

Quantifier elimination for o-minimal structures expanded by a valuational cut

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Abstract

Let R be an o-minimal expansion of a group in a language in which $\text{Th}(R)$ eliminates quantifiers, and let C be a predicate for a valuational cut in R . We identify a condition that implies quantifier elimination for $\text{Th}(R, C)$ in the language of R expanded by C and a small number of constants, and which, in turn, is implied by $\text{Th}(R, C)$ having quantifier elimination and being universally axiomatizable. The condition applies for example in the case when C is a convex subring of an o-minimal field R and its residue field is o-minimal.

Keywords— o-minimality, valuational weakly o-minimal structure, quantifier elimination, universal axiomatization, convex valuation ring

Mathematics subject classification— 03C10, 03C64

1 Introduction

Throughout, we let R be an o-minimal expansion of a group in a language \mathcal{L}_0 in which $\text{Th}(R)$ eliminates quantifiers. We expand our language with a unary predicate C used to define a valuational cut in R .

The main result of this paper concerns cuts C that satisfy a condition we call $*$. We postpone a complete definition until Section 2 (Definition 2.3), but roughly it says that forking is symmetric in Morley sequences in the cut C . We show that

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for such a cut, $Th(R, C)$ eliminates quantifiers, is universally axiomatizable, and has definable Skolem functions in the language $\mathcal{L} := \mathcal{L}_0 \cup \{C\}$ expanded by a small number of constants.

Structures (R, C) as described above are instances of valuations weakly o-minimal structures. Weakly o-minimal structures were first introduced by Dickmann [5]. In [11], Macpherson, Marker, Steinhorn proved a basic dichotomy: Every weakly o-minimal structure is either valuations or non-valuations. While the non-valuations ones are rather well-understood (see for example Bar-Yehuda, Hason, Peterzil [3]), and have been shown to share many desirable properties with o-minimal structures, valuations weakly o-minimal structures have been, at least at a certain level of generality, less well-explored.¹

Examples of valuations weakly o-minimal structures include structures (R, V) , where R is an o-minimal expansion of a real closed field and V is a predicate for a convex subring (equivalently, for the cut of a convex subring). We shall denote the class of all such (R, V) by \mathcal{V} . A particularly well-behaved subclass of \mathcal{V} is the class \mathcal{T} of T-convex structures, in which our results have been known to hold for a long time (see van den Dries, Lewenberg [7] and van den Dries [6]). While having proven to be quite useful, T-convex structures do not include all the cases of interest. For example, if C is the cut of the convex hull of \mathbb{Q} in R , then (R, C) is in general not T-convex. However, this case falls into the class \mathcal{O} , the subclass of \mathcal{V} consisting of all (R, V) with o-minimal residue field. While not being as well-behaved as \mathcal{T} , \mathcal{O} is known to be first-order axiomatizable (see Maříková [13]) and the theories of the structures in \mathcal{O} are model-complete relative to quantifier elimination in R and after expanding the language by a small number of constants (see Ealy, Maříková [9]). One has

$$\mathcal{T} \subsetneq \mathcal{O} \subsetneq \mathcal{V}.$$

We show that all structures in \mathcal{O} satisfy property $*$, and that in $\mathcal{V} \setminus \mathcal{O}$ there are both structures that satisfy $*$ and such that do not.

Our notation and set-up are as follows: Definable shall mean definable with parameters. If M is a structure, then we write M -definable to mean definable in the structure M with parameters from M . Let M be an o-minimal structure. If $M \preceq \mathcal{M}$ and $a \in \mathcal{M} \setminus M$, then we denote by $M\langle a \rangle$ the structure generated by a in \mathcal{M} over M (so $M\langle a \rangle$ is an abbreviation for $\text{dcl}_{\mathcal{M}}(a)$ and we use these notations interchangeably). Note that $M \preceq M\langle a \rangle \preceq \mathcal{M}$.

When M is o-minimal, we follow Marker, Steinhorn [12] (and many others) in saying that a type $q \in S_1(M)$ is a *cut* if there are nonempty disjoint subsets C^- and C^+ of M such that $C^- < C^+$, $C^- \cup C^+ = M$, C^- has no supremum in M , and q consists of the formulas $c < x$ for all $c \in C^-$ and $x < c$ for all $c \in C^+$. (Thus, cuts are precisely the nondefinable 1-types.) We call a cut in M *valuations* if there is $\epsilon \in M^{>0}$ such that C^- is closed under addition by ϵ .

¹In [14], Laskowski, Shaw claim to prove the existence of definable Skolem functions for structures (R, C) , but there is a gap in the proof.

As stated above, we let R be an o-minimal expansion of a group in a language \mathcal{L}_0 . We assume that \mathcal{L}_0 is such that $T_0 = \text{Th}(R)$ eliminates quantifiers in \mathcal{L}_0 , and that \mathcal{L}_0 and T_0 have been expanded by definitions as follows: for each \mathcal{L}_0 -formula $\phi(x_1, \dots, x_n, y)$ such that

$$T_0 \vdash \forall x_1 \dots \forall x_n \exists! y \phi(x_1, \dots, x_n, y)$$

we add a new function symbol f to \mathcal{L}_0 and the axiom

$$\phi(x_1, \dots, x_n, f(x_1, \dots, x_n))$$

to the theory T_0 . Then T_0 is universally axiomatizable.

By C we shall denote a unary predicate for a convex, downward closed set such that the type

$$\{x > r : r \in C\} \cup \{x < r : r > C\}$$

is a cut. In the special case when C is the downward closure of a valuation ring, we shall also use C for the valuation ring itself. We write $C(M)$ for the interpretation of C in a structure M . We consider (R, C) in the language $\mathcal{L} = \mathcal{L}_0 \cup \{C\}$, and we use T to denote the theory of (R, C) (in the language \mathcal{L}). We work inside a monster model (\mathcal{R}, C) of T . In particular, if $M \preceq \mathcal{R}$, then in (M, C) the predicate C is realized by the set $C(\mathcal{R}) \cap M$.

For $A \subseteq \mathcal{R}$, by $S_1(A)$ we mean the 1-types in the (o-minimal!) language \mathcal{L}_0 over A . Similarly, dcl shall always be dcl in the o-minimal language \mathcal{L}_0 .

We will use the following properties of nondividing which hold in all theories (see [1] Lemma 5.2 for a proof):

- (monotonicity) If $A \downarrow_C^d B$, $A_0 \subseteq A$ and $B_0 \subseteq B$, then $A_0 \downarrow_C^d B_0$.
- (base monotonicity) Suppose $D \subseteq C \subseteq B$. If $A \downarrow_D^d B$, then $A \downarrow_C^d B$.
- (left transitivity) Suppose $D \subseteq C \subseteq B$. If $B \downarrow_C^d A$ and $C \downarrow_D^d A$, then $B \downarrow_D^d A$.

Furthermore, we will use that forking and dividing are the same in o-minimal theories. (See the remarks preceding Proposition 2.8 together with Corollary 5.6 in [4] for a proof that forking is the same as dividing in weakly o-minimal theories). This allows us to also use the following:

- (extension) If $A \downarrow_C B$ and $\hat{B} \supseteq B$, then there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}$.

Finally, we note a useful observation about o-minimal structures.

Fact 1.1 *Let $p \in S_1(R)$ be non-isolated. Then there is no \mathcal{L}_0 -definable over R decreasing function mapping the set of realizations of p in \mathcal{R} to itself.*

PROOF: A decreasing function with the above property would have a fixed point a realizing p . But then $a \in \text{dcl}(R)$, a contradiction with p including $\{x \neq r \mid r \in R\}$.

□

2 The (non)-symmetry of forking in Morley sequences in invariant one-types in o-minimal structures

We wish to establish Theorem 3.1, an analogue of Theorem 3.3 [9], which gives a sufficient and necessary condition for a superstructure $(R\langle a \rangle, C)$ of (R, C) to be an elementary extension.

In [9], this theorem is proved with the help of Theorem 2.7 [9], which says that in o-minimal expansions of groups, forking in Morley sequences in invariant one-types is symmetric. Unfortunately, while working towards the results presented here, we realized that the proof of Theorem 2.7 in [9] has a gap, and that the theorem does not hold in the stated generality. While this does not pose a problem for the model completeness result in [9] (Theorem 2.7 holds in the context of [9], namely, when V is a convex subring of an o-minimal field R with o-minimal residue field – see Appendix), here we need to bridge the gap in the proof of Theorem 3.3. We do this by additionally assuming that the cut C has property $*$ (see Definition 2.3).

Definition 2.1 *Given an A -invariant type p over \mathcal{R} , and $B \supseteq A$, a Morley sequence in p over B is any sequence t_1, t_2, \dots in \mathcal{R} constructed as follows: let $t_1 \models p|_B$ and having defined t_1, \dots, t_n , let $t_{n+1} \models p|_{Bt_1 \dots t_n}$.*

Note that in a Morley sequence over B , $t_i \downarrow_B t_1 \dots t_{i-1}$ and moreover the sequence is indiscernible over B .

Proposition 2.2 *Let $p \in S_1(\mathcal{R})$ be the global type $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$, and let t_1, \dots, t_n form a finite Morley sequence in p over R . Let $q \in S_1(\mathcal{R})$ be the global type implied by $r < x$ for all $r \in \mathcal{R}$ with $r < R^{>C(R)}$, and $x < R^{>C(R)}$ (i.e. the R -invariant type at the opposite side of the cut from p). Then the following are equivalent:*

1. *There is no function f , \mathcal{L}_0 -definable over $Rt_2t_3 \dots t_{n-1}$, which maps a cofinal sequence in $C(R)$ onto a coinitial sequence in the convex hull of $R^{>C(R)}$ in \mathcal{R} .*
2. $t_1 \downarrow_R t_2 \dots t_n$.
3. t_n, \dots, t_1 is a finite Morley sequence in $q \in S_1(\mathcal{R})$ over R .

PROOF: First note that for $n = 2$, (1) is always true by Fact 1.1, as an \mathcal{L}_0 -definable over R function mapping a cofinal segment of $C(R)$ to a coinitial segment of the convex hull of $R^{>C(R)}$ in \mathcal{R} would be a decreasing \mathcal{L}_0 -definable over R function mapping a complete type over R to itself. Also, it is always the case that $t_1 \downarrow_R t_2$ and $t_1 \models q|_{Rt_2}$, since if either of these failed to be true, there would be some \mathcal{L}_0 -definable over R function f with $f(t_2) > t_1$. As already noted, such a function cannot be decreasing. But nor can it be increasing, as $f^{-1}(t_1)$ would be less than t_2 , contradicting the choice of t_2 .

So we will assume $n > 2$, and set $\underline{t} = t_2, \dots, t_{n-1}$.

3. \implies 2. Clearly, $t_1 \models q|_{R\bar{t}t_n}$ implies $t_1 \downarrow_R \bar{t}t_n$.
2. \implies 1. Assume that $f_{\bar{t}}$ maps a cofinal sequence in $C(R)$ to a coinital sequence of the convex hull of $R^{>C(R)}$ in \mathcal{R} . Then $t_1 < f_{\bar{t}}(t_n) < R^{>C(R)}$, and thus t_1 is contained in the interval $(t_2, f_{\bar{t}}(t_n))$. This witnesses $t_1 \not\downarrow_R \bar{t}t_n$.
1. \implies 3. Assume that $t_1 \not\models q|_{R\bar{t}t_n}$, and, without loss of generality, assume that n is the least such. Then there is an element $s \in R\langle \bar{t}t_n \rangle$ realizing $q|_R$ with $t_1 < s$. Write $s = f_{\bar{t}}(t_n)$ for some \mathcal{L}_0 -definable over R function f . We claim that $f_{\bar{t}}$ maps a cofinal sequence in $C(R)$ onto a coinital sequence in the convex hull of $R^{>C(R)}$ in \mathcal{R} .

