ULTRAMETRIC DYNAMICS

SIBYLLA PRIESS-CRAMPE AND PAULO RIBENBOIM

Abstract. This paper is concerned with dynamics in general ultrametric spaces. We prove the Fixed Point Theorem, the Stability Theorem, the Common Point Theorem, the Local Fixed Point Theorem, the Attractor Theorem and we derive conditions for a mapping to be surjective. We also discuss tightly continuous mappings and prove a theorem about the transfer of the property of principal completeness by a mapping between ultrametric spaces.

1. Introduction

In short, Dynamics in mathematics is concerned with the effect on points of iterating self-maps on a space. In this context important questions are about the existence of fixed points, of common points to more than one self-map, as well as the existence of stable domains, of attractor points, etc.. In their full generality, ultrametric spaces are just sets endowed with a distance mapping having values in an arbitrary ordered set Γ with a smallest element and satisfying natural conditions (as indicated in the text). In our study, the set Γ is not required to be totally ordered, a fortiori, Γ need not to be contained in the set of non-negative real numbers.

a) The seminal examples of ultrametric spaces are the p-adic fields and their finite degree extensions. The Dynamics in these fields has been studied by numerous authors, see for example the books of Narkiewicz [6] and Silverman [21].

b) Non-classical Functional Analysis started from studying spaces over the p-adic fields. Then the p-adic fields were replaced by fields with valuations of rank one, and now, since about ten years, also spaces with Krull valuations of arbitrary rank are considered (see for example the books of Monna [5] and van Rooij [23] as well as the papers by Ochsenius and Schikhof [7] and Priess-Crampe [9]). For this new direction of non-classical Functional Analysis, it is expected that general theorems of ultrametric spaces will play an important role.

Date: July 24, 2009.

1991 *Mathematics Subject Classification.* [2000]Primary 13F25; Secondary 13A18.

Key words and phrases. ultrametric spaces, fixed points, common points, stability, transfer of spherical completeness.

c) Hardy fields are endowed with a natural valuation which is a Krull valuation of arbitrary rank (see for example [18], [19]). The Fixed Point Theorem is used to determine conditions for the existence of solutions of polynomial differential equations of any order, or even of systems of such equations, see Priess-Crampe and Ribenboim [16], [15].

d) Methods of ultrametric dynamics also find applications in the study of differential equations over rings of power series, as in the work of van der Hoeven, for example see his lecture notes [22].

e) A very different and unexpected application of ultrametric dynamics is found in the determination of solutions of the famous Fermat equation in square matrices with entries in a p-adic field, see [17].

f) Schörner used the Common Point Theorem in a construction involving ternary fields, concerned with the existence of Hahn structures for valued projective planes, see Schörner [20].

g) Programs with positive clauses were shown to have models by means of the fixed point theorem of Knaster and Tarski about monotonic self-maps in a complete lattice. More general programs, involving negation in clauses lead to the ultrametric space of maps from the Herbrand base with values 0, 1; in this space the values of the distance are the subsets of the Herbrand base. The fixed point of the immediate consequence operator gives conditions for the existence of models for the program, see Priess-Crampe and Ribenboim [12], [13] and Hitzler and Seda [2], [1].

The variety of applications calls for a systematic study in full generality of the main theorems of ultrametric dynamics as we do in this paper. The fundamental theorem of arithmetics says already that besides the classical distance associated to the ordinary absolute value, there are all the p-adic distances. The examples mentioned above support our contention that in the mathematical world the ultrametric facets are ubiquitous. It is our hope that specialists in dynamical systems will benefit from an acquaintance with our methods.

It is not superfluous to point out features in our treatment which cannot be found in papers of any other authors.

i) We consider ultrametric spaces with set of values of the distance which need not be totally ordered. Loosely speaking, this more embracing situation is intended to handle the multivariate spaces and ultrametric spaces of functions, as quoted in (e) and (g). The difficulties in this general situation require delicate arguments which are unnecessary in the case when the set of values of the distance is totally ordered.

ii) Whereas in Dynamics one studies a space X and a self-mapping $\varphi : X \to X$, we work most often with configurations $\mathfrak{X} = \langle X, X, X', \varphi, \theta, \psi \rangle$, where X and X' are ultrametric spaces, and $\varphi : X \to X$, $\psi, \theta : X \to X'$ are mappings. This richer situation allows to find applications in a wider variety of contexts, for example the Attractor Theorem.

In §2 we introduce the reader to ultrametric spaces, giving definitions of the relevant concepts and illustrating with examples. These examples will allow the reader to perceive the scope of the theory of such general ultrametric spaces.

Even though we had proved in our earlier papers the Fixed Point Theorem, and some companion theorems, in §3 we prove the Main Theorem for a "configuration" of ultrametric spaces. The Fixed Point Theorem, the Common Point Theorem, the Stability Theorem, the Local Fixed Point Theorem are immediate consequences of the Main Theorem.

And it does not take much work to obtain the Attractor Theorem and consequences about surjective mappings – this is treated in §4.

In the next §5 we discuss what we call tightly continuous mappings. Finally in §6 we prove a theorem of transfer of principal completeness. It is valid without assuming that the set of values of the distance is totally ordered and it requires a delicate and long proof.

2. Ultrametric Spaces

2.1. **Definitions and Relevant Results.** We give the definitions and results which are required in the sequel. For more details, the reader may consult the papers listed in the references.

2.1.1. Let $(\Gamma, \leq)$ be an ordered set with smallest element 0. Let X be a non-empty set. A mapping $d\colon X \times X \to \Gamma$ is called an *ultrametric distance function* when the following properties are satisfied for all $x, y, z \in X$:

(d1) $d(x,y) = 0$ if and only if $x = y$.

(d2) $d(x,y) = d(y,x)$.

