A NON-ARCHIMEDEAN VERSION OF THE TIETZE–URYSOHN THEOREM OVER HENSELIAN VALUED FIELDS

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Abstract. We give a non-Archimedean version of the Tietze–Urysohn theorem for continuous functions definable over Henselian valued fields. Two main ingredients of the proof are resolution of singularities and various consequences of our closedness theorem. Also used is that irreducible simple normal crossing divisors are definable retracts of their clopen neighborhoods.

1. Main result

In the paper, we fix a Henselian valued field K considered in the language $\mathcal{L}$ of Denef–Pas. The ground field K is assumed to be of equicharacteristic zero, not necessarily algebraically closed and with valuation of arbitrary rank. Denote by v, $\Gamma = \Gamma_K$, K°, $K^{\circ\circ}$ and $\widetilde{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. By the K-topology on K^n we mean the topology induced by the valuation v. The word "definable" will usually mean "definable with parameters". Our main purpose is to establish the following non-Archimedean, definable version of the Tietze–Urysohn extension theorem.

Theorem 1.1. *Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed subset A of the projective space $\mathbb{P}^n(K)$ has a continuous extension F to $\mathbb{P}^n(K)$.*

Section 2 contains some preliminary results needed in the proof of Theorem 1.1. It will be given in Section 3, relying on resolution of singularities and on various consequences of our closedness theorem (see [17, 18] and [19] for the analytic non-Archimedean version). Also used is that irreducible simple normal crossing divisors are $\mathcal{L}$-definable retracts of their clopen neighborhoods (Lemma 3.1).

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The classical Tietze–Urysohn extension theorem says that every continuous (and bounded) real valued map on a closed subset of a normal space X can be extended to a continuous (and bounded) function on X. Afterwards the problem of extending maps into metric spaces or locally convex linear spaces was investigated by several mathematicians, i.al. by Hausdorff [12], Dugundji [8], Arens [1] or Michael [16]. Next, Ellis [10] established analogues of their results, concerning the extension of continuous maps defined on closed subsets of zero-dimensional spaces with values in various types of metric spaces. They apply, in particular, to continuous functions from ultranormal spaces into a separable field with non-Archimedean absolute value and to continuous functions from ultraparacompact spaces into an arbitrary complete field with non-Archimedean absolute value. Hence follows his analogue of the Tietze–Urysohn theorem from [9] on extending continuous functions from ultranormal spaces into a locally compact field with non-Archimedean absolute value. Note that ultranormal spaces are precisely those of great inductive dimension zero (cf. [11, Chap. 7]) and that the class of ultraparacompact spaces coincides with that of ultranormal and paracompact spaces.

Finally, let us mention that in this paper we shall not treat the problem of simultaneous extension of continuous functions or, in other words, of the existence of a linear (continuous) extender. This problem, going back to Dugundji [8], was extensively studied by many specialists (see e.g. [6] for references). A non-Archimedean version of the Dugundji extension theorem was given in the paper [13].

2. Preliminary results

We first provide some preliminary results used in the proof of Theorem 1.1. We begin with a presentation of $\mathcal{L}$-definable sets due to van den Dries [7]; see [18, Section 9] for the adaptation to the language of Denef-Pas considered here.

Proposition 2.1. *Every $\mathcal{L}$-definable subset A of K^n is a finite union of intersections of Zariski closed with special open subsets of K^n. A fortiori, A has the following presentation*

$$A = ((V_1 \setminus W_1) \cap G_1) \cup \ldots \cup ((V_s \setminus W_s) \cap G_s), \tag{2.1}$$

where $W_i \subsetneq V_i$ are Zariski closed subsets and G_i are clopen $\mathcal{L}$-definable subsets of K^n, $j = 1, \ldots, s$. □

Remark 2.2. The above proposition is also valid for $\mathcal{L}$-definable subsets A of the projective space $\mathbb{P}^n(K)$ and, more generally, of the K-rational points $V(K)$ of any algebraic variety V.

We still some results concerning the closedness theorem. Below one of its consequences.