We first show that $f_{\bar{t}}$ is strictly decreasing on some interval I containing a cofinal sequence in $C(R)$ and that $f_{\bar{t}}(I \cap C(R))$ is contained in the convex hull of $R^{>C(R)}$ in \mathcal{R} : By o-minimality of $R\langle \bar{t} \rangle$, $f_{\bar{t}}$ is continuous and strictly monotone on some interval I with endpoints in $\text{dcl}(R\bar{t})$ and containing t_n . Since \bar{t}, t_n is a Morley sequence in p over R , the left endpoint of I is contained in the convex hull of $C(R)$ in $R\langle \bar{t} \rangle$. For sufficiently big $c \in C^-$ we have $t_1 < f_{\bar{t}}(c)$, since $t_n \in \{x: f_{\bar{t}}(x) > t_1\}$, and $\{x: f_{\bar{t}}(x) > t_1\}$ is \mathcal{L}_0 -definable over $R\langle \bar{t} \rangle$. Moreover, $f_{\bar{t}}(c) < R^{>C(R)}$ would yield a contradiction with the minimality of n . So we have $f_{\bar{t}}(t_n) < f_{\bar{t}}(c)$, hence $f_{\bar{t}}$ is decreasing on I , and $f_{\bar{t}}(I \cap C(R))$ is contained in the convex hull of $R^{>C(R)}$ in \mathcal{R} .

It remains to show that $f_{\bar{t}}(C(R) \cap I)$ is coinital in the convex hull of $R^{>C(R)}$ in \mathcal{R} . Suppose not. Then there is $c \in R^{>C(R)}$ with $c < f_{\bar{t}}(C(R) \cap I)$. Then $f_{\bar{t}}^{-1}(c) < t_n$ but realizes $p|_R$. This contradicts the definition of t_n .

□

Definition 2.3 Let $p \in S_1(\mathcal{R})$ be the global type $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$. We say that the cut defined by C in R has property $*$ (or (R, C) has property $*$) if whenever $\bar{t} = t_1, \dots, t_n$ form a finite Morley sequence in p over R , then the equivalent conditions of Proposition 2.2 hold.

We use C both for a predicate in the expanded language \mathcal{L} and to denote a cut, i.e. a non-definable 1-type in the o-minimal language \mathcal{L}_0 . Property $*$ is better thought of as a property of an o-minimal 1-type rather than as a property of the predicate C .

Proposition 2.4 Let $p \in S_1(\mathcal{R})$ be a global R -invariant type such that $p|_R$ is the type corresponding to $C(R)$ and assume further that (R, C) has property $*$. Let t_1, \dots, t_n be a finite Morley sequence in p over R . Then

$$t_1 \dots t_k \downarrow_R t_{k+1} \dots t_n \text{ and } t_{k+1} \dots t_n \downarrow_R t_1 \dots t_k.$$

for all k with $1 \leq k < n$.

PROOF: First note that p, q in Proposition 2.2 are the only global R -invariant extensions of the type corresponding to $C(R)$. Now use 2. of Proposition 2.2 and the fact that for any a, b_1, b_2, c , one has $a \downarrow_c b_1 b_2 \implies a \downarrow_{cb_1} b_2$, and $a \downarrow_{cb_1} b_2$ and $b_1 \downarrow_c b_2 \implies ab_1 \downarrow_c b_2$. \square

Proposition 2.5 (R, C) has property $*$ implies that C defines a valuational cut in R .

PROOF: Consider (R, C) with C non-valuational. We denote by $p \in S_1(\mathcal{R})$ the type implied by $C(R) < x$ and $x < \mathcal{R}^{>C(R)}$. Let t_1, t_2, t_3 be a finite Morley sequence in p over R . In particular, $t_3 < t_2 < t_1$.

Since C is non-valuational, $0 < t_1 - t_2 < R^{>0}$, so $t_2 - (t_1 - t_2) \models p|_R$ and thus $t_3 < t_2 - (t_1 - t_2)$. That is, $t_1 < t_2 + (t_2 - t_3)$. Likewise, $0 < t_2 - t_3 < R^{>0}$, so $t_2 + (t_2 - t_3) \models p|_R$. Hence the formula $t_2 < x < t_2 + (t_2 - t_3)$ witnesses $t_1 \not\downarrow_R t_2 t_3$. \square

Proposition 2.5 cannot be strengthened to an equivalence, as the following example shows.

Example 2.6 Let R be the field of the real algebraic numbers and let $R' = R\langle\epsilon\rangle$, the structure generated over the real algebraic numbers by a positive infinitesimal ϵ . Let C be the cut of π . The cut of π is a non-valuational cut in R but a valuational cut in R' . However, the same argument as in Proposition 2.5 shows $t_1 \not\downarrow_{R'} t_2 t_3$.

Later, we shall see that if C is the cut of a convex subring whose residue field is \mathfrak{o} -minimal, then (R, C) has property $*$ (Proposition 5.5). On the other hand, if $C = V$ is a convex subring of R , then the \mathfrak{o} -minimality of the residue field is not a necessary condition for C to have property $*$, as the following example demonstrates.

Example 2.7 Let R be a big elementary extension of the real exponential field, and let $a \in R^{>\mathbb{Q}}$. We set $V = \bigcup_n (-a^n, a^n)$. The residue field of (R, V) is not \mathfrak{o} -minimal, since the inverse function of the exponential induces a map $\ln: \mathbf{k}^{>0} \rightarrow \mathbf{k}$, where $\ln(\mathbf{k}^{>0}) \subseteq \mathbf{k}$ is bounded above but does not have a supremum in \mathbf{k} . However, letting \mathcal{O} be the convex hull of \mathbb{Q} , we see that (R, V) and (R, \mathcal{O}) are interdefinable via the \mathcal{L}_0 -definable function $f(x) = \frac{\ln x}{\ln a}$. The residue field of (R, \mathcal{O}) is \mathbb{R} and is thus \mathfrak{o} -minimal. (In fact, (R, \mathcal{O}) is even T -convex.) Thus, the cut corresponding to \mathcal{O} has property $*$, and therefore so does the cut corresponding to V .

While Example 2.6 shows that (R, C) might not satisfy $*$, one might still hope that $*$ holds whenever C is a predicate for a valuation ring, but this is also not true. Here is an example illustrating this.

Example 2.8 Set $R_0 = \mathbb{R}_{\text{exp}}$, the real exponential field, and let \mathcal{M} be a countably saturated elementary extension of R_0 . We pick $a_1 \in \mathcal{M}^{>\mathbb{Q}}$ and set $R_1 = R_0\langle a_1 \rangle$ and

$V_1 = \bigcup_n (-a_1^n, a_1^n) \subseteq R_1$. We define (R_{k+1}, V_{k+1}) inductively by letting $a_{k+1} \in \mathcal{M}$ be such that $V_k < a_{k+1} < R_k^{>V_k}$, $R_{k+1} = R_k \langle a_{k+1} \rangle$, and $V_{k+1} = \bigcup_n (-a_{k+1}^n, a_{k+1}^n) \subseteq R_{k+1}$. Finally we set $R = \bigcup_k R_k$ and $V = \bigcup_k V_k$. Our aim is to show that V does not have property $*$.

Lemma 2.9 Let $f(x, y) = y^{\frac{\ln y}{\ln x}}$, and let $t_3 < t_2 < t_1$ be a finite Morley sequence in p over R , where $p \in S_1(\mathcal{R})$ is the type $V < x < \mathcal{R}^{>V}$. Then

$$t_1 < f(t_3, t_2) < \mathcal{R}^{>V}.$$

PROOF: For each i , we let $g_i \in R_{i+1}$ be such that $a_i^{g_i} = a_{i+1}$, i.e. $g_i = \frac{\ln a_{i+1}}{\ln a_i}$.

CLAIM 1 : $\mathbb{Q} < g_i < R_i^{>\mathbb{Q}}$

By our choice of a_{i+1} , $g_i > k$ for all $k \in \mathbb{N}$. Moreover, if there was $r \in R_i^{>\mathbb{Q}}$ with $r < g_i$, then $V_i < a_i^r < a_i^{g_i} = a_{i+1}$. But since $a_i^r \in R_i$, this would imply that $a_i^r \in V_i$, a contradiction.

CLAIM 2 : $g_{i+1} < g_i^{\frac{1}{k}}$ for all $k \in \mathbb{N}$.

By Claim 1, $g_{i+1} < g_i$. For any k , $f(x) = x^k$ is increasing and $f(\mathcal{O}) \subseteq \mathcal{O}$, so $g_{i+1}^k \geq g_i$ is impossible.

Note that $\{a_{i+1}^{g_i}\}_{i=1}^\infty$ is a coinital sequence in $\mathcal{R}^{>V}$: If not, then there would be $r \in \mathcal{R}^{>V}$ such that $r < a_{i+1}^{g_i}$ for all i . But $r \in R_j^{>V_j}$ for some j , hence $a_{j+1}^{g_j} < r$, a contradiction. So Claim 3 will complete the proof.

CLAIM 3 : For $i \in \{1, 2\}$, we set $h_i := \frac{\ln t_i}{\ln t_{i+1}}$ (so $t_i = t_{i+1}^{h_i}$). Then, for all i ,

$$t_1 < t_2^{h_2} < a_{i+1}^{g_i} = f(a_i, a_{i+1}).$$

To prove that $t_2^{h_2} < a_{i+1}^{g_i}$ for all i , assume towards a contradiction that i is such that $t_2^{h_2} \geq a_{i+1}^{g_i}$. Then

$$h_2 \geq \frac{g_i \ln a_{i+1}}{\ln t_2} \geq \frac{g_i \ln a_{i+1}}{g_{i+1} \ln a_{i+2}} = \frac{g_i}{(g_{i+1})^2}.$$

By Claim 2, we have $\frac{g_i}{(g_{i+1})^2} > g_i^{\frac{1}{2}}$, so $h_2 > g_i^{\frac{1}{2}}$. It follows that

$$t_2 = t_3^{h_2} > a_{i+1}^{(g_i^{\frac{1}{2}})} \in R_{i+1}^{>V},$$

a contradiction with t_2 realizing $p|_R$.

It is left to show that $t_1 < t_2^{h_2}$. If not, then $t_2^{h_1} > t_2^{h_2}$, so $h_1 > h_2$. We would then be able to find k such that $t_2^{\frac{1}{h_1}} < a_k$, which is equivalent to $t_2^{\frac{\ln t_2}{\ln a_k}} < t_1$. On the other hand, $\frac{\ln t_2}{\ln a_k} > \frac{\ln a_{k+1}}{\ln a_k} = g_k$. So

$$t_2^{\frac{\ln t_2}{\ln a_k}} > t_2^{g_k} > a_{k+1}^{g_k} > V,$$

a contradiction with $t_2^{\frac{\ln t_2}{\ln a_k}} < t_1$. \square

Lemma 2.9 shows that $t_1 \not\downarrow_R t_2 t_3$, and hence V does not have property $*$.

We conclude this section with a lemma showing that property $*$ is preserved by passing to certain superstructures.

Lemma 2.10 *Suppose C defines a cut in R which has property $*$, and let $p \in S(\mathcal{R})$ be the invariant type implied by*

$$\{x > c \mid c \in C(R)\} \cup \{x < r \mid r \in \mathcal{R} \text{ and } r > C(R)\}.$$

and $q \in S(\mathcal{R})$ the type implied by

$$\{x < r \mid r \in R^{>C(R)}\} \cup \{x > r \mid r \in \mathcal{R} \text{ and } r < R^{>C(R)}\}.$$

Then any of the following conditions implies C defines a cut in $R\langle (a_i)_{i < \kappa} \rangle$ which has property $*$:

1. $(a_i)_{i < \kappa}$ forms a Morley sequence in p over R and

$$\text{dcl}_R((a_i)_{i < \kappa} \cap p|_R) > C(\mathcal{R}).$$

2. $(a_i)_{i < \kappa}$ forms a Morley sequence in q over R and

$$\text{dcl}_R((a_i)_{i < \kappa} \cap p|_R) \subseteq C(\mathcal{R}).$$

3. $\text{dcl}_R((a_i)_{i < \kappa} \cap p|_R) = \emptyset$.

