_1/\beta, \dots, \alpha_s/\beta]$$

for some $\alpha_1, \dots, \alpha_s, \beta \in R^{(1)}$. Then there exist $a_{r+1}, \dots, a_{r+s}, b' \in R$ such that $\alpha_i/\beta = \pi(a_{r+i})/\pi(b')$ for each $i, 1 \leq i \leq s$. Consider the local blowing up

$$R \rightarrow R^{(3)}$$

given by

$$R^{(3)} = R''_{\mathfrak{C}_\nu(R''^{(3)})} \text{ for } R'' = R/J(bb')[a_1b'/bb', \dots, a_rb'/bb', a_{r+1}b'/bb', \dots, a_{r+s}b'/bb'].$$

It is straightforward to prove that $R^{(2)} \simeq R^{(3)}$. \square

In view of Lemma 2.3, we will freely use the fact that the composition of finitely many local blowings up is itself a local blowing up without mentioning it explicitly.

For simplicity of notation, we denote the **nilradical** of R by I , i.e.,

$$I = \text{Nil}(R) := \{a \in R \mid a^l = 0 \text{ for some } l \in \mathbb{N}\}.$$

Definition 2.4. We say that $\text{Spec}(R)$ is **normally flat along** $\text{Spec}(R_{\text{red}})$ if I^n/I^{n+1} is an R_{red} -free module for every $n \in \mathbb{N}$.

Since R is noetherian, there exists $N \in \mathbb{N}$ such that $I^n = (0)$ for every $n > N$. Hence, the condition in Definition 2.4 is equivalent to the freeness of the finitely many modules $I/I^2, \dots, I^N/I^{N+1} = I^N$.

Definition 2.5. For a local ring R , a valuation ν centered on R is said to admit **local uniformization** if there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $(R^{(1)})_{\text{red}}$ is regular and $\text{Spec}(R^{(1)})$ is normally flat along $\text{Spec}((R^{(1)})_{\text{red}})$.

Let $\nu = \nu_1 \circ \nu_2$ be a fixed decomposition of ν . For simplicity of notation, we set $\mathfrak{p} := \mathfrak{C}_{\nu_1}(R)$ and for a local blowing up $R \rightarrow R^{(1)}$ we set $\mathfrak{p}^{(1)} := \mathfrak{C}_{\nu_1}(R^{(1)})$. We need to guarantee that the main structure of $R_{\mathfrak{p}}$ and R/\mathfrak{p} are preserved under ν_1 -compatible local blowings up. More precisely, we have to prove the following:

Proposition 2.6. *Let $\pi : R \rightarrow R^{(1)}$ be a ν_1 -compatible local blowing up. Then the canonical maps $R_{\mathfrak{p}} \rightarrow R_{\mathfrak{p}^{(1)}}^{(1)}$ and $R/\mathfrak{p} \rightarrow R^{(1)}/\mathfrak{p}^{(1)}$ induced by π are isomorphisms.*

In order to prove Proposition 2.6 we need the following basic lemma.

Lemma 2.7. *Let S be a multiplicative system of R contained in $R \setminus \mathfrak{C}_{\nu}(R)$. Then the canonical map $\Phi : R_{\mathfrak{C}_{\nu}(R)} \rightarrow (R_S)_{\mathfrak{C}_{\nu}(R_S)}$ given by $\Phi(a/b) = (a/1)/(b/1)$ is an isomorphism.*

Proof. For an element $(a/b)/(c/d) \in (R_S)_{\mathfrak{C}_{\nu}(R_S)}$ we have

$$\nu(b) = \nu(c) = \nu(d) = 0.$$

Consequently, $\nu(bc) = 0$ and $ad/bc \in R_{\mathfrak{C}_{\nu}(R)}$. Then

$$(a/b)/(c/d) = (ad/1)/(bc/1) = \Phi(ad/bc).$$

Suppose that $\Phi(a/b) = 0$. This means that there exists $c/d \in R_S \setminus \mathfrak{C}_{\nu}(R_S)$ such that $ac/d = 0$ in R_S . Thus, there exists $s \in S$ such that $sac = 0$. Moreover, since $c/d \notin \mathfrak{C}_{\nu}(R_S)$ we also have that $c \notin \mathfrak{C}_{\nu}(R)$. This and the fact that $s \in S \subseteq R \setminus \mathfrak{C}_{\nu}(R)$ imply that $sc \notin \mathfrak{C}_{\nu}(R)$. Hence, $a/b = 0$ in $R_{\mathfrak{C}_{\nu}(R)}$ which is what we wanted to prove. \square

Proof of Proposition 2.6. Applying Lemma 2.7 to R (with $S = \{1, b, b^2, \dots\}$) and R' (with $S' = R' \setminus \mathfrak{C}_{\nu}(R')$) and the valuation ν_1 , we obtain that the canonical maps $R_{\mathfrak{p}} \rightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ and $R'_{\mathfrak{C}_{\nu_1}(R')} \rightarrow R_{\mathfrak{p}^{(1)}}^{(1)}$, respectively, are isomorphisms. Hence, in order to prove the first assertion, it is enough to show that the canonical map $(R_b)_{\mathfrak{C}_{\nu_1}(R_b)} \leftarrow R'_{\mathfrak{C}_{\nu_1}(R')}$ is an isomorphism.

Since $R' \subseteq R_b$ and $\mathfrak{C}_{\nu_1}(R') = R' \cap \mathfrak{C}_{\nu_1}(R_b)$ we have that $R'_{\mathfrak{C}_{\nu_1}(R')} \rightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ is injective. On the other hand, any element $(a/b^n)/(c/b^m)$ in $(R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$ can be written as $(ab^m/1)/(cb^n/1)$ which is the image of ab^m/cb^n . Hence the map

$$R'_{\mathfrak{C}_{\nu_1}(R')} \rightarrow (R_b)_{\mathfrak{C}_{\nu_1}(R_b)}$$

is surjective and consequently it is an isomorphism.