Proposition 2.3. *Let X be a closed $\mathcal{L}$-definable subset of $(K^\circ)^n$ and $\{U_i : i = 1, \dots, k\}$ a finite open covering of X. Then there exists a finite clopen $\mathcal{L}$-definable partitioning $\{V_i : i = 1, \dots, k\}$ of X such that $V_i \subset U_i$ for $i = 1, \dots, k$.*

Proof. By induction, it suffices to consider the case $k = 2$. Then

$$(X \setminus U_1) \cap (X \setminus U_2) = \emptyset$$

and it follows from the closedness theorem that

$$(X \setminus U_1) - (X \setminus U_2)$$

is a closed subset of $(K^\circ)^n$. Hence

$$0 \notin (X \setminus U_1) - (X \setminus U_2)$$

and there is an $r \in \Gamma$ such that

$$((X \setminus U_1) + B_n(r)) \cap ((X \setminus U_2) + B_n(r)) = \emptyset,$$

where

$$B_n(r) := \{a \in K^n : v(a) := \min\{v(a_1), \dots, v(a_n)\} > r\}$$

is an n-ball of radius r. The above two sets are open of course, and also closed, again by the closedness theorem. We thus get two clopen subsets

$$X \setminus ((X \setminus U_1) + B_n(r)) \subset U_1 \quad \text{and} \quad X \setminus ((X \setminus U_2) + B_n(r)) \subset U_2$$

which cover X. Putting

$$V_1 := X \setminus ((X \setminus U_1) + B_n(r)) \quad \text{and} \quad V_2 := X \setminus V_1$$

concludes the proof. □

We immediately obtain

Corollary 2.4. *Let X be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^n(K)$ and $\{U_i : i = 1, \dots, k\}$ a finite open covering of X. Then there exists a finite clopen partitioning $\{V_i : i = 1, \dots, k\}$ of X such that $V_i \subset U_i$ for $i = 1, \dots, k$.* □

Let M be a non-singular variety. Recall that a closed subvariety V of X with irreducible components $V_1, \dots, V_s$ has *simple normal crossing* at a point $a \in V$ if, in suitable local coordinates $x_1, \dots, x_n$ at p, each V_j is defined by equations

$$x_{i_1} = \ldots = x_{i_k} = 0, \quad 1 \leq i_1 < \ldots < i_k \leq n,$$

in a Zariski open neighborhood of p. If this holds at every point $p \in V$, we say that V is a *simple normal crossing* subvariety.

Now we are going to combine the above results with resolution of singularities; for references on the latter we refer the reader to e.g. [14, Chap. III].

Proposition 2.5. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^n(K)$ with presentation 2.1 from Proposition 2.1. Without loss of generality, we may assume that V_j, $j = 1, \ldots, s$, are irreducible subvarieties of $\mathbb{P}^n(K)$. Then there exist a composite $\sigma : M \to \mathbb{P}^n(K)$ of blow-ups along smooth centers and a finite clopen partitioning $\{U_i : i = 1, \ldots, k\}$ of M such that*

i) σ is biregular over $\mathbb{P}^n(K) \setminus Z$ for a nowhere dense subvariety Z of the variety $V_1 \cup \ldots \cup V_s$;

ii) each U_i, $i = 1, \ldots, k$, is a chart on M with local coordinates $x_1, \ldots, x_n$;

iii) for each $j = 1, \ldots, s$, the pre-image $X_j := \sigma^{-1}(V_j)$ is a simple normal crossing subvariety of M (including M itself), namely the union of the birational transform $\widetilde{V_j}$ of V_j and a part of the exceptional divisor $E := \sigma^{-1}(Z)$, which is a simple normal crossing divisor. More precisely, on each chart U_i the pre-image $X_j \cap U_i$ is simple normal subvariety of U_i with respect to the local coordinates $x_1, \ldots, x_n$.

Under the circumstances, the pre-image $A^\sigma := \sigma^{-1}(A)$ on each chart U_i is of the form

$$A^\sigma \cap U_i := \sigma^{-1}(A) \cap U_i = (X_1 \cap H_1 \cap U_i) \cup \ldots \cup (X_s \cap H_s \cap U_i) \tag{2.2}$$

with the clopen sets $H_j := \sigma^{-1}(G_j)$.

