## NÉRON DESINGULARIZATION OF EXTENSIONS OF VALUATION RINGS with an appendix by kestutis česnavičius

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ABSTRACT. Zariski's local uniformization, a weak form of resolution of singularities, implies that every valuation ring containing  $\mathbf{Q}$  is a filtered direct limit of smooth  $\mathbf{Q}$ -algebras. Given an immediate extension of valuation rings  $V \subset V'$ containing  $\mathbf{Q}$  we show that V' is a filtered direct limit of smooth V-algebras. This corrects a paper of us [23] where we thought that we may reduce to the case when the value groups are finitely generated. For this correction we use an infinite tower of ultrapowers construction that rests on results from model theory.

### 1. A VERSION OF LOCAL UNIFORMIZATION

Zariski proved in characteristic 0 in [34], that any integral algebraic variety X equipped with a dominant morphism  $v : \operatorname{Spec}(V) \to X$  from a valuation ring V can be "desingularized along V": there exists a proper birational map  $\tilde{X} \to X$  for which the lift  $\tilde{v} : \operatorname{Spec}(V) \to \tilde{X}$  of v supplied by the valuative criterion of properness factors through the regular locus of  $\tilde{X}$ . This implies the following theorem.

**Theorem 1.** (Zariski) Every valuation ring V containing a field K of characteristic zero is a filtered direct limit of smooth K-subalgebras of V (in particular they are regular rings).

A smooth algebra is here finitely presented. A ring map  $A \to A'$  is *ind-smooth* if A' is a filtered direct limit of smooth A-algebras. A filtered direct limit (in other words a filtered colimit) is a limit indexed by a small category that is filtered (see [32, 002V] or [32, 04AX]). A filtered union is a filtered direct limit in which all objects are subobjects of the final colimit, so that in particular all the transition arrows are monomorphisms. The above theorem says that  $K \to V$  is ind-smooth. Actually, Zariski proved that V is a filtered union of smooth K-subalgebras of V. One goal of this paper is to show a weaker statement:

**Theorem 2.** Let  $V \subset V'$  be an immediate extension of valuation rings containing **Q**. Then  $V \subset V'$  is ind-smooth. If dim  $V = \dim V' = 1$  and the residue field extension of  $V \subset V'$  is trivial then  $V \subset V'$  is ind-smooth if and only if the value group extension of  $V \subset V'$  is trivial.

The proof follows from Theorem 21, Lemma 29 and Proposition 38. The above result was stated by mistake in [23] in a more general case. A different proof of Theorem 1 is given in Theorem 35 and the method use some facts from model theory described below. **3.** The method of proof. To achieve the desingularization claimed in Theorem 2. we replace the initial V by the limit V of a certain countable tower of iterated ultrapowers of V, constructed in such a way that V would, in turn, be an immediate extension of a filtered increasing union of valuation rings for which one knows local uniformization. To then conclude, we argue that large immediate extensions and ultrapowers interact well with desingularization.

The techniques we use include extensions of steps of the General Néron desingularization, notably, Lemma 7 that is also key for reductions to complete rank 1 cases. In the purely transcendental case, Kaplansky's classification [12] of Ostrowski's pseudo-convergent sequences plays an important role.

The utility of desingularizing immediate extensions is evident already in the case when V' is complete of rank 1 with a finitely generated value group  $\Gamma'$ . Such a V' has a coefficient field k, so, by choosing a presentation  $\Gamma' \cong \mathbf{Z} \operatorname{val}(x_1) \oplus \cdots \oplus \mathbf{Z} \operatorname{val}(x_n)$ , one obtains the immediate extension  $V' \cap k(x_1, \ldots, x_n) \subset V'$ . To show that  $V = k \subset V'$ is ind-smooth, it remains to observe that a local uniformization of  $V' \cap k(x_1, \ldots, x_n)$ may be constructed using Perron's algorithm in the style of Zariski.

The goal of the tower of ultrapowers argument given in the Appendix is to overcome the obstacle that in general  $\Gamma$  may not be finitely generated and there may not even be a group section  $s: \Gamma \to K^*$  to val:  $K^* \to \Gamma$  (roughly, such an s suffices). Nevertheless, s can always be arranged for any finitely generated submonoid of  $\Gamma$ , and the idea is to then use the following fact from model theory: for a system of equations whose finite subsystems have solutions in V, the entire system has a solution in a well-chosen ultrapower of V (see the Appendix).

This fact, which rests on the Keisler–Kunen theorem about the existence of good ultrafilters, permits us to obtain s at the expense of passing to an ultrapower. However, such a passage replaces  $\Gamma$  by its corresponding ultrapower  $\Gamma^*$  and, in order to extend s to this  $\Gamma^*$ , we then need another ultrapower and some model-theoretic facts about algebraic compactness that ensure that  $\Gamma \to \Gamma^*$  be a split injection. Even though the new ultrapower again enlarges the value group, by repeating the construction countably many times, in the limit we find our final s and can conclude.

We should mention that the proof of the main part of Theorem 2 uses only Lemma 7 and Corollary 19. The last sentence from Theorem 2 needs the method explained above together with some facts from André homology.

We owe thanks to Kestutis Cesnavičius especially for the Appendix, but also for many ideas and his great help on the presentation of the paper. Also we owe thanks to the referees who pointed out several mistakes in a previous version of the paper especially in the former Proposition 20.

### 2. A reduction to the case of complete valuation rings of rank 1

We begin by reviewing the following class of generators of the singular ideal.

For a finitely presented ring map  $A \to B$ , an element  $b \in B$  is standard over A if there exists a presentation  $B \cong A[X_1, \ldots, X_m]/I$  and  $f_1, \ldots, f_r \in I$  with  $r \leq m$ such that b = b'b'' with  $b' = \det((\partial f_i/\partial X_j))_{1 \le i,j \le r} \in A[X_1,\ldots,X_m]$  and a  $\overline{b''} \in A[X_1,\ldots,X_m]$   $A[X_1, \ldots, X_m]$  that kills  $I/(f_1, \ldots, f_r)$  (our standard element is a special power of the standard element from [33, Definition, page 9] given in the particular case of the valuation rings). Any multiple of an element b standard over A is standard over A. The definition is compatible with base change: more precisely, for any morphism  $A \to A'$ , elements of B standard over A map to elements of  $B \otimes_A A'$  standard over A'.

**Lemma 4.** For a finitely presented ring map  $A \to B$ , the loci of vanishing of standard over A elements of B cut out the locus of non-smoothness of  $\text{Spec}(B) \to \text{Spec}(A)$ . The radical of the ideal generated by the elements of B standard over A is  $H_{B/A}$ .

*Proof.* The argument is standard (compare with [8], [33, 4.3]) but we include it due to the lack of a convenient reference.

If  $b \in B$  is standard over A, then  $B_b$  is the localization of the standard smooth A-algebra  $(A[X_1, \ldots, X_n]/(f_1, \ldots, f_r))_{\det((\partial f_i/\partial X_j))_{1 \leq i, j \leq r}}$  (see [32, 00T8]), so is A-smooth. Conversely, if  $B_b$  is the coordinate ring of a smooth neighborhood of a fixed prime  $\mathfrak{p} \subset B$ , then we may choose a presentation  $B[X_1, \ldots, X_m]/I$  and  $f_1, \ldots, f_r \in I$  such that, at the expense of localizing at  $\mathfrak{p}$  further. The module  $(I/I^2)_b$  is a free  $B_b$ -module with a basis given by the classes of  $f_1, \ldots, f_r$  and  $(I/I^2)_b \to \bigoplus_{i=1}^m B_b \cdot dX_i$  is a split injection such that  $dX_{r+1}, \ldots, dX_m$  maps to a basis for the quotient. The first condition and Nakayama's lemma [32, 00DV] then supply an  $i \in I$  with  $(1+i/b^n)I_b \subset (f_1, \ldots, f_r)_b$  for some n > 0. It follows that  $b^N(b^n + i)$  for some N > 0 kills  $I/(f_1, \ldots, f_n)$  and maps to a power of b in B. The second condition implies that  $b' = \det(\partial f_i/\partial X_j))_{1 \leq i, j \leq r}$  is a unit in  $B_b$ , so that b' divides some power of b in B. In conclusion, some power of b is standard over A, as desired.

To stress the relevance of the desingularization lemma 7, we recall the following well-known lemma (see [33, (1.5)] or [32, 07C3]) and definitions, which will be crucial throughout this paper.

**Lemma 5.** For a ring R and a set S of finitely presented R-algebras, an R-algebra R' is a filtered direct limit of elements of S if and only if every R-morphism  $B \to R'$  with B a finitely presented R-algebra factors as  $B \to S \to R'$  for some  $S \in S$ .

By [26, 1.8] (see also [32, 07GC]), a map of Noetherian rings is ind-smooth if and only if A' is A-flat and has geometrically regular A-fibers. In particular, a field extension K'/K is ind-smooth if and only if it is separable.

Concretely, by Lemma 5, a ring map  $A \to A'$  is ind-smooth if and only if every factorization  $A \to B \to A'$  with B finitely presented over A can be refined to  $A \to B \to S \to A'$  with S smooth (or merely ind-smooth) over A. Thus, a finite product or a filtered direct limit of ind-smooth A-algebras is ind-smooth. Evidently, ind-smooth morphisms are stable under base change. They are also stable under compositions, in fact, we have the following slightly finer criterion.

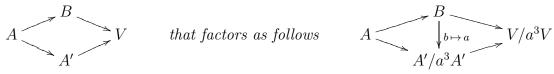
**Lemma 6.** For an ind-smooth map  $A \to A'$  and a map  $A' \to A''$  such that for every factorization  $A \to B \to A''$  with B finitely presented over A the induced factorization  $A' \to A' \otimes_A B \to A''$  can be refined to  $A' \to A' \otimes_A B \to S' \to A''$  for some smooth

A'-algebra S', the map  $A \to A''$  is ind-smooth. In particular, the composition of ind-smooth maps is ind-smooth.

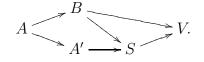
*Proof.* It suffices to argue that the map  $A \to A' \to S'$  is ind-smooth. For this, we express A' as a filtered direct limit of smooth A-algebras  $S_i$ , note that S' descends to a smooth  $S_i$ -algebra  $S'_i$  for some i, and conclude that S' is then the filtered direct limit of the smooth A-algebras  $S'_j = S_j \otimes_{S_i} S'_i$  with  $j \ge i$ .

The following lemma originates in [24, (7.1)] and its variants have appeared, for instance, in [33, 18.1], [26, 7.2], [32, 07CT], [15, Proposition 3], and [28, Proposition 5]. The version below differs in two aspects: we do not assume Noetherianness and do not require the elements a or b to come from the base ring A. The latter improvement is particularly convenient for our purposes—we recall that in the General Néron desingularization arranging for b to come from A is an additional step before one can apply the desingularization lemma (compare with, for instance, [32, 07F4]).

Lemma 7. For a commutative diagram of ring morphisms



with B finitely presented over A, a  $b \in B$  that is standard over A, and a nonzerodivisor  $a \in A'$  that maps to a nonzerodivisor in V that lies in every maximal ideal of V, there is a smooth A'-algebra S such that the original diagram factors as follows:



*Proof.* The finitely presented A'-algebra  $B \otimes_A A'$  comes equipped with a morphism to V and a retraction modulo  $b^3$  to  $A'/a^3A'$  that sends b to a. Moreover, the image of b in  $B \otimes_A A'$  is standard over A'. Thus, by replacing A by A' and B by  $B \otimes_A A'$ , we reduce to the case A = A'.

Since the images of a and b in V agree modulo  $a^3V$ , these images are unit multiples of each other. We write

$$B = A[X_1, \dots, X_m]/I$$
 and  $f_1, \dots, f_r \in I$ 

and choose  $b' = \det((\partial f_i/\partial X_j)_{1 \le i, j \le r}) \in A[X_1, \ldots, X_m]$  and a  $b'' \in A[X_1, \ldots, X_m]$ that kills  $I/(f_1, \ldots, f_r)$  with b = b'b'' in B. In these coordinates, we fix a map

commute, so that  $\tilde{f} \in a^3 A$  for every  $f \in I$ . In particular, the assumption  $b \mapsto a$  gives  $\widetilde{b'b''} \equiv a \mod a^3 A$ . It follows that

$$\widetilde{b'b''} = au$$
 for some  $u \in 1 + a^2 A$ ,

so that, in particular, u maps to a unit in V.

We consider the  $m \times m$  matrix  $\Delta$  given by

$$\begin{pmatrix} \partial f_1 / \partial X_1 & \partial f_1 / \partial X_2 & \dots & \partial f_1 / \partial X_r & \partial f_1 / \partial X_{r+1} & \partial f_1 / \partial X_{r+2} & \dots & \partial f_1 / \partial X_m \\ \partial f_2 / \partial X_1 & \partial f_2 / \partial X_2 & \dots & \partial f_2 / \partial X_r & \partial f_2 / \partial X_{r+1} & \partial f_2 / \partial X_{r+2} & \dots & \partial f_2 / \partial X_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial f_r / \partial X_1 & \partial f_r / \partial X_2 & \dots & \partial f_r / \partial X_r & \partial f_r / \partial X_{r+1} & \partial f_r / \partial X_{r+2} & \dots & \partial f_r / \partial X_m \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

that satisfies  $det(\Delta) = b'$ . We let  $Ad(\Delta)$  denote the adjoint matrix, so that

$$Ad(\Delta) \cdot \Delta = \Delta \cdot Ad(\Delta) = b' \cdot Id_{m \times m}.$$

We let  $x_i$  and  $x'_i$  be the images in V of  $X_i$  and  $\tilde{X}_i$ , respectively, so that, by construction,  $x_i - x'_i \in a^3 V$ . Moreover, a is a nonzerodivisor in V and there we have that

$$\begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \tilde{\Delta} \quad \begin{pmatrix} (x_1 - x'_1)/a^2 \\ \vdots \\ (x_m - x'_m)/a^2 \end{pmatrix} \text{ satisfies } t_i \in aV \text{ and } ab\widetilde{Ad(\Delta)} \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} = u \begin{pmatrix} x_1 - x'_1 \\ \vdots \\ x_m - x'_m \end{pmatrix}.$$

We let  $T_1, \ldots, T_m$  be new variables and set

$$\binom{h_1}{\vdots}_{h_m} = u \binom{X_1 - \tilde{X}_1}{\vdots}_{X_m - \tilde{X}_m} - ab'' \widetilde{Ad(\Delta)} \binom{T_1}{\vdots}_{T_m}, \text{ so that } h_i \in A[X_1, \dots, X_m, T_1, \dots, T_m].$$

By construction, if we map  $T_i$  to  $t_i$  in V, then the  $h_i$  map to 0, so we obtain the map

$$\varphi: A_u[X_1, \dots, X_m, T_1, \dots, T_m]/(h_1, \dots, h_m) \to V$$
 given by  $X_i \mapsto x_i, T_i \mapsto t_i.$ 

Since we have inverted u, the source of this map may be identified with  $A_u[T_1, \ldots, T_m]$ . To proceed further, we will use Taylor's formula to express each  $f_i$  in terms of this identification.

