RAMIFICATION OF VALUATIONS AND LOCAL RINGS IN POSITIVE CHARACTERISTIC

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1. INTRODUCTION

In this paper we consider birational properties of ramification of excellent local rings.

Suppose that K^*/K is a finite separable field extension, S is an excellent local ring of K^* (S has quotient field $\mathrm{QF}(S) = K^*$) and R is an excellent local ring of K such that $\dim S = \dim R$, S dominates R ($R \subset S$ and the maximal ideals m_S of S and m_R of R satisfy $m_S \cap R = m_R$) and ν^* is a valuation of K^* which dominates S (the valuation ring V_{ν^*} of ν^* dominates S). Let ν be the restriction of ν^* to K.

The notation that we use in this paper is explained in more detail in Section 2.

1.1. Local Monomialization.

Definition 1.1. $R \to S$ is monomial if R and S are regular local rings of the same dimension n and there exist regular systems of parameters $x_1, \ldots, x_n$ in R and $y_1, \ldots, y_n$ in S, units $\delta_1, \ldots, \delta_n$ in S and an $n \times n$ matrix $A = (a_{ij})$ of natural numbers with nonzero determinant such that

$$(1) \qquad x_i = \delta_i \prod_{j=1}^n y_j^{a_{ij}} \ \text{for } 1 \leq i \leq n.$$

If R and S have equicharacteristic zero and algebraically closed residue fields, then within the extension $\hat{R} \to \hat{S}$ there are regular parameters giving a form (1) with all $\delta_i = 1$.

More generally, we ask if a given extension $R \to S$ has a local monomialization along the valuation ν^*.

Definition 1.2. A local monomialization of $R \to S$ is a commutative diagram

$$
\begin{array}{ccccc}
R_1 & \to & S_1 & \subset & V_{\nu^*} \\
\uparrow & & \uparrow & & \\
R & \to & S & &
\end{array}
$$

such that the vertical arrows are products of monoidal transforms (local rings of blowups of regular primes) and $R_1 \to S_1$ is monomial.

It is proven in Theorem 1.1 [10] that a local monomialization always exists when K^*/K are algebraic function fields over a (not necessarily algebraically closed) field k of characteristic 0, and $R \to S$ are algebraic local rings of K and K^* respectively. (An algebraic local ring is essentially of finite type over k.)

We can also define the weaker notion of a *weak local monomialization* by only requiring that the conclusions of Definition 1.2 hold with the vertical arrows being required to be birational (and not necessarily factorizable by products of monoidal transforms).

This leads to the following question for extensions $R \to S$ as defined in the beginning of this paper: *When does there exist a local monomialization (or at least a weak local monomialization) of an extension $R \to S$ of excellent local rings dominated by a valuation ν^* ?*

As commented above, the question has a positive answer within algebraic function fields over an arbitrary field of characteristic zero by Theorem 1.1 [10].

For the question to have a positive answer in positive characteristic or mixed characteristic it is of course necessary that some form of resolution of singularities be true. This is certainly true in equicharacteristic zero, and is known to be true very generally in dimension ≤ 2 ([2], [29], [7]) and in positive characteristic and dimension 3 ([4], [11], [8] and [9]). A few recent papers going beyond dimension three are [18], [22], [6], [5], [23], [35], [36], [24] and [37].

The case of two dimensional algebraic function fields over an algebraically closed field of positive characteristic is considered in [14], where it is shown that monomialization is true if $R \to S$ is a *defectless* extension of two dimensional algebraic local rings over an algebraically closed field k of characteristic $p > 0$ (Theorem 7.3 and Theorem 7.35 [14]). We will discuss the important concept of defect later on in this introduction.

In [12], we give an example showing that weak monomialization (and hence monomialization) does not exist in general for extensions of algebraic local rings of dimension ≥ 2 over a field k of char $p > 0$. We prove the following theorem in [12]:

Theorem 1.3. *(Theorem 1.4 [12], Counterexample to local and weak local monomialization) Let k be a field of characteristic $p > 0$ with at least 3 elements and let $n \geq 2$. Then there exists a finite separable extension K^*/K of n dimensional function fields over k, a valuation ν^* of K^* with restriction ν to K and algebraic regular local rings A and B of K and K^* respectively, such that B dominates A, ν^* dominates B and there do not exist regular algebraic local rings A' of K and B' of K^* such that ν^* dominates B', B' dominates A', A' dominates A, B' dominates B and $A' \to B'$ is monomial.*

We have that the defect $\delta(\nu^*/\nu) = 2$ in the example of Theorem 1.3 (with $\nu = \nu^*|K$).

In [10] and [14], a very strong form of local monomialization is established within characteristic zero algebraic function fields which we call strong local monomialization (Theorem 5.1 [10] and Theorem 48 [14]). This form is stable under appropriate sequences of monoidal transforms and encodes the classical invariants of the extension of valuation rings. In [14], we show that strong local monomialization is true for defectless extensions of two dimensional algebraic function fields (Theorem 7.3 and Theorem 7.35 [14]). We give an example in [14] (Theorem 7.38 [14]) showing that strong local monomialization is not generally true for defect extensions of two dimensional algebraic function fields (over a field of positive characteristic).

In this paper, we establish that local monomialization (and strong local monomialization) hold for defectless extensions of two dimensional excellent local rings. We will first state our theorem (which is proven in Section 3), and then we will define and discuss the defect.

Theorem 1.4. *Suppose that R is a 2 dimensional excellent local domain with quotient field $QF(R) = K$. Further suppose that K^* is a finite separable extension of K and S is a two dimensional excellent local domain with quotient field $QF(S) = K^*$ such that S dominates R. Let ν^* be a valuation of K^* which dominates S and let ν be the restriction*

of ν^* to K. Suppose that the defect $\delta(\nu^*/\nu) = 0$. Then there exists a commutative diagram

(2)
$$
\begin{array}{ccccc}
R_1 & \to & S_1 & \subset & V_{\nu^*} \\
\uparrow & & \uparrow & & \\
R & \to & S & &
\end{array}
$$

such that the vertical arrows are products of quadratic transforms along ν^* and $R_1 \to S_1$ is monomial.

The proof of the theorem actually produces stable strong monomialization.

We now define the defect of an extension of valuations. The role of this concept in local uniformization was observed by Kuhlmann [26] and [27]. A good introduction to the role of defect in valuation theory is given in [26]. A brief survey which is well suited to our purposes is given in Section 7.1 of [14]. Suppose that K^*/K is a finite Galois extension of fields of characteristic $p > 0$. The splitting field $K^s(\nu^*/\nu)$ of ν is the smallest field between K and K^* with the property that ν^* is the only extension to K^* of $\nu^*|L$. The defect $\delta(\nu^*/\nu)$ is defined by the identity

$$[K^* : K^s(\nu^*/\nu)] = f(\nu^*/\nu)e(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

(Corollary to Theorem 25 , Section 12, Chapter VI [40]). In the case when K^*/K is only finite separable, we define the defect by

$$\delta(\nu^*/\nu) = \delta(\nu'/\nu) - \delta(\nu'/\nu^*)$$

where ν' is an extension of ν^* to a Galois closure K' of K^* over K.

The defect is equal to zero if the residue field V_ν/m_ν has characteristic zero (Theorem 24, Section 12, Chapter VI [40]) or if V_ν is a DVR (Corollary to Theorem 21, Section 9, Chapter V [39]).

1.2. **Associated graded rings of valuations.** The semigroup of R with respect to the valuation ν is

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}.$$

The group generated by $S^R(\nu)$ is the valuation group Γ_ν of ν which is well understood ([30], [31], [40], [27]); the semigroup can however be extremely complicated and perverse ([17], [15]).

The associated graded ring of ν on R, as defined in [35] and [36], is

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R).$$

Here $P_\gamma(R)$ is the ideal in R of elements of value $\geq \gamma$ and $P_\gamma^+(R)$ is the ideal in R of elements of value $> \gamma$. This ring plays an important role in Teissier's approach to resolution of singularity (it is completely realized for Abhyankar valuations in arbitrary characteristic in [36]).

We always have that $\mathrm{QF}(\mathrm{gr}_\nu(R)) = \mathrm{QF}(\mathrm{gr}_\nu(V_\nu)$ and

$$[\mathrm{QF}(\mathrm{gr}_{\nu^*}(S)) : \mathrm{QF}(\mathrm{gr}_\nu(R))] = f(\nu^*/\nu)e(\nu^*/\nu).$$

In [19] and [20] it is proven that there exists a strong local monomialization $R_1 \to S_1$ for defectless extensions $R \to S$ of two dimensional algebraic local rings in a two algebraic function field over an algebraically closed field, which has the property that the induced extension of associated graded rings along the valuation

(3)
$$\mathrm{gr}_\nu(R) \to \mathrm{gr}_{\nu^*}(S)$$

is of finite type and is even a "toric extension". This result is extended in [16] to the case of two dimensional algebraic function fields over an arbitrary field of characteristic zero. These proofs all make use of the technique of generating sequences of a valuation on a local ring, which is developed by Spivakovsky in [34] for two dimensional regular local rings with algebraically closed residue fields, and is extended in [15] to arbitrary regular local rings of dimension two. Unfortunately, this technique is special to dimension two, and does not extend well to higher dimension local rings, or even to normal local rings of dimension two (the examples of strange semigroups in [17] and [15] show this). An interesting construction of generating sequences within a valuation ring which exhibits the defect of an extension of valuations is given in [38], and a different general construction of generating sequences is given in [32].

In general, the extension (3) is not of finite type, even for equicharacteristic zero algebraic regular local rings of dimension two (Example 9.4 [15]), so blowing up to reach a good stable form is required to obtain that (3) has a good form.

It is not difficult to show that the extension (3) is of finite type and is toric when ν^* is an Abhyankar valuation (equality holds in Abhyankar's inequality (Theorem 1 [1])

$$\operatorname{trdeg}_{S/m_S} V_{\nu^*}/m_{\nu^*} + \dim_{\mathbb{Q}} \Gamma_{\nu^*} \otimes \mathbb{Q} \le \dim S.$$

From a special case of Theorem 5.1 [10] (recalled in Theorem 4.1 of this paper) we give the precise statement of the stable strongly monomial forms $R_1 \to S_1$ obtained by a rational rank 1 valuation ($\dim_{\mathbb{Q}} \Gamma_{\nu^*} \otimes \mathbb{Q} = 1$) in the case when $R \to S$ is an extension of algebraic local rings in an extension of algebraic function fields over an arbitrary field of characteristic zero. In this case, there are regular parameters $x_1, \ldots, x_n$ in R_1 and regular parameters $y_1, \ldots, y_n$ in S and a unit δ in S_1 such that

$$x_1 = \delta y_1^e, x_2 = y_2, \ldots, x_n = y_n$$

where $e = e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu|$. In this paper, we give a simple proof (Theorem 4.2) that in the case when $R \to S$ is an extension of algebraic local rings in an extension of algebraic function fields over an arbitrary field of characteristic zero, and ν has rational rank 1 a strongly monomial extension $R_1 \to S_1$ has the property that

$$(4) \qquad \operatorname{gr}_{\nu^*}(S_1) \cong \left(\operatorname{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1} \right) [Z]/(Z^e - [\delta]^{-1}[x_1]),$$

where $[\gamma_1], [x_1]$ are the respective classes in $\operatorname{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1}$. The degree of the extension of quotient fields of $\operatorname{gr}_\nu(R) \to \operatorname{gr}_{\nu^*}(S)$ is $e(\nu^*/\nu) f(\nu^*/\nu)$, where

$$f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu].$$

In particular, the extension of associated graded rings along the valuation is finite and "toric".

We show in Theorem 4.3 of this paper that the stable strongly monomial forms found in Theorem 1.4 of defectless extensions of two dimensional excellent local rings dominated by a valuation ν^* of rational rank 1 have an extension of associated graded rings along the valuation of the form (4). Since stable forms of Abhyankar valuations have a finite type "toric extension" as commented above, we conclude that stable strongly monomial forms of a defectless extension of two dimensional excellent local rings always has a finite type "toric" extension (3).

In contrast, we do not have such a nice stable form of the extension of associated graded rings along a valuation which has positive defect, as is shown by the example of Theorem 38 [14], analyzed in Section 5 of this paper. Using the notation of Section 7.4

[14], explained in the following subsection on invariants of stable forms along a valuation, it follows from Remark 7.44 [14] that the graded domains $\mathrm{gr}_{\nu^*}(S_n)$ are integral but not finite over $\mathrm{gr}_\nu(R_n)$ for all n, in contrast to the situation when the defect $\delta(\nu^*/\nu)$ is zero in Theorem 4.3. The quotient fields of $\mathrm{gr}_\nu(R_n)$ and $\mathrm{gr}_{\nu^*}(S_n)$ are equal under the extension of Theorem 38 [14], so the degree is $1 = e(\nu^*/\nu)f(\nu^*/\nu)$ as in the conclusions of Theorem 4.3.