Proof. Via resolution of singularities, we can of course find a composite $\sigma : M \to \mathbb{P}^n(K)$ of blow-ups along smooth centers which is biregular off the singular locus Z of the subvariety $V_1 \cup \ldots \cup V_s$ and such that the pre-image

$$X := \sigma^{-1}(V_1 \cup \ldots \cup V_s)$$

is a simple normal crossing subvariety of M; namely the union of the birational transform $\widetilde{V_1} \cup \ldots \widetilde{V_s}$ of $V_1 \cup \ldots \cup V_s$ and of the exceptional divisor $E := \sigma^{-1}(Z)$ being a simple normal crossing divisor. Obviously, M may be regarded as a subvariety of a projective space $\mathbb{P}^N(K)$ for some $N \in \mathbb{N}$. Next, take a finite Zariski open covering of charts $\{U_i : i = 1, \ldots, k\}$ with local coordinates $x_1, \ldots, x_n$ with respect to which $X \cap U_i$ is a simple normal crossing subvariety. Now, by Corollary 2.4, we can replace this covering with a clopen $\mathcal{L}$-definable partitioning. Hence the conclusion of the proposition follows immediately. □

Finally, we recall yet another direct consequence of the closedness theorem, namely a descent property whereby one can apply resolution of singularities in the general non-Archimedean case in much the same way as over locally compact ground fields. It was established in our papers [17, 18] (and inspired by the joint paper [15]).

Proposition 2.6. *Consider a smooth K-variety X and the blow-up $\sigma : Y \longrightarrow X$ along a smooth center. Let D be an $\mathcal{L}$-definable subset of the set $X(K)$ of K-rational points of K. Then the restriction*

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map. Therefore every continuous $\mathcal{L}$-definable function

$$G : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$$

that is constant on the fibers of the blow-up σ descends to a unique continuous $\mathcal{L}$-definable function $F : D \longrightarrow K$. □

3. Proof of the extension theorem

Now, having disposed of the preliminary results of Section 2, we can prove the non-Archimedean version of the Tietze–Urysohn theorem.

Proof of Theorem 1.1. Keep the notation of Proposition 2.5 and proceed with induction on the dimension $\dim A = \dim(V_1 \cup \ldots \cup V_s)$. We adopt, in particular, presentation 2.2 on each clopen chart U_i.

Since $\dim Z < \dim A$, the induction hypothesis allows us to assume that the function f vanishes on $Z \cap A$. We shall have established the theorem once we extend the function $g := f^\sigma = f \circ \sigma$ to a continuous $\mathcal{L}$-definable function G on M which vanishes on Z. Indeed, σ is biregular over $\mathbb{P}^n(K) \setminus Z$ and G vanishes on Z. Therefore G is constant on the fibers of σ. Hence and by the descent property (Proposition 2.6), G descends to a unique continuous $\mathcal{L}$-definable function F on $\mathbb{P}^n(K)$. Clearly, F is an extension of f we are looking for.

Since $\{U_i : i = 1, \ldots, k\}$ is a clopen $\mathcal{L}$-definable partitioning of M, it suffices to find a global extension of g on each fixed chart $U := U_i$ with local coordinates $x_1, \ldots, x_n$. To this end we need the following

Lemma 3.1. *Every divisor $H_i := \{x \in U : x_i = 0\}$ is a retract of a clopen neighborhood Ω_i of H_i in U, which is retracted by the $\mathcal{L}$-definable map*

$$\omega_i : \Omega_i \to H_i, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n),$$

induced by the coordinate map.

Proof. Note that a polynomial map $f : K^N \to K^N$, $f(a) = b$, with coefficients in K° and non-zero Jacobian $e(a) \neq 0$ at a, is an open embedding of the N-ball $B_N(a, v(e(a)))$ onto the N-ball $B_n(b, 2 \cdot v(e(a)))$ (cf. [18, Proposition 2.4]); here

$$B_N(a, r) := \{x \in K^N : v(x - a) > r\}$$

is the ball with center a and radius r.

