SOME RESULTS OF GEOMETRY OVER HENSELIAN FIELDS WITH ANALYTIC STRUCTURE

KRZYSZTOF JAN NOWAK

ABSTRACT. The paper develops non-Archimedean geometry over Henselian valued fields with analytic structure; the case of complete rank one valued fields with the Tate algebra of strictly convergent power series being a classical example. The algebraic case was treated in our previous papers. Here we are going to carry over the research to the general analytic settings. We also prove that certain natural rings of analytic functions are excellent and regular. Several results are established as, for instance, piecewise continuity of definable functions, curve selection for definable sets, several versions of the Łojasiewicz inequality or Hölder continuity of definable functions continuous on closed bounded subsets of the affine space. Likewise as before, at the center of our approach is the closedness theorem to the effect that every projection with closed bounded fiber is a definably closed map. It enables application of resolution of singularities and of transformation to a normal crossing by blowing up (here applied to certain rings of analytic functions) in much the same way as over locally compact ground fields. Here we rely on elimination of valued field quantifiers, term structure of definable functions and b-minimal cell decomposition, due to Cluckers–Lipshitz, as well as on relative quantifier elimination for ordered abelian groups, due to Cluckers–Halupczok. Besides, other two ingredients of the proof of the closedness theorem are existence of the limit (after finite partitioning of the domain) for a definable function of one variable and fiber shrinking, being a relaxed version of curve selection.

1. Introduction

The paper develops non-Archimedean geometry over Henselian valued fields of equicharacteristic zero with analytic structure. This is done in the case of separated analytic structures, whose theory is briefly recalled in Section 2. However, the results established here remain valid in that of strictly convergent analytic structures, because every such a structure can be extended in a definitional way (extension by Henselian functions) to a separated analytic structure (cf. [5]). Complete, rank one valued fields with the Tate algebra of strictly convergent power series are a classical example. Geometry over Henselian valued fields in the algebraic case was treated in our previous articles [18, 19]. We are now going to carry over the research to the general analytic settings.

Throughout the paper, we shall usually assume that the ground valued field K is of equicharacteristic zero, not necessarily algebraically closed. Denote by v, $\Gamma = \Gamma_K$, K°, $K^{\circ\circ}$ and $\widetilde{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. The multiplicative norm corresponding to v will be denoted by $|\cdot|$. By the K-topology on K^n we mean the topology induced by the valuation v. As before, at the center of our approach is the following closedness theorem

Theorem 1.1. *Let K be a Henselian valued field with separated analytic structure in the analytic language $\mathcal{L}$. Given an $\mathcal{L}$-definable subset D of K^n, the canonical projection*

$$\pi : D \times (K^\circ)^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times (K^\circ)^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

We immediately obtain two consequences.

Corollary 1.2. *Let D be an $\mathcal{L}$-definable subset of K^n and $\mathbb{P}^m(K)$ stand for the projective space of dimension m over K. Then the canonical projection*

$$\pi : D \times \mathbb{P}^m(K) \longrightarrow D$$

is definably closed. □

Corollary 1.3. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or of $(K^\circ)^m$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.* □

Theorem 1.1 will be proven in Section 4. The strategy of proof in the analytic settings will generally follow the one in the algebraic case from

our papers [18, 19]. Here we apply elimination of valued field quantifiers for the theory $T_{Hen,\mathcal{A}}$, term structure of definable functions and b-minimal cell decompositions with centers due to Cluckers–Lipshitz [4], as well as relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok [3]. Besides, in the proof of the closedness theorem, we rely on the local behavior of definable functions of one variable and on fiber shrinking, being a relaxed version of curve selection.

Majority of the results, established in the subsequent sections, rely on the closedness theorem. It enables, in particular, application of resolution of singularities and of transformation to a normal crossing by blowing up in much the same way as over locally compact ground fields.

Non-archimedean analytic geometry over Henselian valued fields has a long history (see e.g. [8, 10, 14, 12, 11, 15, 16, 4, 5]). The concept of fields with analytic structure is recalled for the reader's convenience in Section 2 following the last two papers mentioned above. It was introduced by Cluckers–Lipshitz–Robinson [6]. In the next section, we prove that certain natural rings of analytic functions, namely $A_p^{loc}(K)$ and $A_{m,n}^{\ddagger}(K)$, are excellent and regular.

In Section 5, we give two direct applications of the closedness theorem, namely theorems on existence of the limit (Proposition 5.1) and on piecewise continuity (Theorem 5.3). Note that our proof of the closedness theorem makes use of a certain version of the former result (Proposition 4.4).

Section 6 contains several versions of the Łojasiewicz inequality with an immediate consequence, Hölder continuity of definable functions continuous on closed bounded subsets of K^n. We only state the results, because the proofs of the algebraic versions from our papers [18, 19] can be repeated almost verbatim.

Finally, in the last section, we establish a general version of curve selection for definable sets. It differs from the classical one in that the domain of the selected curve is only a definable subset of the unit disk. Its proof relies on the closedness theorem, transformation to a normal crossing by blowing up (applied to excellent regular local rings $A_p^{loc}(K)$ of analytic function germs examined in Section 3), elimination of valued field quantifiers, relative quantifier elimination for ordered abelian groups, and a result from piecewise linear geometry to which fiber shrinking comes down.

2. Fields with analytic structure

In this section, we remind the reader, following the paper [4], the concept of a separated analytic structure.

Let A be a commutative ring with unit and with a fixed proper ideal $I \subsetneq A$; put $\widetilde{A} = A/I$. A *separated (A, I)-system* is a certain system $\mathcal{A}$ of A-subalgebras $A_{m,n} \subset A[[\xi, \rho]]$, $m, n \in \mathbb{N}$; here $A_{0,0} = A$ (op. cit., Section 4.1). Two kinds of variables, $\xi = (\xi_1, \ldots, \xi_m)$ and $\rho = (\rho_1, \ldots, \rho_n)$, play different roles. Roughly speaking, the variables ξ vary over the valuation ring (or the closed unit disc) K° of a valued field K, and the variables ρ vary over the maximal ideal (or the open unit disc) $K^{\circ\circ}$ of K.

For a power series $f \in A[[\xi, \rho]]$, we say that

1) f is *ξ_m-regular of degree d* if f is congruent to a monic polynomial in ξ_m of degree d modulo the ideal

$$I[[\xi, \rho]] + (\rho)A[[\xi, \rho]];$$

2) f is *ρ_n-regular of degree d* if f is congruent to ρ_n^d modulo the ideal

$$I[[\xi, \rho]] + (\rho_1, \ldots, \rho_{n-1}, \rho_n^{d+1})A[[\xi, \rho]].$$

The (A, I)-system $\mathcal{A}$ is called a *separated pre-Weierstrass system* if two usual Weierstrass division theorems hold with respect to division by each $f \in A_{m,n}$ which is ξ_m-regular or ρ_n-regular. By Weierstrass division, units of $A_{m,n}$, being power series regular of degree 0, are precisely elements of the form $c + g$, where c is a unit of A and

$$g \in A_{m,n}^\circ := (I, \rho) \, A_{m,n}.$$

Also introduced is the concept of rings C of *$\mathcal{A}$-fractions* with proper ideal C° (where $C^\circ := I$ if $C = A$) and with rings C_{mn} of separated power series over C; put $C_{m,n}^\circ := (C^\circ, \rho)C_{m,n}$.