1.3. **Invariants of stable forms along a valuation.** Suppose that $R \to S$ is an inclusion of regular two dimensional algebraic local rings within function fields K and K^* respectively, over an algebraically closed field k of positive characteristic $p > 0$, such that K^*/K is finite separable, S dominates R, and there is a valuation ν^* of K^* with restriction ν to K which dominates R and S, such that

1) ν^* dominates S.
2) The residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S.
3) The value group Γ_{ν^*} of ν^* has rational rank 1 (so it is isomorphic as an ordered group to a subgroup of $\mathbb{Q}$).

It is shown in Corollary 7.30 and Theorem 7.33 [14] that there are sequences of quadratic transforms along ν^* (each vertical arrow is a product of quadratic transforms) constructed by the algorithm of Section 7.4 [14], where we have simplified notation, writing R_n for R_{r_n} and S_n for S_{s_n}.

$$
\begin{array}{ccc}
V_\nu & \to & V_{\nu^*} \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
R_n & \to & S_n \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
R_2 & \to & S_2 \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}
$$

(5)

where each R_n is an algebraic regular local ring of K and S_n is an algebraic regular local ring of K^* such that S_n dominates R_n, and no quadratic transform of R_n factors through S_n. For $n \gg 0$, R_n has regular parameters x_n, y_n and S_n has regular parameters u_n, v_n such that there are "stable forms"

$$
(6) \qquad u_n = \gamma_n x_n^{\bar{a}p^{\alpha_n}}, v_n = x_n^{b_n} f_n
$$

where γ_n is a unit in S_n, $d_n = \bar{\nu}_n(f_n \bmod x_n) = p^{\beta_n}$, where $\bar{\nu}_n$ is the natural valuation of the DVR $S_n/x_n S_n$, with $b_n, \alpha_n, \beta_n \geq 0$ and $\alpha_n + \beta_n$ does not depend on n, $\bar{a}$ does not depend on n.

In Theorem 7.38 [14], an example is given where K^*/K is a tower of two Artin Schreier extensions with $\delta(\nu^*/\nu) = 2$, $\bar{a} = 1$, $\alpha_n = 1$ for all n, $\beta_n = 1$ for all n and $b_n = 0$ for all n. In particular, this shows that "strong local monomialization" fails for this extension. However, it is also shown in the example that local monomialization is true for this extension (by considering different sequences of quadratic transforms above R and S).

In Corollary 7.30 and Theorem 7.33 of [14], it is shown that $\alpha_n + \beta_n$ is a constant for $n \gg 0$, where α_n and β_n are the integers defined above which are associated to the stable forms (5) of an extension of valued two dimensional algebraic function fields. If Γ_ν is not p-divisible, it is further shown that α_n and β_n are both constant for $n \gg 0$, and p^{β_n} is the defect of the extension. However, if Γ_ν is p-divisible, then it is only shown that the sum $\alpha_n + \beta_n$ is constant for $n \gg 0$, and that $p^{\alpha_n+\beta_n}$ is the defect of the extension. In Remark 7.34 [14] it is asked if α_n and β_n (and some other numbers computed from the stable forms) are eventually constant in the case when Γ_ν is p-divisible. We give examples in Section 5, equations (31) - (34), showing that this is not the case, even within (defect) Artin Schreier extensions. The examples are found by considering a factorization of the example of Theorem 3.8 [14] into a product of two Artin Schreier extensions, and computing generating sequences on the intermediary rings.

2. Notation and Preliminaries

2.1. **Local algebra.** All rings will be commutative with identity. A ring S is essentially of finite type over R if S is a local ring of a finitely generated R-algebra. We will denote the maximal ideal of a local ring R by m_R, and the quotient field of a domain R by $\mathrm{QF}(R)$. (We do not require that a local ring be Noetherian). Suppose that $R \subset S$ is an inclusion of local rings. We will say that S dominates R if $m_S \cap R = m_R$. If the local ring R is a domain with $\mathrm{QF}(R) = K$ then we will say that R is a local ring of K. If K is an algebraic function field over a field k (which we do not assume to be algebraically closed) and a local ring R of K is essentially of finite type over k, then we say that R is an algebraic local ring of k.

Suppose that $K \to K^*$ is a finite field extension, R is a local ring of K and S is a local ring of K^*. We will say that S lies over R if S is a localization of the integral closure T of R in K^*. If R is a local ring, $\hat{R}$ will denote the completion of R by its maximal ideal m_R.

Suppose that R is a regular local ring. A monoidal transform $R \to R_1$ of R is a local ring of the form $R[\frac{P}{x}]_m$ where P is a regular prime ideal in R (R/P is a regular local ring) and m is a prime ideal of $R[\frac{P}{x}]$ such that $m \cap R = m_R$. R_1 is called a quadratic transform if $P = m_R$.

2.2. **Valuation Theory.** Suppose that ν is a valuation on a field K. We will denote by V_ν the valuation ring of ν:
$$V_\nu = \{f \in K \mid \nu(f) \geq 0\}.$$
We will denote the value group of ν by Γ_ν. Good treatments of valuation theory are Chapter VI of [40] and [3], which contain references to the original papers. If ν is a valuation ring of an algebraic function field over a field k, we insist that ν vanishes on $k \setminus \{0\}$, and say that ν is a k valuation.

If ν is a valuation of a field K and R is a local ring of K we will say that ν dominates R if the valuation ring V_ν dominates R. Suppose that ν dominates R. A monoidal transform $R \to R_1$ is called a monoidal transform along ν if ν dominates R_1.

Suppose that K^*/K is a finite separable extension, ν^* is a valuation of K^* and ν is the restriction of ν to K. The ramification index is
$$e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu|$$
and reduced degree is
$$f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu].$$

The defect $\delta(\nu^*/\nu)$ is defined in the introduction to this paper. Its basic properties are developed in Section 11, Chapter VI [40], [26] and Section 7.1 of [14].

We will call a ring a DVR if it is a valuation ring with value group $\mathbb{Z}$.

2.3. **Galois theory of local rings.** Suppose that K^*/K is a finite Galois extension, R is a local ring of K and S is a local ring of K^* which lies over R. The splitting group $G^s(S/R)$, splitting field $K^s(S/R) = (K^*)^{G^s(S/R)}$ and inertia group $G^i(S/R)$ are defined and their basic properties developed in Section 7 of [3].

2.4. **Galois theory of valuations.** The Galois theory of valuation rings is developed in Section 12 of Chapter VI of [40] and in Section 7 of [3]. Some of the basic results we need are surveyed in Section 7.1 [14]. If we take $S = V_{\nu^*}$ and $R = V_\nu$ where ν^* is a valuation of K^* and ν is the restriction of ν to K, then we obtain the splitting group $G^s(\nu^*/\nu)$, the splitting field $K^s(\nu^*/\nu)$ and the inertia group $G^i(\nu^*/\nu)$. In Section 12 of Chapter VI of [40], $G^s(\nu^*/\nu)$ is written as G_Z and called the decomposition group. $G^i(\nu^*/\nu)$ is written as G_T. The ramification group G_V of ν^*/ν is defined in Section 12 of Chapter VI of [40] and is surveyed in Section 7.1 [14]. We will denote this group by $G^r(\nu^*/\nu)$.

2.5. **Semigroups and associated graded rings of a local ring with respect to a valuation.** Suppose that ν is a valuation of field K which dominates a local ring R of K. We will denote the semigroup of values of ν on S by

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}.$$

Suppose that $\gamma \in \Gamma_\nu$. We define ideals in R

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq 0\}$$

and

$$\mathcal{P}_\gamma^+(R) = \{f \in R \mid \nu(f) > 0\}$$

and define (as in [35]) the associated graded ring of R with respect to ν by

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+.$$

2.6. **Birational geometry of two dimensional regular local rings.** We recall some basic theorems which we will make frequent use of.

Theorem 2.1. *(Theorem 3 [1]) Suppose that K is a field, and R is a regular local ring of dimension two of K. Suppose that S is another 2 dimensional regular local ring of K which dominates R. Then there exists a unique sequence of quadratic transforms of R*

$$R \to R_1 \to \cdots \to R_n = S$$

of R which factor $R \to S$.

Lemma 2.2. *(Lemma 12 [1]) Suppose that A is a two dimensional regular local ring of a field K and ν is a valuation of K which dominates ν. Let*

$$R \to R_1 \to R_2 \to \cdots$$

be the infinite sequence of quadratic transforms along ν. Then

$$V_{\nu^*} = \cup_{i=1}^\infty R_i.$$

We also make use of the fact that "embedded resolution of singularities" is true within a regular local ring of dimension 2 (Theorem 2 [1]), and the fact that resolution of singularities is true for two dimensional excellent local rings ([29], [7]).

3. Local monomialization of two dimensional defectless extensions

In this section we prove Theorem 1.4, establishing local monomialization for defectless extensions of two dimensional excellent local domains. This extends the result for extensions of two dimensional algebraic local rings in two dimensional algebraic function fields over an algebraically closed field in Theorem 7.3 and 7.35 [14].

We indicate the differences between the proof in [14] of the analogue of Theorem 1.4 for algebraic local rings over an algebraically closed field, and the proof of Theorem 1.4 in this paper. The essential case is of rational rank 1 valuations (Theorem 3.3). An essential ingredient in the proof is the computation of complexity in Proposition 3.1, which generalizes Proposition 7.2 [14]. The steps of the proof are the same, but some of the individual calculations require different methods, as we do not have coefficient fields in general in the situation of this paper, and the completions of local rings are no longer extensions of power series rings over a field. In [14], the analogue of Theorem 3.3 is deduced as a consequence of a detailed analysis of stable forms (Theorem 7.33 [14]) which makes essential use of the assumption that there is no residue field extension (the ground field is algebraically closed). In this paper, we give a different, more direct argument to deduce Theorem 3.3.

The proof of Theorem 1.4 actually produces stable strong monomialization. We first establish strong monomialization in the two essential cases of the theorem, and give the proof of Theorem 1.4 at the end of this section. We will make use of the list of good properties of excellent rings given in Scholie 7.8.3 [21].

3.1. Degree formulas.
In this subsection we generalize the formulas of Proposition 7.2 [14].

Proposition 3.1. *Suppose that R and S are two dimensional regular excellent local rings such that S dominates R and $K^* = QF(S)$ is a finite separable extension of $K = QF(R)$, R has a regular system of parameters u, v and S has a regular system of parameters x, y such that there is an expression*

$$u = \gamma x^a, v = x^b f$$

where $a > 0$, $b \geq 0$, γ is a unit in S, $x \nmid f$ in S and f is not a unit in S. Then there exist inclusions

$$R \to R_0 \to S$$

of local rings with the following properties: R_0 is a two dimensional normal local ring of K which is essentially of finite type over R such that S lies above R_0. Let $\bar{\nu}$ be the natural valuation of the DVR S/xS. Then we have

$$[QF(\hat{S}) : QF(\hat{R}_0)] = ad[S/m_S : R/m_R]$$

where $d = \bar{\nu}(f \bmod x)$.

Proof. First suppose that $b > 0$. Let $s = \gcd(a, b)$, I be the ideal which is the integral closure of $(u^{\frac{b}{s}}, v^{\frac{a}{s}})$ in R. Let

$$\varphi = \frac{v^{\frac{a}{s}}}{u^{\frac{b}{s}}}$$

and

$$R_0 = R\left[\frac{1}{u^{\frac{b}{s}}} I\right]_{m_S \cap R\left[\frac{1}{u^{\frac{b}{s}}} I\right]}.$$

8

R_0 is normal since I is integrally closed. (Powers of I are integrally closed by Theorem 2', Appendix 5 [40], so the ring $\bigoplus_{n\geq 0} I^n$ is integrally closed and R_0 is a local ring of the normal scheme $\mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$). The elements $x^a, \varphi \in m_{R_0}S$ so $m_{R_0}S$ is m_S-primary.

By Lemma 9 [1], there exists a rank 2 valuation ν of K^* which dominates S. Let A be the integral closure of R_0 in K^* and let $T = A_{m_S \cap A}$. By Theorem 1 [1], V_ν/m_ν is finite over T/m_T so S/m_S is finite over T/m_T. T is normal of dimension two (since T is excellent and normal). Since $m_T S$ is m_S-primary, we have that $T = S$ by the version of Zariski's main theorem in (10.7) [4]. Thus S lies over R_0.