Set $R_0 = R/J(b)$ and consider the induced map $R_0 \rightarrow R^{(1)}$. Since the canonical map $R \rightarrow R_0$ is surjective, in order to prove the surjectivity of $R \rightarrow R^{(1)}/\mathfrak{p}^{(1)}$, it is enough to show that $R_0 \rightarrow R^{(1)}/\mathfrak{p}^{(1)}$ is surjective. For an element $\alpha \in R^{(1)}$ we write $\alpha = p/q$ where $p = P(a_1/b, \dots, a_r/b)$ and $q = Q(a_1/b, \dots, a_r/b)$ for some

$$P(X_1, \dots, X_r), Q(X_1, \dots, X_r) \in R_0[X_1, \dots, X_r].$$

Set $p_0 = P(0, \dots, 0)$ and $q_0 = Q(0, \dots, 0)$. Then

$$p_1 := p - p_0 = \sum_{i=1}^r a_i/b \cdot P_i(a_1/b, \dots, a_r/b)$$

and

$$q_1 := q - q_0 = \sum_{i=1}^r a_i/b \cdot Q_i(a_1/b, \dots, a_r/b)$$

for some $P_i, Q_i \in R_0[X_1, \dots, X_r]$, $1 \leq i \leq r$. Since $\nu_1(a_i/b) > 0$ we obtain that $\nu_1(p_1) > 0$ and $\nu_1(q_1) > 0$. This implies that

$$(1) \quad \nu_1(q_0) = 0,$$

$$(2) \quad \nu_1(q_0 q) = 0$$

and

$$(3) \quad \nu_1(q_0 p_1 - p_0 q_1) > 0.$$

Therefore,

$$p/q - p_0/q_0 = (q_0 p_1 - p_0 q_1)/q_0 q \in \mathfrak{p}^{(1)}.$$

It remains to prove that $p_0/q_0 \in R_0$. Since $\nu_1(q_1) > 0$, also $\nu(q_1) > 0$. Hence, $\nu(q_0) = \nu(q - q_1) = 0$ and consequently q_0 is a unit in R_0 . Therefore, $p_0/q_0 \in R_0$.

To finish our proof it is enough to show that the kernel of $R \rightarrow R^{(1)}/\mathfrak{p}^{(1)}$ is \mathfrak{p} . This follows immediately from the definition of \mathfrak{p} and $\mathfrak{p}^{(1)}$ as the centers of ν_1 on R and $R^{(1)}$, respectively. \square

Lemmas 2.8 and 2.9 below are generalizations of Lemma 2.18 and Corollary 2.20 of [9], respectively. The proofs presented there can be adapted to our more general case. We present sketches of the proofs for the convenience of the reader.

Lemma 2.8. *For each local blowing up $R_{\mathfrak{p}} \rightarrow \tilde{R}^{(1)}$ with respect to ν_1 , there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $\tilde{R}^{(1)} \simeq R_{\mathfrak{p}^{(1)}}^{(1)}$.*

Proof. We consider the local blowing up $R_{\mathfrak{p}} \rightarrow \tilde{R}^{(1)}$ given by

$$\tilde{R}^{(1)} = \tilde{R}'_{\mathfrak{e}_{\nu_1}(\tilde{R}')} \text{ for } \tilde{R}' = R_{\mathfrak{p}}/J(\beta)[\alpha_1/\beta, \dots, \alpha_r/\beta].$$

Choose $a_1, \dots, a_r, b \in R$ such that for each i , $1 \leq i \leq r$ we have $\Phi(a_i)/\Phi(b) = \alpha_i/\beta$ where $\Phi : R \rightarrow R_{\mathfrak{p}}$ is the canonical map. If $\nu(a_i) < \nu(b)$ for some i , $1 \leq i \leq r$,

then we have $\nu_1(\alpha_i) = \nu_1(\beta)$. Choose i so as to minimize the value $\nu(a_i)$, in other words, so that $\nu(a_i) \leq \nu(a_j)$ for all $j \in \{1, \dots, r\}$. Set

$$\tilde{R}'' := R_{\mathfrak{p}}/J(\alpha_i) \left[\frac{\alpha_1}{\alpha_i}, \dots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{\beta}{\alpha_i}, \frac{\alpha_{i+1}}{\alpha_i}, \dots, \frac{\alpha_r}{\alpha_i} \right].$$

Then $R^{(1)} \simeq \tilde{R}''_{\mathfrak{e}_{\nu_1}(\tilde{R}'')}$. Hence, after a suitable permutation of the set $\{a_1, \dots, a_r, b\}$, we may assume that $\nu(a_i) \geq \nu(b)$ for every i , $1 \leq i \leq r$. Consider the local blowing up

$$R^{(1)} = R'_{\mathfrak{e}_{\nu}(R')} \text{ for } R' = R/J(b)[a_1/b, \dots, a_r/b]$$

with respect to ν . It is straightforward to prove that $R_{\mathfrak{p}^{(1)}}^{(1)} \simeq \tilde{R}^{(1)}$. \square

Lemma 2.9. *For each local blowing up $R/\mathfrak{p} \rightarrow \bar{R}^{(1)}$ with respect to ν_2 , there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $R^{(1)}/\mathfrak{p}^{(1)} \simeq \bar{R}^{(1)}$ and $R_{\mathfrak{p}} \simeq R_{\mathfrak{p}^{(1)}}^{(1)}$.*

Proof. For an element $a \in R$ we denote its image under the canonical map

$$R \rightarrow R/\mathfrak{p}$$

by \bar{a} . Then

$$\bar{R}^{(1)} = \bar{R}'_{\mathfrak{e}_{\nu_2}(\bar{R}')} \text{ with } \bar{R}' = (R/\mathfrak{p})/J(\bar{b})[\bar{a}_1/\bar{b}, \dots, \bar{a}_r/\bar{b}]$$

for some $a_1, \dots, a_r, b \in R \setminus \mathfrak{p}$. Since $\nu_2(\bar{a}_i) \geq \nu_2(\bar{b})$ we have $\nu(a_i) \geq \nu(b)$ for every i , $1 \leq i \leq r$. Then we can consider the local blowing up

$$R^{(1)} = R'_{\mathfrak{e}_{\nu}(R')} \text{ with } R' = R/J(b)[a_1/b, \dots, a_r/b]$$

with respect to ν . It is again straightforward to prove that $R^{(1)}/\mathfrak{p}^{(1)} \simeq \bar{R}^{(1)}$ and $R_{\mathfrak{p}} \simeq R_{\mathfrak{p}^{(1)}}^{(1)}$. \square

3. ASSOCIATED PRIME IDEALS OF R

Let R be a local ring and ν a valuation centered on R . The main result of this section is the following:

Proposition 3.1. *There exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $\text{Nil}(R^{(1)})$ is the only associated prime of $R^{(1)}$.*

In order to prove Proposition 3.1, we need the following result.

