ON THE LOCAL UNIFORMIZATION PROBLEM

JOSNEI NOVACOSKI AND MARK SPIVAKOVSKY

Abstract. In this paper we give a short introduction to the local uniformization problem. This follows a similar line as the one presented by the second author in his talk at ALANT 3. We also discuss our paper on the reduction of local uniformization to the rank one case. In that paper, we prove that in order to obtain local uniformization for valuations centered at objects of a subcategory of the category of noetherian integral domains, it is enough to prove it for rank one valuations centered at objects of the same category. We also announce an extension of this work which was partially developed during ALANT 3. This extension says that the reduction mentioned above also works for noetherian rings with zero divisors (including the case of non-reduced rings).

1. Introduction

Resolution of singularities for an algebraic variety is an important branch of algebraic geometry. Roughly speaking, for an algebraic variety V a resolution of singularities is an algebraic variety V' with no singularities, birationally equivalent to V (a precise definition is provided in Section 2). Local uniformization is the local version of resolution of singularities. Namely, given a valuation ν centered at a point $\mathfrak{p}\in V$ we want to find V' birationally equivalent to V such that the center $\mathfrak{p}'$ of V' is non-singular (see Section 2).

Local uniformization was introduced by Zariski in order to prove resolution of singularities. His approach consists of two steps: proving local uniformization for every valuation and use these local solutions to obtain a resolution of all singularities. In this second step, the quasi-compactness of the Zariski topology on the space of valuations plays an important role. This is because when we obtain local uniformization for a given valuation, every valuation in an open neighbourhood of it is resolved. Hence, using the quasi-compactness of the Zariski space of valuations, it is enough to "glue" only finitely many solutions.

Zariski succeeded in 1940 (see [7]) in proving local uniformization for valuations having a center on any algebraic variety over a field of characteristic zero. He used this to prove resolution of singularities for algebraic varieties of dimension smaller or equal to three over a field of characteristic zero. Abhyankar proved in 1956 (see

2010 *Mathematics Subject Classification.* Primary 13H05 Secondary 13G05, 13A18.

Key words and phrases. Local uniformization, resolution of singularities.

During the realization of this project the first author was supported by a research grant from the program "Ciência sem Fronteiras" from the Brazilian government.

[1]), using Zariski's approach, resolution of singularities for algebraic varieties of dimension 3 and characteristic $p \geq 6$. In 2009, Cossart and Piltant concluded the proof (see [2] and [3]) of resolution of singularities for dimension 3 and any positive characteristic (they also used Zariski's approach).

Resolution of singularities is known for algebraic varieties over a field of characteristic zero. The first full prove of it was given in [4] by Hironaka (for this work he received a Fields Medal in 1970). However, both resolution of singularities and local uniformization are open problems for algebraic varieties of dimension greater than 3 and positive characteristic.

In the cases where local uniformization is known the proof usually proceeds by induction on the rank of the valuation. In [5] we proved that this process does not depend on the proof for the rank one case. Namely, we proved that if every rank one valuation centered in some ring of a given category admits local uniformization, then every valuation centered at objects of this category admits local uniformization (see Section 3 for a more precise statement). In [5] we only dealt with valuations centered at integral domains. However, in our forthcoming paper [6], we extend this result to valuations centered at any type of noetherian local rings (for instance, such rings might have nilpotent elements).

In Section 2, we present some basic definitions from algebraic geometry. We give the background necessary in order to define resolution of singularities and local uniformization. In Section 3 we describe our results on the reduction of local uniformization to the rank one case.

2. Resolution of singularities and local uniformization

Take a field K and a prime ideal $I = (f_1, \ldots, f_r)$ of $K[\underline{X}] := K[X_1, \ldots, X_n]$. Then the **affine algebraic variety** $V(I)$ is defined as the set of zeros of I in K^n, i.e.,

$$V(I) := \{p \in K^n \mid f_i(p) = 0 \text{ for every } i, 1 \leq i \leq r\}.$$

We define the **coordinate ring of** $V := V(I)$ as the ring $K[V] := K[X_1, \ldots, X_n]/I$. Then the **function field** $K(V)$ **of** V is defined as the quotient field of $K[V]$. Set $x_i = X_i + I$ for each $i = 1, \ldots, n$. Then $K[V] = K[\underline{x}] := K[x_1, \ldots, x_n]$ and $K(V) = K(\underline{x}) := K(x_1, \ldots, x_n)$.