(d3) If $d(x,y) \leq \gamma$ and $d(y,z) \leq \gamma$ then $d(x,z) \leq \gamma$, for all $\gamma \in \Gamma$.

(X, d, Γ) is called an *ultrametric space* and $d(x,y)$ is the *ultrametric distance* between x and y. The ultrametric space is *trivial*, if there exists $\gamma \in \Gamma$ such that for all $x, y \in X$, $x \neq y$, $d(x,y) = \gamma$. The space X is said to be *solid* if for every $\gamma \in \Gamma$ and $x \in X$ there exists $y \in X$ such that $d(x,y) = \gamma$. If X is solid then $d(X \times X) = \Gamma$.

If $(\Gamma, \leq)$ is totally ordered, (d3) becomes:

(d3') $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for all $x, y, z \in X$.

If Y is a subset of X, if $d|_Y$ is the restriction of d to $Y \times Y$ and $\Gamma_Y \subseteq \Gamma$ such that $d(Y \times Y) \subseteq \Gamma_Y$, we say that $(Y, d|_Y, \Gamma_Y)$ is a *subspace* of (X, d, Γ).

2.1.2. Let $\gamma \in \Gamma$, let $B_\gamma(x) = \{y \in X \mid d(y,x) \leq \gamma\}$. If $\gamma = 0$ then $B_\gamma(x) = \{x\}$. A set $B \subseteq X$ is called a *ball* if there exists $\gamma \in \Gamma^\bullet = \Gamma \backslash \{0\}$ and $x \in X$ such that $B = B_\gamma(x)$. In this situation x is a *center* of B and γ is a *radius* of B. If $x, y \in X$, $x \neq y$, then $B(x,y) = B_{d(x,y)}(x)$ is said to be a *principal ball.* If X is solid, every ball is principal.

A non-empty subset Y of X is said to be *convex* in X when for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ the principal ball $B(y_1, y_2) \subseteq Y$. It follows that every principal ball is convex in X and furthermore, if $\bigcap_{i \in I} B(x_i, y_i) \neq \varnothing$ then $\bigcap_{i \in I} B(x_i, y_i)$ is convex in X.

2.1.3. Let $\gamma, \delta \in \Gamma^\bullet$.

(1) Let $x, y \in X$.
   (a) If $\gamma \leq \delta$ and $B_\gamma(x) \cap B_\delta(y) \neq \varnothing$ then $B_\gamma(x) \subseteq B_\delta(y)$.
   (b) If $B_\delta(y) \subset B_\gamma(x)$ then $\gamma \not\leq \delta$.

(2) Concerning principal balls, if $x, y, z, u \in X$, $x \neq z$ and $y \neq u$, then:
   (a) $B(x, z) \subseteq B_\delta(y)$ if and only if $d(x, z) \leq \delta$ and $x \in B_\delta(y)$.
   (b) If $B(x, z) \subset B_\delta(y)$ then $d(x, z) < \delta$.
   (c) If $B(x, z) = B(y, u)$ then $d(x, z) = d(y, u)$.

(3) Let X be solid and $x, y \in X$.
   (a) $B_\gamma(x) \subseteq B_\delta(y)$ if and only if $\gamma \leq \delta$ and $x \in B_\delta(y)$.
   (b) If $B_\gamma(x) \subset B_\delta(y)$ then $\gamma < \delta$.
   (c) If $B_\gamma(x) = B_\delta(y)$ then $\gamma = \delta$.

(4) If Γ is totally ordered and $B_\delta(y) \subset B_\gamma(x)$ then $\delta < \gamma$.

2.1.4. A set of balls which is totally ordered by inclusion is said to be a *chain*. Let $\mathcal{C}$ be a chain of balls of X which does not have a smallest ball. Then there exists a limit ordinal λ and a strictly decreasing family of balls $(B_\iota)_{\iota<\lambda}$ such that each $B_\iota \in \mathcal{C}$ and for every ball $C \in \mathcal{C}$ there exists B_ι such that $C \supseteq B_\iota$. Hence $\bigcap \mathcal{C} = \bigcap_{\iota<\lambda} B_\iota$.

2.1.5. An ultrametric space X is said to be *spherically complete* (respectively *principally complete*) when every chain of balls of X (respectively principal balls of X) has a non-empty intersection.

Every spherically complete ultrametric space is principally complete. The converse is true when Γ is totally ordered or the space is solid.

2.1.6. An ultrametric space X is spherically complete (respectively principally complete) if and only if the following property is satisfied: for every limit ordinal λ, every strictly decreasing family $(B_\iota)_{\iota\in\lambda}$ of balls (respectively principal balls) has a non-empty intersection.

2.1.7. If X is principally complete, every subset of X which is convex in X is a principally complete subspace.

2.2. **Examples of Ultrametric Spaces.**

2.2.1. *Example when* $(\Gamma, \leq)$ *is totally ordered.* Let Δ be a totally ordered abelian additive group, let ∞ be a symbol such that $\infty \notin \Delta$, and $\delta + \infty = \infty + \delta = \infty$, $\infty + \infty = \infty$, $\delta < \infty$ for all $\delta \in \Delta$. We denote by 0 the neutral element of Δ, that is $0 + \delta = \delta$ for every $\delta \in \Delta$. Let K be a commutative field, let $v\colon K \to \Delta \cup \{\infty\}$ be a valuation of K, so we have:

(v1) $v(x) = \infty$ if and only if $x = 0$,

(v2) $v(xy) = v(x) + v(y)$,

(v3) $v(x + y) \geq \min\{v(x), v(y)\}$.