Let W^* be the DVR $W^* = S_{(x)}$. $\varphi \in R_0$ is neither a unit nor divisable by x in S. Thus the prime ideal $\mathfrak{p} = xS \cap R_0$ has height one in R_0. Thus

$$W = (R_0)_{\mathfrak{p}} = W^* \cap K$$

is a DVR.

Let t be a regular parameter in W. W is the valuation ring of the valuation ord_t. Since I is a monomial ideal, the value group $\mathbb{Z}$ of ord_t is generated by $\mathrm{ord}_t u$ and $\mathrm{ord}_t v$. Thus $\gcd(\mathrm{ord}_t u, \mathrm{ord}_t v) = 1$. Since $\mathrm{ord}_t \varphi = 0$, we have that $\mathrm{ord}_t u = \frac{a}{s}$ and $\mathrm{ord}_t v = \frac{b}{s}$.

We have that

$$IW = u^{\frac{b}{s}} W = (t^{\frac{a}{s}\frac{b}{s}}).$$

Since

$$IW^* = (x^{\frac{ab}{s}}),$$

we have that $tW^* = (x^s)$, so the ramification index of W^*/W is

$$e(W^*/W) = s.$$

The integrally closed ideal I is generated by all monomials $u^m v^n$ such that

$$\frac{m}{b/s} + \frac{n}{a/s} \geq 1.$$

Since

$$\gcd\left(\frac{a}{s}, \frac{b}{s}\right) = 1,$$

we have that

$$(7) \qquad \mathrm{ord}_x(u^m v^n) = ma + nb > \frac{ab}{s}$$

for such a monomial provided

$$(m, n) \notin \left\{\left(\frac{b}{s}, 0\right), \left(0, \frac{a}{s}\right)\right\}.$$

Thus for $u^m v^n \in I$,

$$\frac{u^m v^n}{u^{\frac{b}{s}}} \in \mathfrak{p}$$

unless

$$u^m v^n = u^{\frac{b}{s}} \text{ or } u^m v^n = v^{\frac{a}{s}}.$$

Since $\overline{\nu}(\varphi) = \frac{ad}{s} > 0$, we have that

$$(8) \qquad\qquad R_0/\mathfrak{p} \cong (R/m_R[\overline{\varphi}])_{(\overline{\varphi})}$$

where $\overline{\varphi}$ is the residue of φ in $R_0/\mathfrak{p}$, and

$$(9) \qquad\qquad R_0/m_{R_0} = R/m_R.$$

9

By Corollary 1 of Section 2, Chapter II of Local Fields [33] and (ii) of Theorem 1, Section 3 of Chapter II [33], and by (9) and the fact that $\overline{\varphi}$ is a regular parameter in $\widehat{R_0/\mathfrak{p}}$ by (8), we have that the inclusion

$$\widehat{R_0/\mathfrak{p}} \to \widehat{S/xS}$$

is a finite extension of complete DVRs, and

$$
\begin{aligned}
[\mathrm{QF}(\widehat{S/xS}) : \mathrm{QF}(\widehat{R_0/\mathfrak{p}})] &= e(\widehat{S/xS}/\widehat{R_0/\mathfrak{p}})f(\widehat{S/xS}/\widehat{R_0/\mathfrak{p}}) \\
&= e(S/xS/R_0/\mathfrak{p})f(S/xS/R_0/\mathfrak{p}) = \tfrac{ad}{s}[S/m_S : R/m_R].
\end{aligned}
$$

We have that $m_R S \subset \mathfrak{p}S$ so xS is the only prime ideal of S lying over $\mathfrak{p}$. Further, $\mathfrak{p}\hat{R}_0$ and $x\hat{S}$ are prime ideals since $R_0/\mathfrak{p}$ and S/xS are regular local rings. $\hat{R}_0$ and $\hat{S}$ are normal since R_0 and S are normal and excellent. Also, $\hat{S}$ is a finite extension of $\hat{R}_0$ by (10.13) and (10.2) of [4], since S lies over R_0 . Thus $\mathrm{QF}(\hat{S})$ is a finite field extension of $\mathrm{QF}(\hat{R}_0)$ and $\hat{S}$ is the integral closure of $\hat{R}_0$ in $\mathrm{QF}(\hat{S})$. We obtain that $\hat{W}^* = \hat{S}_{x\hat{S}}$ is the unique DVR of $\mathrm{QF}(\hat{S})$ which dominates $\hat{W} = (\hat{R}_0)_{\mathfrak{p}\hat{R}_0}$.

Thus, by Theorem 20, page 60 [40],

$$[\mathrm{QF}(\hat{S}) : \mathrm{QF}(\hat{R}_0)] = e(\hat{W}^*/\hat{W})f(\hat{W}^*/\hat{W}).$$

We have that

$$e(\hat{W}^*/\hat{W}) = e(W^*/W) = s$$

since R_0 and S are analytically unramified (as they are excellent), and

$$f(\hat{W}^*/\hat{W}) = [\mathrm{QF}(\hat{S}/x\hat{S}) : \mathrm{QF}(\hat{R}_0/\mathfrak{p}\hat{R}_0)] = \frac{ad}{s}[S/m_S : R/m_R].$$

Thus

$$[\mathrm{QF}(\hat{S}) : \mathrm{QF}(\hat{R}_0)] = ad[S/m_S : R/m_R].$$

Now suppose that $b = 0$. Then taking $R_0 = R$, $W = R_{uR}$ and $W^* = S_{xS}$, a simpler variant of the above proof shows that $e(\hat{W}^*/\hat{W}) = a$, $f(\hat{W}^*/\hat{W}) = d[S/m_S : R/m_R]$ and

$$[\mathrm{QF}(\hat{S}) : \mathrm{QF}(\hat{R})] = ad[S/m_S : R/m_R].$$

$\square$

Proposition 3.2. *Suppose that R and S are two dimensional excellent regular local rings, such that S dominates R and $K^* = QF(S)$ is a finite separable extension of $K = QF(R)$, R has a regular system of parameters u, v and S has a regular system of parameters x, y such that there is an expression*

$$u = \gamma_1 x^a y^c, v = \gamma_2 x^b y^d$$

where γ_1, γ_2 are units in S and $ad - bc \neq 0$. Then there exist inclusions

$$R \to R_0 \to S$$

of local rings with the following properties: R_0 is a two dimensional normal local ring of K which is essentially of finite type over R such that S lies above R_0 and

$$[QF(\hat{S}) : QF(\hat{R}_0)] = |ad - bc|[S/m_S : R/m_R].$$

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Proof. The conclusions of this proposition follow from Proposition 3.1 if one of a, b, c, d is zero, so we may assume that a, b, c, d are all positive. After possibly interchanging x and y, it can be assumed that $ad - bc > 0$. The proof of this proposition is a generalization of the proof of Proposition 3.1, and we give an outline of the proof, indicating the essential differences.

Let $s = \gcd(a, b)$ and $s' = \gcd(c, d)$. Let $I \subset R$ (respectively $I' \subset R$) be the integral closure of the the ideal $(u^{\frac{b}{s}}, v^{\frac{a}{s}})$ (respectively $(u^{\frac{d}{s'}}, v^{\frac{c}{s'}})$). We have that

$$R_0 = R\left[\frac{II'}{u^{\frac{b}{s}} v^{\frac{c}{s'}}}\right]_{m_S \cap R\left[\frac{II'}{u^{\frac{b}{s}} v^{\frac{c}{s'}}}\right]}$$

is a normal local ring (products of integrally closed ideals in a 2-dimensional regular local ring are integrally closed by Theorem 2', Appendix 5 [40]).

Let $\mathfrak{p} = xS \cap R_0$. Let W^* be the DVR $W^* = S_{(x)}$ and let W be the DVR $W = (R_0)_{\mathfrak{p}} = W^* \cap K$. Let

$$\mathfrak{p}' = (xS) \cap R\left[\frac{I}{u^{\frac{b}{s}}}\right]$$

and let

$$\varphi = \frac{v^{\frac{a}{s}}}{u^{\frac{b}{s}}}.$$

$\varphi \notin \mathfrak{p}'$, so $\mathfrak{p}'$ is a height one prime in $R\left[\frac{I}{u^{\frac{b}{s}}}\right]$. Thus

$$W'' = R\left[\frac{I}{u^{\frac{b}{s}}}\right]_{\mathfrak{p}'}$$

is a DVR dominated by W^*. Thus $W = W''$. As in the proof of Proposition 3.1, we conclude that

$$e(W^*/W) = s$$

and that

$$R_0/\mathfrak{p} \cong (R/m_R[\overline{\varphi}])_{(\overline{\varphi})}$$

where $\overline{\varphi}$ is the residue of φ in $R_0/\mathfrak{p}$. We have that

$$\overline{\nu}(\varphi) = \frac{ad - bc}{s}$$

where $\overline{\nu}$ is the natural valuation of S/xS, and $R_0/m_{R_0} \cong R/m_R$. We calculate (as in the proof of Proposition 3.1) that

$$f(\hat{W}^*/\hat{W}) = [\mathrm{QF}(\widehat{S/xS}) : \mathrm{QF}(\widehat{R_0/\mathfrak{p}})] = \frac{ad - bc}{s}[S/m_S : R/m_R]$$

and that

$$[\mathrm{QF}(\hat{S}) : \mathrm{QF}(\hat{R_0})] = e(\hat{W}^*/\hat{W})f(\hat{W}^*/\hat{W}) = (ad - bc)[S/m_S : R/m_R].$$

$\square$

3.2. **Rational rank 1.** Throughout this subsection we will have the following assumptions. Suppose that R is a 2 dimensional excellent local domain with quotient field $\mathrm{QF}(R) = K$. Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field $\mathrm{QF}(S) = K^*$ such that S dominates R.

Suppose that ν^* is a valuation of K^* such that

1) ν^* dominates S.
2) The residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S.
3) The value group Γ_{ν^*} of ν^* has rational rank 1 (so it is isomorphic as an ordered group to a subgroup of $\mathbb{Q}$).

Let ν be the restriction of ν^* to K.

Theorem 3.3. *Suppose that the defect $\delta(\nu^*/\nu) = 0$. Then there exists a commutative diagram*

(10)
$$\begin{array}{ccccc} R_1 & \to & S_1 & \subset & V_{\nu^*} \\ \uparrow & & \uparrow & & \\ R & \to & S & & \end{array}$$

such that the vertical arrows are products of quadratic transforms along ν^* and

1) *R_1 and S_1 are two dimensional regular local rings.*
2) *R_1 has a regular system of parameters u_1, v_1 and S_1 has a regular system of parameters x_1, y_1 and there exists a unit $\gamma_1 \in S_1$ such that*

$$u_1 = \gamma_1 x_1^e \text{ and } v_1 = y_1$$

where $e = e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu|$, and the class of $\nu^*(x_1)$ is a generator of the group $\Gamma_{\nu^*}/\Gamma_\nu \cong \mathbb{Z}_e$.
3) *V_{ν^*}/m_{ν^*} is the join $V_{\nu^*}/m_{\nu^*} = (V_\nu/m_\nu)(S_1/m_{S_1})$ and $[S_1/m_{S_1} : R_1/m_{R_1}] = f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$.*

It will follow from our proof that there exists a diagram (10) such that the conditions 1), 2) and 3) of Theorem 3.3 are stable under further appropriate sequences of quadratic transforms above R and S.

Proposition 3.4. *There exists a local ring R' of K which is essentially of finite type over R, is dominated by ν and dominates R such that if we have a commutative diagram*

(11)
$$\begin{array}{ccc} V_\nu & \to & V_{\nu^*} \\ \uparrow & & \uparrow \\ R_1 & \to & S_1 \\ \uparrow & & \\ R' & & \uparrow \\ \uparrow & & \\ R & \to & S \end{array}$$

where R_1 is a regular local ring of K which is essentially of finite type over R and dominates R, S_1 is a regular local ring of K^* which is essentially of finite type over S and dominates S, R_1 has a regular system of parameters u, v and S_1 has a regular system of parameters x, y such that there is an expression

$$u = \gamma x^a, v = x^b f$$

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where $a > 0$, $b \geq 0$, γ is a unit in S, $x \nmid f$ in S_1 and f is not a unit in S_1, then

(12) $$ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

where $d = \bar{\nu}(f \bmod x)$ with $\bar{\nu}$ being the natural valuation of the DVR S/xS.

Proof. We first prove the proposition with the assumption that K^*/K is Galois with Galois group G.