By Taylor's formula, for any ring R, any section  $R[X_1, \ldots, X_m] \xrightarrow{f \mapsto \tilde{f}} R$ , and any  $f \in R[X_1, \ldots, X_m]$ ,

$$f - \tilde{f} - \sum_{i=1}^{m} (\partial f/\partial X_i) (X_i - \tilde{X}_i) \in (X_1 - \tilde{X}_1, \dots, X_m - \tilde{X}_m)^2 \subset R[X_1, \dots, X_m].$$

In particular, by applying this with  $R = A[T_1, \ldots, T_m]$  and letting d denote the maximal total degree of any monomial that appears in some  $f_i$ , we obtain

$$Q_i \in (T_1, \dots, T_m)^2 \subset A[T_1, \dots, T_m]$$

for which

$$u^{d}f_{i} - u^{d}\tilde{f}_{i} \equiv u^{d-1}a\tilde{b''} \quad ((\partial \widetilde{f_{i}}/\partial X_{1}), \dots, (\partial \widetilde{f_{i}}/\partial X_{m}))\widetilde{Ad(\Delta)} \begin{pmatrix} T_{1} \\ \vdots \\ T_{m} \end{pmatrix} + a^{2}Q_{i} \mod (h_{1}, \dots, h_{m}) \equiv u^{d-1}a\tilde{b''}\tilde{b}T_{i} + a^{2}Q_{i} \equiv a^{2}u^{d}T_{i} + a^{2}Q_{i} \mod (h_{1}, \dots, h_{m})$$

We have  $\tilde{f}_i = a^2 b_i$  for some  $b_i \in aA$ , and for  $1 \leq i \leq r$  we set

$$g_i = u^d b_i + u^d T_i + Q_i \in A[T_1, \dots, T_m], \text{ so that } a^2 g_i \equiv u^d f_i \mod (h_1, \dots, h_m).$$

This achieves the promised expression of  $f_i$  in terms of the identification of the source of  $\varphi$  with  $A[T_1, \ldots, T_m]$  and simultaneously shows that each  $g_i$  vanishes in V, so that  $\varphi$  induces a map

$$\varphi \colon A_u[X_1, \dots, X_m, T_1, \dots, T_m] / (I, g_1, \dots, g_r, h_1, \dots, h_m) \to V.$$

In  $A[X_1, \ldots, X_m]$  the element b'b'' - au lies in the ideal  $(X_1 - \tilde{X}_1, \ldots, X_m - \tilde{X}_m)$ , so in the quotient  $A_u[X_1, \ldots, X_m, T_1, \ldots, T_m]/(h_1, \ldots, h_m)$  it lies in the ideal  $a(T_1, \ldots, T_m)$ . It then follows from the definition of b'' and the fact that after inverting u and modulo  $(h_1, \ldots, h_m)$  the ideal  $(g_1, \ldots, g_r)$  contains  $(f_1, \ldots, f_r)$  that some element from the coset  $a(u + (T_1, \ldots, T_m))$  kills the image of I in

$$A_u[X_1,\ldots,X_m,T_1,\ldots,T_m]/(g_1,\ldots,g_r,h_1,\ldots,h_m).$$

Setting  $u' = \det((\partial g_i/\partial T_j)_{1 \le i,j \le r})$ , we deduce that the same then holds in the localization

$$(A_u[X_1, \dots, X_m, T_1, \dots, T_m]/(g_1, \dots, g_r, h_1, \dots, h_m))_{u'} \cong$$
  
 $(A_u[T_1, \dots, T_m]/(g_1, \dots, g_r))_{u'}.$ 

However, the latter is smooth over A, to the effect that a is a nonzerodivisor in the ring above. It follows that even some element  $u'' \in u + (T_1, \ldots, T_m)$  kills the image of I in the ring above. By construction, both u' and u'' map to units in V and  $\varphi$  factors through the A-smooth algebra

$$S = (A_u[X_1, \dots, X_m, T_1, \dots, T_m] / (g_1, \dots, g_r, h_1, \dots, h_m))_{u'u''}.$$

In some situations, when applying Lemma 7 we will not initially have a map  $A' \to V$ . The following lifting lemma will help to bypass this obstacle. Its key novel aspect is that the elements s, s', and v need not come from the base ring A (compare with [24, (8.1], [33, (17.1)], or [32, 07CP]).

**Lemma 8.** For a ring morphism  $A \to V$  with V local, a smooth A-algebra S, an element  $s \in S$ , a nonunit  $v \in V$ , and a factorization

$$A \to S \xrightarrow{s \mapsto v} V/v^n V$$
 for some  $n \ge 2$ ,

there are a smooth A-algebra S', an element  $s' \in S'$ , and factorizations

 $A \to S' \xrightarrow{s' \mapsto uv} V \quad with \ u \in V^* \quad and \ A \to S \xrightarrow{s \mapsto s'} S'/s'^n S' \to V/v^n V;$ 

if s is the image of an element  $a \in A$ , then one may choose s' = a.

*Proof.* Due to the local structure of smooth and étale morphisms [32, 054L,00UE], by localizing S around the preimage of the maximal ideal of V, we may assume that S is standard étale over a polynomial A-algebra, that is, that

$$S \cong (A[X_1, \ldots, X_d, Y]/(f))_{g \cdot \partial f/\partial Y}$$
 for some  $f, g \in A[X_1, \ldots, X_d, Y]$  with f monic in Y

For a suitable  $n \in \mathbf{N}$ , some unit multiple of  $s \in S$  of the form  $(g \cdot (\partial f/\partial Y))^N \cdot s$  lifts to an  $\tilde{s} \in A[X_1, \ldots, X_d, Y]$ . Letting  $x_1, \ldots, x_d, y$  be some lifts to V of the images of  $X_1, \ldots, X_d, Y$  in  $V/v^n V$ , we find that the A-morphism

 $A[X_1,\ldots,X_d,Y] \xrightarrow{X_i \mapsto x_i, Y \mapsto y} V$ 

maps  $\tilde{s}$  to a unit multiple of v (as may be checked modulo  $v^n$ ), so it maps f to  $\tilde{s}^n w$  for some  $w \in V$ . Thus, we obtain the A-morphism

$$S' = (A[X_1, \dots, X_d, Y, W])_{g \cdot \partial f / \partial Y \cdot \partial (f - \tilde{s}^n W) / \partial Y} / (f - \tilde{s}^n W) \xrightarrow{W \mapsto w} V.$$

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By construction, S' is A-smooth and, setting  $s' = \tilde{s} \cdot (g \cdot \partial f / \partial Y)^{-N}$  in S' we have the identification

$$S'/s'^n S' \cong (A[X_1, \dots, X_d, Y, W])_{g \cdot \partial f/\partial Y \cdot \partial (f - \tilde{s}^n W)/\partial Y}/(f, s'^n) \cong ((S/s^n S)[W])_{\partial (f - \tilde{s}^n W)/\partial Y}$$

with s' corresponding to s and compatibly with the maps to  $V/v^n V$ . The main part of the claim follows, and for the remaining assertion about a note that if s is the image of an  $a \in A$ , then we may choose N = 0 and  $\tilde{s} = s' = a$  above.

For desingularizing valuation rings, the above lemmas will be useful in several different ways. We illustrate this right away with the following results that facilitate passage to completions.

**Proposition 9.** For a ring A, a dense extension of valuation rings (see Section 3)  $V \subset V'$ , K the fraction field of V, a ring morphism  $A \to V$ , a finitely presented A-algebra B, and maps

 $A \rightarrow B \rightarrow V$  such that  $B \rightarrow K$  factors through some A-smooth localization of B

suppose that there exist a smooth A-algebra S' and a factorization  $A \to B \to S' \to V'$ . Then there exist a smooth A-algebra S and a factorization  $A \to B \to S \to V$ . In particular, there exist a smooth A-algebra S and a factorization  $A \to B \to S \to V$  if there exist a smooth A-algebra  $\hat{S}$  and a factorization  $A \to B \to S \to V$  if there exist a smooth A-algebra  $\hat{S}$  and a factorization  $A \to B \to \hat{S} \to \hat{V}$ ,  $\hat{V}$  being the completion of V.

Proof. By hypothesis  $H_{B/A}V \neq 0$  and let  $b \in H_{B/A}V$ ,  $b \neq 0$ . Let  $B \cong A[Y]/I$ ,  $Y = (Y_1 \dots, Y_m)$ , I being a finitely generated ideal. Changing A by A[Z], B by B[Z], the map  $B[Z] \to V$  being given by  $Z \to b$ , we may assume that b comes in fact from A. Indeed, if S is given for B, let us say as in (1) then S[Z] could be taken for B[Z] as in (1). Similarly as in [15, Lemma 4] we may assume that for some polynomials  $f = (f_1, \dots, f_r)$  from I, we have  $b \in NMB$  for some  $N \in ((f) : I)$  and a  $r \times r$ -minor M of the Jacobian matrix  $(\partial f_i/\partial Y_j)$ . Thus we may assume b is standard for B over V, which is necessary later to apply Lemma 7. Note that the composite map  $B \to V \to V/b^3V \cong V'/b^3V'$  factors through a smooth  $A/b^3A$ -algebra. By Lemma 7  $B \to V$  factors through a smooth A-algebra as well.

Indeed, since  $V/b^3V \cong V'/b^3V'$ , Lemma 8 supplies a smooth A-algebra  $S'_0$ , an  $s' \in S'$ , a factorization  $A \to S'_0 \to V$  that sends s' to a unit multiple of b in V, and

a factorization

$$A \xrightarrow{B/b^3 B \to S'/b^3 S'}_{S_0'/s'^3 S_0'} V/b^3 V.$$

The local ring of  $S'_0$  at the preimage of the maximal ideal of V is a domain (see [32, 033C], and s' is nonzero in this local ring, so it is a nonzerodivisor there. Thus, Lemma 7 applies and supplies a smooth  $S'_0$ -algebra S'' with a factorization  $A \to B \to S'' \to V$ . Note that S'' is a smooth A-algebra. 

To draw further consequences, we will use the following well-known result of Nagata (see [20, Theorem 4] or [32, 053E]).

**Lemma 10.** Any finitely generated, flat (equivalently, torsion free) algebra over a valuation ring is finitely presented.

**Corollary 11.** For a local injection  $V \to V'$  of valuation rings that induces a separable extension K'/K of fraction fields, if the map  $V \to \tilde{V'}$  is ind-smooth,  $\tilde{V'}$ being the completion of V', then so is  $V \to V'$ .

Proof. The separability assumption and Lemma 10 imply that Proposition 9 applies to every finite type V-subalgebra  $B \subset V'$ : a limit argument reduces to showing that the smooth locus of B over V is nonempty, which follows from the separability of Frac(B)/K thanks to [10, (6.7.4.1) in IV2] and [10, (17.5.1) in IV4]. It then remains to apply Lemma 5. 

The work above allows us to relate certain "formal desingularization" extensions of valuation rings studied in [25, section 6] to "weak desingularization" (that is, ind-smooth) extensions as follows.

**Proposition 12.** Fix a local injection  $V \rightarrow V'$  of valuation rings with fraction fields  $K \to K'$  such that val $(V \setminus \{0\})$  is cofinal in val $(V' \setminus \{0\})$  and for each  $0 \neq v \in V$ the map  $V/vV \rightarrow V'/vV'$  is ind-smooth, a finitely presented A-algebra B, and maps  $V \to B \to V'$  such that the map  $B \to K'$  factors through some V-smooth localization of B. There is a smooth V-algebra S and a factorization  $V \to B \to S \to V'$ . If, in addition, K'/K is separable, then V' is ind-smooth over V.

*Proof.* In the case when K'/K is separable, B could be any finite type V-subalgebra of V', which is finitely presented by Lemma 10. So the last assertion follows from the rest and Lemma 5. For the assertion about B, we use Lemma 4 to choose an element  $b \in B$  standard over V that does not die in V'. We assume that b is not a unit in V' (or else we may set  $S = B_b$ ) and we choose a  $0 \neq v \in V$  with val(v) > val(b), where val(b) denotes the valuation of b considered in V'. By our assumptions, there are a smooth  $V/v^3V$ -algebra  $\bar{S}$ , an  $s \in \bar{S}$ , and a factorization  $V \to \overline{B} \to \overline{S} \xrightarrow{s \to v} V'/v^3 V'$  such that  $b \mid s$  in  $\overline{S}$ . Thus, since  $\overline{S}$  lifts to a smooth V-algebra (see [3, (1.3.1)] or [32, 07M8]), Lemma 8 supplies a smooth V-algebra S', an  $s' \in S'$ , a factorization  $V \to S' \to V'$  that sends s' to a unit multiple of v, and a factorization  $V \to B \to \overline{S} \xrightarrow{s \mapsto s'} S'/s'^3 S' \to V'/v^3 V'$ . Since  $b \mid s'$  in  $S'/s'^3 S'$ , by replacing S' by its localization by an element of  $1 + s'^2 S'$  if necessary we may ensure that  $a \mid s'$  in S' for some lift  $a \in S'$  of  $b \in S'/s'^3 S'$ . Then we have a factorization

$$V \to B \xrightarrow{b \mapsto a} S'/a^3 S' \to V'/a^3 V'.$$

As in the proof of Lemma 9, Lemma 7 then supplies a smooth V-algebra S.  $\Box$ 

The following localization lemma, a variant of [27, Lemma 2], [33, (12.2)], or [32, 07F9], will permit us to localize our valuation rings when arguing their ind-smoothness.

**Lemma 13.** For ring maps  $A \to B \to V$  with B of finite type over A, a prime  $\mathfrak{P} \subset V$  with preimage  $\mathfrak{p} \subset A$ , and a factorization  $A \to B \to S' \to V_{\mathfrak{P}}$  for a finitely presented  $A_{\mathfrak{p}}$ -algebra S', there are a finitely presented A-algebra S, an  $s \in S$  with  $S_s \otimes_A A_{\mathfrak{p}} \simeq S'[X, X^{-1}]$ , and a factorization

$$A \to B \to S \to V$$
 such that  $S \to V_{\mathfrak{P}}$  factors as  $S \to S_s \otimes_A A_{\mathfrak{p}} \to V_{\mathfrak{P}}$ .