Lemma 3.2. *Let $R' = R/J(b)[a_1/b, \dots, a_r/b] \subseteq R_b$ for some $b, a_1, \dots, a_r \in R$ with $\nu(b) \leq \nu(a_i)$ for every i , $1 \leq i \leq r$. Then for every $c' \in R'$ the ideal $\text{ann}_{R'}(c')$ can be written as $\text{ann}_{R'}(c/1)$ for some $c \in R$. Moreover, if $\text{ann}_{R'}(c')$ is prime, then $\text{ann}_R(b^N c)$ is a prime ideal of R for some $N \in \mathbb{N}$.*

Proof. Choose $c \in R$ such that $c' = c/b^l$ for some $l \in \mathbb{N}$. Fix $a' \in R'$ and write $a' = a/b^m$ for some $m \in \mathbb{N}$ and $a \in R$. Then we have

$$a' \in \text{ann}_{R'}(c') \iff acb^n = 0 \text{ for some } n \in \mathbb{N} \iff a' \in \text{ann}_{R'}(c/1).$$

Now assume that $\text{ann}_{R'}(c')$ is prime and set $R_0 := R/J(b)$. Then

$$\text{ann}_{R_0}(c/1) = \text{ann}_{R'}(c') \cap R_0$$

is also prime. Moreover,

$$(4) \quad \pi^{-1}(\text{ann}_{R_0}(c/1)) = \bigcup_{n=1}^{\infty} \text{ann}_R(b^n c),$$

where $\pi : R \rightarrow R/J(b)$ is the canonical epimorphism. Indeed,

$$\begin{aligned} a \in \pi^{-1}(\text{ann}_{R_0}(c/1)) &\iff ac/1 = 0 \text{ in } R_b \\ &\iff b^n ac = 0 \text{ in } R \text{ for some } n \in \mathbb{N} \\ &\iff a \in \bigcup_{n=1}^{\infty} \text{ann}_R(b^n c). \end{aligned}$$

Since R is noetherian and

$$\text{ann}_R(bc) \subseteq \text{ann}_R(b^2c) \subseteq \cdots \subseteq \text{ann}_R(b^Nc) \subseteq \cdots$$

we have that

$$(5) \quad \text{ann}_R(b^Nc) = \bigcup_{n=1}^{\infty} \text{ann}_R(b^n c) \text{ for some } N \in \mathbb{N}.$$

By (4) and (5) we conclude that $\text{ann}_R(b^Nc)$ is a prime ideal of R . \square

Corollary 3.3. *For a local blowing up $R \rightarrow R^{(1)}$, if $\text{Nil}(R)$ is the only associated prime ideal of R , then $\text{Nil}(R^{(1)})$ is the only associated prime ideal of $R^{(1)}$.*

Proof. Let $R^{(1)} = R'_{\mathfrak{e}_\nu(R')}$ for some R' as in Lemma 3.2. Theorem 6.2 of [8] gives us that $\text{Ass}(R^{(1)}) = \text{Ass}(R') \cap \text{Spec}(R^{(1)})$. This and Lemma 3.2 guarantee that $|\text{Ass}(R^{(1)})| \leq |\text{Ass}(R)| = 1$. Consequently, $R^{(1)}$ has only one associated prime ideal, say \mathfrak{q} . The primary decomposition theorem now gives us that $\mathfrak{q} = \text{Nil}(R^{(1)})$, which is what we wanted to prove. \square

We will use Corollary 3.3 throughout this paper without always mentioning it explicitly.

Proof of Proposition 3.1. Since $\text{supp}(\nu)$ is a minimal prime ideal, there exists at most one associated prime ideal of R contained in (hence equal to) $\text{supp}(\nu)$. We will prove that if $|\text{Ass}(R)| > 1$, then there exists a local blowing up $R \rightarrow R^{(1)}$ such that $|\text{Ass}(R^{(1)})| < |\text{Ass}(R)|$.

Take an associated prime ideal \mathfrak{q} of R such that $\mathfrak{q} \not\subseteq \text{supp}(\nu)$. Write

$$\mathfrak{q} = (b, a_1, \dots, a_r)$$

with $\nu(b) \leq \nu(a_i)$ for every i , $1 \leq i \leq r$. Blowing up R with respect to ν along \mathfrak{q} gives us a local ring

$$R^{(1)} = R'_{\mathfrak{e}_\nu(R')} \text{ where } R' = R/J(b)[a_1/b, \dots, a_r/b].$$

Observe that this is indeed a local blowing up because $\nu(b) \leq \nu(a_i)$ for every i and $\mathfrak{q} \not\subseteq \text{supp}(\nu)$ implies that $b \notin \text{supp}(\nu)$. Since

$$\text{Ass}\left(R^{(1)}\right) = \text{Ass}(R') \cap \text{Spec}\left(R^{(1)}\right) \text{ (see Theorem 6.2 of [8]),}$$

it remains to show that $|\text{Ass}(R')| < |\text{Ass}(R)|$.