Let $\Gamma^\bullet$ be a totally ordered abelian multiplicative group with neutral element 1, let 0 be a symbol such that $0 \notin \Gamma^\bullet$, $0\gamma = \gamma 0 = 0$, $0 \cdot 0 = 0$, $0 < \gamma$ for every $\gamma \in \Gamma^\bullet$. Let $\theta\colon \Delta \cup \{\infty\} \to \Gamma = \Gamma^\bullet \cup \{0\}$ be an order reversing bijection such that $\theta(\infty) = 0$, $\theta(\delta + \delta') = \theta(\delta) \cdot \theta(\delta')$, so $\theta(0) = 1$.

Let $d\colon K \times K \to \Gamma$ be defined by $d(x, y) = \theta(v(x - y))$, then (K, d, Γ) is an ultrametric space which is said to be *associated to the valued field* $(K, v, \Delta \cup \{\infty\})$.

2.2.2. *Another example where* Γ *is totally ordered.* Let Γ be a totally ordered set with smallest element 0, let $\Gamma^\bullet = \Gamma \backslash \{0\}$. Let R be a non-empty set with a distinguished element 0. For each $f\colon \Gamma^\bullet \to R$ let $\operatorname{supp}(f) = \{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq 0\}$ be the *support* of f. Let $R[[\Gamma]]$ be the set of all $f\colon \Gamma^\bullet \to R$ with support which is empty or anti-well ordered. Let $d\colon R[[\Gamma]] \times R[[\Gamma]] \to \Gamma$ be defined by $d(f, f) = 0$ and if $f \neq g$, $d(f, g)$ is the largest element of the set $\{\gamma \in \Gamma^\bullet \mid f(\gamma) \neq g(\gamma)\}$. Then $(R[[\Gamma]], d, \Gamma)$ is an ultrametric space which is solid and spherically complete.

2.2.3. *Examples when* Γ *is not totally ordered.* Let I be a set with at least two elements, let $(X_i)_{i \in I}$ be a family of sets X_i, each one having at least two elements. Let $X = \prod_{i \in I} X_i$. Let $\mathcal{P}(I)$ be the set of all subsets of I, ordered by inclusion. And let $d\colon X \times X \to \mathcal{P}(I)$ be defined by $d(f, g) = \{i \in I \mid f_i \neq g_i\}$, where $f = (f_i)_{i \in I}$ and $g = (g_i)_{i \in I}$. Then $(X, d, \mathcal{P}(I))$ is a solid and spherically complete ultrametric space. If each $X_i = \{0, 1\}$, we obtain the ultrametric space $(\mathcal{P}(I), d, \mathcal{P}(I))$ with $d(A, B) = (A \cup B) \backslash (A \cap B)$ for all $A, B \subseteq I$.

2.2.4. *Other examples.* Let X be a topological space, let Y be a discrete topological space, let $C(X, Y)$ denote the set of continuous functions from X to Y and let $\mathcal{C}\ell(X)$ denote the set of clopen (i.e. closed and open) subsets of X. The mapping $d\colon \mathcal{C}(X, Y) \times \mathcal{C}(X, Y) \to \mathcal{C}\ell(X)$ is defined by $d(f, g) = \{x \in X \mid f(x) \neq g(x)\}$. Then $(C(X, Y), d, \mathcal{C}\ell(X))$ is a solid ultrametric space, and it is spherically complete if $\mathcal{C}\ell(X)$ is a complete sub-Boolean-algebra of $\mathcal{P}(X)$.

3. The Main Theorem

Let X be a non-empty set, let $\varphi\colon X \to X$ be a mapping. The element $x \in X$ is said to be a *fixed point* of φ when $\varphi x = x$.

Let X, X' be non-empty sets, let $\psi_1, \psi_2 \colon X \to X'$ be mappings. The element $\psi_1 x = \psi_2 x$ is said to be a *common point* of ψ_1, ψ_2.

We consider the set $X \neq \varnothing$, the ultrametric space (X', d', Γ') and the following mappings $\varphi \colon X \to X$, $\theta \colon X \to X'$ and $\psi \colon X \to X'$ such that $\theta\varphi(X) \subseteq \psi(X)$. This configuration is abbreviated as $\mathfrak{X} = \langle X, X, X', \varphi, \theta, \psi \rangle$.

For simplicity we shall write d instead of d'.

Two types of configurations will be often considered:

$$\mathfrak{X}_I = \langle X, X, X', \mathrm{id}, \theta, \psi \rangle$$

and

$$\mathfrak{X}_{II} = \langle X, X, X, \varphi, \mathrm{id}, \mathrm{id} \rangle.$$

We shall deal with the properties listed below.

(C) $d(\theta\varphi x, \theta\varphi y) \leq d(\psi x, \psi y)$ for all $x, y \in X$.

(SC) If $\psi x \neq \psi y$ then $d(\theta\varphi x, \theta\varphi y) < d(\psi x, \psi y)$.
If $\psi x = \psi y$ then $\theta\varphi x = \theta\varphi y$.

Clearly, if (SC) is satisfied, so is condition (C).

(SCO) If $x, y \in X$ are such that $\theta\varphi y = \psi x$ and $\theta\varphi y \neq \psi y$ then

$$d(\theta\varphi x, \psi x) < d(\theta\varphi y, \psi y).$$

It is clear that condition (SC) implies condition (SCO).

In configuration $\mathfrak{X}_{II}$ the above conditions become:

(C) $d(\varphi x, \varphi y) \leq d(x, y)$ for all $x, y \in X$.

(SC) If $x \neq y$ then $d(\varphi x, \varphi y) < d(x, y)$.

(SCO) If $\varphi x \neq x$ then $d(\varphi^2 x, \varphi x) < d(\varphi x, x)$.

We say: φ is a *contracting mapping* if it satisfies condition (C); φ is a *strictly contracting mapping* if φ satisfies condition (SC); φ is *strictly contracting on orbits* if φ satisfies condition (SCO).