A pre-Weierstrass system $\mathcal{A}$ is called a *separated Weierstrass system* if the rings C of fractions enjoy the following weak Noetherian property: If

$$f = \sum_{\mu, \nu} c_{\mu,\nu} \xi^\mu \rho^\nu \in C_{m,n} \quad \text{with} \quad c_{\mu,\nu} \in C,$$

then there exist a finite set $J \subset \mathbb{N}^{m+n}$ and elements $g_{\mu,\nu} \in C_{m,n}^\circ$, $(\mu, \nu) \in J$, such that

$$f = \sum_{(\mu,\nu) \in J} c_{\mu,\nu} \, \xi^\mu \, \rho^\nu \, (1 + g_{\mu,\nu}).$$

The above condition is a form of Noetherianity and implies, in particular, that if

$$f = \sum_{\mu,\nu} a_{\mu\nu}\, \xi^\mu \rho^\nu \in A_{m,n},$$

then all the coefficients $a_{\mu\nu}$ are linear combinations of finitely many of them and, moreover, if a coefficient $a_{\mu\nu}$ is "small", so can be the coefficients of such a combination. Consequently, the Gauss norm on each $A_{m,n}$ is defined whenever $A = F^\circ$ and $I = F^{\circ\circ}$ for a valued field F. Moreover, then the weak Noetherian property is equivalent to the condition that for every $f \in A_{m,n}$, $f \neq 0$, there is an element $c \in F$ such that $cf \in A_{m,n}$ and $\|cf\| = 1$.

Let $\mathcal{A}$ be a separated Weierstrass system and K a valued field. A *separated analytic $\mathcal{A}$-structure* on K is a collection of homomorphisms $\sigma_{m,n}$ from $A_{m,n}$ to the ring of K°-valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$, $m, n \in \mathbb{N}$, such that

1) $\sigma_{0,0}(I) \subset K^{\circ\circ}$;

2) $\sigma_{m,n}(\xi_i)$ and $\sigma_{m,n}(\rho_j)$ are the i-th and $(m+j)$-th coordinate functions on $(K^\circ)^m \times (K^{\circ\circ})^n$, respectively;

3) $\sigma_{m+1,n}$ and $\sigma_{m,n+1}$ extend $\sigma_{m,n}$, where functions on $(K^\circ)^m \times (K^{\circ\circ})^n$ are identified with those functions on

$$(K^\circ)^{m+1} \times (K^{\circ\circ})^n \quad \text{or} \quad (K^\circ)^m \times (K^{\circ\circ})^{n+1}$$

which do not depend on the coordinate ξ_{m+1} or ρ_{n+1}, respectively.

It can be shown via Weierstrass division that analytic $\mathcal{A}$-structures preserve composition; more precisely, functions from $A_{k,l}$ may substitute for the variables ξ and functions from $A_{k,l}^\circ$ may substitute for the variables ρ (op. cit., Proposition 4.5.3). When considering a particular field K with analytic $\mathcal{A}$-structure, one may assume that $\ker \sigma_{0,0} = (0)$. Indeed, replacing A by $A/\ker \sigma_{0,0}$ yields an equivalent analytic structure on K with this property. Then $A = A_{0,0}$ can be regarded as a subring of K°.

If the ground field K is trivially valued, then $K^{\circ\circ} = (0)$ and the analytic structure reduces to the algebraic structure given by polynomials. If K is non-trivially valued, then the function induced by a power series from $A_{m,n}$, $m, n \in \mathbb{N}$, is the zero function iff the image in K of each of its coefficients is zero (op. cit., Proposition 4.5.4).

A separated analytic $\mathcal{A}$-structure on a valued field K can be uniquely extended to any algebraic extension K' of K; in particular, to the algebraic closure K_{alg} of K (op. cit., Theorem 4.5.11). (The foregoing properties remain valid for strictly convergent analytic structures

too.) Every valued field with separated analytic structure is Henselian (op. cit., Proposition 4.5.10).

Remark 2.1. From now on, we shall always assume that the ground field K is non-trivially valued and that $\sigma_{0,0}$ is injective. Under the assumptions, one can canonically obtain by extension of parameters a (unique) separated Weierstrass system $\mathcal{A}(K)$ over $(K^\circ, K^{\circ\circ})$ so that K has separated analytic $\mathcal{A}(K)$-structure; a similar extension can be performed for any subfield $F \subset K$ (op. cit., Theorem 4.5.7). (This technique holds for strictly convergent Weierstrass systems as well.)

Now we can describe the analytic language $\mathcal{L}$ of an analytic structure K determined by a separated Weierstrass system $\mathcal{A}$. We begin by defining the semialgebraic language $\mathcal{L}_{Hen}$. It is a two sorted language with the main, valued field sort K, and the auxiliary RV-sort

$$RV = RV(K) := RV^* \cup \{0\}, \quad RV^*(K) := K^\times/(1 + K^{\circ\circ});$$

here $A^\times$ denotes the set of units of a ring A. The language of the valued field sort is the language of rings $(0, 1, +, -, \cdot)$. The language of the auxiliary sort is the so-called inclusion language (op. cit., Section 6.1). The only map connecting the sorts is the canonical map

$$rv : K \to RV(K), \quad 0 \mapsto 0.$$

Since

$$\widetilde{K}^\times \simeq (K^\circ)^\times/(1 + K^{\circ\circ}) \quad \text{and} \quad \Gamma \simeq K^\times/(K^\circ)^\times,$$

we get the canonical exact sequence

$$1 \to \widetilde{K} \to RV(K) \to \Gamma \to 0.$$

This sequence splits iff the valued field K has an angular component map.

The analytic language $\mathcal{L} = \mathcal{L}_{Hen,\mathcal{A}}$ is the semialgebraic language $\mathcal{L}_{Hen}$ augmented on the valued field sort K by the reciprocal function $1/x$ (with $1/0 := 0$) and the names of all functions of the system $\mathcal{A}$, together with the induced language on the auxiliary sort RV (op. cit., Section 6.2). A power series $f \in A_{m,n}$ is construed via the analytic $\mathcal{A}$-structure on their natural domains and as zero outside them. More precisely, f is interpreted as a function

$$\sigma(f) = f^\sigma : (K^\circ)^m \times (K^{\circ\circ})^n \to K^\circ,$$

extended by zero on $K^{m+n} \setminus (K^\circ)^m \times (K^{\circ\circ})^n$.

In the equicharacteristic case, however, the induced language on the auxiliary sort RV coincides with the semialgebraic inclusion language.

It is so because then [4, Lemma 6.3.12] can be strengthen as follows, whereby [4, Lemma 6.3.14] can be directly reduced to its algebraic analogue. Consider a strong unit on the open ball $B = K_{alg}^{\circ\circ}$. Then $rv(E^\sigma)(x)$ is constant when x varies over B. This is no longer true in the mixed characteristic case. There, a weaker conclusion asserts that the functions $rv_n(E^\sigma)(x)$, $n \in \mathbb{N}$, depend only on $rv_n(x)$ when x varies over B; actually, $rv_n(E^\sigma)(x)$ depend only on x mod $(n \cdot K_{alg}^{\circ\circ})$ when x varies over B, as indicated in [5, Remark A.1.12]. Under the circumstances, the residue field $\widetilde{K}$ is orthogonal to the value group Γ_K, whenever the ground field K has an angular component map or, equivalently, when the auxiliary sort RV splits (in a non-canonical way):

$$RV(K) \simeq \widetilde{K} \times \Gamma_K.$$

This means that every definable set in the auxiliary sort $RV(K)$ is a finite union of the Cartesian products of some sets definable in the residue field sort $\widetilde{K}$ (in the language of rings) and in the value group sort Γ_K (in the language of ordered groups). The orthogonality property will often be used in the paper, similarly as it was in the algebraic case treated in our papers [18, 19].