By Lemma 3.2, we obtain that R' has at most $|\text{Ass}(R)|$ many associated prime ideals. Moreover, for the chosen associated prime ideal $\mathfrak{q} = \text{ann}_R(c)$ of R and for every $r \in \mathbb{N}$, the ideal $\text{ann}_{R'}(c/b^r)$ is not prime in R' . Indeed, since $\mathfrak{q} = (b, a_1, \dots, a_r) = \text{ann}_R(c)$ we have $bc = 0$ in R . This means that

$$c/1 = 0/1$$

in R' and consequently $\text{ann}_{R'}(c/1) = R'$ (which is not prime). Therefore

$$|\text{Ass}(R')| < |\text{Ass}(R)|.$$

□

Remark 3.4. If I is the only associated prime ideal of R , then for every $b \notin I$ we have $J(b) = (0)$. In this particular case, we can eliminate the ideal $J(b)$ in the definition of a local blowing up. We will use this throughout this paper without mentioning it explicitly.

4. MAKING R_{red} REGULAR

Let R be a local ring and ν a valuation centered on R . Assume that $\nu = \nu_1 \circ \nu_2$ and denote by \mathfrak{p} the center of ν_1 on R . As usual, we denote by I the nilradical of R and for a local blowing up $R \rightarrow R^{(1)}$ we denote the nilradical of $R^{(1)}$ by $I_{(1)}$. Assume that I is the only associated prime ideal of R . The main goal of this section is to prove the following proposition.

Proposition 4.1. *Assume that $(R_{\mathfrak{p}})_{\text{red}}$ and R/\mathfrak{p} are regular. Then there exists a ν_1 -compatible local blowing up $R \rightarrow R^{(1)}$ such that $(R^{(1)})_{\text{red}}$ is regular. Moreover, for every local blowing up $R^{(1)} \rightarrow R^{(2)}$ along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \dots, a_r \in I_{(1)}$ we have that $(R^{(2)})_{\text{red}}$ is regular.*

In order to prove Proposition 4.1 we will need a few lemmas.

Lemma 4.2. *Assume that $(R_{\mathfrak{p}})_{\text{red}}$ is regular. Then there exists a ν_1 -compatible local blowing up $R \rightarrow R^{(1)}$ such that the $R^{(1)}/\mathfrak{p}^{(1)}$ -module $\mathfrak{p}^{(1)}/\left(\left(\mathfrak{p}^{(1)}\right)^2 + I_{(1)}\right)$ is free. Moreover, if $y_1^{(1)}, \dots, y_r^{(1)}$ are elements of $\mathfrak{p}^{(1)}$ whose images in $\mathfrak{p}^{(1)}/\left(\left(\mathfrak{p}^{(1)}\right)^2 + I_{(1)}\right)$ form a basis of $\mathfrak{p}^{(1)}/\left(\left(\mathfrak{p}^{(1)}\right)^2 + I_{(1)}\right)$ then their images in $\left(R_{\mathfrak{p}^{(1)}}^{(1)}\right)_{\text{red}}$ form a regular system of parameters of $\left(R_{\mathfrak{p}^{(1)}}^{(1)}\right)_{\text{red}}$.*

Lemma 4.3. *Let $\pi : R \rightarrow R^{(1)}$ be a local blowing up along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}$ and $a_1, \dots, a_r \in I$. If $\mathfrak{p}/(\mathfrak{p}^2 + I)$ is a free R/\mathfrak{p} -module, then*

$$\mathfrak{p}^{(1)} / \left(\left(\mathfrak{p}^{(1)} \right)^2 + I_{(1)} \right)$$

is a free $R^{(1)}/\mathfrak{p}^{(1)}$ -module.

Lemma 4.4. *Take $y_1, \dots, y_r \in \mathfrak{p}$ and $x_1, \dots, x_t \in \mathfrak{m} \setminus \mathfrak{p}$ whose images form a regular system of parameters of $(R_{\mathfrak{p}})_{\text{red}}$ and R/\mathfrak{p} , respectively. If $\mathfrak{p}/(\mathfrak{p}^2 + I)$ is an R/\mathfrak{p} -free module with basis $y_1 + (\mathfrak{p}^2 + I), \dots, y_r + (\mathfrak{p}^2 + I)$, then R_{red} is regular.*

Proof of Proposition 4.1, assuming Lemmas 4.2, 4.3 and 4.4. We apply Lemma 4.2 to obtain a ν_1 -compatible local blowing up $R \rightarrow R^{(1)}$ and $y_1^{(1)}, \dots, y_r^{(1)} \in R^{(1)}$ such that their images in $\mathfrak{p}^{(1)} / \left(\left(\mathfrak{p}^{(1)} \right)^2 + I_{(1)} \right)$ form an $(R^{(1)})_{\text{red}}$ -basis and their images in $\left(R_{\mathfrak{p}^{(1)}}^{(1)} \right)_{\text{red}}$ form a regular system of parameters. Moreover, by Proposition 2.6, $R^{(1)}/\mathfrak{p}^{(1)}$ is regular. Also, by Lemma 4.3 and Proposition 2.6, for every local blowing up $R^{(1)} \rightarrow R^{(2)}$ along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \dots, a_r \in I_{(1)}$ the hypotheses of Lemma 4.4 are satisfied for $R^{(2)}$. Hence, we obtain that $(R^{(1)})_{\text{red}}$ and $(R^{(2)})_{\text{red}}$ are regular. \square

We now proceed with the proofs of Lemmas 4.2, 4.3 and 4.4.

Lemma 4.5. *Take generators $y_1, \dots, y_r, y_{r+1}, \dots, y_{r+s}$ of \mathfrak{p} and $b \notin \mathfrak{p}$. Let*

$$\pi : R \rightarrow R^{(1)}$$

be the local blowing up along the ideal (b, y_1, \dots, y_r) . Set

$$y_i^{(1)} = \pi(y_i)/b \text{ for } 1 \leq i \leq r \text{ and } y_{r+k}^{(1)} = \pi(y_{r+k}) \text{ for } 1 \leq k \leq s.$$

Then $\mathfrak{p}^{(1)}$ is generated by $y_1^{(1)}, \dots, y_{r+s}^{(1)}$.