Similarly, for an implicite function $y = f(x)$ given by a finite number of polynomial equations with coefficients in K° and for each point (a, b) of its graph, f is uniquely determined in a polydisk

$$\{(x, y) : \ v(x) > 2 \cdot v(e(a, b)), \ v(y) > v(e(a, b))\},$$

where $e(a, b) \neq 0$ is the suitable minor of the Jacobian matrix of those equations (cf. [18, Proposition 2.5]). Apply these facts to the suitable equations of the non-singular subvariety M of $\mathbb{P}^N(K)$ and to the coordinate map $\phi = (x_1, \ldots, x_n)$ on U. Since the suitable minors do not vanish on U, it follows from the closedness theorem that the valuation of those minors are uniformly bounded from above on U. Hence there is an $r \in \Gamma$ such that for each point $a \in U$ the coordinate map ϕ is injective on $U \cap B_N(a, r)$; here balls are with respect to the ambient projective space $\mathbb{P}^N(K)$.

Now we show that for each $i = 1, \ldots, n$, say $i = n$, there is a $\rho_i \in \Gamma$, $\rho \geq r$, such that ϕ is injective on the ρ-hull

$$H_i(\rho) := (H_i + B_N(0, \rho)) \cap U$$

of H_i in U. Otherwise there would exist distinct points $a, b \in U \subset \mathbb{P}^N(K)$, $a \neq b$, with the n-th coordinate x_n arbitrarily close to 0 and such that

$$\phi(a) = (x_1(a), \ldots, x_n(a)) = \phi(b) = (x_1(b), \ldots, x_n(b)).$$

Consider the set

$$\Sigma := \{(a, b) \in U \times U : \ a \neq b, \ \phi(a) = \phi(b), \ x_n(a) = x_n(b) \neq 0\}$$

and the map

$$\psi : U \times U \to K^2, \quad \psi(a, b) = (x_n(a), x_n(b)).$$

Then $(0, 0)$ would be an accumulation point of the image $\psi(\Sigma)$. Hence and by the closedness theorem, $(0, 0) = \psi(c, d)$ for an accumulation point (c, d) of the set Σ. Then $\phi(c) = \phi(d)$, $x_n(c) = x_n(d) = 0$ and thus $c, d \in H_n$. Since the divisor H_n has no self-intersections on the chart U, we get $c = d$. Therefore every neighborhood of the point (c, c) has points of the set Σ, which contradicts the injectivity of the coordinate map ϕ on $U \cap B_N(c, r)$.

Take $\rho := \max\{\rho_1, \dots, \rho_n\}$. Then every map ω_i, $i = 1, \dots, n$, given by the formula

$$\omega_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$

is a well defined map on the ρ-hull

$$\Omega_i := (H_i + B_N(0, r)) \cap U$$

of the divisor H_i in U. Clearly ω_i makes H_i an $\mathcal{L}$-definable retract of Ω_i which is — again by the closedness theorem — a clopen subset of U, as asserted. □

Repeated application of the above lemma yields the following

Corollary 3.2. *The conclusion of Lemma 3.1 holds also for the coordinate subvarieties*

$$\{x \in U : x_{i_1} = \ldots = x_{i_p} = 0\}, \quad 1 \leq i_1 < \ldots < i_p \leq n,$$

of the chart U. □

At this stage we are able to extend the function g to the chart U. Since every coordinate subvariety is a retract of its clopen $\mathcal{L}$-definable neighborhood in U, our extension process is similar to global extending of continuous functions from a union of coordinate subspaces of K^n. It is done by induction on the dimension $\dim M$ (i.e. the number of local coordinates), and consists in correcting step by step, taking into account the restriction of g to the successive birational transforms $\widetilde{V}_j$. Each such restriction is extended first to $\widetilde{V}_j$ by the induction hypothesis, and next to U through the retracting map and by putting zero outside its clopen domain. This reasoning is rather routine and we finish the proof, leaving the details to the reader. □

We conclude the paper with the following comment.

Remark 3.3. It is very plausible that such a non-Archimedean version of the Tietze–Urysohn theorem will also hold over Henselian valued fields with analytic structure (whose theory was developed in the papers [3, 4, 5]). We are currently working on this problem. A more general context of tame non-Archimedean geometry was recently investigated in the paper [2], devoted to the Lipschitz structure of definable sets as well. It seems, however, that the analogues problem of extending Lipschitz continuous functions is much more complicated and requires an essentially new approach.

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