Let g be the number of extensions of ν to K^*. Writing $f(\nu^*/\nu) = f_0 p^s$ where $\gcd(f_0, p) = 1$ and $e(\nu^*/\nu) = e_0 p^t$ where $\gcd(e_0, p) = 1$, we have by the Corollary to Theorem 25, page 78 [40] that $[G : G^s(\nu^*/\nu)] = g$, $[G^s(\nu^*/\nu) : G^i(\nu^*/\nu)] = f_0$, $[G^i(\nu^*/\nu) : G^r(\nu^*/\nu)] = e_0$ and $|G^r(\nu^*/\nu)| = p^{s+t+\delta(\nu^*/\nu)}$, so that

$$[K^* : K] = ge(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

Since $[K^s(\nu^*/\nu) : K] = g$, we have that

(13) $$[K^* : K^s(\nu^*/\nu)] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}.$$

By the argument at the top of page 86 in [3], there exists R' as above such that for any diagram

(14)
$$
\begin{array}{ccc}
V_\nu & \to & V_{\nu^*} \\
\uparrow & & \uparrow \\
R^* & \to & S^* \\
\uparrow & & \uparrow \\
R' & & \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}
$$

where R^* and S^* are normal local rings of K and K^* respectively such that R^* is essentially of finite type over R' and dominates R' and S^* lies over R^*, we have that

(15) $$G^s(S^*/R^*) = G^s(\nu^*/\nu).$$

Now applying Proposition 3.1 to an extension $R_1 \to S_1$ satisfying (11) (so that R_1 dominates R'), we have a commutative diagram

$$
\begin{array}{ccc}
V_{\nu^*} & \to & V_\nu \\
\uparrow & & \uparrow \\
R^* & \to & S_1 \\
\uparrow & \nearrow & \\
R_1 & &
\end{array}
$$

such that R^* and S_1 are normal local rings such that S_1 lies over R^* and

(16) $$[\mathrm{QF}(\hat{S}_1) : \mathrm{QF}(\hat{R}^*)] = ad[S_1/m_{S_1} : R_1/m_{R_1}].$$

Let $S' = K^s(\nu^*/\nu^*) \cap S_1$. Then $R^* \to S'$ is unramified with $S'/m_{S'} = R^*/m_{R^*}$ by Theorem 1.47 [3], since $K^s(\nu^*/\nu^*) = K^s(S_1/R^*)$ by (15). Thus $\hat{R}^* = \hat{S}'$ by (10.1) [4]. We have that

(17) $$[K^* : K^s(\nu^*/\nu)] = [\mathrm{QF}(\hat{S}') : \mathrm{QF}(\hat{R}^*)]$$

by II of Proposition 1 (page 498) [2], since there is a unique local ring in K^* lying over S'. Now combining (13), (17) and (16), we obtain formula (12). (The proof in [2] is valid in our more general situation since S' is excellent.)

We will now establish the proposition in the general case, when K^*/K is only assumed to be finite and separable. Let K' be a Galois closure of K^*/K, and let ν' be an extension

of ν^* to K'. By the Galois case, there exists a normal local ring R' of K giving the property of Proposition 3.4 within the extension K'/K, and a normal local ring S' of K^* giving the property of Proposition 3.4 within the extension K'/K^*. We can choose R' and S' so that S' lies over R'.

Now suppose that we are given a diagram (11). We then have that $S' \subset S_1$. We have by Proposition 3.1 local rings R_0 of K and S_1 of K^* such that S_1 lies over R_0 and

(18) $$[\mathrm{QF}(\hat{S}_1) : \mathrm{QF}(\hat{R}_0)] = ad[S_1/m_{S_1} : R_1/m_{R_1}].$$

Let T be the integral closure of S_1 in K' and let $T' = T_{m_{\nu'} \cap T}$. By (13) and (17), we have that

$$[\mathrm{QF}(\hat{T}') : \mathrm{QF}(\hat{R}_0)] = e(\nu'/\nu)f(\nu'/\nu)p^{\delta(\nu'/\nu)}$$

and

$$[\mathrm{QF}(\hat{T}') : \mathrm{QF}(\hat{S}_1)] = e(\nu'/\nu^*)f(\nu'/\nu^*)p^{\delta(\nu'/\nu^*)}.$$

Thus

$$[\mathrm{QF}(\hat{S}_1) : \mathrm{QF}(\hat{R}_0)] = \frac{[\mathrm{QF}(\hat{T}') : \mathrm{QF}(\hat{R}_0)]}{[\mathrm{QF}(\hat{T}') : \mathrm{QF}(\hat{S}_1)]} = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

since e, f and p^δ are multiplicative. Now formula (12) follows from (18). $\qquad\square$

Proposition 3.5. *There exists a local ring R'' of K which is essentially of finite type over R, is dominated by ν and dominates R such that if we have a commutative diagram*

(19)
$$
\begin{array}{ccc}
V_\nu & \to & V_{\nu^*} \\
\uparrow & & \uparrow \\
R_1 & \to & S_1 \\
\uparrow & & \uparrow \\
R'' & & \\
\uparrow & & \uparrow \\
R & \to & S
\end{array}
$$

where R_1 is a regular local ring of K which is essentially of finite type over R'', S_1 is a regular local ring of K^* which is essentially of finite type over S and dominates S, R_1 has a regular system of parameters u, v and S_1 has a regular system of parameters x, y such that there is an expression

$$u = \gamma x^a, v = x^b f$$

where $a > 0$, $b \geq 0$, γ is a unit in S, $x \not| f$ in S_1 and f is not a unit in S_1 and there exists a unit $\tau \in S_1$ and $n \in \mathbb{Z}_+$ such that $h = \tau x^n$, then

1) *$\nu^*(x)$ is a generator of $\Gamma_{\nu^*}/\Gamma_\nu \cong Z_{e(\nu^*/\nu)}$,*
2) *$V_{\nu^*}/m_{\nu^*} = (V_\nu/m_\nu)(S_1/m_{S_1})$,*
3)

$$ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

where $d = \bar{(}f \bmod x)$ with $\bar{\nu}$ being the natural valuation of the DVR S/xS.

Proof. Let R' be the local ring of the conclusions of Proposition 3.4 and let $g_1, \ldots, g_s \in V_{\nu^*}$ be such that the classes of $g_1, \ldots, g_s$ in V_{ν^*}/m_{ν^*} are a V_ν/m_ν basis. Let R'' be a regular local ring of K which dominates R' and is essentially of finite type over R' such that $g_1, \ldots, g_s$ are in the integral closure of R'' in K^*.

Suppose that we have a diagram (19). Then $g_1, \ldots, g_s \in S_1$ so conclusion 2) holds.

Since $\nu^*(h) = n\nu^*(x)$ is a generator of $\Gamma_{\nu^*}/\Gamma_\nu$ we have that $\nu^*(x)$ is a generator of $\Gamma_{\nu^*}/\Gamma_\nu$.

Finally, 3) holds by Proposition 3.4, since R'' dominates R'. □

Corollary 3.6. *Let assumptions be as in Proposition 3.5, and further assume that $\delta(\nu^*/\nu) = 0$. Let R'' be the local ring of the conclusions of Proposition 3.5. Suppose that we have a commutative diagram (19). Then*

$$a = e(\nu^*/\nu), d = 1 \text{ and } [S_1/m_{S_1} : R_1/m_{R_1}] = f(\nu^*/\nu).$$

Proof. We have that $a\nu^*(x) \in \Gamma_\nu$ and $\nu^*(x)$ is a generator of $\Gamma_{\nu^*}/\Gamma_\nu \cong \mathbb{Z}_{e(\nu^*/\nu)}$ by 1) of Proposition 3.5 so $e(\nu^*/\nu)$ divides a, and thus

(20) $$a \geq e(\nu^*/\nu).$$

Further,

(21) $$[S_1/m_{S_1} : R_1/m_{R_1}] \geq f(\nu^*/\nu)$$

since $V_{\nu^*}/m_{\nu^*} = (V_\nu/m_\nu)(S_1/m_{S_1})$ by 2) of Proposition 3.5, so S_1/m_{S_1} contains a basis of V_{ν^*}/m_{ν^*} over V_ν/m_ν. Further, we have that

(22) $$ad[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu)f(\nu^*/\nu)$$

by 3) of Proposition 3.5 since $\delta(\nu^*/\nu) = 0$. Thus by equations (20), (21) and (22), we have

$$a[S_1/m_{S_1} : R_1/m_{R_1}] \geq e(\nu^*/\nu)f(\nu^*/\nu) = ad[S_1/m_{S_1} : R_1/m_{R_1}]$$

giving the conclusions of the corollary. □

We now give the proof of Theorem 3.3. Let R'' be the ring of the conclusions of Proposition 3.5. We may assume, after replacing R and S with appropriate sequences of quadratic transforms of R and S, that R and S are regular, R dominates R'' and R has regular parameters u, v and S has regular parameters x, y such that

(23) $$u = \gamma x^a, v = x^b y$$

and $h = \tau x^n$ where $h \in V_{\nu^*}$ is such that $\nu^*(h)$ is a generator of $\Gamma_{\nu^*}/\Gamma_\nu$ and τ is a unit in S. (We have that $d = 1$ by Corollary 3.6). Let $S \to S_1$ be the smallest sequence of quadratic transforms along ν^* such that $\sqrt{xyS_1}$ is a prime ideal (this is possible since ν^* has rational rnak 1 so $\nu^*(x)$ and $\nu^*(y)$ are rationally dependent), and let $R \to R_1$ be the smallest sequence of quadratic transforms along ν such that $\sqrt{uvR_1}$ is a prime ideal. We will show that S_1 dominates R_1 and we have regular parameters u_1, Q in R_1 and x_1, Q in S_1 and a unit γ' in S_1 such that $u_1 = \gamma' x_1^{e(\nu^*/\nu)}$. Since these conclusions will then hold under further sequences of quadratic transforms, we will then have established the conclusions of the theorem, and the remark on stability following the statement of Theorem 3.3.

We have that

$$S_1 = S[x_1, y_1]_{m_{\nu^*} \cap S[x_1, y_1]}$$

where

(24) $$x = x_1^{m_1} y_1^{m_1'}, y = x_1^{n_1} y_1^{n_1'}$$

with $m_1 n_1' - n_1 m_1' = \pm 1$, $\nu^*(x_1) > 0$ and $\nu^*(y_1) = 0$.

$$S[x_1, y_1]/x_1 S[x_1, y_1] \cong S/m_S[y_1]$$

is a polynomial ring over S/m_S. Let $f \in S/m_S[y_1]$ be the monic generator of

$$(m_{\nu^*} \cap S[x_1, y_1])/x_1 S[x_1, y_1].$$

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There exists $P \in S[x_1, y_1]$ such that the residue of P in $S[x_1, y_1]/x_1 S[x_1, y_1]$ is f. Then x_1, P are regular parameters in S_1.