PROOF: Assume (1) holds. Take a Morley sequence $(t_i)_{i < \omega}$ in p over $R\langle (a_i)_{i < \kappa} \rangle$. Then $(a_i)_{i < \kappa} (t_i)_{i < \omega}$ is also a Morley sequence in p over R . Note that $t_1(a_i)_{i < \kappa} \not\downarrow_R t_2 \dots t_n$, because if not, there would be a finite subtuple a_{i_0}, \dots, a_{i_k} with

$$a_{i_0} \dots a_{i_k} t_1 \not\downarrow_R t_2 \dots t_n,$$

and, by indiscernibility of the Morley sequence, we would have

$$t_1 t_2 \dots t_{k+2} \not\downarrow_R t_{k+3} \dots t_{k+n+1},$$

a contradiction with C having property $*$. From $t_1(a_i)_{i < \kappa} \downarrow_R t_2 \dots t_n$, it follows that $t_1 \downarrow_{R\langle (a_i)_{i < \kappa} \rangle} t_2 \dots t_n$.

If (2) holds, one carries out the analogous argument with the Morley sequence $(t_i)_{i < \omega}$ in q .

Now assume (3). Suppose for a contradiction that C does not define a cut in $R\langle (a_i)_{i < \kappa} \rangle$ that has property $*$, and that this is witnessed by a finite Morley sequence $t_k < t_{k-1} < \dots < t_1$ in p over $R\langle (a_i)_{i < \kappa} \rangle$ and an R -definable function f , i.e. $t_1 < f_{\underline{a}}(\underline{t})$, where \underline{a} is a finite subsequence of $(a_i)_{i < \kappa}$, $\underline{t} = t_2, \dots, t_k$ and $f_{\underline{a}}(\underline{t}) \models p|_{R\langle (a_i)_{i < \kappa} \rangle}$. Then the cut defined by C in $R\langle \underline{a} \rangle$ does not have property $*$. Thus, if we could prove that C defining a cut in R with property $*$ implies that C also defines a cut with property $*$ in $R\langle a \rangle$, where a is a singleton and $\text{dcl}_R(a) \cap p|_R = \emptyset$, then by applying this fact finitely many times, we could show C defines a cut with property $*$ in $R\langle \underline{a} \rangle$. So towards a contradiction, we assume C does not define a cut with property $*$ in $R\langle a \rangle$.

Let $t_k < t_{k-1} < \dots < t_1$ be a finite Morley sequence in p over $R\langle a \rangle$ and f an R -definable function with $t_1 < f_a(\underline{t})$, where $\underline{t} = t_2, \dots, t_k$, and $f_a(\underline{t}) \models p|_{R\langle a \rangle}$.

Set $t'_1 = f_a(\underline{t})$ and let $t'_k < \dots < t'_2 < t'_1$ be a finite Morley sequence in p over $R\langle a \rangle$ so that $t'_2 < t_k$. Since $\text{tp}(\underline{t}/R\langle a \rangle) = \text{tp}(\underline{t}'/R\langle a \rangle)$, we have

$$t'_1 = f_a(\underline{t}) < f_a(\underline{t}') < R\langle a \rangle^{>C(R\langle a \rangle)}.$$

We now set $t''_1 = f_a(\underline{t}')$.

CLAIM The sequence t''_1, t'_2, \dots, t'_k is a finite Morley sequence in p over R .

PROOF: This is proved by induction on k . We clearly have $t''_1 \models p|_R$. Now suppose that t''_1, t'_2, \dots, t'_i is a finite Morley sequence in p over R . If $t''_1, t'_2, \dots, t'_{i+1}$ was not, then we would have $C < h(t''_1, t'_2, \dots, t'_i) < t'_{i+1}$. Let $H(x) = h(x, t'_2, \dots, t'_i)$ and consider $H(t''_1)$. The function H is continuous and strictly monotone on some interval, I , with left endpoint in $\text{dcl}_R(t'_2 \dots t'_i)$ and realizing the type $p|_R$, and right endpoint in the convex hull of $R^{>C(R)}$ in \mathcal{R} .

If H is increasing on that interval, then $H(t''_1) < r$ for some $r \in C(R)$, since otherwise t'_1, \dots, t'_{i+1} would not be a Morley sequence in p over R . But if $H(t''_1) < r$, then any $r' > r$, $r' \in C(R)$ would have to be such that $t''_1 < H^{-1}(r') < t''_1$, a contradicting that C defines a cut with property $*$ in R .

If H is decreasing, then a coinital segment of $R^{>C(R)}$ would have to be mapped to a cofinal segment of $C(R)$, since if there were any $r \in R^{>C(R)}$ with $H(r) \models p|_R$ this would necessarily be less than t'_{i+1} (since $H(r) < H(t''_1)$), contradicting t'_1, \dots, t'_{i+1} being a Morley sequence in p over R . If, on the other hand, $r \in C(R)$ is such that $H^{-1}(I \cap R^{>C(R)}) < r$, then $H^{-1}(r) > t''_1$, hence $H^{-1}(r)$ cannot satisfy $p|_R$ because the cut defined by C in R has property $*$. But H mapping a coinital segment of $R^{>C}$ to a cofinal segment of $C(R)$ contradicts C defining a cut in R with property $*$. \square

Note that $a \in \text{dcl}_R(t''_1, \underline{t}')$. Extend t''_1, \underline{t}' to a finite Morley sequence $t''_1, \underline{t}', \underline{t}''$ in p over R . Then $t''_1 < f_a(\underline{t}'')$ and $f_a(\underline{t}'') \models p|_{R\langle a \rangle}$. But $f_a(\underline{t}'') = f_{g(t''_1, \underline{t}')}(\underline{t}'')$ for

some R -definable function g . Hence the function $f(g(\dots)\dots)$ and the finite Morley sequence t''_1, t', t''' in p over R witness the failure of $*$ in the cut defined by C in R , a contradiction. □

3 Elementary extensions

The theorem below is an analog of Theorem 3.3, p. 244 in [9]. The difference between the two statements is that in [9], R is assumed to expand a field and C (called “ V ” in [9]) is assumed to be a proper convex subring of R . Moreover, as already mentioned, Theorem 3.3 in [9] needs an additional assumption to be correct – such as that the residue field of (R, V) is o-minimal, or, more generally, that V has property $*$.

Note that while Theorem 3.3 in [9] is stated as an implication, it is really an equivalence. The easy direction (which went unstated in [9]) being if $(R, C) \preceq (R\langle a \rangle, C)$, then we cannot have $C(R) < a, f(a) < R^{>C(R)}$, where f is an R -definable function and $a \in C(R\langle a \rangle)$ and $f(a) > C(R\langle a \rangle)$. Else,

$$(R\langle a \rangle, C) \models \exists x \in C f(x) > C,$$

hence

$$(R, C) \models \exists x \in C f(x) > C,$$

yielding a contradiction with f being increasing (see Fact 1.1).

Theorem 3.1 *Suppose C is a cut in R with property $*$, and let $a \in \mathcal{R} \setminus R$. Then (R, C) fails to be an elementary submodel of $(R\langle a \rangle, C)$ if and only if there is an element of $C(R\langle a \rangle)$ greater than any element of $C(R)$ (WLOG, we may assume this element is a) and an R -definable function, f , such that*

$$C(R\langle a \rangle) < f(a) < R^{>C(R)}.$$

PROOF: This may be proved in an almost identical fashion to the proof of Theorem 3.3 [9] (which encompasses paragraphs 2 and 3), and below, for the reader familiar with that proof, we outline the minor changes needed.

1. All instances of “ V ” need to be replaced by “ C ”.
2. Replace Theorem 2.7, p. 239 [9] with Corollary 2.4.
3. Corollary 2.9, p. 241 [9] needs the additional assumption that $p|_R$ is the type corresponding to C and C has property $*$.
4. The assumption that (R, V) has property $*$ must be added to the hypotheses of Lemma 3.1 and appeals to Corollary 2.9 are replaced by appeals to this hypothesis.
5. Note that the cut of type p in $R\langle a \rangle$ also has property $*$ by Lemma 2.10 of this paper.

6. In the proof of Lemma 3.1, p. 242, lines -18 to -15: The field assumption is used to reduce 4 cases to 2. But the two remaining cases can simply be proved in the same fashion as Case 2.
7. In the proof of Lemma 3.1, p. 243, lines 1 and 2: “Note that since V is a group, ...” should be replaced with “Note that since $(C, R^{>C})$ is a valuational cut, ...” and on page 243, lines 4,5 “But $\frac{1}{2}\beta$ is also greater than every element of V , ...” should be replaced with “But $\beta - \epsilon$, where $\epsilon > 0$ is such that $r - c > \epsilon$ for all $r \in R^{>C}$ and all $c \in C$, is also greater than every element of C , ...”.
8. The hypothesis of satisfying property $*$ must be added to the hypotheses of Lemma 3.2 and Theorem 3.3 and appeals to Lemma 2.10 (of this paper) added as appropriate to their proofs.

□

For readers not already familiar with the proofs in [9], this may be a significant amount of work, especially since the proofs in [9] themselves require one to go through the proofs in [2] to confirm results implicit, but not stated, therein. We have therefore included a new, simplified proof of Theorem 3.3 of [9] in the remainder of this section which includes an easy proof of the results implicit in [2] that we require. This new proof also contains a result that may be of independent interest, namely Proposition 3.6, a characterization of the divide between valuational and non-valuational cuts in neostability theoretic terms.

In order to prove Theorem 3.1, we will need a result on quantifier elimination for traces (Proposition 3.7) together with a statement about the uniformity of this quantifier elimination (Remark 3.8), both of which may be extracted from the proofs of Baisalov and Poizat in [2].

We shall need the notion of separation which was introduced in [2]:

Definition 3.2 *Let $a \in R^m$, $b \in R^n$, $A \subseteq R$, and let $p \in S_1(A)$. Then a and b are separated in p over A if either*

$$\text{dcl}(aA) \cap p(R) < \text{dcl}(bA) \cap p(R) \text{ or } \text{dcl}(bA) \cap p(R) < \text{dcl}(aA) \cap p(R).$$

(Note that the above condition holds if $\text{dcl}(aA) \cap p(R) = \emptyset$ or $\text{dcl}(bA) \cap p(R) = \emptyset$.) We say that a and b are A -separated if they are separated in all one-types over A . Note that if a and b are separated in p and $\text{dcl}(aA) \cap p(R) \neq \emptyset$ then a and b are A -separated.

We are able to provide short proofs by limiting ourselves to the case where the externally definable set is a trace of a formula whose parameters form a Morley sequence in a cut satisfying $*$, using our result (Proposition 2.4) on symmetry of dividing in Morley sequences, thus easily obtaining separated tuples. (This is also Corollary 2.9 of [9], although there it appeared (incorrectly) without the hypothesis $*$.)