*Proof.* Following the argument of [33, (12.2)], we choose a presentation

$$S' \simeq (B \otimes_A A_{\mathfrak{p}})[X_1, \dots, X_n] / (f_1(X_1, \dots, X_n), \dots, f_m(X_1, \dots, X_n))$$

(see [32, 00F4]) in which the polynomials  $f_i$  have coefficients in B, and we set

$$S = B[X_0, X_1, \dots, X_n] / (X_0^N f_1(X_1/X_0, \dots, X_n/X_0), \dots, X_0^N f_m(X_1/X_0, \dots, X_n/X_0))$$

for a large enough N > 0 for which each  $X_0^N f_i(X_1/X_0, \ldots, X_n/X_0)$  is a (necessarily homogeneous) polynomial in  $X_0, X_1, \ldots, X_n$  of positive degree and coefficients in B. We set  $s = X_0$ , so that a desired isomorphism  $S_s \otimes_A A_{\mathfrak{p}} \simeq S'[X, X^{-1}]$  is induced by the change of variables  $X_0 \mapsto X$  and  $X_i \mapsto XX_i$  for  $1 \leq i \leq n$ . To build the map  $S \to V$ , we first choose  $x_1, \ldots, x_n \in V$  and  $t \in V \setminus \mathfrak{P}$  such that  $X_i$  maps to  $x_i/t \in V_{\mathfrak{P}}$ . Continuing to use abusive notation for homogeneous polynomials, we note that the ("homogeneous" in  $t, x_1, \ldots, x_n$ ) elements  $t^N f_i(x_1/t, \ldots, x_n/t)$  of V die in  $V_{\mathfrak{P}}$ , so they are killed by some  $t' \in V \setminus \mathfrak{P}$ . Thus, the *B*-morphism

$$B[X_0, X_1, \dots, X_n] \to V$$
 given by  $X_0 \mapsto t't, X_1 \mapsto t'x_1, \dots, X_n \mapsto t'x_n$ 

factors through S. By construction, the resulting morphism  $S \to V_{\mathfrak{P}}$  factors through the localization  $S_s \otimes_A A_{\mathfrak{p}}$  of S, as desired.  $\Box$ 

We are ready for the promised reduction to complete, one-dimensional valuation rings.

**Proposition 14.** Consider the following property of a valuation ring V and a subring  $A \subset V$ :

(\*) every  $A \to B \to V$  with B a finite type A-algebra such that  $B \to \text{Frac}(V)$  factors through an A-smooth localization of B has a refinement  $A \to B \to S \to V$  with S smooth over A.

For a finite dimensional valuation ring V with a subfield  $A \subset V$ , if for all consecutive primes  $\mathfrak{q}' \subset \mathfrak{q} \subset V$  the complete height one valuation ring  $(V/\mathfrak{q}')_{\mathfrak{q}}$  satisfies (\*), then so does V.

Proof. We fix a finite type A-algebra B equipped with a factorization  $A \to B \to V$ as in (\*), which we need to factor further as  $A \to B \to S \to V$  for some smooth A-algebra S. When  $B \to V$  itself factors through an A-smooth localization of B, there is nothing to show. Otherwise, since V is of finite height, we may choose the minimal prime  $\mathfrak{q} \subset V$  whose preimage in B does not lie in the A-smooth locus of  $\operatorname{Spec}(B)$  and the largest prime  $\mathfrak{q}' \subsetneq \mathfrak{q} \subset V$  properly contained in  $\mathfrak{q}$  (the assumption in (\*) ensures that  $\mathfrak{q}'$  exists). Thanks to Lemma 13, we may replace V by  $V_{\mathfrak{q}}$  to reduce to the case when  $\mathfrak{q}$  is the maximal ideal (so that  $(V/\mathfrak{q}')_{\mathfrak{q}} = V/\mathfrak{q}'$ ): indeed, once we resolve this case, then, by using Lemma 13, we will be able to refine B to an A-algebra that either is smooth or for which  $\mathfrak{q}$  is strictly larger, and, by iteration, we will then arrive at a desired S.

By Lemma 4, there is an element  $b \in B$  standard over A that maps to  $\mathfrak{q} \setminus \mathfrak{q}'$ . The property from (\*) of  $\widetilde{V/\mathfrak{q}'}$  then supplies a smooth A-algebra S', an element  $s \in S'$  (the image of b), and a factorization

$$A \underbrace{\bigvee_{b \mapsto s}^{B}}_{S'/s^3S'} \widetilde{V/\mathfrak{q}'}/b^3\widetilde{V/\mathfrak{q}'} \cong V/b^3V.$$

Thanks to Lemma 8, we may change S' in order to make sure that the map  $S'/s^3S' \rightarrow V/b^3V$  lifts to an A-morphism  $S' \rightarrow V$ . This puts us in a situation in which we may apply Lemma 7 to obtain a smooth S'-algebra S with a desired factorization  $A \rightarrow B \rightarrow S \rightarrow V$ .

### 3. IND-SMOOTHNESS OF LARGE IMMEDIATE EXTENSIONS OF VALUATION RINGS

Our next goal is to find a large class of extensions of valuation rings that are ind-smooth. The argument combines classical results from valuation theory that go back to Kaplansky, results from [23] (see Lemma 15 and its proof), and the desingularization lemmas from Section 2.

Consider the case when V is not noetherian and its associated valuation has rank one. In the Noetherian case a immediate extension of valuation rings  $V \subset V'$  is dense, but in general case it need not be. If  $V \supset \mathbf{Q}$  the problem is solved by Ostrowski's Defektsatz [22] but when the characteristic of the residue field of V is > 0 the immediate algebraic extensions present extra difficulties.

An inclusion  $V \subset V'$  of valuation rings is an *immediate extension* if it is local as a map of local rings and induces isomorphisms between the value groups and the residue fields of V and V'. For such a  $V \subset V'$ , letting K'/K be the induced fraction field extension, we have  $V = V' \cap K$  (see [5, (4.1) in VI]). Moreover, for any subextension K'/K''/K and the valuation ring  $V'' = V' \cap K''$ , both  $V \subset V''$  and  $V'' \subset V'$  are then also immediate extensions (to check the value group requirement one uses that any  $v'' \in V''$  is a unit if and only if so is its image in V').

For example, for any valuation ring V, the extension  $V \subset \tilde{V}$  is immediate (see [33]),  $\tilde{V}'$  being the completion of V'.

For a valuation ring V with the fraction field K, a sequence  $\{v_i\}_{i < \omega}$  in K indexed by the ordinals *i* less than a fixed limit ordinal  $\omega$  is *pseudo-convergent* if

 $\operatorname{val}(v_i - v_{i''}) < \operatorname{val}(v_{i'} - v_{i''})$  (that is,  $\operatorname{val}(v_i - v_{i'}) < \operatorname{val}(v_{i'} - v_{i''})$ ) for  $i < i' < i'' < \omega$  (see [12], [33]). A (possibly nonunique) *pseudo-limit* of a pseudo-convergent sequence  $\{v_i\}_{i < \omega}$  is an element  $\alpha \in K$  with

 $\operatorname{val}(\alpha - v_i) < \operatorname{val}(\alpha - v_{i'})$  (that is,  $\operatorname{val}(\alpha - v_i) = \operatorname{val}(v_i - v_{i'})$ ) for  $i < i' < \omega$ . A pseudo-convergent sequence  $\{v_i\}_{i < \omega}$  in K is

- (1) algebraic if some  $f \in K[T]$  satisfies  $\operatorname{val}(f(v_i)) < \operatorname{val}(f(v_{i'}))$  for large enough  $i < i' < \omega$ ;
- (2) transcendental if each  $f \in K[T]$  satisfies  $\operatorname{val}(f(v_i)) = \operatorname{val}(f(v_{i'}))$  for large enough  $i < i' < \omega$ .

(Here "large enough" means larger than a fixed ordinal  $\omega' < \omega$  that is allowed to depend on f.) In both cases, [12, Theorems 1, 2] describe the valuation of K' that extends V of K. For instance, in the transcendental case, by [12, Theorem 2], this valuation on K(t) is given by setting

$$\operatorname{val}(((f(t))/(g(t))) = \operatorname{val}(f(v_i)) - \operatorname{val}(g(v_i)) \text{ for large enough } i < \omega.$$

These results lead to [12, Theorem 4]: a valuation ring V has no nontrivial immediate extensions if and only if each pseudo-convergent sequence in its fraction field K has a pseudo-limit in K. If for all  $\gamma \in \Gamma$ , the value group of V, there exists i < i'sufficiently large such that  $\operatorname{val}(v_i - v_{i'}) > \gamma$  then we call  $\{v_i\}_{i < \omega}$  fundamental. As in [23, Lemma 3.2] we get the following lemma.

**Lemma 15.** For an immediate extension  $V \subset V'$  of valuation rings and a transcendental pseudo-convergent sequence  $(v_i)_{i < \omega}$  in K, which has a pseudo-limit v' in K'but no pseudo-limit in K the valuation ring  $V'' = V' \cap K(v')$  is a filtered union of smooth V-subalgebras.

*Proof.* For each i set  $x_i = (v' - v_i)/(v_{i+1} - v_i)$ , so that  $x_i$  is a unit in V'. Let  $\mathfrak{m}'$  be the maximal ideal of V'. We show that for every polynomial  $0 \neq f \in V[t]$  it holds

 $f(v') \in f(v_i) \cdot (1 + \mathfrak{m}' \cap V[x_i])$  for every large enough  $i < \omega$ .

Since  $\{v_i\}_{i < \omega}$  is transcendental, for each  $g(t) \in K[t]$ , the value  $\operatorname{val}(g(v_i))$  is constant for large *i*. Moreover, for large *i* the values  $\operatorname{val}(v' - v_i)$  are strictly increasing as *i* increases. Thus, in the Taylor expansion<sup>1</sup>

$$f(v') = \sum_{n=0}^{\deg f} (D^{(n)}f)(v_i) \cdot (v' - v_i)^n \quad \text{with} \quad D^{(n)}f \in V[t]$$

the values  $\operatorname{val}((D^{(n)}f)(v_i) \cdot (v'-v_i)^n)$  are pairwise distinct for every large enough *i*. Consequently, since  $\operatorname{val}(f(v')) = \operatorname{val}(f(v_i))$  for large *i*, we conclude that

$$\operatorname{val}((D^{(n)}f)(v_i) \cdot (v'-v_i)^n) > \operatorname{val}(f(v_i))$$
 for every  $n > 0$  and large enough  $i < \omega$ .

It remains to note that

$$(v' - v_i)^n = x_i^n \cdot (v_{i+1} - v_i)^n$$
 and  $\operatorname{val}(v' - v_i) = \operatorname{val}(v_{i+1} - v_i),$ 

<sup>&</sup>lt;sup>1</sup>The polynomials  $D^{(n)}f \in R[t]$  for  $f \in R[t]$  make sense for any ring R: indeed, one constructs the Taylor expansion in the universal case  $R = \mathbf{Z}[a_0, \ldots, a_{\deg f}]$  by using the equality  $n! \cdot (D^{(n)}f) = f^{(n)}$ .

which is enough for our claim. In particular, we get that f(v') is transcendental over K. The element  $x_i$  is transcendental over K, so  $V[x_i] \subset V'$  is the polynomial algebra. Moreover, for  $i < i' < \omega$  we have  $x_i = x_{i'} \cdot (v_{i'+1} - v_{i'})/(v_{i+1} - v_i) + (v_{i'} - v_i)/(v_{i+1} - v_i)$ , so  $V[x_i] \subset V[x_{i'}] \subset V'$ . Consequently, we arrive at a nested sequence

 $\{V[x_i]_{\mathfrak{m}'\cap V[x_i]}\}_{i<\omega}$  of ind-smooth V-subalgebras of V',

and, it remains to show that every element of V'' belongs to some  $V[x_i]_{\mathfrak{m}' \cap V[x_i]}$ . In fact, it suffices to show that each  $0 \neq f \in V[t]$  satisfies

$$f(v') \in f(v_i) \cdot (1 + \mathfrak{m}' \cap V[x_i])$$
 for every large enough  $i < \omega$ 

which was done above.

**Lemma 16.** For an immediate extension  $V \subset V'$  of valuation rings containing  $\mathbf{Q}$  with value group  $\Gamma \subset \mathbf{R}$  every algebraic pseudo-convergent sequence  $(v_i)_{i < \alpha}$  in K, which is not a fundamental sequence but has a pseudo-limit v' in K' has also a pseudo-limit in K.

Proof. By [12, Theorem 3] there exists an immediate extension of valued fields  $K \subset K(u)$  such that u is algebraic over K and it is a pseudo-limit of  $(v_i)$  in K(u). As a consequence of Ostrowski's Defektsatz [22, Sect 9, No 55] (see [23, Corollary 4.2], or [25, Corollary 3.10]) we see that  $K \subset K(u)$  is dense, that is, u belongs to the completion of K. Thus  $(v_i)$  has a pseudo-limit in K by [25, Lemma 2.5].  $\Box$ 

**Remark 17.** The above lemma is false if the characteristic of the residue field of a valuation rings is > 0 (see [25, Example 3.13] inspired by [22, Sect 9, No 57]).

**Proposition 18.** For an immediate extension  $V \subset V'$  of valuation rings containing  $\mathbf{Q}$  with value group  $\Gamma \subset \mathbf{R}$ , V' is ind-smooth over V.

Proof. Applying Lemma 15 possible infinitely, even uncountably many, we find a pure transcendental extension  $K'' \subset K'$  of K such that  $V'' = V' \cap K''$  is ind-smooth over V and all transcendental pseudo-convergent sequences in V'' over V having a pseudo limit in V', which are not fundamental sequences, have pseudo-limits in V''. By Lemma 16 we see that this holds also for algebraic pseudo-convergent sequences from V'', which are not fundamental sequences. It follows that the extension  $V'' \subset V'$ is dense using [12, Theorem 1]. Now, it is enough to apply Lemma 7 to see that V' is ind-smooth over V''. Indeed, let  $B \subset V'$  be a finitely generated V''-subalgebra and w its inclusion. By separability,  $w(H_{B/V''}) \neq 0$  and choose an element  $d \in V''$ , which is standard for B over V''. Then the composite map  $B \to V' \to V'/d^3V' \cong V''/d^3V''$ factors obviously to a smooth  $V''/d^3V''$ -algebra. By Lemma 7 w factors through a smooth V''-algebra.

**Corollary 19.** For an extension  $V \subset V'$  of valuation rings containing  $\mathbf{Q}$  with the same value group  $\Gamma \subset \mathbf{R}$ , V' is ind-smooth over V.

*Proof.* By Proposition 9 we may reduce to the case when V, V' are complete and so they are Henselian since dim V = 1, that is, they contain their residue fields k, k'. Let K, K' be the fraction fields of V, resp. V'. Then V' is an immediate extension of  $V'' = V' \cap K(k')$  and so it is ind-smooth by the above proposition. Express k' as

a filtered union of some finitely generated field extensions  $(k_i)$  of k. It is enough to see that  $V_i = V' \cap K(k_i)$  is an ind-smooth extension of V. But  $V_i$  is even essentially smooth over V because  $k_i$  is so over k.