Remark 2.2. Not all valued fields K have an angular component map, but it exists if K has a cross section, which happens whenever K is $\aleph_1$-saturated (cf. [2, Chap. II]). Moreover, a valued field K has an angular component map whenever its residue field $\Bbbk$ is $\aleph_1$-saturated (cf. [22, Corollary 1.6]). In general, unlike for p-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. Since the K-topology is $\mathcal{L}$-definable, the closedness theorem is a first order property. Therefore it can be proven using elementary extensions, and thus one may assume that an angular component map exists.

Denote by $\mathcal{L}^*$ the analytic language $\mathcal{L}$ augmented by all Henselian functions

$$h_m : K^{m+1} \times RV(K) \to K, \quad m \in \mathbb{N},$$

which are defined by means of a version of Hensel's lemma (cf. [4], Section 6).

Let $\mathcal{T}_{Hen,\mathcal{A}}$ be the theory of all Henselian valued fields of characteristic zero with separated analytic $\mathcal{A}$-structure. Two crucial results about analytic structures are Theorems 6.3.7 and 6.3.8 from [4]), stated below.

Theorem 2.3. *The theory $\mathcal{T}_{Hen,\mathcal{A}}$ eliminates valued field quantifiers, is b-minimal with centers and preserves all balls. Moreover, $\mathcal{T}_{Hen,\mathcal{A}}$ has the Jacobian property.*

Theorem 2.4. *Let K be a Henselian field with separated analytic $\mathcal{A}$-structure. Let $f : X \to K$, $X \subset K^n$, be an $\mathcal{L}(B)$-definable function for some set of parameters B. Then there exist an $\mathcal{L}(B)$-definable function $g : X \to S$ with S auxiliary and an $\mathcal{L}^*(B)$-term t such that*

$$f(x) = t(x, g(x)) \quad \text{for all} \;\; x \in X.$$

It follows from Theorem 2.3 that the theory $\mathcal{T}_{Hen,\mathcal{A}}$ admits b-minimal cell decompositions with centers (cf. [7]).

3. Rings of analytic function germs and of overconvergent analytic functions

Keeping the notation from [4], put

$$A^{\dagger}_{m,n}(K) := K \otimes_{K^\circ} A_{m,n}(K).$$

By the weak Noetherian property, we immediately get

$$A^{\dagger}_{m,n}(K)^\circ := \{f \in A^{\dagger}_{m,n}(K) : \|f\| \le 1\} = A_{m,n}(K) =$$

$$\{f \in A^{\dagger}_{m,n}(K) : |f^\sigma(a,b)| \le 1 \;\; \text{for all} \; (a,b) \in (K^\circ_{alg})^m \times (K^{\circ\circ}_{alg})^n\}$$

and

$$A^{\dagger}_{m,n}(K)^{\circ\circ} := (K^{\circ\circ}, \rho)\, A^{\dagger}_{m,n}(K)^\circ = A_{m,n}(K)^\circ =$$

$$\{f \in A^{\dagger}_{m,n}(K) : |f^\sigma(a,b)| < 1 \;\; \text{for all} \; (a,b) \in (K^\circ_{alg})^m \times (K^{\circ\circ}_{alg})^n\}.$$

It is demonstrated in [4, Section 5] that the classical Rückert theory from [1, Section 5.2.5] applies to the rings $A^{\dagger}_{m,0}(K)$ and $A^{\dagger}_{0,n}(K)$. Indeed, for any $f \in A_{m,0}(K)$ or $f \in A_{0,n}(K)$, $f \ne 0$, there is an $a \in K$ such that $af \in A_{m,0}(K)$ or $af \in A_{0,n}(K)$, respectively, $\|af\| = 1$, and af is regular after a Weierstrass change of variables. Hence those rings enjoy many good algebraic properties. The authors conjecture that those properties are shared also by the rings $A_{m,n}(K)$ of separated analytic functions. Their conjecture, however, seems to be a problem yet unsolved. We prove in this section that the rings of analytic function germs and of overconvergent analytic functions, determined by a separated analytic structure, are excellent and regular.

Remark 3.1. For application of resolution of singularities in the proof of curve selection in Section 7, it suffices to make use of local rings of analytic function germs.

Put

$$\Delta_{m,n}(r) := \{(\xi, \rho) \in K^{m+n} : |\xi_i| \le r, \ |\rho_j| < r\} = c \cdot \Delta_{n,m}(1)$$

with $c \in K$, $c \ne 0$, and $r = |c|$. For any $f \in A_{m,n}(K)$, the power series

$$f_c(\xi, \rho) := f(\xi/c, \rho/c) \in K[[\xi, \rho]]$$

determines a function

$$f_c^\sigma : \Delta_{m,n}(r) \longrightarrow K^\circ.$$

Put

$$A_{m,n}(K, r) := \{f^c : f \in A_{m,n}(K)\}, \quad A_{m,n}^\dagger(K, r) := K \otimes_{K^\circ} A_{m,n}(K, r),$$

(obviously, this definition does not depend on the choice of c with $|c| = r$), and let

$$A_{m,n}^{loc}(K) := \bigcup_{r>0} A_{m,n}^r(K), \quad A_{m,n}^\ddagger(K) := \bigcup_{r>1} A_{m,n}^r(K)$$

be the direct limits of the systems of rings

$$A_{m,n}^\dagger(K, r) \subset A_{m,n}^\dagger(K, s), \ s < r,$$

with $r > 0$ and $r > 1$, respectively. Under the assumptions imposed on analytic structures throughout the paper, one can identify a power series f_c with the analytic function f_c^σ. We call $A_{m,n}^{loc}(K)$ and $A_{m,n}^\ddagger(K)$ the rings of analytic function germs (at $0 \in K^{m+n}$) and overconvergent analytic functions, respectively. It is not difficult to check that $A_{m,n}^{loc}(K)$ are local rings with maximal ideal generated by the variables ξ and ρ.

Remark 3.2. In fact, it follows from [4, Prop. 4.5.3] (on preserving composition) that the rings $A_{m,n}^{loc}(K)$ depend only on $p = m+n$. Hence the variables ξ and ρ play locally the same role, and thus the use of double indices m, n is immaterial and refers rather to the names of variables. Therefore it is more natural to denote these local rings of analytic function germs by $A_p^{loc}(K) := A_{m,n}^{loc}(K)$ with $p = m + n$.

Also note that the Gauss norm and supremum norm on $A_{m,n}^\dagger(K)$ coincide whenever the residue field $\widetilde{K}$ is infinite (maximal modulus principle; cf. [4, Remark 5.2.8] or [1, Section 5.1.4, Proposition 3] for the classical case).