Corollary 3.3 *Let $p \in S_1(\mathcal{R})$ be an R -invariant extension of the \mathcal{L}_0 -type over R which states that x realizes the cut C , and assume that latter type has property $*$. If t_1, \dots, t_n is a finite Morley sequence in p over R and $1 \leq k < n$, then t_1, \dots, t_k is R -separated from t_{k+1}, \dots, t_n .*

PROOF: This is simply Proposition 2.4 together with the definition of separation. \square

In addition, we shall use the following fact:

Fact 3.4 *Let a_1, \dots, a_k be such that each a_i realizes a cut in $R\langle a_1, \dots, a_{i-1} \rangle$. Then every element of $R\langle a_1, \dots, a_k \rangle \setminus R$ realizes a cut in R .*

PROOF: This is clear when $k = 1$. Suppose the statement is false, and consider the minimal k for which it fails. Let $b = f(a_1, \dots, a_k)$ witness its failure. Replacing b with $1/b$, if necessary, we may assume that there is a closest element of R to b . Call this element r . Since b lies in a cut in $R\langle a_1, \dots, a_{k-1} \rangle$, there must be elements of $R\langle a_1, \dots, a_{k-1} \rangle$ between b and r . But these elements must therefore realize a definable type over R , contradicting the minimality of k . \square

Finally, we will need to take a global type, p , invariant over a small model R where both the invariant type and its restriction to R are non-definable, build a Morley sequence t_1, \dots, t_n in p over R , and have that $p|_{Rt_1 \dots t_n}$ is also nondefinable. The following example shows that this is not always the case:

Example 3.5 *Let A be the collection of rational numbers less than π and let p be the global type implied by $\{x > a \mid a \in A\} \cup \{x < r \mid r > A\}$. Clearly p is invariant over the empty set and is valutional (as is any non-definable global type invariant over a small set). If, however, R is archimedean, then $p|_R$ is either a non-valutional cut or a non-cut. Let R be an archimedean model. If t_1, t_2, \dots is a Morley sequence in p over R then the only realization of $p|_R$ in $R\langle t_1 \rangle$ is t_1 itself. If $p|_R$ is a non-cut then, clearly, $\text{tp}(t_1/R)$ is definable, and if $p|_R$ is a non-valutional cut, then $\text{tp}(t_2/Rt_1)$ is definable.*

On the other hand, if $p|_A$ is a valutional type and $B \supseteq A$, then $p|_B$ is also valutional (witnessed by the same ϵ). In particular, t_1, \dots, t_k is a Morley sequence in p over A , then $p|_{At_1 \dots t_k}$ is also valutional, and we have the following characterization of valutional cuts:

Proposition 3.6 *If p is a global 1-type invariant over a small set A , and $B \supseteq A$, then the following are equivalent*

1. $p|_B$ is a valutional type.
2. $p|_B$ is a cut and whenever $\text{dcl}(Bc)$ contains $t \models p|_B$, then $\text{dcl}(Bc)$ contains more than one realization of $p|_B$.

3. For all k , if t_1, \dots, t_k is a Morley sequence in p over B then $p|_{Bt_1 \dots t_k}$ is nondefinable.
4. If t_1, t_2 is a Morley sequence in p over B , then $\text{tp}(t_1/B)$ and $\text{tp}(t_2/Bt_1)$ are nondefinable.

PROOF: Since p is a 1-type invariant over B , we may assume, replacing B with an appropriate subset of $\text{dcl}(B)$, that either p is implied by $\{x > b \mid b \in B\} \cup \{x < r \mid r > B\}$ or p is implied by $\{x < b \mid b \in B\} \cup \{x > r \mid r < B\}$. For convenience, we assume the former. The proof in the latter case is identical.

1. \implies 2. Let ϵ witness that $p|_B$ is valuatinal, and let $t \models p|_B$. Then $t - \epsilon \models p|_B$.
2. \implies 1. Suppose $p|_B$ is a cut, and let c be such that $t, \tilde{t} \in \text{dcl}(Bc)$ both realize $p|_B$. Note that $|t - \tilde{t}|$ cannot realize the right infinitesimal neighborhood of 0 in $\text{dcl}(B)$, since $p|_B$ is a cut. So let ϵ be a positive element of $\text{dcl}(B)$ less than $|t - \tilde{t}|$. This ϵ witnesses that $p|_B$ is valuatinal.
1. \implies 3. Since $p|_B$ is valuatinal, it is in particular a cut, and so $p|_{Bt_1 \dots t_k}$ is not the right infinitesimal neighborhood of some element in $\text{dcl}(Bt_1 \dots t_k)$. On the other hand, if ϵ witnesses that $p|_B$ is valuatinal and $a \in \text{dcl}(Bt_1 \dots t_k)$ is such that $a \models p|_B$, then $a - \epsilon \models p|_B$. Thus there is no smallest element of $p|_B \cap \text{dcl}(Bt_1 \dots t_k)$, and so $p|_{Bt_1 \dots t_k}$ cannot be the left infinitesimal neighborhood of some element in $\text{dcl}(Bt_1 \dots t_k)$.
3. \implies 4. Clear.
4. \implies 1. Let t_1, t_2 be a Morley sequence in p over B . Since t_2 and $t_1 - t_2$ are interdefinable over Bt_1 , and $\text{tp}(t_2/Bt_1)$ is nondefinable, $\text{tp}(t_1 - t_2/Bt_1)$ is also nondefinable. In particular, $t_1 - t_2$ does not realize the right infinitesimal neighborhood of 0 in $\text{dcl}(Bt_1)$, and one may choose positive $\tilde{\epsilon} \in \text{dcl}(Bt_1)$ such that $t_2 + \tilde{\epsilon} < t_1$. Since $\text{tp}(t_1/B)$ is nondefinable, $\tilde{\epsilon}$ does not realize the right infinitesimal neighborhood of 0 in $\text{dcl}(B)$, and one may choose positive $\epsilon \in \text{dcl}(B)$ with $\epsilon < \tilde{\epsilon}$. This ϵ witnesses that $p|_B$ is valuatinal.

□

In the proposition below, we let p be a global 1-type which is invariant over the empty set and is such that $p|_R$ has property $*$. Our aim is to prove that if we expand the \mathcal{L}_0 -structure R by a predicate for an externally definable set whose parameters form a finite Morley sequence in p over R , then every set definable in this trace expansion is itself the trace of an externally definable set with parameters from a Morley sequence in p over R .

More precisely, let \tilde{R} be an $|R|^+$ -saturated elementary extension of R . Let S and \tilde{S} be the realizations of $\theta(x_1, \dots, x_k, t_1, \dots, t_n)$ in R and \tilde{R} respectively, where θ is an \mathcal{L}_0 -formula in which the parameters from R are suppressed, and the $t_1, \dots, t_n \in \tilde{R}$ form a Morley sequence in p over R . Up until now, \mathcal{L} had been \mathcal{L}_0 expanded by C . For the remainder of this section we work in more generality: Rather than expanding \mathcal{L}_0 by C , we expand by a predicate S , defining, in a slight abuse of notation, the set S in R , and we denote the language $\mathcal{L}_0 \cup \{S\}$ by \mathcal{L} .

Proposition 3.7 *Let φ be an \mathcal{L} -formula with parameters from R . Then there is an \mathcal{L}_0 -formula, τ_φ , such that $\varphi(R) = \tau_\varphi(\tilde{R}) \cap R^k$, where the parameters of τ_φ are in \tilde{R} and form a Morley sequence in p over R .*

PROOF: Let $\underline{x} = (x_1, \dots, x_k)$. It suffices to show that if $\varphi(x_0, \underline{x})$ is the trace of the \mathcal{L}_0 -formula $\tau_\varphi(x_0, \underline{x}, t_1, \dots, t_n)$, then $\exists x_0 \varphi(x_0, \underline{x})$ is the trace of

$$\tau_{\exists x_0 \varphi} = \exists x_0 (\tau_\varphi(x_0, \underline{x}, t_1, \dots, t_n) \wedge \tau_\varphi(x_0, \underline{x}, t_{n+1}, \dots, t_{2n})),$$

where $t_1, \dots, t_{2n} \in \tilde{R}$ form a Morley sequence in p over R . Suppose $r_1, \dots, r_k \in R$ are such that there is $r_0 \in R$ with $\varphi(r_0, r_1, \dots, r_k)$. Then

$$\tilde{R} \models \tau_\varphi(r_0, r_1, \dots, r_k, t_1, \dots, t_n),$$

and since $t_1, \dots, t_n \equiv_R t_{n+1}, \dots, t_{2n}$, one has

$$\tilde{R} \models \tau_\varphi(r_0, r_1, \dots, r_k, t_{n+1}, \dots, t_{2n})$$

as well.

Now suppose that there is no $r_0 \in R$ such that $\varphi(r_0, r_1, \dots, r_k)$ and assume for a contradiction that there is $\tilde{r}_0 \in \tilde{R}$ with

$$\tilde{R} \models \tau_\varphi(\tilde{r}_0, r_1, \dots, r_k, t_1, \dots, t_n) \wedge \tau_\varphi(\tilde{r}_0, r_1, \dots, r_k, t_{n+1}, \dots, t_{2n}).$$

We consider $\text{tp}(\tilde{r}_0/Rt_1, \dots, t_n)$. Either $\tilde{r}_0 \in \text{dcl}(Rt_1, \dots, t_n)$ or there is an interval (a, b) containing \tilde{r}_0 with $a, b \in \text{dcl}(Rt_1, \dots, t_n)$ (possibly equal to $-\infty, \infty$) such that any $s \in (a, b)$ is such that $\tau_\varphi(s, r_1, \dots, r_k, t_1, \dots, t_n)$. Note that as we are supposing that there is no element of R in that interval, the entire interval realizes a single type $q \in S_1(R)$. Choose $\tilde{r}_1 \in (a, b)$ with $\tilde{r}_1 \in \text{dcl}(Rt_1, \dots, t_n)$. As $p|_R$ is valational, $\text{tp}(t_i/Rt_1 \dots t_{i-1})$ is a cut, by Proposition 3.6. This allows us to apply Proposition 3.4 to \tilde{r}_1 , showing that q is not a definable type. Thus neither a nor b can be in R or equal to $-\infty, \infty$.

Likewise consider $\text{tp}(\tilde{r}_0/Rt_{n+1}, \dots, t_{2n})$. We obtain either $\tilde{r}_0 \in \text{dcl}(Rt_{n+1}, \dots, t_{2n})$ or the existence of an interval (c, d) analogous to (a, b) above.

Now note that we have contradicted the separation of t_1, \dots, t_n from t_{n+1}, \dots, t_{2n} (which follows from Corollary 3.3): For either there is an element not in R common to their definable closures, or else there is an element of the definable closure of one contained in an interval defined over the other. \square

Remark 3.8 *It is easy to see, inspecting the proof of Proposition 3.7, that the construction of τ_φ is uniform in the sense that it is independent of p , R , and \tilde{R} . To be precise, let $R_1 \preceq \tilde{R}_1$ (in the sense of \mathcal{L}_0) with \tilde{R}_1 sufficiently saturated, and let S_1 and \tilde{S}_1 denote the realizations of $\theta(x_1, \dots, x_k, s_1, \dots, s_n)$ in S_1 and \tilde{S}_1 respectively with $s_1, \dots, s_n \in \tilde{S}_1$ a Morley sequence in p_1 over R_1 , where p_1 is a global 1-type invariant over \emptyset such that $p_1|_{R_1}$ has property *. Then given any φ , the proof above creates the same τ_φ for R_1 and p_1 as it does for R and p .*

Note that the uniformity observed in Remark 3.8 translates easily into a fact about elementary substructures. Namely, we have the following corollary.

Corollary 3.9 *Consider R and R_1 as in Remark 3.8 where in addition $R \subseteq R_1$, $\tilde{R} = \tilde{R}_1$. Consider $p = p_1$ invariant over R such that $p|_R$ and $p|_{R_1}$ have property *, and moreover $s_1, \dots, s_n = t_1, \dots, t_n$. Then $R \preceq R_1$ as \mathcal{L} -structures.*

PROOF: We may add constants for R to \mathcal{L} to make p invariant over \emptyset so that we may apply Proposition 3.7. Suppose that $R_1 \models \exists x \varphi(x, \vec{r})$ where $\vec{r} \in R^m$. This happens if and only if $\vec{r} \in \tau_{\exists x \varphi}(\tilde{R}) \cap R_1^m$, where $\tau_{\exists x \varphi}$ is the \tilde{R} -definable set whose trace is $\exists x \varphi(x, \tilde{R}_1^m)$. But the set $\exists x \varphi(x, R^m)$ is also the trace of $\tau_{\exists x \varphi}$. Obviously $\vec{r} \in \tau_{\exists x \varphi}(\tilde{R}) \cap R^m$. Thus $R \models \exists x \varphi(x, \vec{r})$, and we are done by the Tarski-Vaught test. \square

We may now use this corollary to prove a slightly strengthened version (in that we only need to assume the cut is valuatinal, rather than an actual valuation ring) of Theorem 3.3 of [9].