Now substitute (24) into (23), to obtain

$$\begin{aligned} u &= \gamma x_1^{m_1 a} y_1^{m_1' a} \\ v &= x_1^{m_1 b + n_1} y_1^{m_1' b + n_1'}. \end{aligned}$$

Let $s = \gcd(m_1 a, m_1 b + n_1)$. By a sequence of substitutions $\hat{u}_0 = u, \hat{v}_0 = v$, and

$$\hat{u}_i = \hat{u}_{i+1}, \hat{v}_i = \hat{u}_{i+1} \hat{v}_{i+1}$$

or

$$\hat{u}_i = \hat{u}_{i+1} \hat{v}_{i+1}, \hat{v}_i = \hat{v}_{i+1}$$

for $0 \leq i \leq \lambda$ we obtain an expression

$$\begin{aligned} \hat{u}_\lambda &= \gamma^{c_1} x_1^s y_1^{t_1} \\ \hat{v}_\lambda &= \gamma^{c_2} x_1^s y_1^{t_2}. \end{aligned}$$

We have that

$$s|t_1 - t_2| = \left| \mathrm{Det} \begin{pmatrix} s & t_1 \\ s & t_2 \end{pmatrix} \right| = \left| \begin{pmatrix} m_1 a & m_1' a \\ (m_1 b + n_1) & (m_1' b + n_1') \end{pmatrix} \right| = a \left| \begin{pmatrix} m_1 & m_1' \\ n_1 & n_1' \end{pmatrix} \right| = a.$$

We thus have that $t_2 - t_1 \neq 0$. Set

$$u_1 = \hat{u}_\lambda, v_1 = \frac{\hat{v}_\lambda}{\hat{u}_\lambda} \text{ if } t_2 - t_1 > 0,$$

$$u_1 = \frac{\hat{u}_\lambda}{\hat{v}_\lambda}, v_1 = \hat{v}_\lambda \text{ if } t_2 - t_1 < 0.$$

Without loss of generality, we may assume that $t_2 - t_1 > 0$. We then have an expression

$$u = u_1^{m_2} v_1^{m_2'}, v = u_1^{n_2} v_1^{n_2'}$$

with $m_2 n_2' - n_2 m_2' = \pm 1$, $\nu(u_1) > 0$ and $\nu(v_1) = 0$. Now

$$R_1 = R[u_1, v_1]_{m_\nu \cap R[u_1, v_1]}.$$

We have that $R[u_1, v_1]/u_1 R[u_1, v_1] \cong R/m_R[v_1]$ is a polynomial ring over R/m_R. Let $g \in R/m_R[v_1]$ be the monic generator of

$$(m_\nu \cap R[u_1, v_1])/u_1 R[u_1, v_1].$$

There exists $Q \in R[u_1, v_1]$ such that the residue of Q in $R[u_1, v_1]/u_1 R[u_1, v_1]$ is g. u_1, Q are regular parameters in R_1. We have an expression

$$u_1 = (\gamma^{c_1} y_1^{t_1}) x_1^s, v_1 = \gamma^{c_2 - c_1} y_1^{t_2 - t_1}.$$

The inclusion $R \to S$ induces a natural homomorphism

$$R/m_R[v_1] \cong R[u_1, v_1]/u_1 R[u_1, v_1] \to S[x_1, y_1]/x_1 S[x_1, y_1] \cong S/m_S[y_1].$$

Let $0 \neq \gamma_0$ be the residue of γ in S/m_S. Then γ_0 is the residue of γ in $S/m_S[y_1]$ and $\gamma_0^{c_2 - c_1} y_1^{t_2 - t_1}$ is the residue of v_1 in $S/m_S[y_1]$. The residue of Q in $S/m_S[y_1]$ is $g(\gamma_0 y_1^{t_2 - t_1})$ which is nonzero. Thus $b(S_1/R_1) = 0$. We further have that $a(S_1/R_1) = e(\nu^*/\nu)$, $d(S_1/R_1) = 1$ and $[S_1/m_{S_1} : R_1/m_{R_1}] = f(\nu^*/\nu)$ by Proposition 3.5. This completes the proof of Theorem 3.3.

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3.3. **Rational rank 2.** We have the following Theorem 3.7 for rational rank 2 valuations dominating a two dimensional excellent local domain, most of whose simple proof is as on page 40 of [14]. The statement that the extension is defectless follows from the argument of Proposition 3.4 which establishes (12), replacing references to Proposition 3.1 with Proposition 3.2. We obtain the formula

$$|ad - bc|[S_1/m_{S_1} : R_1/m_{R_1}] = e(\nu^*/\nu)f(\nu^*/\nu)p^{\delta(\nu^*/\nu)}$$

instead of (12). Since $e(\nu^*/\nu) = |ad - bc|$, we obtain that $\delta(\nu^*/\nu) = 0$. Related results are proven for valuations of maximal rational rank in algebraic function fields in Theorem 3.1 [25]

Theorem 3.7. *Suppose that R is a 2 dimensional excellent local domain with quotient field $QF(R) = K$. Further suppose that K^* is a finite separable extension of K and S is a 2-dimensional local domain with quotient field $QF(S) = K^*$ such that S dominates R.*
  Suppose that ν^* is a valuation of K^* such that

  1) *ν^* dominates S.*
  2) *The residue field V_{ν^*}/m_{ν^*} of V_{ν^*} is algebraic over S/m_S.*
  3) *The value group Γ_{ν^*} of ν^* has rational rank 2.*

Let ν be the restriction of ν^* to K. Then the defect $\delta(\nu^*/\nu) = 0$, and there exists a commutative diagram

(25)
$$
\begin{array}{ccccc}
R_1 & \to & S_1 & \subset & V_{\nu^*} \\
\uparrow & & \uparrow & & \\
R & \to & S & &
\end{array}
$$

such that the vertical arrows are products of quadratic transforms along ν^* and

  1) *R_1 and S_1 are two dimensional regular local rings.*
  2) *R_1 has a regular system of parameters u_1, v_1 and S_1 has a regular system of parameters x_1, y_1 and there exist units $\gamma_1, \tau_1 \in S_1$ such that*

  $$u_1 = \gamma_1 x_1^a y_1^b \text{ and } v_1 = \tau_1 x_1^c y_1^d$$

  where

  $$e = e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu| = \left| Det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|$$

  and the classes of $\nu^*(x_1), \nu^*(y_1)$ are generators of the group $\Gamma_{\nu^*}/\Gamma_\nu \cong \mathbb{Z}^2/A\mathbb{Z}^2$.
  3) *V_{ν^*}/m_{ν^*} is the join $V_{\nu^*}/m_{\nu^*} = (V_\nu/m_\nu)(S_1/m_{S_1})$ and $[S_1/m_{S_1} : R_1/m_{R_1}] = f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$.*

   There exists a diagram (25) such that the conditions 1), 2) and 3) of Theorem 3.7 are stable under further appropriate sequences of quadratic transforms above R and S.

3.4. **Proof of Theorem 1.4.** We now give the proof of Theorem 1.4.

$$\mathrm{trdeg}_{S/m_S} V_{\nu^*}/m_{\nu^*} + \dim_{\mathbb{Q}} \Gamma_{\nu^*} \otimes \mathbb{Q} \leq 2$$

by Abhyankar's inequality (Theorem 1 [1]).
   If $\mathrm{trdeg}_{R/m_R} S/m_S = 1$, then V_{ν^*} and V_ν are DVRs and algebraic local rings of K^* and K respectively, so that $V_\nu \to V_{\nu^*}$ is a monomial mapping, which is obtained from R, respectively S by a sequence of quadratic transforms along ν^* (there exists a sequence of quadratic transforms $R \to R^*$ along ν such that V_ν/m_ν is algebraic over R/m_R. Thus

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$R^* = V_\nu$ be Zariski's Main Theorem, (10.7) [4]). (For this type of valuation we must have that $\delta(\nu^*/\nu) = 0$ by Theorem 20, Section 11, Chapter VI [40]).

Suppose that $\dim_{\mathbb{Q}} \Gamma_{\nu^*} \otimes \mathbb{Q} = 2$. Then V_{ν^*}/m_{ν^*} is algebraic over S/m_S by Abhyankar's inequality. In this case there exists a monomialization by Theorem 3.7 (we also always have that $\delta(\nu^*/\nu) = 0$ for this type of valuation by Theorem 3.7).

The remaining case is when V_{ν^*}/m_{ν^*} is algebraic over S/m_S, $\dim_{\mathbb{Q}} \Gamma_{\nu^*} \otimes \mathbb{Q} = 1$ and $\delta(\nu^*/\nu) = 0$. The existence of a monomialization in this case follows from Theorem 3.3.

4. Extensions of Associated Graded Rings of Valuations

In this section, we extend the results of [19] and [20] calculating the extension of associated graded rings of a valuation for defectless extensions of 2 dimensional algebraic function fields over an algebraically closed field, and of [15] for 2 dimensional algebraic function fields over a (not necessarily closed) characteristic zero field. We refer to the introduction of this paper for a discussion of this problem.

We recall the following theorem on strong monomialization of rational rank 1 valuations in an extension of characteristic zero function fields from [10].

Theorem 4.1. *(The rational rank 1 case of Theorem 5.1 [10], Theorem 6.1 [14] and Theorem 6.5 [13]) Let K an algebraic function field over a field k of characteristic zero, K^* a finite algebraic extension of K and ν^* a rational rank 1 k valuation of K^*. Suppose that S^* is an algebraic local ring with quotient field K^* which is dominated by ν^* and R^* is an algebraic local ring with quotient field K which is dominated by S^*. Let ν be the restriction of ν^* to K. Then there exists a commutative diagram*

$$
\begin{array}{ccccc}
R & \to & S & \subset & V_{\nu^*} \\
\uparrow & & \uparrow & & \\
R^* & \to & S^*
\end{array}
$$
(26)

such that the vertical arrow are product of monoidal transforms along ν^* and

1) *R and S are regular local rings of dimension equal to $n = trdeg_k V_{\nu^*}/m_{\nu^*}$.*
2) *R has a regular system of parameters $x_1, \ldots, x_n$ and S has a regular system of parameters $y_1, \ldots, y_n$ and there exists a unit $\delta \in S$ such that*

$$x_1 = \delta y_1^e \text{ and } x_i = y_i \text{ for } i \geq 2$$

   where $e = e(\nu^*/\nu) = |\Gamma_{\nu^*}/\Gamma_\nu|$.
3) *The class of $\nu(y_1)$ is a generator of the group $\Gamma_{\nu^*}/\Gamma_\nu$.*
4) *$V_{\nu^*}/m_{\nu^*} = (V_\nu/m_\nu)(S/m_S)$.*
5) *$[S/m_S : R/m_R] = f = f(\nu^*/\nu) = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$*
6) *The conclusions of 1) and 3) - 5) of the theorem continue to hold for $R_1 \to S_1$ whenever there exists a commutative diagram*

$$
\begin{array}{ccccc}
R_1 & \to & S_1 & \subset & V_{\nu^*} \\
\uparrow & & \uparrow & & \\
R & \to & S
\end{array}
$$
(27)

   such that the vertical arrow are product of monoidal transforms along ν^* and 2) holds for $R_1 \to S_1$.

We now show that with the assumptions of the conclusions of Theorem 4.1, we further have a very simple description of the extension of associated graded rings of the valuations.

Theorem 4.2. *Let assumptions be as in Theorem 4.1, and let $R \to S$ be as in the conclusions of Theorem 4.1. Then we have a natural isomorphism of graded rings*

$$\mathrm{gr}_{\nu^*}(S) \cong \left(\mathrm{gr}_\nu(R) \otimes_{R/m_R} S/m_S\right)[Z]/(Z^e - [\delta]^{-1}[x_1]),$$

where $[\gamma_1], [x_1]$ are the respective classes in $\mathrm{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1}$. The degree of the extension of quotient fields of $\mathrm{gr}_\nu(R) \to \mathrm{gr}_{\nu^*}(S)$ is $e(\nu^*/\nu)f(\nu^*/\nu)$.

Proof. Let $\gamma_1, \ldots, \gamma_f \in S$ be such that their residues $\overline{\gamma}_1, \ldots, \overline{\gamma}_f$ in S/m_S are a basis of S/m_S over R/m_R. We now establish the following formula:

(28)   Suppose that $h_1, \ldots, h_f \in R$. Then $\nu^*(\sum_{i=1}^f \gamma_i h_i) = \min\{\nu(h_i)\}$.

Without loss of generality, we may suppose that $\nu(h_1)$ is this minimum. Then $\frac{h_i}{h_1} \in V_\nu$ for all i so we have classes

$$[\frac{h_i}{h_1}] \in V_\nu/m_\nu \subset V_{\nu^*}/m_{\nu^*}.$$

If $\nu^*(\sum_{i=1}^f \gamma_i h_i) > \nu^*(h_1)$, then

$$\sum \overline{\gamma}_i [\frac{h_i}{h_1}] = 0 \text{ in } V_{\nu^*}/m_{\nu^*},$$

which is impossible since $\overline{\gamma}_1, \ldots, \overline{\gamma}_f$ are a basis of V_{ν^*}/m_{ν^*} over V_ν/m_ν, by the assumptions of the theorem. We have thus established formula (28).

Suppose that $z \in S$. Let t be a positive integer such that $t\nu^*(m_S) > \nu^*(z)$. We have an expression

$$z = \left(\sum_{j=0}^{e-1} \sum_{i=1}^f g_{ij}\gamma_i y_1^j\right) + w$$

where $g_{ij} \in R$ for all i, j and $w \in m_S^t$.

We will next establish the following formula:

(29) $$\nu^*(z) = \min\{\nu(g_{ij}) + j\nu^*(y_1)\}.$$

With our assumption on t,

$$\nu^*(z) = \nu^* \left(\sum_{j=0}^{e-1} \sum_{i=1}^f g_{ij}\gamma_i y_1^j\right).$$

Set

$$h_j = \sum_{i=1}^f g_{ij}\gamma_i \text{ for } 0 \leq j \leq e-1.$$

By (28), $\nu^*(h_j) \in \Gamma_\nu$ for all j. For $0 \leq j \leq e-1$, we have

$$\nu^*(h_k y_1^k) - \nu^*(h_j y_1^j) = (k-j)\nu^*(y_1) + \nu^*(h_k) - \nu^*(h_j).$$

Since the class of $\nu^*(y_1)$ has order e in $\Gamma_{\nu^*}/\Gamma_\nu$, we have that

(30) $$\nu^*(h_k y_1^k) \neq \nu^*(h_j y_1^j) \text{ for } j \neq k.$$

Formula (29) now follows from (30) and (28).