We define a topology $\mathcal{Z}(V)$ on V by setting as open sets the sets of the form $V \setminus V(\mathcal{I})$ where $\mathcal{I}$ runs over all the ideals of $K[\underline{X}]$. A map $f : V \longrightarrow K$ is said to be **regular** if there exist $U_f \in \mathcal{Z}(V)$ and $g, h \in K[V]$ such that $h(p) \neq 0$ and $f(p) = g(p)/h(p)$ for every $p \in U_f$. Then every element in $K(V)$ can be seen as the equivalence class of a pair (U_f, f) under the equivalence given by $(U_f, f) \sim (U_{f'}, f')$ if and only if f coincides with f' in $U_f \cap U_{f'}$. A map $\Phi : V \longrightarrow V'$ between the varieties V and V' (over K) is a **morphism** if for every regular map $f : V' \longrightarrow K$ the map $f \circ \Phi : V \longrightarrow K$ is also regular. Let $\mathcal{M}(V, V')$ denote the set of morphisms from V to V'.

A **rational map** is an equivalence class of pairs of the form

$$(U, \Phi) \in \mathcal{Z}(V) \times \mathcal{M}(V, V')$$

under the equivalence $(U, \Phi) \sim (U', \Phi')$ if and only if Φ and Φ' coincide in $U \cap U'$. Hence, every rational map $V \longrightarrow V'$ induces a K-homomorphism $K(V') \longrightarrow K(V)$ of fields. A rational map is said to be **birational** if the induced map on the function fields is an isomorphism.

For a point $p \in V$ we consider the ideal $\mathfrak{p} = \{f(\underline{x}) \in K[V] \mid f(p) \neq 0\}$ of $K[V]$. This is a prime ideal of $K(V)$. Define the local ring of V at p as

$$\mathcal{O}_{V,p} := \{\phi \in K(V) \mid \phi = f(x)/g(x) \text{ such that } g(p) \neq 0\} = K[V]_{\mathfrak{p}}.$$

The point p is said to be **regular** (or **non-singular**) if $\mathcal{O}_{V,p}$ is regular, i.e., if the only maximal ideal of $\mathcal{O}_{V,p}$ is generated by $\dim(\mathcal{O}_{V,p})$ elements.

An algebraic variety V over K is said to admit **resolution of singularities** if there exists a proper birational morphism $\pi : V' \longrightarrow V$ such that every point of V' is regular. Proper means that for every valuation ν of $K(V) = K(V')$ if $\mathcal{O}_{V,p}$ is dominated by $\mathcal{O}_\nu$, then there exists a unique $p' \in \pi^{-1}(p)$ such that $\mathcal{O}_{V',p'}$ is dominated by $\mathcal{O}_\nu$. If ν is a valuation of $K(V)$, then the pair (V, ν) is said to admit **local uniformization** if there exists a proper birational morphism $\pi : V' \longrightarrow V$ from a variety V' to V such that the unique point $p' \in \pi^{-1}(p)$ such that $\mathcal{O}_{V',p'} \subseteq \mathcal{O}_\nu$ is regular.

In the modern language, an algebraic variety over K is replaced by an "integral separated scheme of finite type over K". A scheme is the analogue in algebraic geometry of a manifold. A manifold is a topological space which is the union of open sets which are homeomorphic to $\mathbb{R}^n$. Analogously, a scheme of finite type over K with function field F is the union of objects which are isomorphic (in the category of schemes) to spaces (called affine schemes) of the form $\mathrm{Spec}(A)$ where

$$A = K[x_1, \ldots, x_n] \subseteq F \text{ such that } F = K(x_1, \ldots, x_n).$$

We can think of A as the coordinate ring of an algebraic variety V. Since $\mathrm{Spec}(A)$ consists of all the prime ideals of A, we are extending the definition of point to include also irreducible subvarieties of V. Since local uniformization is a local problem, it is enough to consider only affine varieties. Also, it is natural to consider valuations whose center at such a variety may be a subvariety (rather than only a "classical" point).