First note that if $R\langle a \rangle$ is a superstructure of R (as an \mathcal{L} -structure) and there is an element (which we may assume to be a) in $C^{>R}$ with $f(a)$ in the same cut in R as a but with $f(a)$ greater than C , then we cannot hope to satisfy the hypotheses of Corollary 3.9. In particular, the requirement that $s_1, \dots, s_n = t_1, \dots, t_n$ can not be satisfied. After all, C is the trace of $x < t$ and t must lie between a and $f(a)$, but an R -invariant global type whose restriction to R is the cut of a in R would produce a Morley sequence either less than a or greater than $f(a)$. Thus this Morley sequence could not include t .

However, this is the only obstruction:

Lemma 3.10 [Lemma 3.2 [9]] *There is an R -invariant global type q with $q|_R$ being the cut in R given by C and $q|_{R\langle a \rangle}$ being the cut in $R\langle a \rangle$ given by C , and there is a finite Morley sequence t_1, \dots, t_n in q over $R\langle a \rangle$ (and hence over R) in each of the following cases:*

- a) a realizes the cut in R given by C and $C(R\langle a \rangle)$ is the convex hull of $C(R)$ in $R\langle a \rangle$.
- b) a realizes the cut given by C and $C(R\langle a \rangle) = \{x \in R\langle a \rangle \mid x < R^{>C(R)}\}$.
- c) a is such that $R\langle a \rangle$ does not realize the cut given by C (hence $C(R\langle a \rangle)$ is the convex hull of C in $R\langle a \rangle$ and $C(R\langle a \rangle) = \{x \in R\langle a \rangle \mid x < R^{>C(R)}\}$.)

PROOF: In a), we may take q to be the global type implied by

$$\{x > r \mid r \in C(R)\} \cup \{x < r \mid r \in \mathcal{R} \text{ and } r > C(R)\}.$$

In b), we let q be the global type implied by

$$\{x < r \mid r \in R^{>C(R)}\} \cup \{x > r \mid r \in \mathcal{R} \text{ and } r < R^{>C(R)}\}.$$

In c) we may do either. \square

Now we obtain our criterion for elementary extensions:

Theorem 3.1 [Theorem 3.3 [9]] *Let $a \in \mathcal{R}$ and assume that (R, C) has property $*$. Then $(R\langle a \rangle, C)$ fails to be an elementary extension of (R, C) if and only if there are R -definable one-variable functions f and g with $C(R) < f(a), g(a) < R^{>C(R)}$ and $f(a) \in C$ and $g(a) > C$.*

PROOF: First assume there are no functions f, g as in the statement of the theorem. Then we may apply Lemma 2.10 to obtain that $(R\langle a \rangle, C)$ has property $*$. The lack of such f, g also allows us to apply Lemma 3.10 to find an R -invariant q with $q|_R$ being the cut in R given by C and $q|_{R\langle a \rangle}$ being the cut in $R\langle a \rangle$ given by C . Now we have satisfied the hypotheses of Corollary 3.9, and conclude that $(R, C) \preceq (R\langle a \rangle, C)$.

Now assume that there are such functions f and g , and define h to be $g \circ f^{-1}$. Note that since h maps $f(a)$ to $g(a)$, h maps the cut in R corresponding to C to itself. Thus h is increasing (see Fact 1.1) on an R -definable interval containing this cut. Thus

$$(R\langle a \rangle, C) \models \text{for all sufficiently large } x \in C, h(x) > C,$$

hence

$$(R, C) \models \text{for all sufficiently large } x \in C, h(x) > C.$$

Take $r \in C(R)$ with $h(r) \in R^{>C(R)}$. But $r < f(a)$ while $h(r) > h(f(a))$, a contradiction to h being increasing. \square

4 Substructures of models of T and quantifier elimination

In this section, we obtain our quantifier elimination, universal axiomatization and definable Skolem function results (Theorem 4.12). Since most of the initial lemmas are rather technical, here is a quick road map to how they fit together.

Start with $(R_0, C) \preceq (R, C)$, where (R_0, C) satisfies $*$, and consider a structure (R_1, C) so that $(R_0, C) \subseteq (R_1, C) \subseteq (R, C)$. Then clearly $R_0 \preceq R_1 \preceq R$ (in \mathcal{L}_0). We would like to build up from (R_0, C) to (R_1, C) and thence to (R, C) , one element at a time, with each step along the way being an elementary extension. That will establish model completeness in \mathcal{L} with constants added for R_0 (and then our main results will follow rather easily).

First we show in Lemma 4.1 that given any $a \in R$, $(R_0, C) \preceq (R_0\langle a \rangle, C)$. However, as this uses $(R_0, C) \preceq (R, C)$ and we do not yet know $(R\langle a \rangle, C) \preceq (R_0, C)$, we cannot simply apply Lemma 4.1 repeatedly to achieve our goal.

Next, we extend Lemma 4.1 to any finitely generated extension (R_n, C) of (R_0, C) (working as usual inside (R, C)) and we show that as long as we build up to R_n by adding elements to R_0 in a certain order (roughly, by adding only elements

on the left side of the cut until $C(R_m)$, for some $m \leq n$, is cofinal in $C(R_n)$) then each step is an elementary extension. This is Lemma 4.2.

Then we extend Lemma 4.2 by removing the finitely generated assumption. This is Lemma 4.4.

At that point we still have the restriction that we initially extend by adding elements on the left side of the cut only. For example, choose $a \in C(R)$ with $a > C(R_0)$ and $b \in R$ so that $R_0\langle ab \rangle$ contains no elements realizing the cut over $R_0\langle a \rangle$. Then $(R_0, C) \preceq (R_0\langle a \rangle, C) \preceq (R_0\langle ab \rangle, C)$, and also $(R_0, C) \preceq (R_0\langle b \rangle, C)$. But we do not yet know that $(R_0\langle b \rangle, C) \preceq (R_0\langle ab \rangle, C)$. We establish this in Lemma 4.6 and then we use this in Lemma 4.7 to show $(R_0\langle b \rangle, C) \preceq (R, C)$.

Using this we see that given any $a \in R$, $(R_0, C) \preceq (R_0\langle a \rangle, C)$ and also $(R\langle a \rangle, C) \preceq (R_0, C)$. We use this repeatedly to get the desired result (Proposition 4.9).

Lemma 4.1 *Suppose $(R_0, C) \preceq (R, C)$, C defines a cut in R_0 with property $*$, and $a \in R \setminus R_0$. Then $(R_0, C) \preceq (R_0\langle a \rangle, C)$.*

PROOF: If not, then, by Theorem 3.1, we may assume that a realizes the type

$$C(R_0) < x < R_0^{>C(R_0)},$$

and there is an \mathcal{L}_0 -definable over R_0 function f such that $f(a)$ also realizes the same type, and moreover $a \in C(R_0\langle a \rangle)$ and $f(a) > C(R_0\langle a \rangle)$. Hence

$$(R, C) \models \exists x \in C \ f(x) > C.$$

By Fact 1.1, f is increasing, and so $f(C(R_0)) \subseteq C(R_0)$ and $f(R_0^{>C(R_0)}) \subseteq R_0^{>C(R_0)}$, contradicting $(R_0, C) \preceq (R, C)$. \square

Lemma 4.2 *Suppose $(R_0, C) \preceq (R, C)$, and suppose C defines a cut in R_0 with property $*$. Let $a_1, \dots, a_n \in R$ and set $R_i = R_0\langle a_1, \dots, a_i \rangle$ and $C_i = C(\mathcal{R}) \cap R_i$, for $i \in \{0, \dots, n\}$. Then $(R_0, C) \preceq (R_n, C)$. Moreover, suppose a_1, \dots, a_n are dcl-independent over R_0 , and for some $m \leq n$, $a_{i+1} \in C_{i+1}^{>C_i}$ for each $i \leq m$, and C_m is cofinal in C_n . Then for all i , $(R_i, C) \preceq (R_{i+1}, C)$.*

PROOF: We may as well assume that a_1, \dots, a_n are as in the ‘‘Moreover, ...’’ part of the statement of the lemma. Indeed, suppose C_0 is not cofinal in C_n . Pick $b_1 \in C_n^{>C_0}$. If $C(R_0\langle b_1 \rangle)$ is not cofinal in C_n , then pick $b_2 \in C_n^{>C(R_0\langle b_1 \rangle)}$ so that b_1, b_2 are dcl-independent over R_0 . Continue in the same fashion until $C(R\langle b_1, \dots, b_m \rangle)$ is cofinal in C_n . This is bound to happen after $m \leq n$ many steps. Now choose b_{m+1}, \dots, b_n in such a way that $R_0\langle b_1, \dots, b_n \rangle = R_n$. After replacing a_1, \dots, a_n with b_1, \dots, b_n , we find ourselves in the ‘‘Moreover, ...’’ part of the lemma. Also note that then a_1, \dots, a_m form a Morley sequence in q over R_0 , where q is the global invariant type defined in Lemma 2.10.

We start by showing inductively for $i < m$ that $(R_i, C) \preceq (R_{i+1}, C)$ and (R_{i+1}, C) has property $*$.

Base Step: That $(R_0, C) \preceq (R_1, C)$, is Lemma 4.1. Now by Theorem 3.1 and 2. of Lemma 2.10, C defines a cut in R_1 which has property $*$.

Inductive Step: We assume towards a contradiction that $0 < i < m$ is such that $(R_j, C) \preceq (R_{j+1}, C)$ and (R_{j+1}, C) has property $*$ for all $j < i$, and that there is an R_0 -definable function f such that $f(a_1, \dots, a_i, x_{i+1})$ maps the set of realizations of the type

$$C_i < x < R_i^{>C_i}$$

to itself and $f(a_1, \dots, a_i, a_{i+1})$ is greater than C_{i+1} . Since each $a_{j+1} > C_j$ for $j < i$, we have, by Theorem 3.1, that $R_0^{>C_0}$ is coinitial in $R_i^{>C_i}$.