Proof. Obviously $y_i^{(1)} \in \mathfrak{p}^{(1)}$ for every i , $1 \leq i \leq r+s$. Take an element $p/q \in \mathfrak{p}^{(1)}$. This implies that $p = p(y_1/b, \dots, y_r/b)$ for some $p(X_1, \dots, X_r) \in R[X_1, \dots, X_r]$ (see Remark 3.4). If we set $p_0 = p(0, \dots, 0)$, then

$$p = p_0 + \frac{y_1}{b}p_1 + \dots + \frac{y_r}{b}p_r, \text{ for some } p_1, \dots, p_r \in R'.$$

This implies that $p_0 \in \mathfrak{p}$. Hence, there exist $a_1, \dots, a_{r+s} \in R$ such that $p_0 = a_1y_1 + \dots + a_{r+s}y_{r+s}$. Thus

$$\frac{p}{q} = \sum_{i=1}^r \frac{\pi(ba_i) + p_i}{q} y_i^{(1)} + \sum_{k=1}^s \frac{\pi(a_{r+k})}{q} y_{r+k}^{(1)} \in \left(y_1^{(1)}, \dots, y_{r+s}^{(1)} \right) R^{(1)}.$$

This concludes our proof. \square

Proof of Lemma 4.2. Since $(R_{\mathfrak{p}})_{\text{red}}$ is regular there are elements $y_1, \dots, y_r \in \mathfrak{p}$ such that their images in $(R_{\mathfrak{p}})_{\text{red}}$ form a regular system of parameters. The first step is to reduce to the case when y_1, \dots, y_r generate \mathfrak{p} .

Assume that y_1, \dots, y_r do not generate \mathfrak{p} . Choose $y_{r+1}, \dots, y_{r+s} \in \mathfrak{p}$ such that $y_1, \dots, y_r, y_{r+1}, \dots, y_{r+s}$ generate \mathfrak{p} . For each k , $1 \leq k \leq s$, we can find $b_k \in R \setminus \mathfrak{p}$, $b_{1k}, \dots, b_{rk} \in R$ and $h_k \in (y_1, \dots, y_r)^2$ such that

$$b_k y_{r+k} + b_{1k} y_1 + \dots + b_{rk} y_r + h_k \in I.$$

Consider the local blowing up $\pi : R \rightarrow R^{(1)}$ along (b_1, y_1, \dots, y_r) . It follows that

$$\pi(b_1) \left(y_{r+1}^{(1)} + \pi(b_{11}) y_1^{(1)} + \dots + \pi(b_{r1}) y_r^{(1)} + h_1^{(1)} \right) \in I_{(1)}$$

and

$$\pi(b_k) y_{r+k}^{(1)} + \pi(b_{1k}) y_1^{(1)} + \dots + \pi(b_{rk}) y_r^{(1)} + h_k^{(1)} \in I_{(1)} \text{ for } 2 \leq k \leq s,$$

where $y_i^{(1)} = \pi(y_i)/b_1$ for $1 \leq i \leq r$ and $y_{r+k}^{(1)} = \pi(y_{r+k})$ and some $h_k^{(1)} \in (y_1^{(1)}, \dots, y_r^{(1)})^2$ for $1 \leq k \leq s$. Since $I_{(1)}$ is prime and $\pi(b_1) \notin I_{(1)}$ we obtain that

$$y_{r+1}^{(1)} + \pi(b_{11}) y_1^{(1)} + \dots + \pi(b_{r1}) y_r^{(1)} + h_1^{(1)} \in I_{(1)}.$$

Consequently,

$$\left(y_1^{(1)}, \dots, y_r^{(1)}, y_{r+1}^{(1)}, \dots, y_{r+s}^{(1)} \right) R^{(1)} = \left(y_1^{(1)}, \dots, y_r^{(1)}, y_{r+2}^{(1)}, \dots, y_{r+s}^{(1)} \right) R^{(1)}.$$

We proceed inductively to obtain a ν_1 -compatible local blowing up $R \rightarrow R^{(s)}$ such that

$$\left(y_1^{(s)}, \dots, y_r^{(s)}, y_{r+1}^{(s)}, \dots, y_{r+s}^{(s)} \right) R^{(s)} = \left(y_1^{(s)}, \dots, y_r^{(s)} \right) R^{(s)}.$$

By Lemma 4.5, we have $\mathfrak{p}^{(s)} = \left(y_1^{(s)}, \dots, y_r^{(s)}, y_{r+1}^{(s)}, \dots, y_{r+s}^{(s)} \right) R^{(s)}$ and by Lemma 2.6 the images of $y_1^{(s)}, \dots, y_r^{(s)}$ in $\left(R_{\mathfrak{p}^{(s)}}^{(s)} \right)_{\text{red}}$ form a regular system of parameters. This means that $y_1^{(s)}, \dots, y_r^{(s)}$ generate $\mathfrak{p}^{(s)}$. Thus we have reduced the problem to the case when (y_1, \dots, y_r) generate \mathfrak{p} and will make this assumption from now on.

Now, the only non-trivial fact that remains to be checked is that the images of y_1, \dots, y_r in $\mathfrak{p}/\mathfrak{p}^2 + I$ are R/\mathfrak{p} -linearly independent. Take $a_1, \dots, a_r \in R$ such that

$$a_1 y_1 + \dots + a_r y_r \in \mathfrak{p}^2 + I.$$

Since the images of y_1, \dots, y_r in $(R_{\mathfrak{p}})_{\text{red}}$ form a regular system of parameters, their images in $\mathfrak{p}R_{\mathfrak{p}}/(\mathfrak{p}^2 + I)R_{\mathfrak{p}}$ form an $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ -basis of $\mathfrak{p}R_{\mathfrak{p}}/(\mathfrak{p}^2 + I)R_{\mathfrak{p}}$. This implies that $a_1/1, \dots, a_r/1 \in \mathfrak{p}R_{\mathfrak{p}}$ and consequently $a_1, \dots, a_r \in \mathfrak{p}$.