For $\gamma \in \Gamma_\nu$, we have a homomorphism

$$\mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) \otimes_{R/m_R} S/m_S \to \mathcal{P}_\gamma(S)/\mathcal{P}_\gamma^+(S)$$

defined by $g_i \otimes \gamma_i \mapsto g_i\gamma_i$. This map is 1-1 by the proof of (28). Thus

$$\left(\bigoplus_{\gamma \in \Gamma_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) \right) \otimes_{R/m_R} S/m_S$$

is a graded subalgebra of

$$\bigoplus_{\tau \in \Gamma_{\nu*}} \mathcal{P}_\tau(S)/\mathcal{P}_\tau^+(S).$$

Suppose that $h \in S$. Let $\tau = \nu^*(h)$. By (30), there exists a unique i with $0 \le i \le e-1$ and $\gamma \in \Gamma_\nu$ such that $\tau = \gamma + i\nu^*(y_1)$. Further,

$$h = \left(\sum_{j=1}^{f} \gamma_j g_j \right) y_1^i + h_2$$

where $g_j \in R$ with $\nu(g_j) = \gamma$ for $1 \le j \le f$ and $h_2 \in S$ satisfies $\nu^*(h_2) > \tau$. Thus

$$\mathcal{P}_\tau(S)/\mathcal{P}_\tau^+(S) = \left(\mathcal{P}_\gamma(R)/\mathcal{P}_\gamma^+(R) \otimes_{R/m_R} S/m_S \right)[\bar{y}_1]$$

where $\bar{y}_1$ is the class of y_1. We have that

$$(\bar{y}_1)^e - [\delta]^{-1}[x_1] = 0$$

where $[\delta]$, $[x_1]$ are the classes of δ and x_1. All other relations on $\bar{y}_1$ are divisible by this relation by (29).

$\square$

Theorem 4.3. *Let assumptions be as in Theorem 3.3, and let $R_1 \to S_1$ be as in the conclusions of Theorem 3.3. Then we have a natural isomorphism of graded rings*

$$\mathrm{gr}_{\nu*}(S_1) \cong \left(\mathrm{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1} \right)[Z]/(Z^e - [\gamma_1]^{-1}[u_1]),$$

where $[\gamma_1], [u_1]$ are the respective classes in $\mathrm{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1}$. The degree of the extension of quotient fields of $\mathrm{gr}_\nu(R_1) \to \mathrm{gr}_{\nu*}(S_1)$ is $e(\nu^*/\nu)f(\nu^*/\nu)$.

Proof. The proof is exactly the same as the proof of Theorem 4.2. $\square$

We also obtain from Theorem 3.7 the following result, showing that even in positive and mixed characteristic, the associated graded rings of Abhyankar valuations dominating a stable extension of two dimensional excellent regular local rings have a nice form.

Theorem 4.4. *Let assumptions be as in Theorem 3.7, and let $R_1 \to S_1$ be as in the conclusions of Theorem 3.7. Then we have a natural isomorphism of graded rings*

$$\mathrm{gr}_{\nu*}(S_1) \cong \left(\mathrm{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1} \right)[X,Y]/(X^a Y^b - [\gamma_1]^{-1}[u_1], X^c Y^d - [\tau_1]^{-1}[v_1]),$$

where $[\gamma_1], [\tau_1], [u_1], [v_1]$ are the respective classes in $\mathrm{gr}_\nu(R_1) \otimes_{R_1/m_{R_1}} S_1/m_{S_1}$. The degree of the extension of quotient fields of $\mathrm{gr}_\nu(R_1) \to \mathrm{gr}_{\nu*}(S_1)$ is $e(\nu^*/\nu)f(\nu^*/\nu)$.

5. Non constancy of α_n and β_n

In Corollary 7.30 and Theorem 7.33 of [14], it is shown that $\alpha_n + \beta_n$ is a constant for $n \gg 0$, where α_n and β_n are the integers defined in (6) which are associated to the stable forms (5) of an extension of valued two dimensional algebraic function fields. If Γ_ν is not p-divisible, it is further shown that α_n and β_n are both constant for $n \gg 0$, and p^{β_n} is the defect of the extension. However, if Γ_ν is p-divisible, then it is only shown that the sum $\alpha_n + \beta_n$ is constant for $n \gg 0$, and that $p^{\alpha_n + \beta_n}$ is the defect of the extension. In Remark 7.34 [14] it is asked if α_n and β_n (and some other numbers computed from the stable forms) are eventually constant in the case when Γ_ν is p-divisible. We give examples (31) - (34) here showing that this is not the case, even within defect Artin Schreier extensions.

The example in Theorem 7.38 [14] is a tower K^*/K of two Artin Schreier extensions,

$$K \to K_1 \to K^*,$$

where $K = k(u,v)$, $K_1 = k(x,v)$ and $K^* = k(x,y)$, over an algebraically closed field k of characteristic $p > 0$, with

$$u = \frac{x^p}{1 - x^{p-1}}, v = y^p - x^c y$$

where $p - 1$ divides c. R and S are defined to be $R = k[u,v]_{(u,v)}$ and $S = k[x,y]_{(x,y)}$. The valuation ν^* on K^* is defined by the generating sequence (41) in S and the valuation $\nu = \nu^*|K$ is defined by the generating sequence (46) in R. In Theorem 7.38 [14], it is shown that $\Gamma_{\nu^*} = \Gamma_\nu = \frac{1}{p^\infty}\mathbb{Z}$, $\delta(\nu^*/\nu) = 2$, and in the stable forms $R_n \to S_n$ above $R \to S$, we have $\alpha_n = 1$ and $\beta_n = 1$ for all n.

Let $\nu_1 = \nu^*|K_1$. We define $A = k[x,v]_{(x,v)}$, a local ring of K_1. We will show in this section that $\delta(\nu^*/\nu_1) = \delta(\nu_1/\nu) = 1$ and we have stable forms $R_j \to A_j$ of K_1/K (Theorem 5.7) and $A_j \to S_j$ of K^*/K_1 (Theorem 5.6) such that by the conclusions of Theorem 5.6,

$$(31) \qquad \alpha_j(S_j/A_j) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

and

$$(32) \qquad \beta_j(S_j/A_j) = \begin{cases} 0 & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases}$$

for $A_j \to S_j$, and by Theorem 5.7, we have

$$(33) \qquad \alpha_j(A_j/R_j) = \begin{cases} 0 & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases}$$

and

$$(34) \qquad \beta_j(A_j/R_j) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

for $R_j \to A_j$.

We now prove these statements. We make use of the notation introduced in Section 7.11 [14] in the construction of the example of Theorem 7.38 [14]. The proof makes essential use of the theory of generating sequences of a valuation dominating a two dimensional regular local ring, as developed in [34] and extended in [15].

We define a k-valuation ν_1 of K_1 by prescribing a generating sequence in A, starting with

$$(35) \qquad U_0 = x, U_1 = v, U_2 = v^p - x$$

and for $j \geq 2$,

(36)
$$
\begin{aligned}
U_{j+1} &= U_j^p - x^{p^{2j-2}} U_{j-1} \text{ if } j \text{ is odd} \\
U_{j+1} &= U_j^{p^3} - x^{p^{2j-1}} U_{j-1} \text{ if } j \text{ is even.}
\end{aligned}
$$

We now establish that $\{U_i\}$ determines a unique valuation ν_1 on K_1 such that $\nu_1(x) = 1$. Define $\overline{\gamma}_j$ by

$$
\overline{\gamma}_0 = 1, \overline{\gamma}_1 = \frac{1}{p}
$$

and for $j \geq 2$,

(37)
$$
\overline{\gamma}_j = \begin{cases} \frac{1}{p}(p^{2j-2} + \overline{\gamma}_{j-1}) & \text{if } j \text{ is odd} \\ \frac{1}{p^3}(p^{2j-1} + \overline{\gamma}_{j-1}) & \text{if } j \text{ is even} \end{cases}
$$

By induction on $j \geq 0$, we have that

(38)
$$
\overline{\gamma}_{j+1} = \begin{cases} p^{2j-2}(\sum_{j'=0}^{j} \frac{1}{p^{4j'}}) & \text{if } j \text{ is odd} \\ p^{2j-1}(\sum_{j'=0}^{j} \frac{1}{p^{4j'}}) & \text{if } j \text{ is even} \end{cases}
$$

Let

$$
n_i = \begin{cases} p & \text{if } j \text{ is odd} \\ p^3 & \text{if } j \text{ is even} \end{cases}
$$

Let Γ_j be the group generated by $\overline{\gamma}_0, \ldots, \overline{\gamma}_j$. By Remark 7.171 [14] or Theorem 1.1 [15] and its proof, $\{U_i\}$ is a generating sequence of a unique valuation ν_1 on K_1 such that $\nu_1(U_i) = \overline{\gamma}_i$ for all i if

(39)
$$
n_i = [\Gamma_i : \Gamma_{i-1}]
$$

for $i \geq 1$ and

(40)
$$
\overline{\gamma}_{i+1} > n_i \overline{\gamma}_i
$$

for $i \geq 1$.

By (37), $n_i \overline{\gamma}_i \in \Gamma_{i-1}$, so $[\Gamma_i : \Gamma_{i-1}] = n_i$ if $\overline{\gamma}_i$ has order precisely n_i in Γ_i / Γ_{i-1}.

By (38), we have that

$$
\Gamma_{i-1} = \begin{cases} \frac{1}{p^{2i-2}} \mathbb{Z} & \text{if } i \text{ is odd} \\ \frac{1}{p^{2i-3}} \mathbb{Z} & \text{if } i \text{ is even} \end{cases}
$$

for $i \geq 1$, so that by (38), γ_i has order n_i in Γ_i / Γ_{i-1}. Since (40) holds by (38), we have that $\{U_i\}$ is a generating sequence in A which determines a valuation ν_1 of K_1.

The following Propositions 5.1 and 5.2 are a little stronger than Proposition 7.40 [14].

Proposition 5.1. *Let*
$$
S_1 = S \to S_2 \to \cdots
$$
be the sequence of quadratic transforms such that $S_i = S_{r_i'}$ in the notation of Definition 7.11 [14] so that $V_{\nu^*} = \cup S_i$. Let Q_i be the generating sequence of S of (71) of [14] determining ν^* (Proposition 7.40 [14])

(41)
$$
\begin{aligned}
Q_0 &= x \\
Q_1 &= y \\
Q_2 &= y^{p^2} - x \\
Q_{j+1} &= Q_j^{p^2} = x^{p^{2j-2}} Q_{j-1} \text{ for } j \geq 2.
\end{aligned}
$$

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Then there exist generating sequences $\{Q(k)_i\}$ in S_k such that

$$x_{S_2} = Q(2)_0 = Q_1 = y, \, y_{S_2} = Q(2)_1 = \frac{Q_2}{Q_0} = \frac{Q_2}{x}$$

are regular parameters in S_2 such that $Q(2)_0 = 0$ is a local equation of the exceptional locus of $Spec(S_2) \to Spec(S)$ and for $k \geq 2$,

$$x_{S_{k+1}} = Q(k+1)_0 = Q(k)_1, \, y_{S_{k+1}} = Q(k+1)_1 = \frac{Q_{k+1}}{x^{p^{2k-2}}Q_{k-1}}$$

are regular parameters in S_{k+1}, such that $Q(k+1)_0 = 0$ is a local equation of the exceptional locus of $Spec(S_{k+1}) \to Spec(S)$.

In S_2, the generating sequence $\{Q(2)_i\}$ is defined by

(42)
$$Q(2)_0 = Q_1, \, Q(2)_i = \frac{Q_{i+1}}{x^{p^{2(i-1)}}}$$

for $i \geq 1$, and for $k \geq 3$, the generating sequence $\{Q(k)_j\}$ in S_k is defined by

(43)
$$Q(k)_0 = Q(k-1)_1 = \frac{Q_{k-1}}{x^{p^{2(k-3)}}Q_{k-3}}$$

and

(44)
$$Q(k)_i = \frac{Q_{i+k-1}}{x^{p^{2(i+k-3)}}Q_{k-2}^{p^{2(i-1)}}}$$

for $i \geq 1$.