Consider now any $f \in A_{m,n}(K)$ and $c \in K^{\circ\circ}$, $c \ne 0$. Similarly as in the classical case of strictly convergent power series elaborated in [1, Section 5.2.3 and 5.2.4], the power series $af_{1/c}$ is, after a Weierstrass change of variables, regular with respect to any of the variables ξ or ρ for some $a \in K$. Therefore Weierstrass preparation and division, as

well as the Weierstrass finiteness theorem hold for the rings $A_p^{loc}(K)$ and $A_{m,n}^{\ddagger}(K)$. The latter enables induction on the number of indeterminates. In particular, for any maximal ideal of the rings under study, its residue field is a finite extension of the ground field K. Hence also applicable here is Rückert's theory (op. cit., Section 5.2.5) We can thus state the following

Theorem 3.3. *The rings $A_p^{loc}(K)$ and $A_{m,n}^{\ddagger}(K)$, $p, m, n \in \mathbb{N}$, are Noetherian factorial Jacobson rings. Furthermore, there is a one-to-one correspondence between the maximal ideals of $A_{m,n}^{\ddagger}(K)$ and the orbits of $\left(K_{alg}^{\circ}\right)^{m+n}$ under the Galois group of K_{alg} over K, and the rings $A_{m,n}^{\ddagger}(K)$ satisfy the Nullstellensatz.* □

Hence we obtain two corollaries.

Corollary 3.4. *The ring $A_p^{loc}(K)$, $p \in \mathbb{N}$, is an excellent regular local ring of dimension p.*

Proof. The dimension of $A_p^{loc}(K) = A_{m,n}^{loc}(K)$ is $\geq p = m + n$ because one has the chain of prime ideals

$$(0) \subset (\xi_1) \subset \ldots \subset (\xi_1, \ldots, \xi_m, \rho_1) \subset \ldots \subset (\xi_1, \ldots, \xi_m, \rho_1, \ldots, \rho_n).$$

Since the maximal ideal is generated by $p = m + n$ variables ξ and ρ, the converse inequality and the regularity of $A_p^{loc}(K)$ follows immediately. Finally, observe that $A_p^{loc}(K)$ contains its residue field K of characteristic 0, and that

$$\partial/\partial\xi_1, \ldots, \partial/\partial\xi_m, \partial/\partial\rho_1, \ldots, \partial/\partial\rho_n$$

are derivatives of $A_p^{loc}(K)$ over K such that

$$\partial\xi_i/\partial\xi_j = \delta_{ij} \quad \text{and} \quad \partial\rho_i/\partial\rho_j = \delta_{ij}.$$

Hence and by the Jacobian criterion for excellence [17, Theorem 102], the ring $A_p^{loc}(K)$ is excellent, concluding the proof. □

Corollary 3.5. *The ring $A_{m,n}^{\ddagger}(K)$, $m, n \in \mathbb{N}$, is an excellent regular ring of dimension $m + n$.*

Proof. Taking into account that each maximal ideal of $A_{m,n}^{\ddagger}(K)$ comes from a point in $\left(K_{alg}^{\circ}\right)^{m+n}$, the foregoing proof can be repeated mutatis mutandi. □

We still recall [4, Remark 5.2.8]. Given an ideal J of $A_{m,n}^{\dagger}(K)$, the elements of $A_{m,n}^{\dagger}(K)/J$ define functions on

$$V(J) := \left\{ x \in \left(K_{alg}^{\circ}\right)^m \times \left(K_{alg}^{\circ\circ}\right)^n : f(x) = 0 \ \text{ for all } \ f \in J \right\}.$$

Since K_{alg} admits quantifier elimination in the language of valued fields with the multiplicative inverse and names of the elements of $\bigcup_{m,n} A_{m,n}^{\dagger}$, the supremum norm

$$\|f\|_{sup} := \sup\{|f(x)| : x \in V(J)\} \ \text{ for } \ f \in A_{m,n}^{\dagger}(K)/J$$

is well defined, takes values in $|K_{alg}|$ and is a norm provided that $A_{m,n}^{\dagger}(K)/J$ is reduced. Therefore the rings $A_{m,n}^{\dagger}(K)$, $m, n \in \mathbb{N}$, satisfy the Nullstellensatz and there is a one-to-one correspondence between the maximal ideals of $A_{m,n}^{\dagger}(K)$ and the orbits of $\left(K_{alg}^{\circ}\right)^m \times \left(K_{alg}^{\circ\circ}\right)^n$ under the Galois group of K_{alg} over K.

Finally, note that resolution of singularities applies to excellent regular schemes (cf. [24, Theorem 1.1.3] or [13], Corollary 3 on p. 146 along with the explanations on p. 161, for the classical case of excellent regular local rings). In particular, it applies to the scheme $X := \mathrm{Spec}\,(A_p^{loc}(K))$. Below we state a version which refers, in fact, to transformation to a normal crossing by blowing up.

Theorem 3.6. *Let $f_1, \ldots, f_k \in A_p^{loc}(K)$. Then there exists a finite composite of blow-ups $\pi : \widetilde{X} \to X$ along smooth centers such that $f_1 \circ \pi, \ldots, f_k \circ \pi$ are simple normal crossing divisors on $\widetilde{X}$.*   □

Remark 3.7. It is well known that in the conclusion one can require

$$f_1 \circ \pi, \ldots, f_k \circ \pi$$

to be linearly ordered with respect to divisibility relation at each point in $\widetilde{X}$.

It is possible to adapt here Serre's concept of analytic manifolds ([23], Part II, Chap. III) with respect to the class of analytic function germs induced (via translations) by $A_p^{loc}(K)$. Though this concept is quite weak, it is sufficient for our further applications. For simplicity, we shall denote by the same symbol a function germ or a set germ and its representative. This convention will not lead to confusion. Clearly, $X, \widetilde{X}$ and the map $\pi : \widetilde{X} \to X$ correspond respectively to

1) a polydisk $\Delta_p(r) := \Delta_{p,0}(r)$ and a closed (in the K-topology) analytic submanifold $\widetilde{X}_0$ of $\Delta_p(r) \times \mathbb{P}^N(K)$, for some $0 < r < 1$ and $N \in \mathbb{N}$;

2) the restriction to $\widetilde{X}_0$ of the projection of $\Delta_p(r) \times \mathbb{P}^N(K)$ onto the first factor.

We may regard X_0 and $\widetilde{X}_0$ as the sets of "K-rational points" of X and $\widetilde{X}$, respectively. Summing up, we obtain

Corollary 3.8. *Let $f_1, \ldots, f_k \in A_p^{loc}(K)$. Then there exists a finite composite of blow-ups $\pi_0 : \widetilde{X}_0 \to \Delta_p(r)$ along smooth analytic submanifolds such that the pull-backs $f_1 \circ \pi_0, \ldots, f_k \circ \pi_0$ are simple normal crossing divisors on $\widetilde{X}_0$. The map π_0 is surjective and definably closed by virtue of the closedness theorem.*

Moreover, one can require

$$f_1 \circ \pi_0, \ldots, f_k \circ \pi_0$$

to be linearly ordered with respect to divisibility relation at each point in $\widetilde{X}_0$. It means that at each point $a \in \widetilde{X}_0$ there are local analytic coordinates $x = (x_1, \ldots, x_p)$ with $x(a) = 0$ such that

$$f_i \circ \pi_0(x) = u_i(x) \cdot x^{\alpha_i}, \quad i = 1, \ldots, k,$$

where the u_i are analytic units at a, $u_i(a) \neq 0$, $\alpha_i \in \mathbb{N}^p$, and the monomials x^{α_i}, $i = 1, \ldots, k$, are linearly ordered by divisibility relation.

4. Proof of the closedness theorem

In the algebraic case, the proofs of the closedness theorem given in our papers [18, 19]) make use of the following three main tools:

• elimination of valued field quantifiers and cell decomposition due to Pas;

• fiber shrinking ([18, 19, Proposition 6.1]);

• and the theorem on existence of the limit ([18, Proposition 5.2] and [19, Theorem 5.1]).