### 4. EXTENSIONS OF VALUATION RINGS

The following proposition is an extension of Corollary 19.

**Proposition 20.** Let  $V \subset V'$  be an extension of valuation rings containing  $\mathbf{Q}$ . Suppose dim  $V < \infty$  and the value group extension of  $V \subset V'$  is trivial. Then V' is ind-smooth over V.

*Proof.* For the proof we apply Lemma 5.

Let E be a V-algebra of finite presentation, let us say  $E \cong V[Y]/I$ ,  $Y = (Y_1 \ldots, Y_m)$ , I being a finitely generated ideal. Let  $w : E \to V'$  be a V-morphism. We will show that w factors through a smooth V-algebra. E is finitely generated and so it is  $\operatorname{Im} w$ . By Lemma 10 we see that  $\operatorname{Im} w$  is finitely presented. So we may replace E by  $\operatorname{Im} w$ , that is we may assume w injective. By separability we have  $w(H_{E/V}) \neq 0$ , let us assume that  $w(H_{E/V})V' \supset zV'$  for some  $z \in V, z \neq 0$ . Replacing z by a power of it we may assume that  $z = \sum_i^s b_i b'_i$  for some  $b_i = \det(\partial f_{ij}/\partial Y_{j_i})$  for some systems of polynomials  $f_i$  from I and  $b''_i \in V[Y]$  which kills  $I/(f_i)$ . Similarly as in [15, Lemma 4] we may assume that we can take s = 1, that is for some polynomials  $f = (f_1, \ldots, f_r)$  from I, we have  $z \in NME$  for some  $N \in ((f) : I)$  and a  $r \times r$ -minor M of the Jacobian matrix  $(\partial f_i/\partial Y_j)$  (since V' is a valuation ring this reduction is much easier). Thus we may assume z is standard over V (see the beginning of Section 2), which is necessary later to apply Lemma 7. Let  $q'_2 \in \operatorname{Spec} V'$ , be the minimal prime ideal of zV' and  $q_2 = q'_2 \cap V$ . As the value group extension of  $V \subset V'$  is trivial we have  $q'_2 = q_2V'$ .

Let  $q_1 \in \operatorname{Spec} V$ ,  $q_1 \subset q_2$  be the greatest prime ideal of V not containing z. Then  $q_1 \neq q_2$ . The extension  $V_{q_2}/q_1 V_{q_2} \subset V'_{q'_2}/q_1 V_{q'_2}$  has the trivial value group extension. and so it is ind-smooth by Corollary 19. The composite map  $E \xrightarrow{w} V' \rightarrow V'$  $V'_{q'_2}/q_2 V'_{q'_2}$  factors by a smooth  $V_{q_2}/q_1 V_{q_2}$ -algebra G, let us say it is the composite map  $E \xrightarrow{\alpha} G \xrightarrow{\beta} V'_{q_2}/q_2 V'_{q_2}$ . We may assume that  $G = (V_{q_2}/q_1 V_{q_2})[U]_{g'h}/(g)$ , with  $U = (U_1, \ldots, U_l), g' = \partial g / \partial U_1, g, h \in V[U]$  by [33, Theorem 2.5] and let  $\beta$  be given by  $U + (g) \to u + q_1 V'_{q'_2}$  for some  $u \in (V'_{q'_2})^l$ . Note that [33, Theorem 2.5] gives just that a localization of G has the form a localization of  $C = (V_{q_2}/q_1V_{q_2})[U]_{g'}/(g)$ and so the above composite map factors through a  $C_h$  for some  $h \in V[Y]$ . Then  $g(u) \equiv 0 \mod q_1 V'_{q'_2}$  and in particular  $g(u) \equiv 0 \mod z^3 V'_{q'_2}$ . Then  $g(u) = z^3 t$  for some  $t \in V'_{q'_2}$ . Note that the composite map  $E \to V' \to V'_{q'_2}$  factors through the smooth  $V_{q_2}$ -algebra  $D = (V_{q_2}[U,T]/(g-z^3T))_{g'h}$  modulo  $z^3$ , where  $T \to t$ . By Lemma 7 we see that  $E \to V'_{q'_2}$  factors through a smooth *D*-algebra *D'* which is also smooth over  $V_{q_2}$ . Using Lemma 13 we see that w factors through a finitely presented V-algebra E'', let us say through a map  $w'': E'' \to V'$  with  $w''(H_{E''/V}) \not\subset q'_2$ . More precisely, by Lemma 13 there exist a finitely presented V-algebra E'' and  $c \in E''$ with  $E''_c \otimes_V V_q \cong D'[X, X^{-1}]$  and a factorization  $V \to E \to E'' \to V'$  such that

 $E' \to V'_{q'}$  factors through  $E' \to E'_c \otimes_V V_q \to V'_{q'}$ . Note that dim  $V' = \dim V < \infty$  because V, V' have the same value group. We arrive in finite steps using induction on dim V'/zV' in the case when z is a unit, that is we can embed E in a smooth V-algebra. This is enough by Lemma 5.

**Theorem 21.** Let  $V \subset V'$  be an immediate extension of valuation rings containing **Q**. Then V' is ind-smooth over V.

Proof. Let  $K \subset K'$  be the fraction field of  $V \subset V'$  and  $K'' \subset K'$  a pure transcendental extension of K generated by a transcendental basis of K'/K, that is K'/K is algebraic. Applying Lemma 15 possible infinitely and even uncountably many, as in Proposition 18 we see that  $V'' = V' \cap K''$  is ind-smooth over V and all transcendental pseudo-convergent sequences in V'' over V having a pseudo limit in V', which are not fundamental sequences, have pseudo-limits in V''. Thus we reduce to show that V' is ind-smooth over V when K'/K is algebraic. Actually, it is enough to assume K'/K finite because V' is the filtered union of  $V' \cap L$  for all subfields  $L \subset K'$  which are finite extension over K.

Let E = V[Y]/I,  $Y = (Y_1, \ldots, Y_n)$  be a finitely generated V-subalgebra of V' (so finitely presented by Lemma 10) and a map  $w: E \to V'$ . By Lemma 5 it is enough to show that w factors through a smooth V-algebra. Consider  $H_{E/V}$  and a standard element  $z \in V$  for E over V, so  $w(z) \in w(H_{E/V})V'$ , as in the proof of Proposition 20. If  $V \subset V'$  is dense we may apply Proposition 9 to see that w factors through a smooth V-algebra (note that  $w(H_{E/V}) \neq 0$  says that the composite map  $E \to V' \to K'$  factors through a smooth V-algebra). In the remaining case the factorization is constructed in some steps: for the standard element z one chooses adjacent prime ideals  $q_1 \subset q$  of V such that  $w(z) \in qV' \setminus q_1V'$  and construct a factorization  $E \to E' \xrightarrow{w'} V'$  such that  $w'(H_{E'/V}) \not\subset qV'$ , where E' is finitely presented over V. If after finite steps we get a factorization  $E \to E^{(n)} \xrightarrow{w^{(n)}} V'$  such that  $w^{(n)}(H_{E^{(n)}/V})V' = V'$  the goal is reached. A hard problem is to show that we can find such  $E^{(n)}$  in finite steps. For this we will consider a finite partition  $\mathcal{P}_i, i = 1, \ldots, s$  of Spec V corresponding to those  $q \in \operatorname{Spec} V$  which have the same dimension  $f_i = f_q \leq f = [K': K]$  of the fraction field extension  $K_q \subset K'_q$  of  $V/q \subset V'/qV'$ . We will see that to each construction  $q_1$  change from one  $\mathcal{P}_j$  to another one from  $\mathcal{P}_i$  with j < i and  $f_i < f_j$ . Finally we arrive in finite steps to the case  $f_{q_1} = 1$ , which is done easily as in the dense case.

Assume  $V \subset V'$  is not dense and  $q, q_1, f_{q_1}, (\mathcal{P}_i)_{1 \leq i \leq s}$  as above. More precisely, let  $q' \in \operatorname{Spec} V'$  be the minimal prime ideal of w(z)V' and  $q = q' \cap V$ . Thus qV' = q' because  $V \subset V'$  is immediate. Let  $q'_1 \in \operatorname{Spec} V'$  be the prime ideal corresponding to the maximal ideal of the fraction ring of V' with respect to the multiplicative system generated by z. Then  $q'_1$  is the biggest prime ideal of V' contained strictly in q' and so height $(q'/q'_1) = 1$ . Set  $q_1 = q'_1 \cap V$ . We have  $f_q \leq f_{q_1}$ . Let  $x_q$  be a primitive element of the separable finite extension  $K'_q/K_q$  and  $g_q \in V'$ .

Let  $x_q$  be a primitive element of the separable finite extension  $K'_q/K_q$  and  $g_q \in V/q[X]$  be a primitive polynomial multiple of  $Irr(x_q, K_q)$  by a nonzero constant of K. Note that if  $q, q \in \mathcal{P}_i, q \subset q$  then  $f_q = f_q = f_i$  and  $g_q$  remains irreducible over V/q. Clearly,  $f_s = 1$  because  $f_m = 1$  for the maximal ideal  $\mathfrak{m}$  of V, the extension

 $V \subset V'$  being immediate. A set  $\mathcal{P}_i$  has a maximum element for inclusion namely  $\mathfrak{p}_i = \bigcup_{\mathfrak{q} \in \mathcal{P}_i} \mathfrak{q}$ . Indeed,  $\mathfrak{p}_i$  is clearly a prime ideal and if  $f_{\mathfrak{p}_i} < f_i$  then  $f_{\mathfrak{q}} < f_i$  for some  $\mathfrak{q} \in \mathcal{P}_i$ , which is false.

Assume  $q_1 \in \mathcal{P}_j$ . If  $q_1 \neq \mathfrak{p}_j$  then  $(V/q_1)_{\mathfrak{p}_j} \subset (V'/q_1V')_{\mathfrak{p}_jV'}$  is in fact a localization of  $(V/q_1)[X]/(g_{q_1})$  because  $g'_{q_1} = \partial g_{q_1}/\partial X$  corresponds to a unit in  $(V'/q_1V')_{\mathfrak{p}_j}$  and so the composite map  $E \to V' \to (V'/q_1V')_{\mathfrak{p}_jV'}$  factors through an etale  $V/q_1$ algebra of the form  $((V/q_1)[X]/(g_{q_1})_{g'_{q_1}h}$  for some  $h \in V[X]$ . In particular  $E \to$  $V' \to (V'/(z^3))_{\mathfrak{p}_jV'}$  factors through an etale  $V/(z^3)$ -algebra and by Lemma 7 the map  $E \to V' \to V'_{\mathfrak{p}_jV'}$  factors through a smooth V-algebra. Using Lemma 13 we see that w factors through a finitely presented V-algebra E', let us say through a map  $w' : E' \to V'$  with  $w'(H_{E'/V}) \not\subset \mathfrak{p}_jV'$ . Changing E by E' we see that the new q belongs to  $\mathcal{P}_i$  for some i > j. Moreover, the new  $q_1$  belongs also to  $\mathcal{P}_i$  for some i > j, because otherwise we get  $q_1 = \mathfrak{p}_j$ .

If  $q_1 = \mathfrak{p}_j$  then  $q \in \mathcal{P}_{j+1}$  and we apply Corollary 19. Then we see that  $(V/q_1)_q \subset (V'/q'_1)_{q'}$  is ind-smooth and as above we see that the composite map  $E \to V' \to (V'/q'_1)_{q'}$  factors through a smooth  $V/q_1$ -algebra and finally by Lemmas 7 and 13 we get that w factors through a finitely presented V-algebra E', let us say through a map  $w' : E' \to V'$  with  $w'(H_{E'/V}) \not\subset qV'$ . Now the new  $q_1$ , that is the old q, belongs to  $\mathcal{P}_{j+1}$ . In some steps (at most s) we arrive to the case when  $f_{q_1} = 1$ 

If  $f_{q_1} = 1$  then we get  $f_{q''_1} = 1$  for all  $q''_1 \in \text{Spec } V$  containing  $q_1$ . Actually, we get V/q = V'/q' and so in particular  $V/q \subset V'/q'$  is ind-smooth. Using Lemma 7 we see that w factors through a smooth (in fact etale) V-algebra. Applying Lemma 5 we are done.

## **Proposition 22.** Let $V \subset V'$ be an extension of valuation rings. Suppose that

- (1) V is a discrete valuation ring extending  $\mathbf{Z}_{(p)}$  with  $\pi$  its local parameter, and p a prime number.
- (2)  $\pi V'$  is the maximal ideal of V',
- (3) the residue field extension of  $V \subset V'$  is separable.

Then  $V \to V'$  is ind-smooth.

Proof. Let E, w,  $H_{E/V}$ , z be as in Proposition 20 and we may assume K' is the fraction field of  $\operatorname{Im} w$ . choose  $q'_2 \in \operatorname{Spec} V'$  a minimal prime ideal of  $w(H_{E/V})V'$ . If  $q'_2 \neq \pi V'$  then using Zariski's Uniformization Theorem we may change E, w with some E', w' such that  $w'(H_{E'/V})V' \not\subset q'_2$ . Step by step we arrive to the case when either  $w(H_{E/V})V' = V'$ , or  $w(H_{E/V})V'$  is a  $\pi V'$ -primary ideal. In the first case, w factors through a localization of E which is smooth. In the second case,  $q'_1 = \bigcap_{i \in \mathbb{N}} \pi^i V'$  is a prime ideal and the composite map  $V \to V' \to V'/q'_1$  is a regular map of discrete valuation rings and so an ind-smooth map by the classical Néron Desingularization. The proof ends by using Lemma 5.

**Corollary 23.** Let V be a discrete valuation ring extending  $\mathbf{Z}_{(p)}$  with p a prime number and V' an ultrapower of V with respect to a nonprincipal ultrafilter on N. Then  $V \subset V'$  is ind-smooth.

For the proof note that the maximal ideal of V generates the maximal ideal of V'and apply the above proposition. **Proposition 24.** Let V be a discrete valuation ring extending  $\mathbf{Z}_{(p)}$  with p a prime number and  $V \subset V'$  an extension of valuation rings such that

- (1) p is a local parameter of V,
- (2) pV' is a  $\mathfrak{m}'$ -primary ideal of V', where  $\mathfrak{m}'$  is the maximal ideal of V',
- (3) the residue field extension of  $V \subset V'$  is separable.

Then V' is a filtered direct limit of regular local rings essentially of finite type over V.