For every $x \in X$ let $\pi_x = d(\theta\varphi x, \psi x)$ and let $B_x = B'_{\pi_x}(\psi x) \cap \psi(X)$. (Here B' is used to refer to the ultrametric space X'). If $\pi_x \neq 0$ then B_x is a ball of $\psi(X)$ containing $\theta\varphi x \in \psi(X)$. If $\pi_x = 0$ then $B_x = \{\psi x\}$. Similarly $B_{\varphi x} = B'(\theta\varphi^2 x, \psi\varphi x) \cap \psi(X)$.

In configuration $\mathfrak{X}_{II}$, if $x \in X$ then $\pi_x = d(\varphi x, x)$ and $B_x = B(\varphi x, x)$.

If $x \in X$ and $\pi_x \neq 0$, we say that the ball B_x is *stable* when the following condition is satisfied:

(St) If $x, y \in X$, $\pi_x \neq 0$ and $\psi y \in B_x$ then $B_y = B_x$.

The condition (St) may be written in the following equivalent form:

(St) If $x, y \in X$, $\pi_x \neq 0$ and $\psi y \in B_x$ then $\pi_y = \pi_x$.

The equivalence of the two forms of (St) follows from (2.1.3), because $\theta\varphi(X) \subseteq \psi(X)$, so B_x and B_y are principal balls of $\psi(X)$.

From the above definition, it follows at once that a stable ball does not contain any common point of ψ, $\theta\varphi$.

We also consider the following condition:

(D) If $x, y \in X$ and $\psi x \in B_{\varphi y}$ then $B_x \subseteq B_y$.

Lemma 1.

(1) *Assume that condition (D) is satisfied. If* $y \in X$ *then* $B_{\varphi y} \subseteq B_y$ *and in configuration* $\mathfrak{X}_{II}$, $\varphi(B_{\varphi y}) \subseteq B_y$.
(2) *Assume that condition (C) is satisfied. If* $x, y \in X$ *and* $\psi x \in B_y$ *then* $B_x \subseteq B_y$.
(3) *Assume that condition (C) is satisfied. For configurations* $\mathfrak{X}_I$ *and* $\mathfrak{X}_{II}$ *, if* $y \in X$ *then* $B_{\varphi y} \subseteq B_y$ *and condition (D) is satisfied.*
(4) *If condition (SCO) is satisfied, there is no element* $t \in X$ *such that* B_t *is a stable ball.*

Proof. (1): We have $\varphi y \in X$ and $\psi\varphi y \in B_{\varphi y}$. By (D) $B_{\varphi y} \subseteq B_y$. In configuration $\mathfrak{X}_{II}$ let $x \in B_{\varphi y}$, by (D) $B_x \subseteq B_y$. Then $\varphi x \in B_y$ and therefore $\varphi(B_{\varphi y}) \subseteq B_y$.

(2): Let $x, y \in X$ and assume that $\psi x \in B_y$. It follows from (C) that $d(\theta\varphi x, \theta\varphi y) \leq d(\psi x, \psi y) \leq d(\psi y, \theta\varphi y)$, hence $\pi_x = d(\theta\varphi x, \psi x) \leq d(\theta\varphi y, \psi y) = \pi_y$. Since $\psi x \in B_x \cap B_y$, by (2.1.3) $B_x \subseteq B_y$.

(3): For configuration $\mathfrak{X}_I$, we have $B_{\varphi y} = B_y$. For configuration $\mathfrak{X}_{II}$, $\psi\varphi y \in B_y$, hence by (1), $B_{\varphi y} \subseteq B_y$. For both configurations, if $\psi x \in B_{\varphi y}$, by (1), $B_x \subseteq B_{\varphi y} \subseteq B_y$, which proves condition (D).

(4): Let $y \in X$ and let B_y be a ball of $\psi(X)$, so $\theta\varphi y \neq \psi y$. There exists $x \in X$ such that $\theta\varphi y = \psi x$, so $\psi x \in B_y$. By (SCO) $\pi_x = d(\theta\varphi x, \psi x) < d(\theta\varphi y, \psi y) = \pi_y$, so B_y is not stable. □

We prove the following Main Theorem.

Theorem 1. *We assume that* $\psi(X)$ *is a principally complete subspace of* X'.

(1) *If condition (D) is satisfied, there exists a common point* $\theta\varphi x = \psi x$*, or there exists* $t \in X$ *such that* B_t *is a stable ball.*
(2) *If conditions (D) and (SCO) are satisfied, then there is no element* $t \in X$ *such that* B_t *is a stable ball, hence there exists a common point* $\theta\varphi x = \psi x$.
(3) *If condition (SC) is satisfied, there is no element* $t \in X$ *such that* B_t *is a stable ball and if* ψ *or* $\theta\varphi$ *is injective, there is at most one common point* $\theta\varphi x = \psi x$*. In configuration* $\mathfrak{X}_I$ *there is a common point* $\theta x = \psi x$*, in configuration* $\mathfrak{X}_{II}$ *there is a fixed point* $\varphi x = x$.

Proof. (1): We assume that $\psi x \neq \theta\varphi x$ for all $x \in X$, so B_x is a principal ball of $\psi(X)$, because $\theta\varphi x \in \psi(X)$.

The set $\mathcal{B}$ of principal balls $\mathcal{B} = \{B_z \mid z \in X\}$ is ordered by inclusion. Let $\mathbb{F}$ be the set of chains $\mathcal{C}$ in $\mathcal{B}$ satisfying the following condition:

if $B_z \in \mathcal{C}$ then $B_{\varphi z} \in \mathcal{C}$.