In this paper, we apply analytic versions of quantifier elimination, cell decomposition and term structure (Theorems 2.3 and 2.4) due to Cluckers–Lipshitz.

Now recall the concept of fiber shrinking. Let A be an $\mathcal{L}$-definable subset of K^n with accumulation point

$$a = (a_1, \ldots, a_n) \in K^n$$

and E an $\mathcal{L}$-definable subset of K with accumulation point a_1. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),$$

i.e. for any neighbourhood U of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$,

$t \neq a_1$. When $n = 1$, A is itself a fiber shrinking for the subset A of K at an accumulation point $a \in K$.

Proposition 4.1. *(Fiber shrinking) Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

Its proof was reduced, by means of elimination of valued field quantifiers, to Lemma 4.2 below ([19, Lemma 6.2]), which, in turn, was obtained via relative quantifier elimination for ordered abelian groups. That approach can be repeated verbatim in the analytic settings.

Lemma 4.2. *Let Γ be an ordered abelian group and P be a definable subset of Γ^n. Suppose that $(\infty, \ldots, \infty)$ is an accumulation point of P, i.e. for any $\delta \in \Gamma$ the set*

$$\{x \in P : x_1 > \delta, \ldots, x_n > \delta\} \neq \emptyset$$

is non-empty. Then there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \ldots, r_n t + \gamma_n) : t \in \Gamma, \ t \geq 0\} \quad \text{with} \ \ r_1, \ldots, r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1, \ldots, \gamma_n) \in P$ and such that $(\infty, \ldots, \infty)$ is an accumulation point of the intersection $P \cap L$ too. □

In a similar manner, one can obtain the following

Lemma 4.3. *Let P be a definable subset of Γ^n and*

$$\pi : \Gamma^n \to \Gamma, \quad (x_1, \ldots, x_n) \mapsto x_1$$

be the projection onto the first factor. Suppose that ∞ is an accumulation point of $\pi(P)$. Then there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \ldots, r_n t + \gamma_n) : t \in \Gamma, \ t \geq 0\} \ \ \text{with} \ r_1, \ldots, r_n \in \mathbb{N}, \ r_1 > 0,$$

passing through a point $\gamma = (\gamma_1, \ldots, \gamma_n) \in P$ and such that ∞ is an accumulation point of $\pi(P \cap L)$ too.

The above two lemmas will be often used in further reasonings.

As for the theorem on existence of the limit, here we first prove, using Theorem 2.4, a weaker version given below. The full analytic version (Proposition 5.1) will be established in Section 4 by means of the closedness theorem.

Proposition 4.4. *Let $f : E \to K$ be an $\mathcal{L}$-definable function on a subset E of K and suppose 0 is an accumulation point of E. Then there is an $\mathcal{L}$-definable subsets $F \subset E$ with accumulation point 0 and a point $w \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \to 0} f|F(x) = w.$$

Moreover, we can require that

$$\{(x, f(x)) : x \in F\} \subset \{(x^r, \phi(x)) : x \in G\},$$

where r is a positive integer and ϕ is a definable function, a composite of some functions induced by series from $\mathcal{A}$ and of some algebraic power series (coming from the implicit function theorem). Then, in particular, the definable set

$$\{(v(x), v(f(x))) : \ x \in (F \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$$

is contained in an affine line with rational slope

$$q \cdot l = p \cdot k + \beta,$$

with $p, q \in \mathbb{Z}$, $q > 0$, $\beta \in \Gamma$, or in $\Gamma \times \{\infty\}$.

Proof. In view of Remark 2.1, we may assume that K has separated analytic $\mathcal{A}(K)$-structure. We apply Theorem 2.4 and proceed with induction with respect to the complexity of the term t. Since an angular component map exists (cf. Remark 2.2), the sorts $\widetilde{K}$ and Γ are orthogonal in

$$RV(K) \simeq \widetilde{K} \times \Gamma_K.$$

Therefore, after shrinking F, we can assume that $\overline{ac}\,(F) = \{1\}$ and the function g goes into $\{\xi\} \times \Gamma^s$ with a $\xi \in \widetilde{K}^s$, and next that $\xi = (1, \ldots, 1)$; similar reductions were considered in our papers [18, 19]. For simplicity, we look at g as a function into Γ^s. We shall briefly explain the most difficult case where

$$t(x, g(x)) = h_m(a_0(x), \ldots, a_m(x), (1, g_0(x))),$$

assuming that the theorem holds for the terms $a_0, \ldots, a_m$; here g_0 is one of the components of g. In particular, each function $a_i(x)$ has, after partitioning, a limit, say, $a_i(0)$ when x tends to zero.

By Lemma 4.3, we can assume that

(4.1) $$pv(x) + qg_0(x) + v(a) = 0$$

for some $p, q \in \mathbb{N}$ and $a \in K \setminus \{0\}$. By the induction hypothesis, we get

$$\{(x, a_i(x)) : x \in F\} \subset \{(x^r, \alpha_i(x)) : x \in G\}, \quad i = 0, 1, \ldots, m,$$

for some power series $\alpha_i(x))$ as stated in the theorem. Put

$$P(x, T) := \sum_{i=0}^{m} a_i(x)T^i.$$

By the very definition of h_m and since we are interested in the vicinity of zero, we may assume that there is an $i_0 = 0, \ldots, m$ such that

$$\forall\, x \in F \;\exists\, u \in K \quad v(u) = g_0(x), \quad \overline{ac}\, u = 1,$$

and the following formulas hold

(4.2) $$v(a_{i_0}(x)u^{i_0}) = \min\{v(a_i(x)u^i),\; i = 0, \ldots, m\},$$

$$v(P(x, u)) > v(a_{i_0}(x)u^{i_0}), \quad v\left(\frac{\partial P}{\partial T}(x, u)\right) = v(a_{i_0}(x)u^{i_0 - 1}).$$

Then $h_m(a_0(x), \ldots, a_m(x), (1, g_0(x)))$ is a unique $b(x) \in K$ such that

$$P(x, b(x)) = 0, \quad v(b(x)) = g_0(x), \quad \overline{ac}\, b(x) = 1.$$

By [19, Remarks 7.2, 7.3], the set F contains the set of points of the form $c^r t^{Nqr}$ for some $c \in K$ with $\overline{ac}\, c = 1$, a positive integer N and all $t \in K^\circ$ small enough with $\overline{ac}\, t = 1$. Hence and by equation 4.1, we get

$$g_0(c^r t^{Nqr}) = g_0(c^r) - v(t^{Npr}).$$

Take $d \in K$ such that $g_0(c^r) = v(d)$ and $\overline{ac}\, d = 1$. Then

$$g_0(c^r t^{Nqr}) = v(dt^{-Npr}).$$

Thus the homothetic change of variable

$$Z = T/dt^{-Npr} = t^{Npr}T/d$$

transforms the polynomial

$$P(c^r t^{Nqr}, T) = \sum_{i=0}^{m} \alpha_i(ct^{Nq})T^i$$

into a polynomial $Q(t, Z)$ to which Hensel's lemma applies (cf. [21, Lemma 3.5]):

(4.3) $$P(c^r t^{Nqr}, T) = P(c^r t^{Nqr}, dt^{-Npr}Z) =$$

$$\sum_{i=0}^{m} \alpha_i(ct^{Nq}) \cdot (dt^{-Npr}Z)^i = (\alpha_{i_0}(ct^{Nq}) \cdot (dt^{-Npr})^{i_0} \cdot Q(t, Z).$$