It follows from Proposition 5.1 that $\nu^*(Q(k)_0) = p^2\nu^*(Q(k)_1)$, so we have regular parameters $\overline{x}_{S_{k+1}}, \overline{y}_{S_{k+1}}$ in S_{k+1} defined by

(45)
$$Q(k)_0 = \overline{x}_{S_{k+1}}^{p^2}(\overline{y}_{S_{k+1}} + 1), \, Q(k)_1 = \overline{x}_{S_{k+1}}.$$

Proposition 5.2. *Let*

$$R_1 = R \to R_2 \to \cdots$$

be the sequence of quadratic transforms such that $R_i = R_{r_i'}$ in the notation of Definition 7.11 [14] so that $V_\nu = \cup R_i$. Let P_i be the generating sequence of R of (76) of [14] determining ν (Corollary 7.41 [14])

(46)
$$
\begin{aligned}
P_0 &= u \\
P_1 &= v \\
P_2 &= v^{p^2} - u \\
P_{i+1} &= P_i^{p^2} - u^{p^{2i-2}}P_{i-1} \text{ for } i \geq 2.
\end{aligned}
$$

Then there exist generating sequences $\{P(k)_i\}$ in R_k determining ν such that

$$u_{R_2} = P(2)_0 = P_1 = v, \, v_{R_2} = P(2)_1 = \frac{P_2}{P_0} = \frac{P_2}{u}$$

are regular parameters in R_2 such that $u_{R_2} = 0$ is a local equation of the exceptional locus of $Spec(S_2) \to Spec(S)$ and for $k \geq 2$,

$$u_{R_{k+1}} = P(k+1)_0 = P(k)_1, \, v_{R_{k+1}} = P(k+1)_1 = \frac{P_{k+1}}{u^{p^{2k-2}}P_{k-1}}$$

are regular parameters in R_{k+1}, such that $P(k+1)_0 = 0$ is a local equation of the exceptional locus of $Spec(R_{k+1}) \to Spec(R)$.

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In R_2, the generating sequence $\{P(2)_j\}$ is defined by

$$(47) \qquad P(2)_0 = P_1, P(2)_i = \frac{P_{i+1}}{u^{p^{2(i-1)}}}$$

for $i \geq 1$, and for $k \geq 3$, the generating sequence $\{P(k)_j\}$ in R_k is defined by

$$(48) \qquad P(k)_0 = P(k-1)_1 = \frac{P_{k-1}}{u^{p^{2(k-3)}} P_{k-3}}$$

and

$$(49) \qquad P(k)_i = \frac{P_{i+k-1}}{u^{p^{2(i+k-3)}} P_{k-2}^{p^{2(i-1)}}}$$

for $i \geq 1$.

It follows from Proposition 5.2 that we have regular parameters $\overline{u}_{R_{k+1}}, \overline{v}_{R_{k+1}}$ in R_{k+1} defined by

$$(50) \qquad P(k)_0 = \overline{u}_{R_{k+1}}^{p^2}(\overline{v}_{R_{k+1}} + 1), P(k)_1 = \overline{u}_{R_{k+1}}.$$

Proposition 5.3. *The sequence $\{U_j\}$ is a generating sequence in A of a unique k-valuation ν_1 of K_1 such that $\nu_1(x) = 1$. Let*

$$A_1 = A \to A_2 \to \cdots$$

be the sequence where $A_i = A_{a'_i}$ in the notation of Definition 7.11 [14], so that $V_{\nu_1} = \cup A_i$. Then there exist generating sequences $\{U(k)_i\}$ in A_k determining ν_1 such that

$$x_{A_2} = U(2)_0 = U_1 = v, v_{A_2} = U(2)_1 = \frac{U_2}{U_0} = \frac{U_2}{x}$$

are regular parameters in A_2 such that $U(2)_0 = 0$ is a local equation of the exceptional locus of $Spec(A_2) \to Spec(A)$. For $j \geq 2$,

$$(51) \qquad x_{A_{j+1}} = U(j+1)_0 = U(j)_1, v_{A_{j+1}} = U(j+1)_1 = \begin{cases} \frac{U_{j+1}}{x^{p^{2j-2}} U_{j-1}} & \text{if } j \text{ is odd} \\ \frac{U_{j+1}}{x^{p^{2j-1}} U_{j-1}} & \text{if } j \text{ is even} \end{cases}$$

are regular parameters in A_{j+1}, such that $U(j+1)_0 = 0$ is a local equation of the exceptional locus of $Spec(A_{j+1}) \to Spec(A)$.

In A_2, the generating sequence $\{U(2)_j\}$ is defined by

$$(52) \qquad U(2)_0 = U_1$$

and for $j \geq 1$,

$$(53) \qquad U(2)_j = \begin{cases} \frac{U_{j+1}}{x^{p^{2j-2}}} & \text{if } j \text{ is odd} \\ \frac{U_{j+1}}{x^{p^{2j-1}}} & \text{if } j \text{ is even} \end{cases}$$

and for $k \geq 3$, the generating sequence $\{U(k)_j\}$ in A_k is defined by

$$(54) \qquad U(k)_0 = U(k-1)_1 = \begin{cases} \frac{U_{k-1}}{x^{p^{2k-6}} U_{k-3}} & \text{if } k \text{ is odd} \\ \frac{U_{k-1}}{x^{p^{2k-5}} U_{k-3}} & \text{if } k \text{ is even} \end{cases}$$

and for k odd, $j \geq 1$,

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$$(55) \qquad U(k)_j = \begin{cases} \dfrac{U_{j+k-1}}{x^{p^{2(j+k)-5}} U_{k-2}^{p^{2j-2}}} & \text{if } j \text{ is odd} \\[2ex] \dfrac{U_{j+k-1}}{x^{p^{2(j+k)-6}} U_{k-2}^{p^{2j-3}}} & \text{if } j \text{ is even} \end{cases}$$

and for k even, $j \geq 1$,

$$(56) \qquad U(k)_j = \begin{cases} \dfrac{U_{j+k-1}}{x^{p^{2(j+k)-6}} U_{k-2}^{p^{2j-2}}} & \text{if } j \text{ is odd} \\[2ex] \dfrac{U_{j+k-1}}{x^{p^{2(j+k)-5}} U_{k-2}^{p^{2j-1}}} & \text{if } j \text{ is even} \end{cases}$$

Further, there exist units $\delta(k)_j$ in A_k with $\delta(k)_j \equiv 1 \bmod m_{A_k}$ such that if k is odd, then

$$(57) \qquad U(k)_2 = U(k)_1^p - \delta(k)_0 U(k)_0$$

and for $j \geq 2$,

$$(58) \qquad U(k)_{j+1} = \begin{cases} U(k)_j^p - \delta(k)_{j+1} U(k)_0^{p^{2j-2}} U(k)_{j-1} & \text{if } j \text{ is odd} \\[1ex] U(k)_j^{p^3} - \delta(k)_{j+1} U(k)_0^{p^{2j-1}} U(k)_{j-1} & \text{if } j \text{ is even} \end{cases}$$

and if k is even, then

$$(59) \qquad U(k)_2 = U(k)_1^{p^3} - \delta(k)_0 U(k)_0$$

and for $j \geq 2$,

$$(60) \qquad U(k)_{j+1} = \begin{cases} U(k)_j^{p^3} - \delta(k)_{j+1} U(k)_0^{p^{2j-2}} U(k)_{j-1} & \text{if } j \text{ is odd} \\[1ex] U(k)_j^p - \delta(k)_{j+1} U(k)_0^{p^{2j-3}} U(k)_{j-1} & \text{if } j \text{ is even} \end{cases}$$

Proof. The fact that the sequence $\{U_j\}$ is a generating sequence of a unique k-valuation ν_1 of K_1 was shown before Proposition 5.1.

The remainder of the proposition is proved by induction on k. By (35) and (36) if $k = 1$ and (57) - (60) if $k > 1$ and by Theorem 7.1 [15], there exists a generating sequence $\{\tilde{U}(k+1)\}$ in A_{k+1} defined if k is odd by

$$(61) \qquad \tilde{U}(k+1)_0 = U(k)_1, \tilde{U}(k+1)_1 = \frac{U(k)_2}{U(k)_1^p}$$

and for $j \geq 2$,

$$(62) \qquad \tilde{U}(k+1)_j = \begin{cases} \dfrac{U(k)_{j+1}}{U(k)_1^{p^{2j-1}}} & \text{if } j \text{ is odd} \\[2ex] \dfrac{U(k)_{j+1}}{U(k)_1^{p^{2j}}} & \text{if } j \text{ is even} \end{cases}$$

and if k is even, then

$$(63) \qquad \tilde{U}(k+1)_0 = U(k)_1, \tilde{U}(k+1)_1 = \frac{U(k)_2}{U(k)_1^{p^3}}$$

and for $j \geq 2$,

$$(64) \qquad \tilde{U}(k+1)_j = \begin{cases} \dfrac{U(k)_{j+1}}{U(k)_1^{p^{2j+1}}} & \text{if } j \text{ is odd} \\ \dfrac{U(k)_{j+1}}{U(k)_1^{p^{2j}}} & \text{if } j \text{ is even} \end{cases}$$

To prove that the conclusions of the proposition hold for $\{U(k+1)_i\}$, we use the induction assumption and the appropriate equations (66), (68) or (69) which are stated below to first verify that there are units $\lambda(k+1)_j$ in A_{k+1} with $\lambda(k+1)_j \equiv 1 \bmod m_{A_{k+1}}$ such that

$$(65) \qquad U(k+1)_j = \lambda(k+1)_j \tilde{U}(k+1)_j$$

for all j, verifying that $\{U(k+1)_j\}$ is a generating sequence in A_{k+1} for ν_1, and then that the appropriate equations (57) - (60) hold and finally that the appropriate equations (52) and (53), or (54) - (56) hold.

We now state and prove the equations (66), (68) and (69).

There exists a unit $\varepsilon(2)$ in A_2 with $\varepsilon(2) \equiv 1 \bmod m_{A_2}$ such that

$$(66) \qquad U_1^p = \varepsilon(2)x.$$

Equation (66) follows from (35) and the fact (Theorem 7.1 [15]) that A_2 has regular parameters $\overline{x}_{A_2}$ and $\overline{v}_{A_2}$ defined by

$$(67) \qquad U(1)_0 = x = \overline{x}_{A_2}^p(\overline{v}_{A_2}+1), U(1)_1 = v = \overline{x}_{A_2}$$

There exists a unit $\varepsilon(k+1) \in A_{k+1}$ with $\varepsilon(k+1) \equiv 1 \bmod m_{A_{k+1}}$ such that

$$(68) \qquad U(k)_1^p = \varepsilon(k+1)\frac{U_{k-1}}{U_{k-2}^p}$$

if $k \geq 2$ is odd and

$$(69) \qquad U(k)_1^{p^3} = \varepsilon(k+1)\frac{U_{k-1}}{U_{k-2}^{p^3}}$$

if $k \geq 2$ is even.