This completes the proof of the Lemma. \square

Proof of Lemma 4.3. Take $y_1, \dots, y_s \in \mathfrak{p}$ such that their images form an R/\mathfrak{p} -basis of $\mathfrak{p}/(\mathfrak{p}^2 + I)$. We claim that the images of $\pi(y_1), \dots, \pi(y_s)$ form an $R^{(1)}/\mathfrak{p}^{(1)}$ -basis of $\mathfrak{p}^{(1)}/\left((\mathfrak{p}^{(1)})^2 + I_{(1)} \right)$. Take an element $\alpha \in \mathfrak{p}^{(1)}$. Then $\alpha = p/q$ where $p, q \in R' := R[a_1/b, \dots, a_r/b]$ with $\nu_1(p) > 0$ and $\nu_1(q) = 0$. Set $p_0 = p(0, \dots, 0)$ and write

$$p = p_0 + \frac{a_1}{b} p_1 + \dots + \frac{a_r}{b} p_r \text{ for some } p_1, \dots, p_r \in R'.$$

This implies that $p_0 \in \mathfrak{p}$. By our assumption, there exist $c_1, \dots, c_s \in \mathfrak{p}$, $g \in \mathfrak{p}^2$ and $h \in I$ such that

$$p_0 = c_1 y_1 + \dots + c_s y_r + g + h.$$

Consequently,

$$\alpha = \frac{\pi(c_1)}{q} \pi(y_1) + \dots + \frac{\pi(c_s)}{q} \pi(y_s) + \frac{\pi(g)}{q} + \frac{\pi(h)}{q} + \frac{a_1 p_1}{b q} + \dots + \frac{a_r p_r}{b q}.$$

Since $a_1, \dots, a_r, h \in I$, we have that

$$\frac{\pi(h)}{q} + \frac{a_1 p_1}{b q} + \dots + \frac{a_r p_r}{b q} \in I_{(1)}.$$

This and the fact that $\pi(g)/q \in (\mathfrak{p}^{(1)})^2$ imply that the images of $\pi(y_1), \dots, \pi(y_r)$ generate $\mathfrak{p}^{(1)}/\left((\mathfrak{p}^{(1)})^2 + I_{(1)}\right)$.

Now assume that there exists $\alpha_i = (a_i/b^{l_i})/(c_i/b^{m_i}) \in R^{(1)}$, $1 \leq i \leq r$, such that

$$\alpha_1 \pi(y_1) + \dots + \alpha_r \pi(y_r) \in \left(\mathfrak{p}^{(1)}\right)^2 + I_{(1)}.$$

Then there exists $n \in \mathbb{N}$ such that

$$a_1 b^n y_1 + \dots + a_r b^n y_r \in \mathfrak{p}^2 + I.$$

This implies that $a_i b^n \in \mathfrak{p}$ for every i , $1 \leq i \leq r$. Since $b \notin \mathfrak{p}$, this implies that $a_1, \dots, a_r \in \mathfrak{p}$. Therefore, $\alpha_1, \dots, \alpha_r \in \mathfrak{p}^{(1)}$, which concludes our proof. \square

Proof of Lemma 4.4. Set $\mathfrak{p}' = \{a + I \in R_{\text{red}} \mid a \in \mathfrak{p}\}$. Since the images of the y_i 's in $\mathfrak{p}/(\mathfrak{p}^2 + I)$ form a basis of $\mathfrak{p}/(\mathfrak{p}^2 + I)$, we conclude that $(y_1, \dots, y_r) + \mathfrak{p}^2 + I = \mathfrak{p}$. Applying Nakayama's Lemma (corollary of Theorem 2.2 of [8]) we conclude that $(y_1, \dots, y_r) + I = \mathfrak{p}$ and consequently $y_1 + I, \dots, y_r + I$ generate \mathfrak{p}' .

Since the images of $y_1, \dots, y_r, x_1, \dots, x_t$ in R_{red} generate $\mathfrak{m}' = \{a + I \in R_{\text{red}} \mid a \in \mathfrak{m}\}$ we conclude that $r + t \geq \dim R_{\text{red}}$. Also, since $r = \dim (R_{\mathfrak{p}})_{\text{red}} = \text{ht}(\mathfrak{p}')$ and $t = \dim (R/\mathfrak{p}) = \text{ht}(\mathfrak{m}/\mathfrak{p}) = \text{ht}(\mathfrak{m}'/\mathfrak{p}')$ we have

$$\dim(R_{\text{red}}) = \text{ht}(\mathfrak{m}') \geq \text{ht}(\mathfrak{p}') + \text{ht}(\mathfrak{m}'/\mathfrak{p}') = r + t \geq \dim(R_{\text{red}}).$$

Therefore, $r + t = \dim(R_{\text{red}})$ and hence R_{red} is regular. \square

5. MAKING I^n/I^{n+1} FREE

Let R be a local ring and ν a valuation centered on R . Assume that

$$\nu = \nu_1 \circ \nu_2$$

and denote by \mathfrak{p} the center of ν_1 on R . As usual, we set $I = \text{Nil}(R)$ and $I_{\mathfrak{p}} := \text{Nil}(R_{\mathfrak{p}})$. Also, for a local blowing up $R \rightarrow R^{(k)}$ we set $I_{(k)} = \text{Nil}(R^{(k)})$ and $I_{\mathfrak{p}^{(k)}} := \text{Nil}(R_{\mathfrak{p}^{(k)}}^{(k)})$. Assume that I is the only associated prime ideal of R . The main goal of this section is to prove the following proposition.

Proposition 5.1. *Assume that $I_{\mathfrak{p}}^n/I_{\mathfrak{p}}^{n+1}$ is an $(R_{\mathfrak{p}})_{\text{red}}$ -free module for every $n \in \mathbb{N}$. Then there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}$ and $a_1, \dots, a_r \in I$ such that the $(R^{(1)})_{\text{red}}$ -module $I_{(1)}^n/I_{(1)}^{n+1}$ is free for every $n \in \mathbb{N}$.*

In order to prove Proposition 5.1, we will need some preliminary results.

Lemma 5.2. *Take elements $y_1, \dots, y_{r+s} \in I^n$ such that their images in I^n/I^{n+1} generate I^n/I^{n+1} as an R_{red} -module. Consider the local blowing up $\pi : R \rightarrow R^{(1)}$ along the ideal (b, y_1, \dots, y_r) for some $b \in R \setminus I$. Set*

$$y_i^{(1)} = \pi(y_i)/b \text{ for } 1 \leq i \leq r \text{ and } y_{r+k}^{(1)} = \pi(y_{r+k}) \text{ for } 1 \leq k \leq s.$$

Then the images of $y_1^{(1)}, \dots, y_{r+s}^{(1)}$ in $I_{(1)}^n/I_{(1)}^{n+1}$ form a set of generators of this module.