Indeed, formulas 4.2 imply that the coefficients $b_i(t)$, $i = 0, \ldots, m$, of the polynomial Q are power series (of order ≥ 0) in the variable t, and that

$$v(Q(t, 1)) > 0 \quad \text{and} \quad v\left(\frac{\partial Q}{\partial Z}(t, 1)\right) = 0$$

fot $t \in K^0$ small enough. Hence

$$v(Q(0, 1)) > 0 \quad \text{and} \quad v\left(\frac{\partial Q}{\partial Z}(0, 1)\right) = 0.$$

But, for $x(t) = c^r t^{Nqr}$, the unique zero $T(t) = b(x(t))$ of the polynomial $P(x(t), T)$ such that

$$v(b(x(t))) = v(dt^{-Npr}) \ \ \text{and} \ \ \overline{ac}\, b(x(t)) = 1$$

corresponds to a unique zero $Z(t)$ of the polynomial $Q(t, Z)$ such that

$$v(Z(t)) = v(1) \ \ \text{and} \ \ \overline{ac}\, Z(t) = 1.$$

Therefore the conclusion of the theorem can be directly obtained via the implicit function theorem (cf. [19, Proposition 2.5]) applied to the polynomial

$$P(A_0, \ldots, A_m, Z) = \sum_{i=0}^{m} A_i Z^i$$

in the variables A_i substituted for $a_i(x)$ at the point

$$A_0 = b_0(0), \ \ldots, \ A_m = b_m(0), \ Z = 1.$$

$\square$

Now we can readily proceed with the

Proof of the closedness theorem (Theorem 1.1). We must show that if B is an $\mathcal{L}$-definable subset of $D \times (K^\circ)^n$ and a point a lies in the closure of $A := \pi(B)$, then there is a point b in the closure of B such that $\pi(b) = a$. As before (cf. [19, Section 8]), the theorem reduces easily to the case $m = 1$ and next, by means of fiber shrinking (Proposition 4.1), to the case $n = 1$. We may obviously assume that $a = 0 \notin A$.

By b-minimal cell decomposition, we can assume that the set B is a relative cell with center over A. It means that B has a presentation of the form

$$\Lambda : B \ni (x, y) \to (x, \lambda(x, y)) \in A \times RV(K)^s,$$

where $\lambda : B \to RV(K)^s$ is an $\mathcal{L}$-definable function, such that for each $(x, \xi) \in \Lambda(B)$ the pre-image $\lambda_x^{-1}(\xi) \subset K$ is either a point or an open ball; here $\lambda_x(y) := \lambda(x, y)$. In the latter case, there is a center, i.e. an $\mathcal{L}$-definable map $\zeta : \Lambda(B) \to K$, and a (unique) map $\rho : \Lambda(B) \to RV(K) \setminus \{0\}$ such that

$$\lambda_x^{-1}(\xi) = \{y \in K : rv\,(y - \zeta(x, \xi)) = \rho(x, \xi)\}.$$

Again, since the sorts $\widetilde{K}$ and Γ are orthogonal in $RV(K) \simeq \widetilde{K} \times \Gamma$, we can assume, after shrinking the sets A and B, that

$$\lambda(B) \subset \{(1, \ldots, 1)\} \times \Gamma^s \subset \widetilde{K}^s \times \Gamma^s;$$

let $\tilde{\lambda}(x, y)$ be the projection of $\lambda(x, y)$ onto Γ^s. By Lemma 4.3, we can assume once again, after shrinking the sets A and B, that the set

$$\{(v(x), v(y), \tilde{\lambda}(x, y)) : \; (x, y) \in B\} \subset \Gamma^{s+2}$$

is contained in an affine semi-line with integer coefficients. Hence $\lambda(x, y) = \phi(v(x))$ is a function of one variable x. We have two cases.

Case I. $\lambda_x^{-1}(\xi) \subset K^\circ$ is a point. Since each λ_x is a constant function, B is the graph of an $\mathcal{L}$-definable function. The conclusion of the theorem follows thus from Proposition 4.4.

Case II. $\lambda_x^{-1}(\xi) \subset K^\circ$ is a ball. Again, application of Lemma 4.3 makes it possible, after shrinking the sets A and B, to arrange the center

$$\zeta : \Lambda(B) \ni (x, \xi) \to \zeta(x, v(x)) = \zeta(x) \in K$$

and the function $\rho(x, \xi) = \rho(v(x))$ as functions of one variable x. Likewise as it was above, we can assume that the set

$$P := \{(v(x), \rho(v(x))) : x \in A\} \subset \Gamma^2$$

is contained in an affine line $pv(x) + q\rho(v(x)) + v(c) = 0$ with integer coefficients p, q, $q \neq 0$; furthermore, that P contains the set

$$Q := \{(v(ct^{qN}), \rho(v(ct^{qN}))) : \; t \in K^\circ\}$$

for a positive integer N. Then we easily get

$$\rho(v(ct^{qN})) = \rho(c) - pNv(t) = v(ct^{-pN}).$$

Hence the set B contains the graph

$$\{(ct^{qN}, \zeta(ct^{qN}) + ct^{-pN}) : \; t \in K^\circ\}.$$

As before, the conclusion of the theorem follows thus from Proposition 4.4, and the proof is complete. $\qquad\square$

5. Direct applications

The framework of b-minimal structures provides cell decomposition and a good concept of dimension (cf. [7]), which in particular satisfies the axioms from the paper [9]. For separated analytic structures, the zero-dimensional sets are precisely the finite sets, and also valid is the following dimension inequality, which is of great geometric significance:

(5.1) $$\dim \partial E < \dim E;$$

here E is any $\mathcal{L}$-definable subset of K^n and $\partial E := \overline{E} \setminus E$ denotes the frontier of A.

We first apply the closedness theorem to obtain the following full analytic version of the theorem on existence of the limit.

Proposition 5.1. *Let $f : E \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function on a subset E of K, and suppose that 0 is an accumulation point of E. Then there is a finite partition of E into $\mathcal{L}$-definable sets $E_1, \ldots, E_r$ and points $w_1 \ldots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \to 0} f|E_i(x) = w_i \quad for \quad i = 1, \ldots, r.$$

Proof. We may of course assume that $0 \notin E$. Put

$$F := \mathrm{graph}\,(f) = \{(x, f(x) : x \in E\} \subset K \times \mathbb{P}^1(K);$$

obviously, F is of dimension 1. It follows from the closedness theorem that the frontier $\partial F \subset K \times \mathbb{P}^1(K)$ is non-empty, and thus of dimension zero by inequality 5.1. Say

$$\partial F \cap (\{0\} \times \mathbb{P}^1(K)) = \{(0, w_1), \ldots, (0, w_r)\}$$

for some $w_1, \ldots, w_r \in \mathbb{P}^1(K)$. Take pairwise disjoint neighborhoods U_i of the points w_i, $i = 1, \ldots, r$, and set

$$F_0 := F \cap \left(E \times \left(\mathbb{P}^1(K) \setminus \bigcup_i^r E_i \right) \right).$$

Let

$$\pi : K \times \mathbb{P}^1(K) \longrightarrow K$$

be the canonical projection. Then

$$E_0 := \pi(F_0) = f^{-1} \left(\mathbb{P}^1(K) \setminus \bigcup_i^r E_i \right).$$