Proof. As in Proposition 22 we may consider  $w : E \to V'$  and we may reduce to the case when  $w'(H_{E'/V})V'$  is  $\mathfrak{m}'$ -primary ideal. We may assume that  $p^s$  is a standard for E over V for some  $s \in \mathbb{N}$  and as in the proof of [30, Theorem 3.6] there exists a local essentially smooth V-algebra G and  $b \in G$  such that the map  $E/p^{3s}E \to V'/p^{3s}V'$  factors through  $G/(p^{3s}, p - b)$ . Then a variant of Lemma 7 in the idea of [30, Proposition 3.4] shows that w factors through a local essentially smooth D = G/(p-b)-algebra D'. This D' is regular local since D is so. Now apply Lemma 5.

## 5. Structure of equicharacteristic valuation rings possessing a cross-section

Modulo all the reductions and simplifications that go into the overall proof of Theorem 2, our ultimate source of expressions of valuation rings as filtered direct limits of smooth rings is Lemma 26 below. This lemma describes some valuations on an affine space for which local uniformizations can be constructed by successively blowing up regular centers as in [31, 4.5, 4.19] following Perron's algorithm (whose relevance to the resolution of singularities was explained already in [34]). We present a more direct argument for this uniformization that is close to [23, Lemma 4.6] and rests on the following lemma that captures the "combinatorial" part of local uniformization.

We will need the following lemma (see [7, 2.2], or [23, 4.6.1], or [9, 6.1.30]).

**Lemma 25.** For a totally ordered abelian group  $\Gamma$ , the submonoid  $\Gamma_{\geq 0} \subset \Gamma$  of nonnegative elements is a filtered union of its finite free submonoids isomorphic to  $\mathbf{Z}_{\geq 0}^r$ , where  $r \in \mathbf{Z}_{\geq 0}$  need not be constant.

We include a mixed characteristic version of the following lemma because it requires virtually no additional effort in comparison to the equicharacteristic case that we will use below.

- **Lemma 26.** (1) For a field  $\mathbf{F}$ , a valuation ring  $\mathbf{F} \subset V$  with fraction field  $\mathbf{F}(x_1, \ldots, x_n)$  such that  $\operatorname{val}(x_1), \ldots, \operatorname{val}(x_n)$  are  $\mathbf{Z}$ -linearly independent is a countable direct union of essentially smooth  $\mathbf{F}$ -algebras.
  - (2) For a discrete valuation ring  $\Lambda$  with uniformizer  $\pi$  and fraction field  $\mathbf{F}$ , a valuation ring  $\Lambda \subset V$  that dominates  $\Lambda$  and has fraction field  $\mathbf{F}(x_1, \ldots, x_n)$  such that  $\operatorname{val}(\pi), \operatorname{val}(x_1), \ldots, \operatorname{val}(x_n)$  are  $\mathbf{Z}$ -linearly independent is a countable direct union of regular local  $\Lambda$ -algebras of the form

$$(\Lambda[Y_1,\ldots,Y_{n+1}]/(\pi-Y_1^{b_1}\cdots Y_{n+1}^{b_{n+1}}))_{(Y_1,\ldots,Y_{n+1})} \quad with \quad \gcd(b_1,\ldots,b_{n+1}) = 1.$$

*Proof.* To avoid repeating the argument, we will prove both claims simultaneously, so in (1) we set  $\Lambda = \mathbf{F}$  and  $\pi = 0$  and in both parts we set  $p = \text{Char}(\Lambda/(\pi))$ . By [5, Theorem 1 in VI (10.3)],

$$\gamma_1 = \operatorname{val}(x_1), \quad \dots, \quad \gamma_n = \operatorname{val}(x_n), \quad \gamma_{n+1} = \operatorname{val}(\pi) \quad (\text{resp. } \gamma_{n+1} = 0 \text{ if } \pi = 0)$$

satisfy  $\Gamma \cong \mathbf{Z}\gamma_1 \oplus \cdots \oplus \mathbf{Z}\gamma_{n+1}$ , where  $\Gamma$  is the value group of V. We set N = n + 1(resp., N = n if  $\pi = 0$ ) and use Lemma 25 to find a countable sequence  $\Gamma_0 \subset \Gamma_1 \subset \cdots$ of submonoids of  $\Gamma_{\geq 0}$  with  $\Gamma_i \simeq \mathbf{Z}_{\geq 0}^N$  for each i and  $\Gamma_{\geq 0} = \bigcup_{i\geq 0} \Gamma_i$ . We fix a  $\mathbf{Z}_{\geq 0}$ -basis  $\nu_{i1}, \ldots, \nu_{iN}$  of  $\Gamma_i$  with  $(\nu_{01}, \ldots, \nu_{0N}) = (\gamma_1, \ldots, \gamma_N)$ , so that the elements  $\nu_{i1}, \ldots, \nu_{iN}$ are  $\mathbf{Z}$ -linearly independent in  $\Gamma$ , and we express them in terms of the fixed  $\mathbf{Z}$ -basis:  $\nu_{ij} = d_{ij1}\gamma_1 + \cdots + d_{ijN}\gamma_N$  for unique  $d_{ij1}, \ldots, d_{ijN} \in \mathbf{Z}$  and every  $j = 1, \ldots, N$ . We set  $x_{n+1} = \pi$  and note that, by construction, for each  $i \geq 0$  and  $1 \leq j \leq N$ , the element

$$y_{ij} = x_1^{d_{ij1}} \cdots x_N^{d_{ijN}} \in \mathbf{F}(x_1, \dots, x_n)$$
 has valuation  $\nu_{ij}$ .

Since  $\Gamma_{i'} \subset \Gamma_i$  for i' < i, each  $y_{i'j}$  is in a unique way a monomial in the elements  $y_{i1}, \ldots, y_{iN}$ :

if we express  $\nu_{i'j} = b_{i'i1}\nu_{i1} + \cdots + b_{i'iN}\nu_{iN}$  with  $b_{i'ij} \in \mathbb{Z}_{\geq 0}$ , then  $y_{i'j} = y_{i1}^{b_{i'i1}} \cdots y_{iN}^{b_{i'iN}}$ . Since the valuations of  $y_{i1}, \ldots, y_{iN}$  are Z-linearly independent, the  $\Lambda$ -subalgebra  $\Lambda[y_{i1}, \ldots, y_{iN}]$  of  $\mathbf{F}(x_1, \ldots, x_n)$  is the regular ring

 $\Lambda[Y_{i1}, \ldots, Y_{iN}]/(\pi - Y_{i1}^{b_1} \cdots Y_{iN}^{b_N})$  with  $b_i = b_{0Ni}$  (resp.  $\Lambda[Y_{i1}, \ldots, Y_{iN}]$  if  $\pi = 0$ ), where  $gcd(b_1, \ldots, b_N) = 1$  because  $val(\pi)$  is assumed to be a primitive element of  $\Gamma$ 

when  $\pi \neq 0$ . In particular, we obtain a nested sequence of  $\Lambda$ -subalgebras  $R_i = \Lambda[y_{i1}, \ldots, y_{iN}]_{(y_{i1}, \ldots, y_{iN})} \subset V$ 

that are regular (resp., essentially smooth if  $\pi = 0$ ) and it remains to argue that every  $f \in V$  belongs to some  $R_i$ . For this, we first express f as a rational function as follows:

$$f = (\sum \lambda_{s_1, \dots, s_N} x_1^{s_1} \cdots x_N^{s_N}) / (\sum \lambda'_{r_1, \dots, r_N} x_1^{r_1} \cdots x_N^{r_N}) \text{ with } \lambda_{s_1, \dots, s_N}, \lambda'_{r_1, \dots, r_N} \in \Lambda^{\times} \cup \{0\}.$$

The linear independence of the  $\gamma_i$  ensures that the valuations of the monomials that appear in the numerator (resp., denominator) are all distinct. Thus, by taking out the monomials with minimal valuations, we reduce to showing that every  $x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ with  $\alpha_j \in \mathbf{Z}$  and  $\alpha_1 \gamma_1 + \cdots + \alpha_N \gamma_N > 0$  is a product of nonnegative powers of the elements  $y_{i1}, \ldots, y_{iN}$  for some  $i \ge 0$ . For this, it suffices to note that  $\alpha_1 \gamma_1 + \cdots + \alpha_n \gamma_n$ lies in some  $\Gamma_i$ , and then to express it as a  $\mathbf{Z}_{\ge 0}$ -linear combination of the  $\nu_{i1}, \ldots, \nu_{iN}$ : more precisely, if  $\alpha_1 \gamma_1 + \cdots + \alpha_N \gamma_N = c_1 \nu_{i1} + \cdots + c_N \nu_{iN}$  with  $c_j \in \mathbf{Z}_{\ge 0}$ , then

$$x_1^{\alpha_1}\cdots x_N^{\alpha_N} = y_{i1}^{c_1}\cdots y_{iN}^{c_N}$$

 $\square$ 

For a valuation ring V with the value group  $\Gamma$  and the fraction field K, a crosssection of V is a section

 $s \colon \Gamma \to K^*$  in the category of abelian groups of the valuation map val:  $K^* \to \Gamma$ .

(see for details in the Appendix).

**Proposition 27.** An equicharacteristic valuation ring V that has a cross-section  $s: \Gamma \to K^*$  and a subfield  $k \subset V$  lifting the residue field is an immediate extension

$$V_0 = \bigcup_i V_i \subset V$$

of a filtered union of valuation subrings  $V_i \subset V$  dominated by V such that each  $V_i$  has a finitely generated value group, is a countable increasing union of localizations of smooth k-subalgebras of V so  $V_0$  is ind-smooth over k, and has the restriction of s as a cross-section.

Proof. By Lemma 25, the submonoid  $\Gamma_{\geq 0} \subset \Gamma$  of positive elements is a filtered union  $\Gamma_{\geq 0} = \bigcup_i \Gamma_i$  of submonoids  $\Gamma_i \simeq \mathbf{Z}_{\geq 0}^{d_i}$  with  $d_i \geq 0$ . Thus, the cross-section s gives rise to the filtered system of subfields  $k_i = k(s(\gamma) \mid \gamma \in \Gamma_i)$  of the field of fractions K of V. By choosing a  $\mathbf{Z}_{\geq 0}$ -basis for  $\Gamma_i$  and applying [5, Theorem 1, in VI section 10] we see that each  $k_i$  is a purely transcendental extension of k and that the value group of the valuation subring  $V_i = V \cap k_i$  of V is  $\mathbf{Z}^{d_i} \simeq \mathbf{Z}\Gamma_i \subset \Gamma$ . By construction, s restricts to a cross-section of  $V_i$  and, by Lemma 26, each  $V_i$  is a filtered union of localizations of k-subalgebras. The construction ensures that V is an immediate extension of the resulting  $V_0$ .

### 6. Counterexamples when the value groups are finitely generated

**Lemma 28.** Let  $V \subset V'$  be an extension of valuation rings which is ind-smooth. Then  $\Omega_{V'/V}$ , that is  $H_0(V, V', V')$  in terms of Andre-Quillen homology, is a flat V'-module and  $H_1(V, V', V') = 0$  (the last homology is usually denoted by  $\Gamma_{V'/V}$ ).

Proof. Assume that V' is the filtered direct limit of some smooth V -algebras  $B_i$ ,  $i \in I$ . Then  $\Omega_{B_i/V}$  is projective over  $B_i$  and  $H_1(V, B_i, B_i) = 0$  by e.g [33, Theorem 3.4]. But  $\Omega_{V'/V}$ , and  $H_1(V, V', V')$  are filtered direct limits of  $V' \otimes_{B_i} \Omega_{B_i/V}$  resp.  $V \otimes_{B_i} H_1(V, B_i, B_i)$  by [33, Lemma 3.2], which is enough.  $\Box$ 

**Lemma 29.** Let  $V \subset V'$  be an extension of valuation rings of dimension one with the same residue field and let  $\Gamma \subsetneq \Gamma'$  be their value group extension. Assume that  $\Gamma'/\Gamma$  has torsion. Then the extension  $V \subset V'$  is not ind-smooth.

Proof. Let  $\gamma \in \Gamma' \setminus \Gamma$  be such that  $n\gamma \in \Gamma$  for some positive integer n. Choose an element  $x \in V'$  such that  $\operatorname{val}(x) = \gamma$ . Then  $x^n = zt$  for some  $z \in V$  and an unit  $t \in V'$ . Thus the system S of polynomials  $X^n = zT$ , TT' = 1 over V has a solution in V'. If V' is ind-smooth over V then S has a solution in a smooth V-algebra and so one  $(\tilde{x}, \tilde{t}, \tilde{t}')$  in the completion of V. But then  $\gamma = \operatorname{val}(z)/n = \operatorname{val}(\tilde{x})$  must be in  $\Gamma$  which is false.

**Lemma 30.** Let  $V \subset V'$  be an extension of valuation rings of dimension one containing  $\mathbf{Q}$  having the same residue field k. Assume that V contains k and its value group  $\Gamma \subset \mathbf{R}$  is dense in  $\mathbf{R}$ . Also assume that the value group  $\Gamma' \subset \mathbf{R}$  of V' is finitely generated (that is finitely generated over 0),  $\Gamma \neq \Gamma'$  and  $\Gamma'/\Gamma$  has no torsion. Then the extension  $V \subset V'$  is not ind-smooth. Proof. Since  $\Gamma$  is free over  $\mathbf{Z}$  we may take a basis of positive elements  $\gamma_1, \ldots, \gamma_m$  of  $\Gamma$  which may be completed with some positive elements  $\gamma_{m+1}, \ldots, \gamma_n \in \Gamma'$  to a basis of  $\Gamma'$ . Choose  $x_1, \ldots, x_m \in V$  and  $x_{m+1}, \ldots, x_n \in V'$  such that  $\operatorname{val}(x_i) = \gamma_i$ . Let  $V_0 = V \cap k(x_1, \ldots, x_m)$  and  $V'_0 = V' \cap k(x_1, \ldots, x_n)$ .

We will show that  $\Omega_{V'_0/V_0}$  has torsion. **First assume** that n = m + 1. We will use the proof of [23, Lemma 7.2]. By Lemma 25  $\Gamma_+ = \bigcup_{j \in \mathbf{N}} \Gamma_j$  for some monoids  $\Gamma_j \subset \Gamma_+$  generated by bases of  $\Gamma$ , the union being filtered. We consider as in the quoted lemma two real sequences  $(u_i), (v_i)$  which converge in  $\mathbf{R}$  to  $\gamma_n$  and such that 1)  $u_j, v_j \in \Gamma_j$  and  $u_{j+1} - u_j, v_j - v_{j+1} \in \Gamma_{j+1}$ ,

2)  $v_j - u_j$  is an element of the basis  $\nu_j$  of  $\Gamma$  generating  $\Gamma_j$ , we may assume  $v_j - u_j = \nu_{j1}$ .