We now simultaneously verify the equations (68) and (69). We first verify that A_{k+1} has regular parameters $\overline{x}_{A_{k+1}}$ and $\overline{v}_{A_{k+1}}$ defined if k is odd by

$$(70) \qquad U(k)_0 = \overline{x}_{A_{k+1}}^p(\overline{v}_{A_{k+1}}+1), U(k)_1 = \overline{x}_{A_{k+1}}$$

and if k is even by

$$(71) \qquad U(k)_0 = \overline{x}_{A_{k+1}}^{p^3}(\overline{v}_{A_{k+1}}+1), U(k)_1 = \overline{x}_{A_{k+1}}.$$

If k is odd, we have that $p\nu_1(U(k)_1) = \nu_1(U(k)_0)$ by (57) and if k is even we have that $p^3\nu_1(U(k)_1) = \nu_1(U(k)_0)$ by (59), so (70) and (71) follow from (57), (59) and Theorem 7.1 [15]. Thus for $j \leq k$, U_j is a power of $U(k+1)_0$ times a unit in A_{k+1} which is equivalent to 1 mod $m_{A_{k+1}}$. Thus by (36), there exists a unit $\varepsilon(k+1)$ in A_{k+1} with $\varepsilon(k+1) \equiv 1 \bmod m_{A_{k+1}}$ such that

$$U_k^p = \varepsilon(k+1)x^{p^{2k-2}}U_{k-1} \text{ if } k \text{ is odd}$$

and

$$U_k^{p^3} = \varepsilon(k+1)x^{p^{2k-1}}U_{k-1} \text{ if } k \text{ is even.}$$

Thus if k is odd,

$$
\begin{aligned}
U(k)_1^p &= \left(\frac{U_k}{x^{p^{2k-3}}U_{k-2}}\right)^p = \frac{U_k^p}{x^{p^{2k-2}}U_{k-2}^p} \\
&= \frac{\varepsilon(k+1)x^{p^{2k-2}}U_{k-1}}{x^{p^{2k-2}}U_{k-2}^p} = \varepsilon(k+1)\frac{U_{k-1}}{U_{k-2}^p}.
\end{aligned}
$$

if k is even,

$$
\begin{aligned}
U(k)_1^{p^3} &= \left(\frac{U_k}{x^{p^{2k-4}}U_{k-2}}\right)^{p^3} = \frac{U_k^{p^3}}{x^{p^{2k-1}}U_{k-2}^{p^3}} \\
&= \frac{\varepsilon(k+1)x^{p^{2k-1}}U_{k-1}}{x^{p^{2k-1}}U_{k-2}^{p^3}} = \varepsilon(k+1)\frac{U_{k-1}}{U_{k-2}^{p^3}}.
\end{aligned}
$$

$\square$

Lemma 5.4. *We have that*

$$(72) \qquad\qquad U_0 = x = Q_0, U_1 = Q_1^p - x^c y,$$

and for $j \geq 1$, if j is odd, then

$$(73) \qquad\qquad U_{j+1} = Q_{j+1} + x^{p^{2j-2}(\sum_{j'=0}^{\frac{j-1}{2}}\frac{1}{p^{4j'}})}f_{j+1}(x,y)$$

where $f_{j+1} \in k[[x]][y]$, x divides f_{j+1} in $k[[x]][y]$, and $\deg_y f_{j+1} = p^{2j-1}$.
  If j is even, then

$$(74) \qquad\qquad U_{j+1} = Q_{j+1}^p + x^{p^{2j-1}(\sum_{j'=0}^{\frac{j}{2}-1}\frac{1}{p^{4j'}})}f_{j+1}(x,y)$$

where $f_{j+1} \in k[[x]][y]$, x divides f_{j+1} in $k[[x]][y]$, and $\deg_y f_{j+1} = p^{2j}$.

Proof. We have that

$$
\begin{aligned}
U_1 &= v = y^p - x^c y = Q_1^p - x^c y, \\
U_2 &= v^p - x = Q_2 - x^{cp}y^p,
\end{aligned}
$$

verifying (73) for $j = 1$ and

$$U_3 = U_2^{p^3} - x^{p^3}U_1 = Q_3^p - x^{cp^4}y^p + x^{p^3+c}y$$

verifying (74) for $j = 2$.
   We prove the equations (73) and (74) for $j \geq 3$ by induction. First assume that j is odd. We have that

$$
\begin{aligned}
U_{j+1} &= U_j^p - x^{p^{2j-2}}U_{j-1} \\
&= Q_{j+1} + x^{p^{2j-2}(\sum_{j'=0}^{\frac{j-3}{2}}\frac{1}{p^{4j'}})}f_j^p - x^{p^{2j-2}+p^{2j-6}(\sum_{j'=0}^{\frac{j-3}{2}}\frac{1}{p^{4j'}})}f_{j-1}.
\end{aligned}
$$

The formula (73) then follows since

$$p^{2j-2}(\sum_{j'=0}^{\frac{j-1}{2}}\frac{1}{p^{4j'}}) = p^{2j-2}(\sum_{j'=0}^{\frac{j-3}{2}}\frac{1}{p^{4j'}}) + 1$$

and

$$p^{2j-2} + p^{2j-6}(\sum_{j'=0}^{\frac{j-3}{2}}\frac{1}{p^{4j'}}) = p^{2j-2}(\sum_{j'=0}^{\frac{j-1}{2}}\frac{1}{p^{4j'}}).$$

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Now assume that $j \geq 3$ is even. We have that

$$
\begin{aligned}
U_{j+1} &= U_j^{p^3} - x^{p^{2j-1}} U_{j-1} \\
&= Q_{j+1}^p + x^{p^{2j-1}(\sum_{j'=0}^{\frac{j}{2}-1} \frac{1}{p^{4j'}})} f_j^{p^3} - x^{p^{2j-1}+p^{2j-5}(\sum_{j'=0}^{\frac{j}{2}-2} \frac{1}{p^{4j'}})} f_{j-1}.
\end{aligned}
$$

The formula (74) then follows since

$$
p^{2j-1} + p^{2j-5}\left(\sum_{j'=0}^{\frac{j}{2}-2} \frac{1}{p^{4j'}}\right) = p^{2j-1}\left(\sum_{j'=0}^{\frac{j}{2}-1} \frac{1}{p^{4j'}}\right).
$$

$\square$

Proposition 5.5. *We have $V_{\nu_1} = V_{\nu^*} \cap K_1$ and $\nu^*|K_1 = \nu_1$. Further, ν^* is the unique extension of ν_1 to K^*.*

Proof. Since $\nu_1(x) = \nu^*(x) = 1$, it suffices to show that $V_{\nu_1} \subset V_{\nu^*}$. From (51) and the fact that $V_{\nu_1} = \cup A_j$, we need only show that $v_{A_{j+1}} \in V_{\nu^*}$ for all j. From equation (72) of [14], we obtain that

$$
\text{(75)} \qquad \nu^*(Q_{j+1}) = \overline{\beta}_{j+1} < p^{2j-2}\left(\sum_{j'=0}^{\frac{j-1}{2}} \frac{1}{p^{4j'}}\right) + 1
$$

if j is odd, and

$$
\text{(76)} \qquad \nu^*(Q_{j+1}^p) = p\overline{\beta}_{j+1} < p^{2j-1}\left(\sum_{j'=0}^{\frac{j}{2}-1} \frac{1}{p^{4j'}}\right) + 1
$$

if j is even. Thus by Lemma 5.4,

$$
\text{(77)} \qquad \nu^*(U_{j+1}) = \begin{cases} \nu^*(Q_{j+1}) & \text{if } j \text{ is odd} \\ p\nu^*(Q_{j+1}) & \text{if } j \text{ is even} \end{cases}
$$

Thus by (51) and Proposition 7.40 [14],

$$
\nu^*(v_{A_{j+1}}) = \begin{cases} \nu^*(y_{S_{j+1}}) & \text{if } j \text{ odd} \\ p\nu^*(y_{S_{j+1}}) & \text{if } j \text{ even} \end{cases}
$$

The fact that ν^* is the unique extension of ν to K^* follows since $\nu_1|K = \nu^*|K = \nu$ by Proposition 7.40 [14], and ν^* is the unique extension of ν to K^* by (1) of Theorem 7.38 [14]. $\square$

Theorem 5.6. *For all j, S_j is a finite extension of A_j. If j is odd, we have expressions*

$$
\text{(78)} \qquad x_{A_{j+1}} = \tau_{j+1} x_{S_{j+1}}^p, v_{A_{j+1}} = \gamma_{j+1} y_{s_{j+1}} + x_{S_{j+1}} \Omega_{j+1}
$$

where τ_{j+1}, γ_{j+1} are units in S_{j+1} and $\Omega_{j+1} \in S_{j+1}$.
If j is even, we have expressions

$$
\text{(79)} \qquad x_{A_{j+1}} = \tau_{j+1} x_{S_{j+1}}, v_{A_{j+1}} = \gamma_{j+1} y_{S_{j+1}}^p + x_{S_{j+1}} \Omega_{j+1}
$$

where τ_{j+1}, γ_{j+1} are units in S_{j+1} and $\Omega_{j+1} \in S_{j+1}$.

Proof. For $i \leq j$, Q_i are units in S_{j+1} times a power of $x_{S_{j+1}}$. By (73), (74) and (75) and (76), we then have that U_i are units in S_{j+1} times a power of $x_{S_{j+1}}$ and for $i \leq j$,

$$(80) \qquad U_i = \begin{cases} \alpha Q_i^p & \text{where } \alpha \in S_{j+1} \text{ is a unit if } i \text{ is odd} \\ \alpha Q_i & \text{where } \alpha \in S_{j+1} \text{ is a unit if } i \text{ is even.} \end{cases}$$

We have from (75) that
(81)

$$\nu^*(Q_{j-1}) = \overline{\beta}_{j-1} = p^{2j-6}(\sum_{j'=0}^{j-2} \frac{1}{p^{4j'}}) < p^{2j-6}(\sum_{j'=0}^{\frac{j-1}{2}-1} \frac{1}{p^{4j'}}) + 1 = p^{2j-2}(\sum_{j'=0}^{\frac{j-1}{2}} \frac{1}{p^{4j'}}) + 1 - p^{2j-2}$$

if j is odd and by (76),
(82)

$$p\nu^*(Q_{j-1}) = p\overline{\beta}_{j-1} = p^{2j-5}(\sum_{j'=0}^{j-2} \frac{1}{p^{4j'}}) < p^{2j-5}(\sum_{j'=0}^{\frac{j}{2}-2} \frac{1}{p^{4j'}}) + 1 = p^{2j-1}(\sum_{j'=0}^{\frac{j}{2}-1} \frac{1}{p^{4j'}}) + 1 - p^{2j-1}$$

if j is even.

If j is odd, by equations (51), (73), (80) and (81) and Proposition 5.1, we have an expression

$$v_{A_{j+1}} = \frac{U_{j+1}}{x^{p^{2j-2}} U_{j-1}} = \frac{Q_{j+1} + x^{p^{2j-2}(\sum_{j'=0}^{\frac{j-1}{2}} \frac{1}{p^{4j'}})} f_{j+1}}{\alpha x^{p^{2j-2}} Q_{j-1}} = \gamma y_{S_{j+1}} + x_{S_{j+1}} \Omega$$

for some unit $\gamma \in S_{j+1}$ and $\Omega \in S_{j+1}$.

If j is even, by equations (51), (74), (80) and (82) and Proposition 5.1, we have an expression

$$v_{A_{j+1}} = \frac{U_{j+1}}{x^{p^{2j-1}} U_{j-1}} = \frac{Q_{j+1}^p + x^{p^{2j-1}(\sum_{j'=0}^{\frac{j}{2}-1} \frac{1}{p^{4j'}})} f_{j+1}}{\alpha x^{p^{2j-1}} Q_{j-1}^p} = \gamma y_{S_{j+1}}^p + x_{S_{j+1}} \Omega$$

for some unit $\gamma \in S_{j+1}$ and $\Omega \in S_{j+1}$.

By (80), we have that

$$x_{A_{j+1}} = \frac{U_j}{x^{p^{2j-3}} U_{j-2}} = \frac{\alpha Q_j^p}{x^{p^{2j-3}} Q_{j-2}^p} = \alpha x_{S_{j+1}}^p$$

for some unit $\alpha \in S_{j+1}$ if j is odd and

$$x_{A_{j+1}} = \frac{U_j}{x^{p^{2j-4}} U_{j-2}} = \frac{\alpha Q_j}{x^{p^{2j-4}} Q_{j-2}} = \alpha x_{S_{j+1}}$$

for some unit $\alpha \in S_{j+1}$ if j is even.

The extension $A_j \to S_j$ is finite for all j since each $A_j \to S_j$ is quasi finite and ν^* is the unique extension of ν to K^*.

$\square$

Theorem 5.7. *For all j, A_j is a finite extension of R_j. If j is odd, we have expressions*

$$(83) \qquad u_{R_{j+1}} = \sigma_{j+1} x_{A_{j+1}}, \quad v_{R_{j+1}} = \lambda_{j+1} v_{A_{j+1}}^p + x_{A_{j+1}} \Lambda_{j+1}$$

where $\sigma_{j+1}, \lambda_{j+1}$ are units in A_{j+1} and $\Lambda_{j+1} \in A_{j+1}$.

If j is even, we have expressions

$$(84) \qquad u_{R_{j+1}} = \sigma_{j+1} x_{A_{j+1}}^p, v_{R_{j+1}} = \lambda_{j+1} v_{A_{j+1}} + x_{A_{j+1}} \Lambda_{j+1}$$

where $\sigma_{j+1}, \lambda_{j+1}$ are units in A_{j+1} and $\Lambda_{j+1} \in A_{j+1}$.

Proof. We have that $R_{j+1} \subset A_{j+1}$ and $R_{j+1} \to A_{j+1}$ is finite since $R_{j+1} \to S_{j+1}$ and $A_{j+1} \to S_{j+1}$ are finite, A_{j+1} is normal and $\mathrm{QF}(R_{j+1}) \subset \mathrm{QF}(A_{j+1})$. The expressions (83) and (84) follow from (2) of Theorem 7.38 [14] and Theorem 5.6. $\qquad \square$

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