We note that for every $z \in X$ there is a chain $\mathcal{C}_z \in \mathbb{F}$ which is defined as follows:

$$\mathcal{C}_z : B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$$

where

$$B_0 = B_z, \quad B_1 = B_{\varphi z}\,, \quad B_2 = B_{\varphi^2 z}\,, \text{ etc... } .$$

Indeed, by (D) $\mathcal{C}_z$ is a chain of balls. So $\mathbb{F} \neq \varnothing$. The set $\mathbb{F}$ is ordered as follows: $\mathcal{C} \leq \mathcal{C}'$, when $\mathcal{C} \subseteq \mathcal{C}'$ (as subsets of $\mathcal{B}$). With this order, it is immediate that $\mathbb{F}$ is inductive. By Zorn's Lemma, $\mathbb{F}$ has a maximal element, denoted by $\mathcal{C}$. Since $\psi(X)$ is principally complete, there exists $y \in X$ such that $\psi y \in \bigcap \mathcal{C}$. We show that $B_y \subseteq B_x$ for every $B_x \in \mathcal{C}$. If $B_x \in \mathcal{C}$ then $B_{\varphi x} \in \mathcal{C}$, and by assumption $\psi y \in B_{\varphi x}$. By (D) $\psi y \in B_y \subseteq B_x$; so B_y is contained in each ball of $\mathcal{C}$. Hence $\mathcal{C} \cup \mathcal{C}_y$ is a chain which belongs to $\mathbb{F}$. By the maximality of $\mathcal{C}$, $\mathcal{C}_y \subseteq \mathcal{C}$, hence B_y, $B_{\varphi y} \in \mathcal{C}$. Therefore, B_y is the smallest ball in $\mathcal{C}$. By (D), $B_{\varphi y} \subseteq B_y$ and from $B_{\varphi y} \in \mathcal{C}$ then $B_{\varphi y} = B_y$. We show that B_y is a stable ball. Let $z \in X$ be such that $\psi z \in B_{\varphi y}$. By (D), $B_z \subseteq B_y$. Therefore the chain of balls $\mathcal{C} \cup \mathcal{C}_z$ belongs to $\mathbb{F}$. Since $\mathcal{C}$ is maximal, then $\mathcal{C}_z \subseteq \mathcal{C}$. In particular $B_z \in \mathcal{C}$ and necessarily $B_z = B_y$, because B_y is the smallest ball in $\mathcal{C}$. This proves that B_y is a stable ball.

(2): By Lemma 1 $\psi(X)$ does not contain any stable ball. By (1) there must exist $x \in X$ such that $\theta\varphi x = \psi x$.

(3): Now we assume that condition (SC) is satisfied. Hence both conditions (C) and (SCO) are satisfied. By Lemma 1 there is no element $t \in X$ such that B_t is a stable ball.

We observe that if $\theta\varphi$ is injective then ψ is injective. Indeed, if $x \neq y$ by (C) $0 < d(\theta\varphi x, \theta\varphi y) \leq d(\psi x, \psi y)$. So it suffices to assume that ψ is injective. Let $x \neq y$ be such that $\theta\varphi x = \psi x$ and $\theta\varphi y = \psi y$. From $\psi x \neq \psi y$ it follows by (SC) that $d(\theta\varphi x, \theta\varphi y) < d(\psi x, \psi y) = d(\theta\varphi x, \theta\varphi y)$, which is absurd. By Lemma 1, in both configurations $\mathfrak{X}_I$ and $\mathfrak{X}_{II}$, (C) implies condition (D). By (1) for $\mathfrak{X}_I$ there exists $x \in X$ such $\theta x = \psi x$, and for configuration $\mathfrak{X}_{II}$ there exists $x \in X$ such that $\psi\varphi x = \psi x$. □

The Original Common Point Theorem is a special case of the Main Theorem.

Theorem 2. *Let θ, ψ be mappings from X to X' such that $\theta(X) \subseteq \psi(X)$ and assume that $\psi(X)$ is principally complete.*

1. *If conditions (C) and (SCO) are satisfied, then θ and ψ have a common point $\theta x = \psi x$.*
2. *If condition (SC) is satisfied and θ or ψ is injective, then θ and ψ have a unique common point.*

Proof. We are in configuration $\mathfrak{X}_I$. By Lemma 1, (C) implies (D), hence the result follows at once from the Main Theorem. □

The Original Fixed Point Theorem combined with the Stability Theorem is also a special case of the Main Theorem.

Theorem 3. *Let $\varphi\colon X \to X$ be a mapping and assume that X is principally complete.*

(1) *If condition (C) is satisfied, then φ has a fixed point $x \in X$ or there exists $y \in X$ such that B_y is a stable ball.*
(2) *If conditions (C) and (SCO) are satisfied, then φ has a fixed point $x \in X$, and X does not contain any stable ball B_y (with $y \in X$).*
(3) *If condition (SC) is satisfied, then φ has a unique fixed point and there is no element $y \in X$ such that B_y is a stable ball.*

Proof. We are in configuration $\mathfrak{X}_{II}$. As seen in Lemma 1, (C) implies (D), so the theorem follows from the Main Theorem. □

The following corollary contains a statement which deserves to be called the *Local Fixed Point Theorem.*

Theorem 4. *Let X be an ultrametric space and let $\varphi\colon X \to X$.*

(1) *Assume that φ satisfies conditions (C) and (SCO).*
   (a) *If $z \in X$, $z \neq \varphi z$ and if B_z is pincipally complete, then B_z contains a fixed point $x = \varphi x$.*
   (b) *If X is principally complete, for every $z \in X$ such that $z \neq \varphi z$, B_z contains a fixed point of φ.*
(2) *Assume that φ satisfies the condition (SC) and that X is principally complete.*
   (a) *φ has a unique fixed point and it belongs to every ball B_z (with $z \neq \varphi z$).*
   (b) *If $z, t \in X$ and $d(z, \varphi z) = d(t, \varphi t) \neq 0$ then $B_z = B_t$.*

Proof. (1) (a): From condition (C) it follows by Lemma 1 that $\varphi(B_z) \subseteq B_z$. The restriction $\varphi|_{B_z}$ satisfies condition (C) and (SCO). Since B_z is principally complete, by the Fixed Point Theorem, there exists $x \in B_z$ such that $\varphi x = x$.