3)  $u_j < \gamma_n < v_j$  for all j.

We may also suppose that  $u_{j+1} - u_j = v_j - v_{j+1}$  if necessary restricting to a subsequence of  $(\Gamma_j)$ . Let  $a_j$ ,  $b_j$  be in V with values  $u_j$ , resp.  $v_j$ , and take  $y_{jn} = x_n/a_j$ and  $z_{jn} = b_j/x_n$  in V'. As in the proof of Lemma 4.2 a), we have  $\nu_{ji} = d_{ji1}\gamma_1 + \dots + d_{jim}\gamma_m$  and set  $y_{ji} = x_1^{d_{ji1}} \cdots x_m^{d_{jim}} \in V$  which has valuation  $\nu_{ji}$ ,  $i \in [m]$ . Then  $V_0$  is a filtered union of localizations  $B_j$  of  $k[y_{j1}, \dots, y_{jm}]$  and  $V'_0$  is a filtered union of localizations  $C_j$  of  $B_j[Z_j, Z'_j]/(Z_jZ'_j - y_{j1}) \cong B_j[z_j, z'_j]$ , where  $z_j = x_n/a_j$  and  $z'_j = b_j/x_n$  in V'. Note that the map  $C_j \to C_{j+1}$  is given by  $Z_j \to (a_{j+1}/a_j)Z_{j+1}$ ,  $Z'_j \to (b_j/b_{j+1})Z'_{j+1}$ .

We claim that the map  $f_j : C_{j+1} \otimes_{C_j} \Omega_{C_j/B_j} \to \Omega_{C_{j+1}/B_{j+1}}$  given by  $dz_j \to (a_{j+1}/a_j)dz_{j+1}, dz'_j \to (b_j/b_{j+1})dz'_{j+1}$  is injective. Indeed, an element from Ker  $f_j$  induced by  $w = \alpha \otimes dz_j + \beta \otimes dz'_j, \alpha, \beta \in C_{j+1}$  must go by  $f_j$  in

$$\alpha(a_{j+1}/a_j)dz_{j+1} + \beta(b_j/b_{j+1})dz'_{j+1} \in \langle z'_{j+1}dz_{j+1} + z_{j+1}dz'_{j+1} \rangle$$

in  $C_{j+1}dz_{j+1} \oplus C_{j+1}dz'_{j+1}$ . So  $\alpha(a_{j+1}/a_j) = \mu z'_{j+1}$  and  $\beta(b_j/b_{j+1}) = \mu z_{j+1}$  for some  $\mu \in C_{j+1}$ . It follows that

$$w = (\mu(b_{j+1}/b_j)(a_{j+1}/a_j)) \otimes z'_j dz_j + (\mu(b_{j+1}/b_j)(a_{j+1}/a_j)) \otimes z_j dz'_j$$

belongs to  $\langle z'_j dz_j + z_j dz'_j \rangle$ , which shows our claim.

We may assume that  $\operatorname{val}(z_j) \leq \operatorname{val}(z'_j)$ . For some  $j' \geq j$  we have  $z'_j = t_j z_j$  in  $C_{j'}$ for some  $t_j$  in  $C_{j'}$ . We have  $dz_j, dz'_j$  in  $\Omega_{C_{j'}/B_{j'}}$  and  $z_j w'_j = 0$ , for  $w'_j = dz'_j + t_j dz_j$ . So  $C_{j'} \otimes_{C_j} \Omega_{C_j/B_j}$  is not torsion free. Here we should point out that the localizations are given by elements from  $\rho + ((y_{j'i})_i, z_{j'}, z'_{j'}), \rho \in k, \rho \neq 0$ , which cannot kill  $w'_j$ . Since  $f_j$  are injective we see that  $\Omega_{V'_0/V_0}$  has torsion.

Now **assume** that n > m + 1 and consider  $V''_0 = V' \cap k(x_1, \ldots, x_{n-1})$ . Apply induction on n-m, the case n-m = 1 having been considered above. By induction hypothesis, we assume that  $\Omega_{V''_0/V_0}$  has torsion. As above  $V''_0$  is a filtered direct limit of some localizations  $\tilde{B}_j$  of  $k[\tilde{y}_{j1}, \ldots, \tilde{y}_{j,n-1}]$  and  $V'_0$  is the filtered direct limit of some localizations  $\tilde{C}_j$  of  $\tilde{B}_j[\tilde{z}_j, \tilde{z}'_j]$ . Set  $I_j = (Z_j Z'_j - \tilde{y}_{j1}) \subset \tilde{B}_j[Z_j, Z'_j]$ . By definition we have the following exact sequence

$$0 \to H_1(\tilde{B}_j, \tilde{C}_j, \tilde{C}_j) \to I_j/I_j^2 \xrightarrow[19]{d} \tilde{C}_j d\tilde{z}_j \oplus \tilde{C}_j d\tilde{z}'_j \to \Omega_{\tilde{C}_j/\tilde{B}_j} \to 0.$$

If  $h \in I_j$  induces an element in Ker d then we get

$$(\partial h/\partial Z_j)(\tilde{z}_j, \tilde{z}'_j)d\tilde{z}_j \oplus (\partial h/\partial Z'_j)(\tilde{z}_j, \tilde{z}'_j)d\tilde{z}'_j = 0.$$

But  $h = \tilde{h}(Z_j Z'_j - \tilde{y}_{j1})$  for some  $\tilde{h} \in \tilde{B}_j[Z_j, Z'j]$  and it follows  $\tilde{h}(\tilde{z}_j, \tilde{z}'_j)\tilde{z}'_j = 0$ , that is  $\tilde{h}(\tilde{z}_j, \tilde{z}'_j) = 0$  and so  $\tilde{h} \in I_j$ . Hence d is injective and  $H_1(\tilde{B}_j, \tilde{C}_j, \tilde{C}_j) = 0$ .

In the Jacobi-Zariski sequence ([33, Theorem 3.3] applied to  $V_0 \to V_0'' \to V_0''$ 

$$0 = H_1(V_0'', V_0', V_0') \to V_0' \otimes_{V_0''} \Omega_{V_0''/V_0} \xrightarrow{\lambda} \Omega_{V_0'/V_0} \to \Omega_{V_0'/V_0''} \to 0$$

we see that the map  $\lambda$  is injective. It follows that  $\Omega_{V'_0/V_0}$  has torsion which proves our claim.

By Proposition 18 the immediate extension  $V_0 \subset V$  is ind-smooth. Assume, aiming for contradiction, that V' is ind-smooth over V. Then V' is ind-smooth over  $V_0$  and by the above lemma we get  $\Omega_{V'/V_0}$  flat over V'. Again by Proposition 18 we have V' ind-smooth over  $V'_0$ . As in the above lemma we obtain that  $\Omega_{V'/V'_0}$  is a flat module over V'. In the Jacobi-Zariski sequence applied to  $V_0 \to V'_0 \to V'$ 

$$H_1(V'_0, V', V') \to V' \otimes_{V'_0} \Omega_{V'_0/V_0} \to \Omega_{V'/V_0} \to \Omega_{V'/V'_0} \to 0$$

we have  $H_1(V'_0, V', V') = 0$  and the last two modules are flat by the above lemma. We obtain that  $V' \otimes_{V'_0} \Omega_{V'_0/V_0}$  is flat also over V' and so  $\Omega_{V'_0/V_0}$  is also flat, which is not possible because it has torsion. Thus V' is not ind-smooth over V.

**Remark 31.** In the above proof the main point was to show that when  $\Gamma'/\Gamma \neq 0$  has no torsion then  $\Omega_{V'_0/V_0}$  has torsion.

**Lemma 32.** Let  $V \subset V'$  be an extension of valuation rings of dimension one containing  $\mathbf{Q}$ . Assume that V contains its residue field and its value group  $\Gamma \subset \mathbf{R}$  is dense in  $\mathbf{R}$ . Also assume that the value group  $\Gamma' \subset \mathbf{R}$  of V' is finitely generated and  $\Gamma'/\Gamma$ ,  $\Gamma \neq \Gamma'$  has no torsion. Then the extension  $V \subset V'$  is not ind-smooth.

Proof. In the proof of Lemma 30 take  $W = V' \cap \operatorname{Frac}(V)(x_{m+1}, \ldots, x_n)$ . Then the extension  $V \subset W$  has the same residue field but the value group extension is  $\Gamma \subset \Gamma'$ . Then  $\Omega_{W/V}$  has torsion as in the proof of the quoted lemma. In the Jacobi-Zariski sequence applied to V, W, V'

$$H_1(W, V', V') \to V' \otimes_W \Omega_{W/V} \to \Omega_{V'/V}$$

we see that the left module is zero because the valuation extension  $W \subset V'$  is indsmooth (see Lemma 28), having the same value group (see Corollary 19). It follows that  $\Omega_{V'/V}$  has torsion. But if  $V \subset V'$  is ind-smooth then  $\Omega_{V'/V}$  is torsion free. So  $V \subset V'$  is not ind-smooth.

**Lemma 33.** Let  $V \subset V'$  be an extension of valuation rings of dimension one containing  $\mathbf{Q}$  with value groups  $\Gamma \subset \Gamma'$  and having the same residue field k. Assume that V contains k and its value group  $\Gamma \subset \mathbf{R}$  is dense in  $\mathbf{R}$ . Also assume that the value group  $\Gamma'/\Gamma \neq 0$  has no torsion and V' has a cross-section  $s : \Gamma' \to K'^*$  such that  $s(\Gamma) \subset K^*$ , K, K' being the fraction fields of V, resp. V'. Then the extension  $V \subset V'$  is not ind-smooth. Proof. We follow the proof of the above lemma. Take  $V_0 = V \cap k(s(\Gamma))$  and  $V'_0 = V' \cap k(s(\Gamma'))$ . For every finitely generated  $\Gamma_1 \subset \Gamma$  and  $\Gamma'_1 \subset \Gamma'$  such that  $\Gamma_1 \subset \Gamma'_1 \not\subset \Gamma'_1 \not \subset \Gamma'_1 \not = \Gamma'_1 \not \subset \Gamma'_1 \not = \Gamma'_1 \not = \Gamma'_1 \not = \Gamma'_1$ 

**Lemma 34.** Let  $V \subset V'$  be an extension of valuation rings of dimension one containing  $\mathbf{Q}$  with value groups  $\Gamma \subset \Gamma'$ . Assume that V contains its residue field and its value group  $\Gamma \subset \mathbf{R}$  is dense in  $\mathbf{R}$ . Also assume that the group  $\Gamma'/\Gamma \neq 0$  has no torsion and V' has a cross-section  $s : \Gamma' \to K'^*$  such that  $s(\Gamma) \subset K^*$ , K, K' being the fraction fields of V, resp. V'. Then the extension  $V \subset V'$  is not ind-smooth.

The proof follows as in Lemma 33 using now Lemma 32.

#### 7. The case when the value group is not finitely generated

A weaker form of Theorem 1 is given below with an independent proof (note that in the proof of Theorem 21 we do not use the Zariski's Uniformization Theorem.

**Theorem 35.** Every valuation ring V containing its residue field k of characteristic zero is a filtered direct limit of smooth k-algebras  $(R_i)_i$ , that is the inclusion  $k \subset V$  is ind-smooth (in particular, all the  $R_i$  are regular rings).

Proof. Let  $\Gamma$  be the value group of V, K the fraction field of V and  $\tilde{k}$ ,  $\tilde{\Gamma}$ ,  $\tilde{V}$ ,  $\tilde{K}$ ,  $\tilde{s} : \tilde{\Gamma} \to \tilde{K}^*$  be given as in Theorem A.10. Note that in the proof of Theorem A.10 the ultrapoduct  $k_1 \subset V_1$  of  $k \subset V$  gives an inclusion in  $V_1$  of its residue field. More precisely, given a map  $f : A \to B$  the ultrapower of f is the map between the ultrapowers of A and B given by  $[a_i] \to [f(a_i)]$ . By induction we see that the ultrapoduct  $k_n \subset V_n$  of  $k_{n-1} \subset V_{n-1}$  gives an inclusion in  $V_n$  of its residue field. Thus  $\tilde{V} = \bigcup_{n \in \mathbf{N}} V_n$  contains its residue field  $\tilde{k} = \bigcup_{n \in \mathbf{N}} k_n$ .

By Proposition 27 we see that  $\tilde{V}$  is an immediate extension of a valuation ring  $\tilde{V}_0$  which is a filtered union of localizations of smooth  $\tilde{k}$ -algebras. By Theorem 21 we get  $\tilde{V}_0 \subset \tilde{V}$  ind-smooth. Hence  $k \subset \tilde{V}$  is ind-smooth because  $k \subset \tilde{k}$  is separable and so ind-smooth (see Lemma 6). It follows that  $k \subset V$  is ind-smooth too. Indeed, let E = k[Y]/I,  $Y = (Y_1, \ldots, Y_n)$  be a finitely generated k-algebra and  $w : E \to V$  be a morphism of k-algebras. For the result we will apply Lemma 5 if we show that w factors through a smooth k-algebra. But the composite map  $E \to V \to \tilde{V}$  factors through a smooth k-algebra S = k[Z]/(h),  $Z = (Z_1, \ldots, Z_s)$  for a system of polynomials h from k[Z], thus it is the composite map  $E \xrightarrow{\psi} \tilde{V}$ , where  $\tilde{w}$  is given by  $Z \to \tilde{z} \in \tilde{V}^s$  and t is induced by  $Y \to g \in k[Z]^n$ . Then  $\tilde{z}$  is a solution of h and of the system P of polynomial equations g(Z) = w(Y) over V. Actually,  $\tilde{z}$  belongs to some  $V_n$  and so h, P has also a solution  $z_{n-1}$  in  $V_{n-1}$  because  $V_n$  is an ultrapower of  $V_{n-1}$ . By induction we get a solution  $z \in V$  of h, P. Therefore, w factors through the map  $S \to V, Z \to z$ .

**Lemma 36.** Let B be an A-algebra and  $A^n$ ,  $B^n$  be the product of n-copies of A, resp. B. Then  $\Omega_{B^n/A^n} \cong \Omega_{B/A}^n$  and  $H_1(A^n, B^n, B^n) \cong H_1(A, B, B)^n$ .