Note that we cannot repeat the argument of Lemma 4.1 exactly, for while $(R_0, C) \preceq (R_i, C)$, we do not know $(R_i, C) \preceq (R, C)$. But as $(R_0, C) \preceq (R, C)$, we have that for any $r_{01}, \dots, r_{0i} \in C_0$,

$$\begin{aligned} (R, C) \models & \exists x_1 \in C^{>r_{01}} \dots \exists x_i \in C^{>r_{0i}} \ x_1 < \dots < x_i \\ & \exists y \in C^{>x_i} \ \forall x_{i+1} \in C^{>y} \ f(x_1, \dots, x_i, x_{i+1}) > C \\ & \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}. \end{aligned}$$

Since $(R_0, C) \preceq (R, C)$, (R_0, C) satisfies the same \mathcal{L}_{R_0} -formulas. Thus it also satisfies

$$\begin{aligned} (R_0, C) \models & \forall z_1 \in C \dots \forall z_i \in C \ \exists x_1 \in C^{>z_1} \dots \exists x_i \in C^{>z_i} \ x_1 < \dots < x_i \\ & \exists y \in C^{>x_i} \ \forall x_{i+1} \in C^{>y} \ f(x_1, \dots, x_i, x_{i+1}) > C \\ & \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}. \end{aligned}$$

Now since $(R_0, C) \preceq (R_1, C)$, (R_1, C) satisfies the same sentence. So we may choose $a'_1 \in C_1^{\geq a_1}$ (and thus $\text{tp}_{\mathcal{L}_0}(a'_1/R_0) = \text{tp}_{\mathcal{L}_0}(a_1/R_0)$) such that

$$\begin{aligned} (R_1, C) \models & \forall z_2 \in C \dots \forall z_i \in C \ \exists x_2 \in C^{>z_2} \dots \exists x_i \in C^{>z_i} \ a'_1 < x_2 < \dots < x_i \\ & \exists y \in C^{>x_i} \ \forall x_{i+1} \in C^{>y} \ f(a'_1, x_2, \dots, x_i, x_{i+1}) > C \\ & \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}. \end{aligned}$$

Similarly, we may find $a'_2 \in C_2^{\geq a_2}$ (and thus $\text{tp}_{\mathcal{L}_0}(a'_1 a'_2/R_0) = \text{tp}_{\mathcal{L}_0}(a_1 a_2/R_0)$) such that

$$\begin{aligned} (R_2, C) \models & \forall z_3 \in C \dots \forall z_i \in C \ \exists x_3 \in C^{>z_3} \dots \exists x_i \in C^{>z_i} \ a'_1 < a'_2 < \dots < x_i \\ & \exists y \in C^{>x_i} \ \forall x_{i+1} \in C^{>y} \ f(a'_1, a'_2, x_3, \dots, x_i, x_{i+1}) > C \\ & \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}. \end{aligned}$$

Continuing in this fashion, we find that

$$\begin{aligned} (R_i, C) \models & \exists y \in C^{>a'_i} \ \forall x_{i+1} \in C^{>y} \ f(a'_1, \dots, a'_i, x_{i+1}) > C \\ & \text{and } f \text{ is increasing as a function of } x_{i+1} \text{ on } C^{>y}. \end{aligned}$$

Note that $\text{tp}_{\mathcal{L}_0}(\underline{a}'/R_0) = \text{tp}_{\mathcal{L}_0}(\underline{a}/R_0)$, where $\underline{a}' = (a'_1, \dots, a'_i)$ and $\underline{a} = (a_1, \dots, a_i)$. Furthermore, the above sentence is witnessed by $y = h(\underline{a}')$ for some R_0 -definable function h .

Let g be another R_0 -definable function such that $h(\underline{a}') < g(\underline{a}')$ and $g(\underline{a}') \in C_i$ (so f is increasing at $g(\underline{a}')$). Since $f(\underline{a}', g(\underline{a}')) > C_i$, $f(\underline{a}', g(\underline{a}')) > r_0$ for some $r_0 \in R_0^{>C_0}$ by coinitality of $R_0^{>C_0}$ in $R_i^{>C_i}$. Then $g(\underline{a}) < a_{i+1}$ (again by coinitality of $R_0^{>C_0}$ in $R_i^{>C_i}$) but $f(\underline{a}, a_{i+1}) < r_0 < f(\underline{a}, g(\underline{a}))$, a contradiction with f being increasing in x_{i+1} on an interval containing $g(\underline{a})$ and a_{i+1} . So $(R_i, C) \preceq (R_{i+1}, C)$, and by a use of Theorem 3.1 and 2. of Lemma 2.10, we get that C defines a cut in R_{i+1} with property $*$. This finishes the inductive argument.

Having shown that $(R_i, C) \preceq (R_{i+1}, C)$ for $i < m$, we complete the proof of the lemma by noting that $(R_i, C) \preceq (R_{i+1}, C)$ whenever $m \leq i < n$, by Theorem 3.1. \square

Remark 4.3 *Note that instead of choosing $a_1, \dots, a_m \in C_n$ successively larger realizations of the type $C_0 < x < R_0^{>C_0}$ until C_m was cofinal in C_n , we could just as easily have chosen successively smaller realizations a_1, \dots, a_k of $C_0 < x < R_0^{>C_0}$, until $R_k^{>C_k}$ was coinital in $R_n^{>C_n}$. Then the inductive argument above is easily modified to yield $(R_i, C) \preceq (R_{i+1}, C)$ for $i < k$, and Lemma 3.1 shows $(R_k, C) \preceq (R_n, C)$.*

We will use the above lemma to show that if one builds up from (R_0, C) to (R, C) by taking successively larger elements of $C(R)$, each realizing the cut determined by C in the previous model, until one has a cofinal subset of $C(R)$ and then builds the rest of the way to R in any fashion, then each step from R_0 to R is an elementary extension. More precisely:

Lemma 4.4 *Suppose $(R_0, C) \preceq (R, C)$ and suppose C defines a cut in R_0 with property $*$. Let $(a_\alpha)_{\alpha < \kappa}$ be such that $R = R_0 \langle (a_\alpha)_{\alpha < \kappa} \rangle$, and set $R_\alpha = R_0 \langle (a_\beta)_{\beta < \alpha} \rangle$ and $C_\alpha = R_\alpha \cap C(\mathcal{R})$. Assume further that*

1. *there is $\gamma \leq \kappa$ such that $a_\alpha \in C_{\alpha+1}^{>C_\alpha}$ for all $\alpha < \gamma$, and C_γ is cofinal in $C(R)$.*
2. *there is $\delta \leq \kappa$ such that $R_\delta^{>C_\delta}$ is coinital in $R^{>C(R)}$ and for all α with $\gamma \leq \alpha < \delta$, $a_\alpha \in R_{\alpha+1}^{>C_{\alpha+1}}$ but $a_\alpha < R_\alpha^{>C_\alpha}$.*

Then

$$(R_\alpha, C) \preceq (R_{\alpha+1}, C) \preceq (R, C)$$

for each $\alpha < \kappa$.

PROOF: Let γ be as indicated in the lemma. Note that $(a_\alpha)_{\alpha < \gamma}$ is a Morley sequence in q (in the sense of Lemma 2.10 with R_0 playing the role of R).

By Lemma 4.1, $(R_0, C) \preceq (R_1, C)$. Then by Lemma 2.10, C defines a cut in R_1 with property $*$. Suppose $\alpha_0 < \gamma$ is the first ordinal such that $(R_{\alpha_0}, C) \not\preceq (R_{\alpha_0+1}, C)$.

By applying Lemma 4.1 and then Theorem 3.1 at successor ordinals less than α_0 , we see that $(a_\alpha)_{\alpha < \alpha_0}$ satisfies condition (2) of Lemma 2.10. Thus in (R_{α_0}, C) , one has that C_{α_0} satisfies property $*$ and we may apply Theorem 3.1 to obtain $f(a_{\alpha_0})$ not in C_{α_0+1} but smaller than $R_{\alpha_0}^{>C_{\alpha_0}}$. Suppose the parameters in f are $a_{\beta_0}, \dots, a_{\beta_n}$, where each $\beta_i < \alpha_0$. Then

$$R_0 \langle a_{\beta_0}, \dots, a_{\beta_n} \rangle \not\leq R_0 \langle a_{\beta_0}, \dots, a_{\beta_n}, a_{\alpha_0} \rangle,$$

contradicting Lemma 4.2.

Note $(a_\alpha)_{\gamma \leq \alpha < \delta}$ is a Morley sequence in p (in the sense of Lemma 2.10) over R_γ , showing that the cut C_α in R_α has property $*$ for each $\gamma < \alpha < \delta$, by condition (1) of Lemma 2.10. Also note that $(a_\alpha)_{\delta \leq \alpha}$ satisfies condition (3) of Lemma 2.10 (where R_δ plays the role of R in Lemma 2.10), showing that property $*$ holds for the cut C_α in R_α for each $\delta < \alpha$.

For any $\alpha \geq \gamma$, the fact that C_γ is cofinal in $C(R)$ means that Theorem 3.1 implies $(R_\alpha, C) \preceq (R_{\alpha+1}, C)$. Taking unions at limit ordinals yields $(R_\beta, C) \preceq (R, C)$ for any $\beta < \kappa$. \square

Remark 4.5 *Suppose instead of having the sequence $(a_\alpha)_{\alpha < \kappa}$ in the order described in the hypotheses of Lemma 4.4, one has*

1. $\gamma \leq \kappa$ such that $R_\gamma^{>C_\gamma}$ is coinitial in $R^{>C(R)}$ and for all α with $\alpha < \gamma$, $a_\alpha \in R_{\alpha+1}^{>C_{\alpha+1}}$ but $a_\alpha < R_\alpha^{>C_\alpha}$, and
2. $\delta \leq \kappa$ such that $a_\alpha \in C_{\alpha+1}^{>C_\alpha}$ for all $\gamma \leq \alpha < \delta$, and C_γ is cofinal in $C(R)$.

then one can see that the conclusion of Lemma 4.4 still holds using the same proof except flipping the use of condition (1) and condition (2) of Lemma 2.10 and obtaining a contradiction with Remark 4.3 instead of Lemma 4.2.

Lemma 4.6 *Suppose $(R_1, C) \preceq (R_2, C)$ and suppose that (R_1, C) has property $*$. Let $C_1 = C(R_1)$. Suppose that $a \in R_2$ with $C_1 < a < R^{>C_1}$ and $C(a)$. Suppose that $b \in R_2 \setminus R_1$ and there is no $\tilde{b} \in \text{dcl}_{R_1}(b)$ with $C_1 < \tilde{b} < R^{>C_1}$. Then $(R_1 \langle b \rangle, C) \preceq (R_1 \langle ab \rangle, C)$.*

PROOF: First note that by Lemma 4.1, $(R_1, C) \preceq (R_1 \langle a \rangle, C)$ and $(R_1, C) \preceq (R_1 \langle b \rangle, C)$, and applying Lemma 2.10, we see that $(R_1 \langle a \rangle, C)$ and $(R_1 \langle b \rangle, C)$ have property $*$. Also note that a can be taken to be the beginning of a sequence as in the hypotheses of Lemma 4.4, and so, applying that lemma, we see that $(R_1 \langle a \rangle, C) \preceq (R_2, C)$. Thus applying Lemma 4.1, we see that $(R_1 \langle a \rangle, C) \preceq (R_1 \langle ab \rangle, C)$.

Applying Theorem 3.1 to $(R_1 \langle a \rangle, C) \preceq (R_1 \langle ab \rangle, C)$, we see that, *a priori*, there are three possibilities:

1. $\text{dcl}_{R_1 \langle a \rangle}(b)$ contains no elements greater than $C(R_1 \langle a \rangle)$ and less than $R_1 \langle a \rangle^{>C(R_1 \langle a \rangle)}$,
2. The elements of $\text{dcl}_{R_1 \langle a \rangle}(b)$ that are greater than $C(R_1 \langle a \rangle)$ and less than $R_1 \langle a \rangle^{>C(R_1 \langle a \rangle)}$ are all in C ,

3. The elements of $\text{dcl}_{R_1\langle a \rangle}(b)$ that are greater than $C(R_1\langle a \rangle)$ and less than $R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$ are all greater than C .

In the first two cases, Theorem 3.1 implies that $(R_1\langle b \rangle, C) \preceq (R_1\langle ab \rangle)$, while in the third case it implies that $(R_1\langle b \rangle, C) \not\preceq (R_1\langle ab \rangle)$.

Thus, for a contradiction, we assume that there is a R_1 -definable function $f(x, y)$ with $C(R_1\langle a \rangle) < f(a, b) < R_1\langle a \rangle^{>C(R_1\langle a \rangle)}$. Let $B_x = \{y \mid f(x, y) > C\}$. Suppose first that $f(a, y)$ is increasing as a function of y . Then B_a , realized as a subset of $R_1\langle a \rangle$, includes everything to the right of the cut of b in $R_1\langle a \rangle$, since $f(a, b) > C(R_1\langle a \rangle)$. (And note that $\text{tp}(b/R_1\langle a \rangle)$ is a cut since $f(a, y)$ maps this type to the cut corresponding to C in $R_1\langle a \rangle$.) B_a cannot include any $d \in R_1$ to the left of the cut of b , since then

$$C(R_1\langle a \rangle) < f(a, d) < f(a, b) < R_1\langle a \rangle^{>C(R_1\langle a \rangle)},$$

which is impossible as $f(a, d) \in R_1\langle a \rangle$. And since $R_1\langle b \rangle$ contains nothing realizing $\text{tp}(a/R_1)$, $R_1\langle a \rangle$ can contain no points realizing $\text{tp}(b/R_1)$. Thus if B_a included anything to the left of the cut of b in $R_1\langle a \rangle$, it would contain a element of R_1 . Similarly if $f(a, y)$ is decreasing, B_a contains everything to the left of the cut of b and contains nothing to its right.