A natural way of solving singularities is by blowing ups. Take $V = \mathrm{Spec}(A)$ and consider a valuation ν of $F := \mathrm{Quot}(A)$ having a center $\mathfrak{p}'$ on V (i.e., $A \subseteq \mathcal{O}_\nu$ and $\mathfrak{p}' := A \cap \mathfrak{m}_\nu$). We can describe a blowing up of V at $\mathfrak{p}'$ along ν in the following way. Let R be a noetherian local domain (e.g., $R = A_{\mathfrak{p}'}$) and a valuation ν of $\mathrm{Quot}(R)$ centered at R (i.e., such that R is dominated by $\mathcal{O}_\nu$). A **local blowing up of R with respect to ν** is an inclusion map $R \longrightarrow R^{(1)}$ where

$$R^{(1)} := R[a_1, \cdots, a_l]_{\mathfrak{m}_\nu \cap R[a_1, \cdots, a_l]} \text{ for some } a_1, \ldots, a_l \in \mathcal{O}_\nu.$$

Hence, we modify the definition of local uniformization in this more modern language to the following: for a noetherian local domain R and a valuation ν of $F = \mathrm{Quot}(R)$ centered at R, we say that the pair (R, ν) admits local uniformization if there exists a local blowing up $R \longrightarrow R^{(1)}$ of R with respect to ν such that $R^{(1)}$ is regular.

3. Reduction of local uniformization to the rank one case

An important step in most of the proofs of local uniformization is to reduce local uniformization to the case of rank one valuations. This step appears, for instance, in Zariski's proof for the characteristic zero case and in Cossart and Piltant's proof for positive characteristic and dimension at most 3. Let us formulate this reduction more precisely.

Let $\mathcal{N}$ denote the category of all noetherian local domains. Let $\mathcal{N}'$ be a subcategory of $\mathcal{N}$. We will say that $\mathcal{N}'$ admits **reduction to rank one** if the following holds: if every rank one valuation centered at any object of $\mathcal{N}'$ admits local uniformization, then all the valuations centered at objects of $\mathcal{N}'$ admit local uniformization. In [5], we prove the following:

Theorem 3.1. *Let $\mathcal{N}'$ be a subcategory of $\mathcal{N}$ which is closed under taking homomorphic images and localizing at a prime ideal any finitely generated birational extension. Then $\mathcal{N}'$ admits reduction to rank one.*

In [5], we also consider stronger forms of local uniformization. Two of those are what we called **weak embedded local uniformization** and **embedded local uniformization**. Recently, it was noted to us by Schoutens (in an e-mail by Kuhlmann), that these two concepts are equivalent.

During ALANT 3 we have made important developments in our forthcoming paper [6]. The aim of this paper is to generalize Theorem 3.1 to the case where objects of $\mathcal{N}$ are not necessarily integral domains. For that case we have to adapt our definitions.

Take a noetherian local ring R (R may have zero divisors and even nilpotent elements) and an abelian group Γ. Take ∞ to be an element not in Γ and set Γ_∞ to be $\Gamma \cup \{\infty\}$ with extensions of addition and order as usual.

Definition 3.2. A **valuation on** R is a map $\nu : R \longrightarrow \Gamma_\infty$ such that the following holds:

- **(V1):** $\nu(ab) = \nu(a) + \nu(b)$ for every $a, b \in R$,
- **(V2):** $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for every $a, b \in R$,
- **(V3):** $\nu(1) = 0$ and $\nu(0) = \infty$,
- **(V4):** $\mathrm{supp}(\nu) := \{a \in R \mid \nu(a) = \infty\}$ is a minimal prime ideal of R.