Proof. We treat only the case n = 2. If B = A[X]/I,  $X = (X_i)_i$  then  $B^2 = A^2[X]/J$ , where J is given by polynomials of the form  $h_{f,g} = \sum_{j=(j_1,\ldots,j_s)} (a_j, b_j) X^j$  for some polynomials  $f = \sum a_j X_j$ ,  $g = \sum b_j X^j$  from I. Then  $\Omega_{B^2/A^2}$  is the cokernel of the map  $d : J/J^2 \to \bigoplus_i B^2 dX_i$  given by  $h_{f,g} \to \sum_i (\partial f/\partial X_i, \partial g/\partial X_i) dX_i = \sum_i (\partial f/\partial X_i dX_i, \partial g/\partial X_i dX_i)$ . Also  $H_1(A^2, B^2, B^2)$  is the kernel of d and note that  $d(h_{(f,g)}) = 0$  if and only if  $d_0(f) = 0$  and  $d_0(g) = 0$ ,  $d_0$  being the map  $I/I^2 \to \bigoplus_i B dX_i$ . Hence  $\Omega_{B^2/A^2} \cong \Omega_{B/A}^2$  and  $H_1(A^2, B^2, B^2) \cong H_1(A, B, B)^2$ .

**Lemma 37.** Let *B* be an *A*-algebra,  $\mathcal{U}$  and ultrafilter on a set *U* and  $\hat{B}$ , resp.  $\hat{A}$ the ultrapowers of *B* (see the Appendix for the details), resp. *A* with respect to  $\mathcal{U}$ . Then  $\Omega_{\tilde{B}/\tilde{A}}$  (resp.  $H_1(\tilde{A}, \tilde{B}, \tilde{B})$ ) is the corresponding ultrapower of  $\Omega_{B/A}$  (resp.  $H_1(A, B, B)$ ) with respect to  $\mathcal{U}$ . In particular,  $\Omega_{B/A}$  has torsion if and only if  $\Omega_{\tilde{B}/\tilde{A}}$ has torsion and  $H_1(\tilde{A}, \tilde{B}, \tilde{B}) = 0$  if and only if  $H_1(A, B, B) = 0$ .

For the proof note for example that  $\Omega_{\tilde{B}/\tilde{A}}$  is the direct limit of  $\Pi_{u \in \mathcal{U}}(\Omega_{B/A})_u$ , where  $(\Omega_{B/A})_u = \Omega_{B/A}$  (see the Appendix).

**Proposition 38.** Let  $V \subset V'$  be an extension of valuation rings of dimension one containing  $\mathbf{Q}$  such that its residue field extension is trivial. Assume that the value groups  $\Gamma, \Gamma' \subset \mathbf{R}$  of V respectively V' are dense in  $\mathbf{R}$  and the factor of the value groups  $\Gamma'/\Gamma \neq 0$  has no torsion. Then the extension  $V \subset V'$  is not ind-smooth.

Proof. Using Proposition 9 we may suppose that V, V' are complete. So V contains its residue field. By Variant A.11 we find an extension of valuation rings  $\tilde{V} \subset \tilde{V}'$ such that there exists a cross-section  $\tilde{s} : \tilde{\Gamma}' \to (\tilde{K}')^*$  such that  $\tilde{s}(\tilde{\Gamma}) \subset \tilde{K}$ . We remind that we wrote  $\Gamma'$  as a filtered union of finitely generated subgroups  $(\Gamma'_i)_{i\in I}$  and set  $\Gamma_i = \Gamma'_i \cap \Gamma$ . Set  $V_{0i} = V \cap s'(\Gamma_i) = V' \cap s'(\Gamma_i)$ ,  $V'_{0i} = V' \cap s'(\Gamma'_i)$ . By Remark 31 the modules  $\Omega_{V'_{0i}/V_{0i}}$ ,  $i \in I$  have torsion. Note that the filtered union  $V_{01}$  of  $V_{0i}$ ,  $i \in I$  is a valuation ring with value group  $\Gamma$ , and similarly consider  $V'_{01}$  which has the value group  $\Gamma'$ . Clearly,  $\Omega_{V'_{01}/V_{01}}$  has torsion since it is the limit of  $V'_{01} \otimes_{V'_{01}} \Omega_{V'_{0i}/V_{0i}}$ . Set  $\tilde{V}_0 = \tilde{V} \cap s'(\tilde{\Gamma})$ ,  $\tilde{V}'_0 = \tilde{V}' \cap s'(\tilde{\Gamma}')$ . By iteration we define the extensions  $V_{0n} \subset V_n$ and  $V'_{0n} \subset V'_n$  with the same value group  $\Gamma_n$ ,  $\Gamma'_n$  obtained taking *n*-ultrapowers of  $\Gamma$ , resp.  $\Gamma'$  and we see that  $\Omega_{V'_{0n}/V_{0n}}$  has torsion by Lemma 37. Then  $\Omega_{\tilde{V}'_0/\tilde{V}_0}$  has torsion since it is the limit of  $\tilde{V}'_0 \otimes_{V'_{0n}} \Omega_{V'_{0n}/V_{0n}}$ .

Assume that  $V \subset V'$  is ind-smooth. In the Jacobi-Zariski sequence applied to  $\tilde{V}_0, \tilde{V}, \tilde{V}'$ 

$$H_1(\tilde{V}, \tilde{V}', \tilde{V}') \to \tilde{V}' \otimes_{\tilde{V}} \Omega_{\tilde{V}/\tilde{V}_0} \to \Omega_{\tilde{V}'/\tilde{V}_0} \to \Omega_{\tilde{V}'/\tilde{V}} \to 0$$

we claim that the left module is zero and the last module has no torsion by Lemmas 28, 37 because  $\Omega_{V'/V}$  has no torsion and  $H_1(V, V', V') = 0$ ,  $V \subset V'$  being indsmooth. More precisely, we see that  $\Omega_{V'_n/V_n}$  has no torsion and  $H_1(V_n, V'_n, V'_n) = 0$  for all n using Lemma 28 and by iteration Lemma 37. At the limit we get our claim.

Also  $\Omega_{\tilde{V}/\tilde{V}_0}$  is flat (so it has no torsion) since the extension  $\tilde{V}_0 \subset \tilde{V}$  is ind-smooth having the same value group (see Proposition 20). It follows that  $\Omega_{\tilde{V}'/\tilde{V}_0}$  has also no torsion. Now, in the Jacobi-Zariski sequence applied to  $\tilde{V}'_0, \tilde{V}'_0, \tilde{V}'$ 

$$H_1(\tilde{V}_0, \tilde{V}', \tilde{V}') \to \tilde{V}' \otimes_{\tilde{V}_0'} \Omega_{\tilde{V}_0'/\tilde{V}_0} \to \Omega_{\tilde{V}'/\tilde{V}_0}$$

we see that the left module is zero by Lemma 28 because  $\tilde{V}'_0 \subset \tilde{V}'$  is ind-smooth by Proposition 20. As above  $\Omega_{\tilde{V}'_0/\tilde{V}_0}$  has torsion and so  $\Omega_{\tilde{V}'/\tilde{V}_0}$  has torsion too, which is false.

**Acknowledgements:** This work has been partially elaborated in the frame of the International Research Network ECO-Math.

# Appendix. Cross-sections via infinite towers of ultrapowers by Kęstutis Česnavičius<sup>2</sup>

The goal of this Appendix is to show that by replacing a valuation ring by the limit of an infinite tower of its suitable ultrapowers one may arrange the valuation map val:  $V \setminus \{0\} \rightarrow \Gamma$  to admit a multiplicative section (see Theorem A.10). For this, we use techniques from model theory, specifically, the Keisler–Kunen theorem about the existence of good ultrafilters:<sup>3</sup> the idea is that constructing a section amounts to solving a system of equations for which any finite subsystem has a solution, and such systems always have solutions in well-chosen ultrapowers. For instance, if the system is countable, then solutions exists in any nonprincipal ultrafilter on **N**, and in general the main subtlety is in constructing the ultrafilter (within ZFC).

A.1. Cross-sections of valuation rings. For a valuation ring V with the value group  $\Gamma$  and the fraction field K, a *cross-section* of V is a section

 $s \colon \Gamma \to K^*$  in the category of abelian groups to the valuation map val:  $K^* \to \Gamma$ .

Concretely, s is a group homomorphism such that  $\operatorname{val}(s(\gamma)) = \gamma$  for  $\gamma \in \Gamma$ . For a submonoid  $M \subset \Gamma$ , a partial cross-section defined on M is a monoid morphism  $s: M \to K^*$  (concretely, s(0) = 1 and s(m + m') = s(m)s(m')) with  $\operatorname{val}(s(m)) = m$ for  $m \in M$ . Evidently, partial cross-sections exist for  $M \simeq \mathbb{Z}_{\geq 0}^r$  and correspond to choices of tuples of elements of  $K^*$  whose valuations form the standard basis of  $\mathbb{Z}_{\geq 0}^r$ .

**Example A.2.** Cross-sections exist when  $\Gamma$  is free as a **Z**-module, for instance, when it is finitely generated. As we now explain, they also exist when V is strictly Henselian of residue characteristic p and there is a free subgroup  $\Gamma_0 \subset \Gamma$  such that  $\Gamma/\Gamma_0$  is torsion with  $(\Gamma/\Gamma_0)[p^{\infty}] = 0$ . Indeed, we first define s on  $\Gamma_0$  and then, by Zorn's lemma, reduce to the situation in which s is already defined on some subgroup  $\Gamma' \supset \Gamma_0$  and needs to be extended to a  $\Gamma'' \supseteq \Gamma'$  with  $\Gamma''/\Gamma'$  cyclic of order n prime to p. For the latter, we first choose an  $x \in V$  such that val(x) lies in  $\Gamma''$  and generates the quotient  $\Gamma''/\Gamma'$ , which gives the following equation in V:

 $x^n = u \cdot s(n \cdot \operatorname{val}(x))$  for some  $u \in V^*$ .

Since  $p \nmid n$ , Hensel's lemma [10, IV, 18.5.17] (which is the Implicit Function Theorem in this context) now implies that the equation  $X^n = u$  has a solution in V, so we may adjust x to assume that u = 1. Granted this, s then extends to  $\Gamma''$  by setting

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I thank Dorin Popescu for helpful discussions and for hosting me during a visit to Romania in March 2019. I thank Matthias Aschenbrenner for helpful comments. This project has received funding from l'Agence Universitaire de la Francophonie and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 851146).

<sup>&</sup>lt;sup>3</sup>After this appendix was written, we learned of a much simpler way to deduce sharper versions of Theorem A.10 and Variant A.11 from the results reviewed in A.8, see [1, 3.3.39, 3.3.40]. We left this appendix in place in case the method used here would prove useful for other purposes.

s(val(x)) = x: indeed, any relation  $N \cdot val(x) = \gamma$  with  $N \in \mathbb{Z}$  and  $\gamma \in \Gamma'$  must be a multiple of such a relation with N = n, so  $s(N \cdot val(x)) = s(\gamma)$ .

**A.3. Ultrafilters and ultraproducts.** We recall that an *ultrafilter* on a nonempty set U is a set  $\mathscr{U}$  of subsets of U that is closed under finite intersections, closed under taking supersets, does not contain the empty set, and for every  $U' \subset U$  contains either U' or  $U \setminus U'$ . Such a  $\mathscr{U}$  is *principal* if it consists of all the subsets containing some fixed  $u \in U$ , and is *nonprincipal* otherwise. An ultrafilter  $\mathscr{U}$  is *countably incomplete* if some countable collection of elements of  $\mathscr{U}$  has an empty intersection. Such a  $\mathscr{U}$  is also nonprincipal and it exists whenever U is infinite (see [4, §A.3, 8.4]). We view any ultrafilter  $\mathscr{U}$  as a partially ordered set, where  $U' \leq U''$  if  $U' \supseteq U''$ .

For any category  $\mathcal{C}$  that has small products and filtered direct limits, the *ultra-product* of a set  $\{C_u\}_{u\in U}$  of objects of  $\mathcal{C}$  with respect to an ultrafilter  $\mathscr{U}$  on U, which we denote abusively by  $\prod_{\mathscr{U}} C_u$ , is

 $\varinjlim_{U'\in\mathscr{U}}(\prod_{u\in U'}C_u) \quad \text{where transition maps are projections onto partial products}$ (the limit is filtered because  $\mathscr{U}$  is closed under finite intersections). In the case when all the  $C_u$  are the same object  $C \in \mathcal{C}$ , we call  $\prod_{\mathscr{U}} C$  an *ultrapower* of C.

A.4. Ultraproducts of valuation rings. We will work with ultraproducts of rings or modules. For instance, an ultraproduct of fields is again a field: every nonzero element is invertible (thanks to the axiom that  $U' \in \mathscr{U}$  or  $U \setminus U' \in \mathscr{U}$ ). Likewise, an ultraproduct  $\prod_{\mathscr{U}} V_u$  of valuation rings  $\{V_u\}_{u \in U}$  with fraction fields  $\{K_u\}_{u \in U}$  is a valuation ring with fraction field  $\prod_{\mathscr{U}} K_u$ : for any nonzero element v of the latter, either v or  $v^{-1}$  lies in  $\prod_{\mathscr{U}} V_u$ . We see similarly that

- (1) the maximal ideal of  $\prod_{\mathscr{U}} V_u$  is the ultraproduct  $\prod_{\mathscr{U}} \mathfrak{m}_u$  of the maximal ideals;
- (2) the residue field of  $\prod_{\mathscr{U}} V_u$  is the ultraproduct  $\prod_{\mathscr{U}} k_u$  of the residue fields;
- (3) the value group of  $\prod_{\mathscr{U}} V_u$  is the ultraproduct  $\prod_{\mathscr{U}} \Gamma_u$  of the value groups;
- (4) the monoid of nonnegative elements  $(\prod_{\mathscr{U}} \Gamma_u)_{\geq 0}$  is identified with  $\prod_{\mathscr{U}} (\Gamma_u)_{>0}$ .

The existence of "well-chosen" ultrapowers mentioned above rests on the Keisler– Kunen theorem from model theory that we recall in the following lemma. Keisler proved it in [13] assuming the Generalized Continuum Hypothesis and Kunen gave an unconditional proof in [16, Theorem 3.2].

**Lemma A.5** ([6], Theorem 6.1.4). For an infinite set U, there is a countably incomplete ultrafilter  $\mathscr{U}$  on U such that for any inclusion-reversing function

 $f: \{ \text{finite subsets of } U \} \to \mathscr{U}, \text{ there is a function } f_0: \{ \text{finite subsets of } U \} \to \mathscr{U}$ that is also inclusion-reversing and satisfies

 $f_0(U') \subset f(U')$  and  $f_0(U' \cup U'') = f_0(U') \cap f_0(U'')$  for all finite subsets  $U', U'' \subset U$ .

Of course, the requirement that  $f_0$  be inclusion-reversing is superfluous: it is a special case of the requirement that  $f_0$  transform finite unions into intersections.

We now verify that the ultrapowers that result from the ultrafilters supplied by Lemma A.5 have the promised property of solvability of systems of equations.