Note that $f(x, b)$ maps $\text{tp}(a/R_1)$ to itself. Thus $f(x, b)$ is increasing on an definable neighborhood of a . So there is an interval I defined over R_1 and including b so that for all $y \in I$, $f(x, y)$ is increasing in x . We restrict ourselves to such an I .

Since $f(x, y)$ is increasing as a function of x , $x_1 < x_2$ implies that $B_{x_1} \subseteq B_{x_2}$. Choose $\tilde{a} \in C(R_1\langle a \rangle)$ with $\tilde{a} > a$. $B_a \subseteq B_{\tilde{a}}$ so $B_{\tilde{a}}$ also includes everything on one side of the cut of b , and we claim that $B_{\tilde{a}}$ also cannot contain any d on the other side of the cut of b . If it did contain a d on the other side of b , then we could choose such a d in R_1 . Then $(R_1\langle a \rangle, C)$ would satisfy the sentence “there is x in C such that B_x contains d ”, and (R_1, C) , being an elementary submodel, would satisfy the same sentence. Thus R_1 would contain an r in C such that $d \in B_r$. Thus $f(r, d) > C$, but $f(r, d) < f(a, d) < f(a, b)$, so $f(r, d) \in R_1$ would realize the cut of a in R_1 .

So for any $\tilde{a} > a$ in $C(R_1\langle a \rangle)$, $B_{\tilde{a}} = B_a$. Thus

$$(R_1\langle a \rangle, C) \models \exists x_0 \bigcup_{x \in C} B_x = B_{x_0}.$$

Since $(R_1, C) \preceq (R_1\langle a \rangle, C)$, we see that $(R_1, C) \models \exists x_0 \bigcup_{x \in C} B_x = B_{x_0}$.

Say $a_0 \in R_1$ is such an x_0 . (R_1, C) satisfies $\bigcup_{x \in C} B_x = B_{a_0}$, so $(R_1\langle a \rangle, C)$ does as well. Thus $(R_1\langle a \rangle, C) \models B_a = B_{a_0}$. Since $(R_1\langle a \rangle, C) \preceq (R_1\langle ab \rangle, C)$, $B_a = B_{a_0}$ holds in $(R_1\langle ab \rangle, C)$ as well. In particular, $f(a_0, b) > C$. Since $f(a_0, b) > C$ is quantifier free, it holds in $(R_1\langle b \rangle, C)$ as well. As we also have $f(a_0, b) < f(a, b)$, we see that $C_1 < f(a_0, b) < R_1^{>C_1}$, contradicting our choice of b . \square

Lemma 4.7 *Suppose $(R_0, C) \preceq (R, C)$, C defines a cut in R_0 with property $*$, and $b \in R \setminus R_0$. Then $(R_0\langle b \rangle, C) \preceq (R, C)$.*

PROOF:

Case 1: There is an element of $\text{dcl}_{R_0}(b)$ in $C(R)^{>C(R_0)}$: We may assume that this element is b . We may find $(a_\alpha)_{\alpha < \kappa}$ as in the hypotheses of Lemma 4.4 with $a_0 = b$ to obtain $(R_0\langle b \rangle, C) \preceq (R, C)$.

Case 2: There is an element of $\text{dcl}_{R_0}(b)$ greater than $C(R_0\langle b \rangle)$ but less than $R_0^{>C(R_0)}$: As in Case 1, but using Remark 4.5.

Case 3: No element of $\text{dcl}_{R_0}(b)$ is greater than $C(R_0)$ but less than $R_0^{>C(R_0)}$. Choose $(a_\alpha)_{\alpha < \kappa}$ as in the hypotheses of Lemma 4.4 (and let R_α and γ be as in the proof of Lemma 4.4 as well) with the additional requirement that if at any stage α there is an element of $\text{dcl}_{R_0}(b(a_\beta)_{\beta < \alpha})$ that could be chosen as a_α then one chooses such an element. If this happens at some stage prior to γ , let β denote that stage. Otherwise, let $\beta = \gamma$. Thus one has:

$$\begin{array}{ccccccc} (R_0\langle b \rangle, C) & \xleftarrow{\rho_1} & (R_1\langle b \rangle, C) & \xleftarrow{\rho_2} & \dots & \xleftarrow{\rho_\beta} & (R_\beta\langle b \rangle, C) & \xleftarrow{\rho} & (R, C) \\ \sigma_0 \uparrow & & \sigma_1 \uparrow & & & & \sigma_\beta \uparrow & & \\ (R_0, C) & \xleftarrow{\tau_1} & (R_1, C) & \xleftarrow{\tau_2} & \dots & \xleftarrow{\tau_\beta} & (R_\beta, C) & & \end{array}$$

where each τ_i is an elementary embedding by Lemma 4.4, each σ_i is an elementary embedding by Lemma 4.1, and ρ is an elementary embedding since either $\beta < \gamma$ and ρ continues the chain of elementary embeddings $(R_\alpha, C) \preceq (R_{\alpha+1}, C)$, or $\beta = \gamma$ and $C(R_\beta\langle b \rangle)$ is cofinal in $C(R)$.

Note that $R_0\langle b \rangle$ contains no x with $C(R_0) < x < R_0^{C(R_0)}$, so Lemma 4.6 shows that ρ_1 is an elementary embedding. Inductively, except for possibly $i = \beta$, $R_i\langle b \rangle$ contains no x with $C(R_i) < x < R_i^{C(R_i)}$, so Lemma 4.6 shows that ρ_{i+1} is an elementary embedding. Thus composing the ρ_i and ρ , one sees that $(R_0\langle b \rangle, C) \preceq (R, C)$.

□

Remark 4.8 *Up to now, we have worked inside a fixed monster model of $\text{Th}(R, C)$, and so we have treated property $*$ as a property of a particular substructure even though its definition depends on parameters from outside the structure. In what follows, we will take a substructure of (R, C) and embed it into a model of a potentially different theory, raising the question of whether this could potentially change whether the substructure satisfies $*$.*

But it is easy to see that this will not happen. For if a substructure $(R_0, C) \subseteq (R, C)$ does not satisfy $$ as a substructure of (\mathcal{R}, C) , then there is a tuple t and a function $f(x, t)$ mapping a cofinal sequence in $C(R_0)$ to a sequence coinitial in $R_0^{>C(R_0)}$, and this can be seen to be a property of the \mathcal{L}_0 -type of t over R_0 . If (R_0, C) is also a substructure of a sufficiently saturated (\mathcal{R}_1, C) , then as the \mathcal{L}_0 -theories of \mathcal{R}_1 and \mathcal{R} are the same, this type is also realized in \mathcal{R}_1 , and (R_0, C) also fails to satisfy $*$ as a substructure of \mathcal{R}_1 .*

Proposition 4.9 *Suppose (R_0, C) is a substructure of (R, C) such that (R, C) can be embedded over R_0 into an elementary extension (\tilde{R}, C) of (R_0, C) , and suppose C defines a cut in R_0 with property $*$. Then $(R_0, C) \preceq (R, C) \preceq (\tilde{R}, C)$.*

PROOF: This follows from Lemmas 4.1 and 4.7 by transfinite induction. \square

The above shows that, after adding constants for an elementary submodel of (R, C) where the cut C has property $*$, any substructure of (R, C) is an elementary submodel. This easily leads to quantifier elimination in the language with the new constants.

One may ask whether these constants are really necessary. Certainly one cannot obtain the final goal of quantifier elimination, universal axiomatization, and definable Skolem functions without adding any constants. Consider a simple example: Let $R = \mathbb{R}\langle t \rangle$ where $t > \mathbb{Q}$, C is the convex hull of \mathbb{Q} in R , and let \mathcal{L}_0 be the language of ordered rings. Here one needs to add at least one constant, e.g. t . Otherwise the reals form a substructure of R and hence $Th(R, C)$ is not universally axiomatizable. On the other hand, once one adds t , one has both terms which are cofinal in C , as well as terms coinital in $R^{>C}$. We show that in our case, having such terms is all that is needed.

We use the following fact, found for example in Hodges [10], p. 294, Theorem 6.5.1 (where his B and \bar{a} are both A below).

Fact 4.10 *Let A be a substructure of C , and assume that every existential formula with parameters from A that is true in C is also true in A . Then there is an embedding of C into an elementary extension, D , of A , and this embedding can be chosen to be the identity on A .*

Lemma 4.11 *Fix a model (R, C) where the cut C has property $*$. Choose a sequence $(c_i)_{i \in I}$ cofinal in C and a sequence $(d_j)_{j \in J}$ coinital in $R^{>C}$. Let R_0 be $\text{dcl}((c_i)_{i \in I} (d_j)_{j \in J})$. Then (R_0, C) has property $*$, and (R, C) can be embedded over R_0 into an elementary extension (\tilde{R}, C) of (R_0, C) .*

PROOF: Note that by Proposition 2.2 (1), (R_0, C) clearly satisfies $*$. By Fact 4.10, it suffices to show that any existential statement with parameters from R_0 true in (R, C) is also true in (R_0, C) . Note that any quantifier free formula in \mathcal{L}_C is a finite disjunction of formulas of the form

$$\theta(x) \wedge C(f_1(x)) \wedge \cdots \wedge C(f_n(x)) \wedge \cdots \wedge \neg C(g_1(x)) \wedge \cdots \wedge \neg C(g_m(x))$$

where θ , each f_i , and each g_j are \mathcal{L}_0 -definable over R_0 , and where x may be a tuple. By setting

$$f(x) = \max(f_1(x), \dots, f_n(x)) \text{ and } g(x) = \min(g_1(x), \dots, g_m(x)),$$

we see that every quantifier free formula is a finite disjunction of formulas

$$\varphi(x) = \theta(x) \wedge C(f(x)) \wedge \neg C(g(x)).$$

Note that in (R, C) , $C(x)$ iff $\bigvee_{i \in I} x < c_i$, and $\neg C(x)$ iff $\bigvee_{j \in J} x > d_j$. Thus,

$$\varphi(R) = \theta(R) \cap \bigcup_{i \in I} f^{-1}(R^{<c_i}) \cap \bigcup_{j \in J} g^{-1}(R^{>d_j}).$$

Now assume that $(R, C) \models \exists x \varphi(x)$ is witnessed by r . In particular, there are i, j such that r satisfies the formula $\theta(x) \wedge f(x) < c_i \wedge g(x) > d_j$. As this set is \mathcal{L}_0 -definable over R_0 , and as R_0 considered as an \mathcal{L}_0 -structure has definable Skolem functions, there is $r_0 \in \text{dcl}(R_0(c_i)_{i \in I}(d_j)_{j \in J})$ also contained in this set, and thus $(R_0, C) \models \exists x \varphi(x)$. \square

Theorem 4.12 *Fix a model (R, C) where the cut C has property $*$. Choose a set Δ consisting of the elements of a sequence $(c_i)_{i \in I}$ cofinal in C and the elements of a sequence $(d_j)_{j \in J}$ coinital in $R^{>C}$. Let R_0 be $\text{dcl}(\Delta)$ and let \mathcal{L}_Δ be the language \mathcal{L} with constant symbols for Δ . We let T_Δ be the theory of (R, C) in \mathcal{L}_Δ . Then T_Δ is universally axiomatizable, has quantifier elimination and definable Skolem functions.*

PROOF: By Lemma 4.11, (R_0, C) has property $*$. So we may apply Lemma 4.11 to embed a copy of (R, C) into an elementary extension (\tilde{R}, C) of (R_0, C) . Then we apply Proposition 4.9 to see that (R_0, C) is an elementary substructure of the copy of (R, C) , showing that the \mathcal{L}_Δ -theory of (R_0, C) is T_Δ .