Proof. Take an element $p/q \in I_{(1)}^n$. As in proof of the Lemma 4.5, we can write

$$p = p_0 + \frac{y_1}{b}p_1 + \dots + \frac{y_r}{b}p_r, \text{ for some } p_1, \dots, p_r \in R'$$

with $p_0 \in I^n$. This means that there exists $a_1, \dots, a_{r+s} \in R$ such that $p_0 - a_1y_1 - \dots - a_{r+s}y_{r+s} = y_0 \in I^{n+1}$. Consequently,

$$\frac{p}{q} - \sum_{i=1}^r \frac{\pi(ba_i) + p_i}{q} y_i^{(1)} - \sum_{i=r+1}^{r+s} \frac{\pi(a_i)}{q} y_i^{(1)} = \frac{\pi(y_0)}{q} \in I_{(1)}^{n+1}.$$

This concludes our proof. \square

Lemma 5.3. *Under the same assumptions as in the previous lemma, if the images of y_1, \dots, y_r in I^n/I^{n+1} are R_{red} -linearly independent, then the images of $a_1^{(1)}, \dots, a_r^{(1)}$ in $I_{(1)}^n/I_{(1)}^{n+1}$ are $(R^{(1)})_{\text{red}}$ -linearly independent.*

Proof. Take elements $\alpha_1, \dots, \alpha_r \in R^{(1)}$ such that

$$(6) \quad \alpha_1 y^{(1)} + \dots + \alpha_r y^{(1)} \in I_{(1)}^{n+1}.$$

We have to show that $\alpha_1, \dots, \alpha_r \in I_{(1)}$. For each i , $1 \leq i \leq r$, we write $\alpha_i = (a_i/b^{r_i})/(c_i/b^{s_i})$ for some $a_i, c_i \in R$ and $r_i, s_i \in \mathbb{N}$. Then equation (6) implies that there exists $l \in \mathbb{N}$ and $c \in R \setminus \mathfrak{p}$ such that

$$a_1 b^l c y_1 + \dots + a_r b^l c y_r \in I^{n+1}.$$

Since $y_1 + I^{n+1}, \dots, y_r + I^{n+1}$ are R_{red} -linearly independent, this implies that

$$a_i b^l c \in I \quad \text{for every } i, 1 \leq i \leq r.$$

Since I is prime (this is a consequence of the fact that it is the only associated prime ideal of R) and $b, c \in R \setminus I$, we obtain that $a_1, \dots, a_r \in I$. Consequently, $\alpha_1, \dots, \alpha_r \in I_{(1)}$, which concludes our proof. \square

Proof of Proposition 5.1. By assumption, we have that $I_{\mathfrak{p}}^n/I_{\mathfrak{p}}^{n+1}$ is $(R_{\mathfrak{p}})_{\text{red}}$ -free for every $n \in \mathbb{N}$. Hence, by Proposition 2.6 for every ν_1 -compatible local blowing up $R \rightarrow R^{(1)}$ we have that $I_{\mathfrak{p}^{(1)}}^n/I_{\mathfrak{p}^{(1)}}^{n+1}$ is $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\text{red}}$ -free for every $n \in \mathbb{N}$. Therefore, it is enough to show that for a fixed $n \in \mathbb{N}$, there exists a local blowing up $R \rightarrow R^{(1)}$ along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}$ and $a_1, \dots, a_r \in I$ such that $I_{(1)}^n/I_{(1)}^{n+1}$ is $(R^{(1)})_{\text{red}}$ -free.

Take elements $y_1/b_1, \dots, y_r/b_r \in I_{\mathfrak{p}}^n$, $y_1, \dots, y_r \in R$ and $b_1, \dots, b_r \in R \setminus \mathfrak{p}$ such that

$$y_1/b_1 + I_{\mathfrak{p}}^{n+1}, \dots, y_r/b_r + I_{\mathfrak{p}}^{n+1}$$

form a basis of $I_{\mathfrak{p}}^n/I_{\mathfrak{p}}^{n+1}$. We observe first that since I is prime and $y_i/b_i \in I_{\mathfrak{p}}^n$, we have $y_i \in I^n$ for each i , $1 \leq i \leq r$. We claim that if

$$y_1 + I^{n+1}, \dots, y_r + I^{n+1}$$

generate I^n/I^{n+1} as an R_{red} -module, then this module is free. Indeed, if there exists $a_i + I \in R_{\text{red}}$ such that $a_1 y_1 + \dots + a_r y_r \in I^{n+1}$, then

$$a_1 b_1 / 1 \cdot y_1 / b_1 + \dots + a_r b_r / 1 \cdot y_r / b_r = (a_1 y_1 + \dots + a_r y_r) / 1 \in I_{\mathfrak{p}}^{n+1}.$$

This implies that for each i , $1 \leq i \leq r$, $a_i b_i / 1 \in I_{\mathfrak{p}}$ and consequently $a_i b_i c_i \in I$ for some $c_i \in R \setminus \mathfrak{p}$. Since I is prime and $b_1 c_1, \dots, b_r c_r \in R \setminus I$, we conclude that $a_1, \dots, a_r \in I$, which is what we wanted to prove.