(b): By assumption, X is principally complete. By (2.1.7) each principal ball B_z is principally complete, so (b) follows from (a).

(2) (a): By the Fixed Point Theorem φ has a unique fixed point $x = \varphi x$. Since (SC) implies (C) and (SCO), by (1)(b), $x \in B_z$ for every z such that $z \neq \varphi z$.

(b): By (a) above, the balls B_z and B_t contain the unique fixed point x of φ. Since $d(t, \varphi t) = d(z, \varphi z) \neq 0$ then $B_t = B_z$, by (2.1.3). □

Special case when Γ is totally ordered

Lemma 2. *Let (X, d, Γ) be an ultrametric space with Γ totally ordered. The following conditions are equivalent:*

(a) X is principally complete.

(b) Every strictly contracting mapping $\varphi\colon X \to X$ has a fixed point.

Proof.

(a) $\Rightarrow$ (b): This was proved in Theorem 3.

(b) $\Rightarrow$ (a): We assume that X is not principally complete, so there exists a chain $\mathcal{C}$ of principal balls such that $\bigcap \mathcal{C} = \varnothing$. Hence $\mathcal{C}$ does not have a smallest ball and therefore the coinitial type λ of $\mathcal{C}$ is a limit ordinal. Then there exists a strictly decreasing family $(B_\iota)_{\iota<\lambda}$ of balls $B_\iota \in \mathcal{C}$ such that $\bigcap\limits_{\iota<\lambda} B_\iota = \bigcap \mathcal{C} = \varnothing$. We write $B_\iota = B_{\gamma_\iota}(a_\iota)$ and we define $\varphi\colon X \to X$. If $x \in X$ there exists the smallest $\kappa = \kappa(x) < \lambda$ such that $x \notin B_\kappa$; we define $\varphi x = a_\kappa$.

We show that φ is strictly contracting. Let $x, y \in X$, $x \neq y$. If $\kappa(x) = \kappa(y)$ then $0 = d(\varphi x, \varphi y) < d(x, y)$. If $\kappa(x) \neq \kappa(y)$, say $\kappa(x) < \kappa(y)$, from $B_{\kappa(x)} \supset B_{\kappa(y)}$ and $x \notin B_{\kappa(x)}$, $y \in B_{\kappa(x)}$ we get $d(x, y) > \gamma_{\kappa(x)} \geq d(\varphi x, \varphi y)$. So φ is strictly contracting.

From the definition of φ it is obvious that φ does not have a fixed point. $\square$

4. Attractors and Surjective Mappings

Let $\psi\colon X \to X'$ be a mapping between ultrametric spaces.

The mapping $\varphi\colon X \to X$ is called an *approximator* of $z' \in X'$ *along* ψ when the following condition is satisfied:

(At1) (i) If $\psi x = z'$ then $\varphi x = x$.

(ii) If $\psi x \neq z'$ then $d(\psi\varphi x, z') < d(\psi x, z')$.

We note that for the configuration $\mathfrak{X}_{II} = \langle X, X, X, \varphi, \mathrm{id}, \mathrm{id}\rangle$, if $t \in X$ then $B_t = B_{d(t,\varphi t)}(t)$ and the conditions (SCO) and (D) are expressed as follows:

(SCO) If $x \in X$ and $x \neq \varphi x$ then $d(\varphi^2 x, \varphi x) < d(\varphi x, x)$.

(D) If $x, z \in X$ and $x \in B_{\varphi z}$ then $B_x \subseteq B_z$.

We say that $z' \in X'$ is an *attractor of* X *along* ψ if there is an approximator $\varphi\colon X \to X$ of z' along ψ such that the conditions (SCO) and (D) are satisfied in the configuration $\mathfrak{X}_{II}$.

We remark that if $z' \in \psi(X)$ then z' is an attractor of X along ψ. Indeed, let $Z = \{z \in X \mid \psi z = z'\}$, so $Z \neq \varnothing$; let $z_0 \in Z$. Let $\varphi\colon X \to X$ be defined as follows. For every $z \in Z$, $\varphi z = z$, for every $x \in X \setminus Z$ let $\varphi x = z_0$. We observe that $\varphi^2 x = \varphi x$ for all $x \in X$. It is very easy to verify that φ satisfies the conditions (At1), (SCO) and (D), so z' is an attractor of X along ψ.

Now we prove the *Attractor Theorem*:

Theorem 5. *Let $\psi\colon X \to X'$ and assume that X is principally complete. If $z' \in X'$ is an attractor of X along ψ, then $z' \in \psi(X)$.*

Proof. Let $\varphi\colon X \to X$ be an approximator attached to the attractor z', such that the conditions (SCO) and (D) are satisfied by the configuration $\mathfrak{X}_{II} = \langle X, X, X, \varphi, \mathrm{id}, \mathrm{id}\rangle$. By the Main Theorem 1, there exists $x \in X$ such that $\varphi x = x$. By (At1), $z' = \psi x \in \psi(X)$. □

Corollary 1. *Let $\psi\colon X \to X'$ and assume that X is principally complete. If every $x' \in X'$ is an attractor of X along ψ, then ψ is surjective.*

Proof. The corollary follows at once from the Attractor Theorem. □

Corollary 2. *Let Y be a principally complete subspace of X. If $x \in X$ is an attractor of Y along the inclusion mapping $Y \to X$, then $x \in Y$.*

Proof. This is just a special case of the Attractor Theorem. □

Corollary 3. *Let $\psi : X \to X'$ and assume that z' is an attractor of X along ψ, with approximator $\varphi\colon X \to X$. If X is principally complete, for every $x \in X$ there exists $z \in B_x = B(x, \varphi x)$ such that $\psi z = z'$.*