We observe that if S is a multiplicative set of R, contained in $R \setminus \mathrm{supp}(\nu)$, then ν extends to a valuation on R_S (which we denote again by ν) by setting $\nu(a/b) = \nu(a) - \nu(b)$. A valuation ν on R is said to have a center if $\nu(a) \geq 0$ for every $a \in R$. In this case the center is defined as $\mathfrak{c}_R(\nu) := \{a \in R \mid \nu(a) > 0\}$. Moreover, if R is a local ring with unique maximal ideal $\mathfrak{m}$, then a valuation ν on R is said to be centered at R if $\nu(a) \geq 0$ for every $a \in R$ and $\nu(a) > 0$ for every $a \in \mathfrak{m}$. We observe that if ν is a valuation having a center at R, then the extension of ν to $R_{\mathfrak{c}_R(\nu)}$ is centered at $R_{\mathfrak{c}_R(\nu)}$.

Take an element $b \in R \setminus \mathrm{supp}(\nu)$. If we consider the natural map $\Phi : R \longrightarrow R_b$ given by $\Phi(a) = a/1$, then

$$J(b) := \ker \Phi = \bigcup_{i=1}^{\infty} \mathrm{ann}_R(b^i).$$

Hence, we have a natural embedding $R/J(b) \subseteq R_b$. Take $a_1, \ldots, a_r \in R$ such that $\nu(a_i) \geq \nu(b)$ for each i, $1 \leq i \leq r$. Consider the subring $R' := R/J(b)[a_1/b, \ldots, a_r/b]$ of R_b. Then ν has a center $\mathfrak{c}_{R'}(\nu)$ on R'. Consider the ring $R^{(1)} := R'_{\mathfrak{c}_{R'}(\nu')}$ and the extension $\nu^{(1)}$ of ν to $R^{(1)}$.

Definition 3.3. A local blowing up of R along ν is an inclusion map $R \longrightarrow R^{(1)}$ of the form described above.

Let $I = \sqrt{(0)}$ (I is the **nilradical of** R).

Definition 3.4. We say that $\mathrm{Spec}(R)$ is **normally flat along** $\mathrm{Spec}(R_{\mathrm{red}})$ if I^n/I^{n+1} is an R_{red}-free module for every $n \in \mathbb{N}$.

Since $I^n = (0)$ for every $n > N$ for some $N \in \mathbb{N}$, this condition is equivalent to the freeness of the finitely many modules $I/I^2, \ldots, I^N/I^{N+1} = I^N$.

A pair (R, ν) as above is said to admit **local uniformization** if there exists a local blowing up $R \longrightarrow R^{(1)}$ along ν such that $R^{(1)}_{\mathrm{red}}$ is regular and $\mathrm{Spec}(R^{(1)})$ is normally flat along $\mathrm{Spec}(R^{(1)}_{\mathrm{red}})$.

Definition 3.5. We say that ν is of **rank one** if its value group Γ embeds as an ordered additive subgroup into the group $\mathbb{R}$ of real numbers.

Consider the category $\mathcal{M}$ of all noetherian rings (not necessarily integral domains). Again, we will say that $\mathcal{M}'$ admits **reduction to rank one** if the following holds: if every rank one valuation centered at any member of $\mathcal{M}'$ admits local uniformization, then all the valuations centered at members of $\mathcal{M}'$ admit local uniformization. The following is the main theorem of [6].

Theorem 3.6. *Let $\mathcal{M}'$ be a subcategory of $\mathcal{M}$ which is closed under taking homomorphic images and localizing at a prime any finitely generated birational extension. Then $\mathcal{M}'$ admits reduction to rank one.*

References

[1] S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*, Pure and Applied Mathematics **24**, Academic Press, New York and London 1966.

[2] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*, J. Algebra **320** no. 3 (2008), 1051–1082.

[3] V. Cossart and O. Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** no. 7 (2009), 1836–1976.

[4] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79** (1964), 109–326.

[5] J. A. Novacoski and M. Spivakovsky, *Reduction of local uniformization to the rank one case*, Proceedings of the Second International Conference on Valuation Theory, EMS Series of Congress Reports (2014), 404–431.

[6] J. A. Novacoski and M. Spivakovsky, *Local uniformization for rings which are not domains*, in preparation.

[7] O. Zariski, *Local uniformization theorem on algebraic varieties*, Ann. of Math. **41** (1940), 852 – 896.

Josnei Novacoski
CAPES Foundation
Ministry of Education of Brazil
Brasília/DF 70040-020
Brazil
E-mail address: jan328@mail.usask.ca