Clearly, the closure $\overline{F}_0$ of F_0 in $K \times \mathbb{P}^1(K))$ and $\{0\} \times \mathbb{P}^1(K))$ are disjoint. Hence and by the closedness theorem, $0 \notin \overline{E_0}$, the closure of E_0 in K. The set E_0 is thus irrelevant with respect to the limit at $0 \in K$. Therefore it remains to show that

$$\lim_{x \to 0} f|E_i(x) = w_i \quad for \quad i = 1, \ldots, r.$$

Otherwise there is a neighborhood $V_i \subset U_i$ such that 0 would be an accumulation point of the set

$$f^{-1}(U_i \setminus V_i) = \pi(F \cap (E \times (U_i \setminus V_i))).$$

Again, it follows from the closedness theorem that $\{0\} \times \mathbb{P}^1(K)$ and the closure of $F \cap (E \times (U_i \setminus V_i))$ in $K \times \mathbb{P}^1(K))$ would not be disjoint. This contradiction finishes the proof. $\square$

Remark 5.2. Let us mention that Proposition 5.1 can be strengthened as stated below (cf. the algebraic versions [18, Proposition 5.2] and [19, Theorem 5.1]):

Moreover, perhaps after refining the finite partition of E, there is a neighbourhood U of 0 such that each definable set

$$\{(v(x), v(f(x))): \ x \in (E_i \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \ \{\infty\}), \ i = 1, \ldots, r,$$

is contained in an affine line with rational slope

$$q \cdot l = p_i \cdot k + \beta_i, \ i = 1, \ldots, r,$$

with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

Now we turn to a second application, namely the following theorem on piecewise continuity.

Theorem 5.3. *Let $A \subset K^n$ and $f : A \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.*

Proof. Consider an $\mathcal{L}$-definable function $f : A \to \mathbb{P}^1(K)$ and its graph

$$E := \{(x, f(x)) : x \in A\} \subset K^n \times \mathbb{P}^1(K).$$

We shall proceed with induction with respect to the dimension

$$d = \dim A = \dim E$$

of the source and graph of f.

Observe first that every $\mathcal{L}$-definable subset E of K^n is a finite disjoint union of locally closed $\mathcal{L}$-definable subsets of K^n. This can be easily proven by induction on the dimension of E by means of inequality 5.1. Therefore we can assume that the graph E is a locally closed subset of $K^n \times \mathbb{P}^1(K)$ of dimension d and that the conclusion of the theorem holds for functions with source and graph of dimension $< d$.

Let F be the closure of E in $K^n \times \mathbb{P}^1(K)$ and $\partial E := F \setminus E$ be the frontier of E. Since E is locally closed, the frontier ∂E is a closed subset of $K^n \times \mathbb{P}^1(K)$ as well. Let

$$\pi : K^n \times \mathbb{P}^1(K) \longrightarrow K^n$$

be the canonical projection. Then, by virtue of the closedness theorem, the images $\pi(F)$ and $\pi(\partial E)$ are closed subsets of K^n. Further,

$$\dim F = \dim \pi(F) = d$$

and

$$\dim \pi(\partial E) \leq \dim \partial E < d;$$

the last inequality holds by inequality 5.1. Putting

$$B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A,$$

we thus get

$$\dim B = d \quad \text{and} \quad \dim (A \setminus B) < d.$$

Clearly, the set

$$E_0 := E \cap (B \times \mathbb{P}^1(K)) = F \cap (B \times \mathbb{P}^1(K))$$

is a closed subset of $B \times \mathbb{P}^1(K)$ and is the graph of the restriction

$$f_0 : B \longrightarrow \mathbb{P}^1(K)$$

of f to B. Again, it follows immediately from the closedness theorem that the restriction

$$\pi_0 : E_0 \longrightarrow B$$

of the projection π to E_0 is a definably closed map. Therefore f_0 is a continuous function. But, by the induction hypothesis, the restriction of f to $A \setminus B$ satisfies the conclusion of the theorem, whence so does the function f. This completes the proof. $\square$

We immediately obtain

Corollary 5.4. *The conclusion of the above theorem holds for any $\mathcal{L}$-definable function $f : A \to K$.*

6. The Łojasiewicz inequalities

Algebraic non-Archimedean versions of the Łojasiewicz inequality, established in our papers [18, 19], can be carried over to the analytic settings considered here with proofs repeated almost verbatim. We thus state only the results (Theorems 11.2, 11.5 and 11.6, Proposition 11.3 and Corollary 11.4 from [19]). Let us mention that the main ingredients of the proof are the closedness theorem, elimination of valued field quantifiers, the orthogonality of the auxiliary sorts and relative quantifier elimination for ordered abelian groups. They allow us to reduce the problem under study to that of piecewise linear geometry. We first state the following version, which is closest to the classical one.

Theorem 6.1. *Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geq v((g_1(x), \ldots, g_m(x)))$$

for all $x \in A$. Equivalently, there is a $C \in |K|$ such that

$$|f(x)|^s \leq C \cdot \max \{|g_1(x)|, \ldots, |g_m(x)|\}$$

for all $x \in A$.

A direct consequence of Theorem 6.1 is the following result on Hölder continuity of definable functions.

Proposition 6.2. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.*

$$s \cdot v(f(x) - f(z)) + \beta \geq v(x - z)$$

for all $x, z \in A$. Equivalently, there is a $C \in |K|$ such that

$$|f(x) - f(z)|^s \leq C \cdot |x - z|$$

for all $x, z \in A$.

We immediately obtain

Corollary 6.3. *Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed bounded subset $A \subset K^n$ is uniformly continuous.*

Now we formulate another, more general version of the Łojasiewicz inequality for continuous definable functions of a locally closed subset of K^n.

Theorem 6.4. *Let $f, g : A \to K$ be two continuous $\mathcal{L}$-definable functions on a locally closed subset A of K^n. If*

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Finally, put

$$\mathcal{D}(f) := \{x \in A : f(x) \neq 0\} \quad \text{and} \quad \mathcal{Z}(f) := \{x \in A : f(x) = 0\}.$$

The following theorem may be also regarded as a kind of the Łojasiewicz inequality, which is, of course, a strengthening of Theorem 6.4.

Theorem 6.5. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f)$ to a (unique) continuous $\mathcal{L}$-definable function on A.*

7. Curve selection

Consider a Henselian field K with a separated analytic $\mathcal{A}$-structure. In this section, we establish a general version of curve selection for $\mathcal{L}$-definable sets. Note that the domain of the selected curve is, unlike in the classical version, only an $\mathcal{L}$-definable subset of the unit disk.

Proposition 7.1. *Let A be an $\mathcal{L}$-definable subset of K^p. If a point $a \in K^p$ lies in the closure (in the K-topology) $\mathrm{cl}\,(A\backslash\{a\})$ of $A\backslash\{a\}$, then there exist an $\mathcal{L}$-definable map $\varphi : K^\circ \longrightarrow K^p$ given by power series from $A_p^{loc}(K)$, and an $\mathcal{L}$-definable subset E of K° with accumulation point 0 such that*

$$\varphi(0) = a \quad and \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

Proof. We call the problem under study curve selection for the couple (A, a). We may assume without loss of generality that $a = 0 \in K^p$. By elimination of valued field quantifiers, the set $A \setminus \{a\}$ is a finite union of sets defined by conditions of the form

$$(v(t_1(x)), \ldots, v(t_r(x))) \in P, \quad (\overline{ac}\,\tau_1(x), \ldots, \overline{ac}\,\tau_s(x)) \in Q,$$

where t_i, τ_j are terms of the separated analytic structure $\mathcal{A}(K)$, and P and Q are definable subsets of Γ^r and $\Bbbk^s$, respectively. Here it is convenient to deal with a local concept of term, i.e. a finite composite of functions analytic near a given point (in some local analytic coordinates) and the reciprocal function $1/x$.