Proof. Since z' is an attractor of X along ψ, with approximator φ, then the condition (D) holds in the configuration $\mathfrak{X}_{II}$. By Lemma 1, $B_{\varphi x} \subseteq B_x$. Then $(B_{\varphi^n x})_{n\geq 1}$ is a decreasing chain of subsets of B_x. If there exists $n \geq 1$ such that $\varphi^{n+1}x = \varphi^n x$ then $\psi(\varphi^n x) = z'$ with $z = \varphi^n x \in B_x$. If $\varphi^{n+1}x \neq \varphi^n x$ for every $n \geq 1$, then $B_{\varphi^n x}$ is a principal ball, hence a convex subset of X. Since X is principally complete, then $C = \bigcap\limits_{n\geq 1} B_{\varphi^n x} \neq \varnothing$ and C is convex, hence by 2.1.2, C is a principally complete subset of B_x. By Lemma 1 $\varphi(B_{\varphi^{n+1}x}) \subseteq B_{\varphi^n x}$, hence $\varphi(C) = \varphi(\bigcap\limits_{n\geq 1} B_{\varphi^n x}) = \varphi(\bigcap\limits_{n\geq 1} B_{\varphi^{n+1}x}) \subseteq \bigcap\limits_{n\geq 1} \varphi(B_{\varphi^{n+1}x}) \subseteq \bigcap\limits_{n\geq 1} B_{\varphi^n x} = C$.

Let φ^*, respectively ψ^*, be the restriction of φ, respectively ψ, to C. Then φ^* is an approximator of z' along ψ^*, and conditions (D) and (SCO) hold in the configuration $\langle C, C, C, \varphi^*, id, id\rangle$. So z' is an attractor of C along ψ^*. By Theorem 5 there exists $z \in C \subseteq B_x$ such that $\psi z = z'$. □

In the next lemma, we give sufficient conditions for $z' \in X'$ to be an attractor of X along ψ.

Lemma 3. *Let $\psi\colon X \to X'$ be a mapping, let $z' \in X'$ and assume that z' has an approximator $\varphi\colon X \to X$ along ψ satisfying the following conditions:*

(At2) $\psi(B(x, \varphi x)) \subseteq B'(\psi x, z')$.

(At3) If $x, t \in X$, if $d(\psi t, z') < d(\psi x, z')$ and $B(t, \varphi t) \cap B(x, \varphi x) \neq \varnothing$, then $d(t, \varphi t) < d(x, \varphi x)$.

Then z' is an attractor of X along ψ.

Proof. We show that conditions (SCO) and (D) in $\mathfrak{X}_{II}$ are satisfied.

Proof of (SCO): Let $\varphi x \neq x$. If $\varphi^2 x = \varphi x$ then $d(\varphi^2 x, \varphi x) = 0 < d(\varphi x, x)$. Let $\varphi^2 x \neq \varphi x$. Since $\varphi x \neq x$, by (At1) $d(\psi\varphi x, z') < d(\psi x, z')$. $B(\varphi x, \varphi^2 x)$ is a ball and $\varphi x \in B(\varphi x, \varphi^2 x) \cap B(x, \varphi x)$. By (At3) $d(\varphi^2 x, \varphi x) < d(\varphi x, x)$.
Proof of (D): Let $x \in B_{\varphi z}$. If $\varphi^2 z = \varphi z$ then $x = \varphi z$, so $\varphi x = \varphi^2 z = \varphi z = x$, hence $B_x = \{x\} = \{\varphi z\} \subseteq B_z$. Now let $\varphi^2 z \neq \varphi z$, so $\varphi z \neq z$. By (SCO) $d(\varphi^2 z, \varphi z) < d(\varphi z, z)$. From $\varphi z \in B(z, \varphi z) \cap B(\varphi z, \varphi^2 z)$, then by (2.1.3) $B(\varphi z, \varphi^2 z) \subseteq B(z, \varphi z)$. From $x \in B(\varphi z, \varphi^2 z)$, by (At2) $\psi x \in \psi(B(\varphi z, \varphi^2 z)) \subseteq B'(\psi\varphi z, z')$. Hence $d(\psi x, z') \leq d(\psi\varphi z, z') < d(\psi z, z')$, the latter inequality by (At1), because $\varphi z \neq z$. From $x \in B(x, \varphi x) \cap B(\varphi z, \varphi^2 z) \subseteq B(x, \varphi x) \cap B(z, \varphi z)$, by (At3) $d(x, \varphi x) < d(z, \varphi z)$, hence by (2.1.3) $B(x, \varphi x) \subseteq B(z, \varphi z)$. □

We remark that if $X = X'$, if ψ is the identity mapping, if condition (At1) is satisfied by $\varphi\colon X \to X$ for $z = z' \in X$, then condition (At2) is also satisfied. Indeed, if $x = z$ then by (At1) $\varphi x = x$, hence condition (At2) holds trivially. If $x \neq z$ then $d(\varphi x, z) < d(x, z)$, hence $d(x, \varphi x) \leq d(x, z)$ and this means that $B(x, \varphi x) \subseteq B(x, z)$.

Special case when Γ is totally ordered

Now we shall assume that Γ is totally ordered.

Lemma 4. *Let $\psi\colon X \to X'$, $z' \in X'$ and let $\varphi\colon X \to X$ be an approximator of z' along ψ satisfying the condition (At2). If Γ is totally ordered, then φ satisfies also condition (At3), hence z' is an attractor of ψ.*

Proof. We show that z' satisfies the condition (At3). Let $x, t \in X$ be such that $d(\psi t, z') < d(\psi x, z')$ and $B(t, \varphi t) \cap B(x, \varphi x) \neq \varnothing$. If $d(x, \varphi x) \leq d(t, \varphi t)$ then $B(x, \varphi x) \subseteq B(t, \varphi t)$ by (2.1.3). By (At2) $\psi x \in \psi(B(x, \varphi x)) \subseteq \psi(B(t, \varphi t)) \subseteq B'(\psi t, z')$, so $B'(\psi x, z') \subseteq B'(\psi t, z')$. Therefore by (2.1.3) $d(\psi x, z') \leq d(\psi t, z')$, which is a contradiction.