**Proposition A.6.** For an infinite cardinal  $\kappa$ , every ultrafilter  $\mathscr{U}$  supplied by Lemma A.5 for a set U of cardinality  $\kappa$  is such that: for any ring R (resp., and any left R-module M), any polynomial (resp., linear) system of equations

$$\{g_i(\{X_\sigma\}_\sigma)=0\}_{i\in I} \quad (resp., \quad \{\sum_{\sigma} r_{i,\sigma}X_\sigma=m_i\}_{i\in I}) \quad with \quad \#I \le \kappa$$

in variables  $\{X_{\sigma}\}_{\sigma}$  and coefficients in  $\prod_{\mathscr{U}} R$  (resp.,  $r_{i,\sigma} \in \prod_{\mathscr{U}} R$  and  $m_i \in \prod_{\mathscr{U}} M$ ) has a solution in  $\prod_{\mathscr{U}} R$  (resp.,  $\prod_{\mathscr{U}} M$ ) as soon as so do all its finite subsystems.

*Proof.* The assertion is a concrete case of the model-theoretic [6, Theorem 6.1.8], and the latter is sharper in multiple aspects. For convenience, we recall the argument.

For brevity, we denote the system in question by  $\{g_i = 0\}_{i \in I}$  and we lift it to a system  $\{\tilde{g}_i = 0\}_{i \in I}$  with coefficients in  $\prod_{u \in U} R$  (resp., and in  $\prod_{u \in U} M$ ) and the same variables  $\{X_{\sigma}\}_{\sigma}$  by lifting the nonzero coefficients along the surjection

$$\prod_{u \in U} R \twoheadrightarrow \prod_{\mathscr{U}} R \quad (\text{resp., and along} \quad \prod_{u \in U} M \twoheadrightarrow \prod_{\mathscr{U}} M \quad \text{for the } m_i).$$

Since  $\mathscr{U}$  is countably incomplete, we may fix a decreasing sequence

$$U \supset U_0 \supset U_1 \supset U_2 \supset \cdots$$
 of sets in  $\mathscr{U}$  with  $\bigcap_{n \ge 0} U_n = \emptyset$ 

We then define a function

 $f: \{ \text{finite subsets of } I \} \to \mathscr{U}$ 

by letting  $pr_u$  denote the projection onto the u-th factor of  $\prod_{u \in U}$  and setting

 $f(I') := U_{\#I'} \cap \{ u \in U \mid \text{the system } \{ \operatorname{pr}_u(\widetilde{g}_i) = 0 \}_{i \in I'} \text{ is solvable in } R \text{ (resp., } M) \}.$ 

The well-definedness of f follows from the solvability of the subsystem  $\{g_i = 0\}_{i \in I'}$  in  $\prod_{\mathscr{U}} R$  (resp.,  $\prod_{\mathscr{U}} M$ ) and from the stability of  $\mathscr{U}$  under supersets. By construction,  $f(I') \supset f(I'')$  whenever  $I' \subset I''$ , so, since  $\#I \leq \#U$ , Lemma A.5 supplies a function

 $f_0\colon \{\text{finite subsets of } I\} \to \mathscr{U} \text{ such that } f_0(I') \subset f(I'), \ f_0(I' \cup I'') = f_0(I') \cap f_0(I'')$ 

for all finite subsets  $I', I'' \subset I$  (technically, to apply Lemma A.5 we first embed I into U as a subset and then extend f to finite subsets  $U' \subset U$  by the rule  $U' \mapsto f(U' \cap I)$ ).

For each  $u \in U$ , we set

$$I_u := \{ i \in I \mid u \in f_0(\{i\}) \}.$$

Whenever,  $i_1, \ldots, i_n \in I_u$  are pairwise distinct, we have

$$u \in f_0(\{i_1\}) \cap \dots \cap f_0(\{i_n\}) = f_0(\{i_1, \dots, i_n\}) \subset f(\{i_1, \dots, i_n\}) \subset U_n,$$

so, since the  $U_n$  have empty intersection, each  $I_u$  is finite. Then the preceding display applied to an enumeration of  $I_u$  shows that  $u \in f(I_u)$ , to the effect that the system  $\{\operatorname{pr}_u(\widetilde{g}_i) = 0\}_{i \in I_u}$  has a solution  $\{x_{\sigma,u}\}_{\sigma}$  in R (resp., M).

We claim that  $\{x_{\sigma} := (x_{\sigma,u})_{u \in U}\}_{\sigma}$  gives a solution in  $\prod_{\mathscr{U}} R$  (resp.,  $\prod_{\mathscr{U}} M$ ) to the system  $\{g_i = 0\}_{i \in I}$ . Indeed, for every  $i \in I$  we have  $f_0(\{i\}) \in \mathscr{U}$  and for every  $u \in f_0(\{i\})$  we have  $i \in I_u$ , so  $\widetilde{g}_i(\{x_{\sigma}\}_{\sigma}) = 0$  in the projection on  $\prod_{u \in f_0(\{i\})}$ .  $\Box$ 

The argument is not specific to rings or modules, and it also shows the following.

**Variant A.7.** For an infinite cardinal  $\kappa$ , every ultrafilter  $\mathscr{U}$  supplied by Lemma A.5 for a set U of cardinality  $\kappa$  is such that: for any monoid G, any system

$$\{g_i(\{X_\sigma\}_\sigma) = g'_i(\{X_\sigma\}_\sigma)\}_{i \in I} \quad with \quad \#I \le \kappa$$

of monomial equations in variables  $\{X_{\sigma}\}_{\sigma}$  and coefficients in  $\prod_{\mathscr{U}} G$  has a solution in  $\prod_{\mathscr{U}} G$  as soon as so does each of its finite subsystems.

As we now review, Proposition A.6 supplies algebraically compact ultrapowers.

**A.8.** Algebraic compactness. We fix a ring R and recall that a map  $M \to M'$  of left R-modules is *pure* if the map  $M'' \otimes_R M \to M'' \otimes_R M'$  is injective for every right R-module M''. An R-module M is algebraically compact (or *pure-injective*) if every pure map  $M \to M'$  of R-modules is a split injection. For example, if M is an algebraically compact abelian group (so  $R = \mathbb{Z}$ ), then every short exact sequence

 $0 \to M \to M' \to M'/M \to 0$  of abelian groups with  $(M'/M)_{\text{tors}} = 0$  splits.

A concrete criterion for algebraic compactness is given by [11, 7.1 (with 6.5)]: a left *R*-module *M* is algebraically compact if every system of equations

$$\{\sum_{\sigma} r_{i,\sigma} X_{\sigma} = m_i\}_{i \in I}$$
 with  $r_{i,\sigma} \in R$  and  $m_i \in M$ 

has a solution in M as soon as so do all its finite subsystems. Moreover, by [11, 7.28, 7.29], it suffices to consider systems with cardinality  $\#I \leq \max(\#R, \#\mathbf{Z})$ . In particular, thanks to Proposition A.6, there is an ultrafilter  $\mathscr{U}$  such that for any R-module M, the R-module  $\prod_{\mathscr{U}} M$  is algebraically compact.

With model-theoretic input in place, we turn to the tower of ultrapowers argument in Theorem A.10. The final input is the following lemma proved in [7, 2.2], [23, 4.6.1], or [9, 6.1.30] that captures the "combinatorial" part of local uniformization.

**Lemma A.9.** For a totally ordered abelian group  $\Gamma$ , the submonoid  $\Gamma_{\geq 0} \subset \Gamma$  of nonnegative elements is a filtered increasing union of its finite free submonoids isomorphic to  $\mathbf{Z}_{\geq 0}^r$  (where  $r \in \mathbf{Z}_{\geq 0}$  need not be constant).

**Theorem A.10.** For a valuation ring V with value group  $\Gamma$ , there is a countable sequence of ultrafilters  $\mathscr{U}_1, \mathscr{U}_2, \ldots$  on some respective sets  $U_1, U_2, \ldots$  for which the valuation rings  $\{V_n\}_{n\geq 0}$  defined inductively by  $V_0 := V$  and  $V_{n+1} := \prod_{\mathscr{U}_{n+1}} V_n$  are such that the valuation ring

$$\widetilde{V} := \underline{\lim}_{n \ge 0} V_n$$
 has a cross-section  $\widetilde{s} : \widetilde{\Gamma} \to \widetilde{K}^*$ ,

where  $\widetilde{K}$  and  $\widetilde{\Gamma}$  are the fraction field and the value group of  $\widetilde{V}$ .

*Proof.* We let  $K_n$  and  $\Gamma_n$  denote the fraction field and the value group of  $V_n$ , so that  $\Gamma_{n+1} \cong \prod_{\mathscr{U}_{n+1}} \Gamma_n$  and  $K_{n+1} = \prod_{\mathscr{U}_{n+1}} K_n$  (see §A.4) with

$$\widetilde{\Gamma} \cong \underline{\lim}_{n \ge 0} \Gamma_n$$
 and  $\widetilde{K} \cong \underline{\lim}_{n \ge 0} K_n$ .

The idea is to build ultrafilters  $\mathscr{U}_n$  one by one using Lemma A.5 in such a way that a desired cross-section

 $\widetilde{s}: \widetilde{\Gamma} \to \widetilde{K}^*$  would be the limit of compatible partial cross-sections  $s_n: \Gamma_n \to K_{n+1}^*$ .

For this, as an initial step, we replace V by a suitable ultrapower to ensure that the abelian group  $\Gamma$  is algebraically compact (see §A.8). Granted this, it suffices to carry out the inductive step: setting  $\Gamma_{-1} := 0$  for convenience and assuming that we have already constructed  $s_{n-1}$  and  $V_n$  for some  $n \ge 0$  in such a way that the abelian groups  $\Gamma_{n-1}$  and  $\Gamma_n$  are algebraically compact, it suffices to construct  $V_{n+1}$ with  $\Gamma_{n+1}$  algebraically compact in such a way that  $s_{n-1}$  extends to an  $s_n$ .

The role of algebraic compactness is to split the map  $\Gamma_{n-1} \hookrightarrow \Gamma_n \cong \prod_{\mathscr{U}_n} \Gamma_{n-1}$ whose cokernel is torsion free:

$$\Gamma_n \cong \Gamma_{n-1} \oplus G$$
 for some subgroup  $G \subset \Gamma_n$ .

Thanks to this splitting, we only need to build an ultrafilter  $\mathscr{U}_{n+1}$  and a partial cross-section  $s_G : G \to (\prod_{\mathscr{U}_{n+1}} K_n)^*$  such that  $\prod_{\mathscr{U}_{n+1}} \Gamma_n$  is algebraically compact. In fact, we let  $\mathscr{U}_{n+1}$  be any ultrafilter as in Lemma A.5 applied to the cardinal  $\max(\#\Gamma_n, \#\mathbf{Z})$ . Then  $\prod_{\mathscr{U}_{n+1}} \Gamma_n$  is necessarily algebraically compact by the criterion reviewed in §A.8 and Proposition A.6.

The subgroup G inherits a total order from  $\Gamma_n$ , and any partial cross-section

 $s_{G_{\geq 0}}: G_{\geq 0} \to (\prod_{\mathscr{U}_{n+1}} V_n) \setminus \{0\}$  will give rise to a desired  $s_G$ .

For each  $g \in G_{>0}$ , we fix a  $v_g \in V_n$  with  $\operatorname{val}(v_g) = g$ . Then  $s_{G_{\geq 0}}$  amounts to a solution in  $\prod_{\mathscr{U}_{n+1}} V_n$  of the following system of equations in variables  $\{X_g, U_g, U'_g\}_{g \in G_{>0}}$ :

$$\{X_{g+g'} = X_g X_{g'}, \ X_g U_g = v_g, \ U_g U'_g = 1\}_{g,g' \in G_{>0}}.$$

Likewise, for any submonoid  $G' \subset G_{\geq 0}$ , the restriction of  $s_{G_{\geq 0}}|_{G'}$ , that is, a partial cross-section defined on G', amounts to a solution in  $\prod_{\mathscr{U}_{n+1}} V_n$  of the subsystem consisting of those equations that only involve the variables  $\{X_g, U_g, U'_g\}_{g \in G'}$ . However, a partial cross-section  $G' \to \prod_{\mathscr{U}_{n+1}} V_n$  (and even  $G' \to V_n$ ) certainly exists if  $G' \simeq \mathbf{Z}^d_{\geq 0}$ , and, by Lemma A.9, the monoid  $G_{\geq 0}$  is a filtered increasing union of such G'. This implies that every finite subsystem of the above system has a solution in  $\prod_{\mathscr{U}_{n+1}} V_n$  (and even in  $V_n$ ). Then, by Proposition A.6, the entire system has a solution in  $\prod_{\mathscr{U}_{n+1}} V_n$ , which completes the inductive step.  $\Box$ 

**Variant A.11.** For every faithfully flat map  $V \subset V'$  of valuation rings with value groups  $\Gamma \subset \Gamma'$  such that  $\Gamma'/\Gamma$  is torsion free, there is a countable sequence of ultrafilters  $\mathscr{U}_1, \mathscr{U}_2, \ldots$  on some respective sets  $U_1, U_2, \ldots$  for which the valuation rings  $\{V_n\}_{n\geq 0}$  and  $\{V'_n\}_{n\geq 0}$  defined inductively by  $V_0 := V$  and  $V'_0 := V'$  with  $V_{n+1} := \prod_{\mathscr{U}_{n+1}} V_n$  and  $V'_{n+1} := \prod_{\mathscr{U}_{n+1}} V'_n$  are such that the valuation ring

$$\widetilde{V}' = \varinjlim_{n \ge 0} V'_n \quad with \quad \widetilde{K}' := \operatorname{Frac}(\widetilde{V}') \quad has \ a \ cross-section \quad \widetilde{s} : \widetilde{\Gamma}' \to \widetilde{K}'^*$$

whose restriction to the value group  $\widetilde{\Gamma}$  of  $\widetilde{V} := \underset{n \geq 0}{\lim} V_n$  lands in  $\widetilde{K} := \operatorname{Frac}(\widetilde{V})$ .

*Proof.* An ultrapower of an ultrapower is itself an ultrapower [6, 6.5.2], so we may make an initial replacement of V and V' by suitable large ultrapowers and use §A.8 with Proposition A.6 to ensure that  $\Gamma$  is algebraically compact and later absorb the appearing initial ultrafilter into  $\mathcal{U}_1$  (alternatively, we could simply insert this initial ultrafilter as  $\mathcal{U}_1$  without using loc. cit.). Then, thanks to the torsion-freeness assumption on  $\Gamma'/\Gamma$ , the inclusion  $\Gamma \subset \Gamma'$  splits. A choice of a splitting induces a compatible splitting on any ultrapower, so the proof of Theorem A.10 continues to give the claimed variant granted that we take advantage of the splitting to build the cross-section in such a way that its restriction to  $\tilde{\Gamma}$  lands in  $\tilde{K}$ .

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