By choosing a sufficiently saturated (\tilde{R}, C) , universal axiomatizability follows from showing that every substructure of (\tilde{R}, C) is a submodel, and model completeness follows from showing every submodel of (\tilde{R}, C) is an elementary submodel. Since any \mathcal{L}_Δ -substructure contains R_0 , both of these follow from Proposition 4.9.

A model complete theory eliminates quantifiers precisely when T^\forall has the amalgamation property. Thus, model completeness together with universal axiomatizability implies quantifier elimination. Furthermore, quantifier elimination together with uniform axiomatizability implies the existence of definable Skolem functions. \square

Corollary 4.13 *If $f: R \rightarrow R$ is (R, C) -definable, then there are R -definable functions $f_1, \dots, f_k: R \rightarrow R$ such that for each $a \in R$, $f(a) = f_i(a)$ for some $i \in \{1, \dots, k\}$.*

PROOF: This is a consequence of T_Δ having definable Skolem functions and a universal axiomatization, where Δ is chosen as in the previous theorem. \square

Finally, we have a partial converse to Theorem 4.12.

Proposition 4.14 *Suppose that the theory of (R, C) has quantifier elimination and is universally axiomatizable. Then any sufficiently saturated elementary extension (\tilde{R}, C) has property $*$.*

PROOF: Consider $R \preceq \mathcal{R}$ but without choosing an interpretation of the predicate C in \mathcal{R} . Choose $a \in \mathcal{R}$ realizing the cut corresponding to $C(R)$ and define $C(R\langle a \rangle)$ to be the convex hull of $C(R)$ in $R\langle a \rangle$. Repeat this process until one has built (R_1, C) with the coinitiality of $R_1^{>C(R_1)}$ greater than the cofinality of $C(R_1)$ (which stay equal to the cofinality of $C(R)$). (Or if the coinitiality of $R^{>C(R)}$ was uncountable, repeat only countably many times to obtain (R_1, C) with the coinitiality of $R_1^{>C(R_1)}$ countable.)

Since the cofinality of $C(R_1)$ and the coinality of $R_1^{>C(R_1)}$ differ, Proposition 2.3 implies that (R_1, C) has property $*$. (R, C) is a substructure of (R_1, C) , and thus by quantifier elimination and universal axiomatizability, $(R, C) \preceq (R_1, C)$. Moreover, (R_1, C) embeds into any sufficiently saturated elementary extension (\tilde{R}, C) of (R, C) , and we may build up to (\tilde{R}, C) from (R_1, C) with each step preserving property $*$. \square

5 Appendix – convex valuations with o-minimal residue field

Here we show that if C is a predicate for a convex subring (hence valuation ring) of R with o-minimal residue field, then the cut defined by C has property $*$.

In this section, we assume that $C = V$ is realized as a proper convex subring of its ambient o-minimal field. For an o-minimal field S we denote by \bar{x} the residue of $x \in V(S)$, and we let

$$I_S = \{x \in S : 0 \leq x \leq 1\}.$$

For $X \subseteq S^{1+n}$, we set

$$X(a) := \{\underline{x} \in S^n : (a, \underline{x}) \in X\},$$

where $\underline{x} = (x_1, \dots, x_n)$. We assume that $p \in S_1(\mathcal{R})$ is the global R -invariant type

$$\{r < x : r \in V(R)\} \cup \{x < r : r \in \mathcal{R}^{>V(R)}\}.$$

Definition 5.1 *Let S be an o-minimal field. We say that $(S, V) \models \Sigma(n)$ if for every $X \subseteq I_S^{1+n}$ definable in S (considered as an o-minimal field) there is ϵ_0 in $\mathfrak{m}^{>0}$, the maximal ideal of $V(S)$, such that for each $\epsilon \in \mathfrak{m}^{>\epsilon_0}$,*

$$\overline{X(\epsilon_0)} = \overline{X(\epsilon)}.$$

Observe that then by cell decomposition, $\Sigma(1)$ is equivalent to: For every $f: I_S \rightarrow I_S$ definable in the o-minimal field S there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that for all $\epsilon \in \mathfrak{m}^{\geq \epsilon_0}$,

$$\overline{f(\epsilon)} = \overline{f(\epsilon_0)}.$$

We shall use the following two facts.

Fact 5.2 ([13], **Theorem 1.2**) *The residue field with structure induced from R (equivalently, from (R, V) – see [9], *Theorem 4.6*) is o-minimal iff $(R, V) \models \Sigma(1)$.*

Fact 5.3 (M., van den Dries [8], **Theorem 1.2**) *For any n ,*

$$(R, V) \models \Sigma(1) \Rightarrow \Sigma(n).$$

Lemma 5.4 *Suppose $(R, V) \models \Sigma(1)$, t realizes $p|_R$, and let $R_t = R(t)$. Suppose that for each $a \in R_t$ with $V(R) < a < R^{>V(R)}$ one has $V(a)$. Then $(R_t, V) \models \Sigma(1)$.*

PROOF: We shall denote by \mathfrak{m} and \mathfrak{m}_t the maximal ideal of $V(R)$ and of $V(R_t)$ respectively, and note that \mathfrak{m}_t is the convex hull of \mathfrak{m} in R_t . Assume to the contrary $(R_t, V) \models \neg\Sigma(1)$. Then there is an R -definable function $f: I_{R_t}^2 \rightarrow I_{R_t}$ such that for all $\epsilon \in \mathfrak{m}_t$ there is $\delta_\epsilon \in \mathfrak{m}_t^{>\epsilon}$ with

$$\overline{f\left(\frac{1}{t}, \epsilon\right)} \neq \overline{f\left(\frac{1}{t}, \delta_\epsilon\right)}.$$

By Fact 5.3, we can find $\epsilon_0 \in \mathfrak{m}^{>0}$ such that

$$\overline{(\Gamma f \cap (I_R \times \{\epsilon\} \times I_R))} = \overline{(\Gamma f \cap (I_R \times \{\epsilon_0\} \times I_R))} \in V(R)/\mathfrak{m} \quad (1)$$

for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. We shall show that then $\overline{f\left(\frac{1}{t}, \epsilon_0\right)} = \overline{f\left(\frac{1}{t}, \epsilon\right)} \in V(R_t)/\mathfrak{m}_t$ for all $\epsilon \in \mathfrak{m}_t^{>\epsilon_0}$.

Fix $\epsilon \in \mathfrak{m}_t^{>\epsilon_0}$ such that $\overline{f\left(\frac{1}{t}, \epsilon_0\right)} \neq \overline{f\left(\frac{1}{t}, \epsilon\right)}$ (where both residues are elements of $V(R_t)/\mathfrak{m}_t$). Since \mathfrak{m}_t is the convex hull of \mathfrak{m} in R_t , we may assume that $\epsilon \in \mathfrak{m}$. Since $(R, V) \models \Sigma(1)$, there are $a \in \mathfrak{m}^{>0}$ and $b \in I_R^{>\mathfrak{m}}$ such that

$$\begin{aligned} &\overline{\Gamma f \cap ([a, b] \times \{\epsilon\} \times I_R)} \text{ and } \overline{\Gamma f \cap ([a, b] \times \{\epsilon_0\} \times I_R)} \\ &\text{are graphs of functions } [0, \bar{b}] \rightarrow V(R)/\mathfrak{m}. \end{aligned} \quad (2)$$

Define $g(x) = |f(x, \epsilon) - f(x, \epsilon_0)|$ for $x \in [a, b] \subseteq R_t$. By 1 and 2, $g(x) \in \mathfrak{m}$ for all $x \in [a, b] \subseteq R$. After possibly shrinking $[a, b] \subseteq R$ subject to the condition that $a < \frac{1}{t} < b$, we may moreover assume that g is continuous and monotone on (a, b) , contradicting $g\left(\frac{1}{t}\right) > \mathfrak{m}$. \square

Proposition 5.5 *If $(R, V) \models \Sigma(1)$, then the cut corresponding to V has property $*$.*

PROOF: Assume to the contrary that $\underline{t} = t_1, \dots, t_n$ is a finite Morley sequence in p over R and $f: R^{n+1} \rightarrow R$ is \mathcal{L}_0 -definable over R such that $f_{\underline{t}}$ takes a cofinal segment of $V(R)$ to a coinitial segment of the convex hull of $R^{>V(R)}$ in \mathcal{R} . Note that we may replace \underline{t} with any other tuple satisfying the same o-minimal type over R , and thus we may assume that, letting $R_{\underline{t}} = R(\underline{t})$, one has $V(a)$ for each $a \in R_{\underline{t}}$ with $V(R) < a < R^{>V(R)}$. Thus by Lemma 5.4, $(R_{\underline{t}}, V) \models \Sigma(1)$.

After replacing $f_{\underline{t}}$ (restricted to a suitable interval) with $\frac{1}{f_{\underline{t}}^{-1}} \circ \frac{1}{x}$, we obtain a function that maps a cofinal segment of \mathfrak{m} to a coinitial segment of $V(R)^{>\mathfrak{m}}$, where \mathfrak{m} is the maximal ideal of $V(R)$. The function $f_{\underline{t}}$ is definable over $R_{\underline{t}}$, decreasing, and maps a cofinal segment of $\mathfrak{m}_{\underline{t}}$ (the maximal ideal of $V(R_{\underline{t}})$) to a proper convex subset of $V(R_{\underline{t}})^{>\mathfrak{m}_{\underline{t}}}$ and, after composition with the residue map, we obtain a function definable in $R_{\underline{t}}$ whose image is not eventually constant on $\mathfrak{m}_{\underline{t}}$, contradicting $(R_{\underline{t}}, V) \models \Sigma(1)$. □

By Theorem 6.3, p. 5470 [11], there is a canonical way in which (R, C) defines a valuation ring. In Example 2.6, this valuation ring is the convex hull of \mathbb{Q} in R' , so its residue field is o-minimal by Corollary 2.4, p. 244 [9]. Example 2.6 therefore shows that it is not possible to weaken the assumption in Proposition 5.5 to: “the residue field of the canonical valuation associated to (R, C) is o-minimal”.

We now turn to Corollary 3.4 [9] in which we claim that $(R, V) \models \Sigma(1)$ whenever V is the convex hull of \mathbb{Q} . Our proof of this now requires property $*$.

Theorem 5.6 [Corollary 3.4 [9]] *Consider (R, V) where V is the convex hull of \mathbb{Q} . If (R, V) has property $*$, then $(R, V) \models \Sigma(1)$.*

PROOF: Work in a sufficiently saturated elementary extension (\mathcal{R}, V) . Note that each cut in \mathbb{Q} is realized in \mathcal{R} . Take an element $a \in \mathcal{R}$ realizing such a cut. If $R\langle a \rangle$ contains no element \tilde{a} with $V(R) < \tilde{a} < R^{>V(R)}$ then note that in $(R\langle a \rangle, V)$, V is again the convex hull of \mathbb{Q} . If $R\langle a \rangle$ contains \tilde{a} with $V(R) < \tilde{a} < R^{>V(R)}$ then there is an R -definable function, f , mapping the cut of a to the cut in R corresponding to V . If $f(a) > V$, then note that $V(R\langle a \rangle)$ is again the convex hull of \mathbb{Q} (by Theorem 3.1 and Lemma 4.1). If $V(f(a))$, choose $b \in \mathcal{R}$ with $V(\mathcal{R}) < b < R^{>V(R)}$, and replace a with $f^{-1}(b)$. Again, $V(R\langle a \rangle)$ is the convex hull of \mathbb{Q} .

By Lemma 4.7, $(R\langle a \rangle, V) \preceq (\mathcal{R}, V)$. Thus we may repeatedly add realizations of cuts in \mathbb{Q} while keeping V the convex hull of \mathbb{Q} until we have built a model (R_1, V) which is an elementary extension of (R, V) and which has \mathbb{R} as a residue field. This residue field is necessarily o-minimal, and hence $(R_1, V) \models \Sigma(1)$, as does (R, V) . □

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