If $y_1 + I^{n+1}, \dots, y_r + I^{n+1}$ do not generate I^n/I^{n+1} (as an R_{red} -module), then we take $y_{r+1}, \dots, y_{r+s} \in I^n$ such that $y_1 + I^{n+1}, \dots, y_{r+s} + I^{n+1}$ generate I^n/I^{n+1} . For each k , $1 \leq k \leq s$, since $y_{r+k} \in I^n$ there exist $b_k \in R \setminus \mathfrak{p}$, such that

$$(7) \quad b_k y_{r+k} - b_{1k} y_1 - \dots - b_{rk} y_r \in I^{n+1},$$

for some $b_{1k}, \dots, b_{rk} \in R$. Consider now the local blowing up along the ideal (b_1, y_1, \dots, y_r) . Set

$$y_i^{(1)} := \pi(y_i)/b_1 \in R^{(1)} \quad \text{for each } i, 1 \leq i \leq r$$

and

$$y_{r+k}^{(1)} := \pi(y_{r+k}) \in R^{(1)} \quad \text{for each } k, 1 \leq k \leq s.$$

From equation (7) we obtain that

$$y_{r+1}^{(1)} - \pi(b_{11}) y_1^{(1)} - \dots - \pi(b_{r1}) y_r^{(1)} \in I_{(1)}^{n+1},$$

and

$$\pi(b_k) y_{r+k}^{(1)} - \pi(b_{1k}) y_1^{(1)} - \dots - \pi(b_{rk}) y_r^{(1)} \in I_{(1)}^{n+1},$$

for every k , $2 \leq k \leq s$. Consequently, $y_{r+1}^{(1)} + I_{(1)}^{n+1}$ is generated in the $(R^{(1)})_{\text{red}}$ -module $I_{(1)}^n/I_{(1)}^{n+1}$ by $y_1^{(1)} + I_{(1)}^{n+1}, \dots, y_r^{(1)} + I_{(1)}^{n+1}$. Moreover, using Lemma 5.2, we obtain that $I_{(1)}^n/I_{(1)}^{n+1}$ is generated as an $R_{\text{red}}^{(1)}$ -module by the images of

$$y_1^{(1)}, \dots, y_r^{(1)}, y_{r+2}^{(1)}, \dots, y_{r+s}^{(1)}.$$

Also, by Lemma 5.3, the images of $y_1^{(1)}, \dots, y_r^{(1)}$ in $I_{(1)}^n/I_{(1)}^{n+1}$ are $(R^{(1)})_{\text{red}}$ -linearly independent.

We proceed inductively to obtain a local blowing up $R \rightarrow R^{(s)}$ such that the $(R^{(s)})_{\text{red}}$ -module $I_{(s)}^n/I_{(s)}^{n+1}$ is generated by the images of $y_1^{(s)}, \dots, y_r^{(s)}$ and the images of $y_1^{(s)}, \dots, y_r^{(s)}$ in $I_{(s)}^n/I_{(s)}^{n+1}$ are $(R^{(s)})_{\text{red}}$ -linearly independent. \square

6. PROOF OF THE MAIN THEOREM

In this section we present the proof of our main theorem.

Proof of Theorem 1.1. We will prove the assertion by induction on the rank. Since all rank one valuations admit local uniformization by assumption, we fix $n \in \mathbb{N}$ and will prove that if all valuations of rank smaller than n admit local uniformization, then also valuations of rank n admit local uniformization.

Let ν be a valuation centered in the local ring $R \in \text{Ob}(\mathcal{M})$ such that $\text{rk}(\nu) = n$. By Lemma 3.1, there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $\text{Nil}(R^{(1)})$ is the only associated prime ideal of $R^{(1)}$. Hence, replacing R by $R^{(1)}$, we may assume that the only associated prime ideal of R is $\text{Nil}(R)$.

Decompose ν as $\nu = \nu_1 \circ \nu_2$ for valuations ν_1 and ν_2 with rank smaller than n . By assumption, we know that ν_1 and ν_2 admit local uniformization. Since ν_1 admits local uniformization, by use of Lemma 2.8, there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $R_{\mathfrak{p}^{(1)}}^{(1)}$ is regular and $I_{\mathfrak{p}^{(1)}}^n/I_{\mathfrak{p}^{(1)}}^{n+1}$ is $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\text{red}}$ -free for every $n \in \mathbb{N}$. Replacing R by $R^{(1)}$ we may assume that $(R_{\mathfrak{p}})_{\text{red}}$ is regular and $I_{\mathfrak{p}}^n/I_{\mathfrak{p}}^{n+1}$ is $(R_{\mathfrak{p}})_{\text{red}}$ -free for every $n \in \mathbb{N}$.

Since ν_2 admits local uniformization, we can use Lemma 2.9 to obtain that there exists a local blowing up $R \rightarrow R^{(1)}$ with respect to ν such that $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\text{red}}$ and $R^{(1)}/\mathfrak{p}^{(1)}$ are regular and $I_{\mathfrak{p}^{(1)}}^n/I_{\mathfrak{p}^{(1)}}^{n+1}$ is $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\text{red}}$ -free for every $n \in \mathbb{N}$. Replacing R by $R^{(1)}$ we can assume that $(R_{\mathfrak{p}})_{\text{red}}$ and R/\mathfrak{p} are regular and that $I_{\mathfrak{p}}^n/I_{\mathfrak{p}}^{n+1}$ is $(R_{\mathfrak{p}})_{\text{red}}$ -free for every $n \in \mathbb{N}$.

Since $(R_{\mathfrak{p}})_{\text{red}}$ and R/\mathfrak{p} are regular, we apply Proposition 4.1 to obtain a ν_1 -compatible local blowing up $R \rightarrow R^{(1)}$ such that $(R^{(1)})_{\text{red}}$ is regular. Using Proposition 2.6, we have that $I_{\mathfrak{p}^{(1)}}^n/I_{\mathfrak{p}^{(1)}}^{n+1}$ is a free $(R_{\mathfrak{p}^{(1)}}^{(1)})_{\text{red}}$ -module for every $n \in \mathbb{N}$. By Proposition 5.1, there exists a $\mathfrak{p}^{(1)}$ -compatible local blowing up $R^{(1)} \rightarrow R^{(2)}$ such that $I_{(2)}^n/I_{(2)}^{n+1}$ is an $(R^{(2)})_{\text{red}}$ -free module for every $n \in \mathbb{N}$. Moreover, since this local blowing up is along an ideal (b, a_1, \dots, a_r) with $b \notin \mathfrak{p}^{(1)}$ and $a_1, \dots, a_r \in I_{(1)}$, we conclude using Proposition 4.1 that $(R^{(2)})_{\text{red}}$ is regular. This concludes our proof. \square

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