One can, of course, assume that A is just a set of this form. We shall proceed with induction on the complexity of these terms. Its lowering is possible via successive transformations to a normal crossing by means of Corollary 3.8 and the three straightforward observations below.

Observation 1. Consider a finite composite of blow-ups

$$\pi_0 : \widetilde{X}_0 \to \Delta_p(r)$$

from Corollary 3.8 and put $B := \pi_0^{-1}(A \setminus \{a\})$. Since π_0 is a surjective, definably closed map by Corollary 1.2 to the closedness theorem, there is a point $b \in \mathrm{cl}\,(B)\backslash B$ such that $\pi_0(b) = a$. Clearly, if the couple (B, b) satisfies the conclusion of Proposition 7.1, so does the couple (A, a).

Observation 2. Suppose that a finite number of $\mathcal{L}$-terms t_i, $i = 1, \ldots, l$, have been already transformed to normal crossing divisors with respect to some local analytic coordinates $x = (x_1, \ldots, x_p)$ near a point a. Next consider a finite number of other functions f_j, $j = 1, \ldots, q$, analytic near a. After simultaneous transformation of the functions

f_j and the coordinates x_k to a normal crossing (possibly taking into account divisibility relation), all the terms t_i and functions f_j become normal crossing divisors (along the fiber over the point a).

Observation 3. Let $t(x)$ be a term of the form

$$h(f_1/g_1(x), \ldots, f_k/g_k(x)),$$

where f_i, g_i are analytic functions near a point $a = 0 \in K^p$ and h is the interpretation of a function of the language $\mathcal{L}$ (extended by zero off its natural domain). Consider simultaneous transformation to a normal crossing of the functions f_i, g_i which takes into account linear ordering with respect to divisibility relation and a point b such that $\pi_0(b) = a$. Then we can assume without loss of generality that the quotients $f_i/g_i \circ \pi_0$ are normal crossing divisors at b. Otherwise the term $t \circ \pi_0$ would vanish near b, and then we would pass to a term of lower complexity.

Now it is not difficult to reduce curve selection for the initial couple (A, a) to curve selection for a couple (B, b) where B is a set defined by conditions of the form

$$(v(t_1(y)), \ldots, v(t_r(y))) \in P, \quad (\overline{ac}\,\tau_1(y), \ldots, \overline{ac}\,\tau_s(y)) \in Q,$$

where y are suitable local analytic coordinates near b, each of the t_i, τ_j is either a normal crossing $u(y) \cdot y^\alpha$ at b, or a reciprocal normal crossing $u(y) \cdot 1/y^\alpha$ at b, where $u(b) \neq 0$ and $\alpha \in \mathbb{N}^p$, or vanishes near b.

Since the valuation map and the angular component map composed with a continuous function are locally constant near any point at which this function does not vanish, the conditions which describe the set B near b can be easily expressed in the form

$$(v(y_1), \ldots, v(y_p)) \in \widetilde{P}, \quad (\overline{ac}\,y_1, \ldots, \overline{ac}\,y_p) \in \widetilde{Q},$$

where $\widetilde{P}$ and $\widetilde{Q}$ are definable subsets of Γ^p and $\widetilde{K}^p$, respectively.

We thus achieved the same reduction as in the algebraic case studied in our papers [18, 19]. In this manner, we can repeat verbatim the remaining part of the proof given in those papers. The main ingredient of the further reasoning is Lemma 4.2 ([19, Lemma 6.2]) which, in turn, relies on relative quantifier elimination for ordered abelian groups. $\square$

We conclude the paper with the following comment.

Remark 7.2. We established in the recent paper [20] a non-Archimedean version of the Tietze–Urysohn extension theorem for continuous functions definable over Henselian valued fields of equicharacteristic zero.

It is very plausible that such a version will also hold over Henselian valued fields with analytic structure.

References

[1] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedian Analysis: a systematic approach to rigid analytic geometry*, Grundlehren der math. Wiss. **261**, Springer-Verlag, Berlin, 1984.

[2] G. Cherlin, *Model Theoretic Algebra, Selected Topics*, Lect. Notes Math. **521**, Springer-Verlag, Berlin, 1976.

[3] R. Cluckers, E. Halupczok, *Quantifier elimination in ordered abelian groups*, Confluentes Math. **3** (4) (2011), 587–615.

[4] R. Cluckers, L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. **13** (2011), 1147–1223.

[5] R. Cluckers, L. Lipshitz, *Strictly convergent analytic structures*, J. Eur. Math. Soc. **19** (2017), 107–149.

[6] R. Cluckers, L. Lipshitz, Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Sci. École Norm. Sup. (4) **39** (2006), 535–568.

[7] R. Cluckers, F. Loeser, *b-minimality*, J. Math. Logic **7** (2) (2007), 195–227.

[8] J. Denef, L. van den Dries, *p-adic and real subanalytic sets*, Ann. Math. **128** (1988), 79–138.

[9] L. van den Dries, *Dimension of definable sets, algebraic boundedness and Henselian fields*, Ann. Pure Appl. Logic **45** (1989), 189–209.

[10] L. van den Dries, *Analytic Ax–Kochen–Ershov theorems*, Contemporary Mathematics **131** (1992), 379–392.

[11] L. van den Dries, D. Haskell, D. Macpherson, *Onr dimensional p-adic subanalytic sets*, J. London Math. Soc. **56** (1999), 1–20.

[12] L. van den Dries, A. Macintyre, D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. **140** (1994), 183–205.

[13] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. **79** (1964), 109–326.

[14] L. Lipshitz, *Rigid subanalytic sets*, Amer. J. Math. **115** (1993), 77-108.

[15] L. Lipshitz, Z. Robinson, *Rings of Separated Power Series and Quasi-Affinoid Geometry*, Astérisque **264** (2000).

[16] L. Lipshitz, Z. Robinson, *Uniform properties of rigid subanalytic sets*, Trans. Amer. Math. Soc. **357 (2005), 4349–4377.**

[17] **H. Matsumura, *Commutative Algebra*, Benjamin/Cummings Publ. Co., London, 1980.**

[18] **K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. 23 (2017), 455–495.**

[19] **K.J. Nowak, *A closedness theorem and applications in geometry of rational points over Henselian valued fields*, arXiv:1706.01774 [math.AG] (2017).**

[20] **K.J. Nowak, *A non-Archimedean version of the Tietze–Urysohn theorem over Henselian valued fields*, arXiv:1808.09782 [math.AG] (2018).**

[21] **J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. 399 (1989), 137–172.**

[22] J. Pas, *On the angular component map modulo p*, J. Symbolic Logic 55 (1990), 1125–1129.

[23] J-P. Serre, *Lie Algebras and Lie Groups*, Lect. Notes in Math. 1500, Springer, 2006.

[24] M. Temkin, *Functorial desingularization over $\mathbb{Q}$: boundaries and the embedded case*, Israel J. Math. 224 (2018), 455–504.

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Profesora S. Łojasiewicza 6
30-348 Kraków, Poland
E-mail address: nowak@im.uj.edu.pl