Since Γ is totally ordered, then $d(t, \varphi t) < d(x, \varphi x)$. This proves (At3). By Lemma 3 z' is an attractor of X along ψ. □

Corollary 4. *Let $\psi\colon X \to X'$, assume that Γ is totally ordered and X is principally complete. Assume also that every $x' \in X'$ has an approximator $\varphi_{x'}$ satisfying the condition (At2). Then $\psi(X) = X'$.*

Proof. By Lemma 4 each $x' \in X'$ is an attractor of X. By Corollary 1 each element $x' \in X'$ belongs to $\psi(X)$, so $\psi(X) = X'$. □

Corollary 5. *Let X be a subspace of X'. Assume that Γ is totally ordered and X is principally complete. Assume also that for every $x' \in X'$ there exists an approximator $\varphi_{x'}\colon X \to X$ along the inclusion mapping $X \to X'$. Then $X = X'$.*

Proof. It is easy to verify that condition (At2) is satisfied by every approximator along the inclusion mapping. By Lemma 4 each $x' \in X'$ is an attractor of X along the inclusion mapping ψ. By Corollary 2, ψ is surjective, that is $X = X'$. $\square$

5. Continuity of Mappings

The notion of continuity for mappings between ultrametric spaces may be conceived in different ways. We have chosen the definitions below. For each $x \in X$ we denote by $\mathcal{B}(x)$ the set of balls $B_\gamma(x)$. If there exists $\gamma \in \Gamma^\bullet$ such that $B_\gamma(x) = \{x\}$ we say that x is an *isolated point* of X.

We say that $\psi\colon X \to X'$ is *continuous* at x when for every principal ball $B'(\psi x, z')$ there exists a ball $B \in \mathcal{B}(x)$ such that $\psi(B) \subseteq B'(\psi x, z')$.

Clearly, if x is an isolated point of X, then ψ is continuous at x.

If ψ is continuous at every $x \in X$, we say that ψ is a *continuous mapping.*

We introduce the more useful related concept: the mapping $\psi\colon X \to X'$ is said to be *tightly continuous at* x when the following condition holds:

For every principal ball $P' = B'(\psi x, z')$ there exists $B \in \mathcal{B}(x)$ satisfying:

i) $\psi(B) \subseteq P'$ and

ii) for every $y' \in P'$ there exists $y \in B$ such that $d(\psi y, y') < d(\psi x, z')$.

Thus if ψ is tightly continuous at x then ψ is continuous at x.

We say that $\psi\colon X \to X'$ is *tightly continous* if ψ is tightly continuous at every $x \in X$; then ψ is continuous.

Lemma 5. *Let $\psi\colon X \to X'$ be a tightly continuous mapping. Then every $z' \in X'$ has an approximator $\varphi_{z'}\colon X \to X$ satisfying the condition (At2) $\psi(B(x, \varphi_{z'} x)) \subseteq B'(\psi x, z')$ for all $x \in X$.*

Proof. Let $z' \in X'$, we shall define a mapping $\varphi_{z'}\colon X \to X$. Let $x \in X$, if $\psi x = z'$, we define $\varphi_{z'} x = x$. Now let $\psi x \neq z'$. By assumption there exists $B \in \mathcal{B}(x)$ such that $\psi(B) \subseteq B'(\psi x, z')$, and moreover there exists $\bar{x} \in B$ such that $d(\psi\bar{x}, z') < d(\psi x, z')$. We choose an element $\bar{x}$ with the above property and define $\varphi_{z'} x = \bar{x}$. Thus $d(\psi\varphi_{z'} x, z') < d(\psi x, z')$, so $\varphi_{z'}$ is an approximator.

Proof of (At2): From $\varphi_{z'} x \in B$ it follows that $B(x, \varphi_{z'} x) \subseteq B$. Hence if $t \in B(x, \varphi_{z'} x)$, then $\psi t \in \psi(B) \subseteq B'(\psi x, z')$. $\square$

Special case when Γ is totally ordered

Now we consider the special case when Γ is totally ordered.

Theorem 6. *Let $\psi\colon X \to X'$ be tightly continuous and assume that Γ is totally ordered.*

(1) *Every $z' \in X'$ is an attractor.*
(2) *If X is principally complete, then ψ is surjective.*

Proof. By Lemma 5, every $z' \in X'$ has an approximator satisfying condition (At2); so by Lemma 4, since Γ is totally ordered, z' is an attractor. If moreover X is principally complete, by Corollary 4 $\psi(X) = X'$. □

6. Transfer of Principal Completeness

Let $\psi\colon X \to X'$. Our main result indicates a sufficient condition on ψ which insures that if X is principally complete then X' is also principally complete.

Let $\psi\colon X \to X'$. We say that X' *attracts* X *along* ψ when every $z' \in X'$ has an approximator $\varphi_{z'}\colon X \to X$ and the following conditions are satisfied for all $x, t \in X$ and $z', y' \in X'$.

(At2) $\psi(B(x, \varphi_{z'}\, x)) \subseteq B'(\psi x, z')$.

(At4) If $d(\psi t, y') < d(\psi x, z')$ and $B(t, \varphi_{y'}\, t) \cap B(x, \varphi_{z'}\, x) \neq \varnothing$ then $d(t, \varphi_{y'}\, t) < d(x, \varphi_{z'}\, x)$.

Taking $y' = z'$ in (At4) we see that condition (At3) is satisfied. So every $z' \in X'$ is an attractor.

Theorem 7. *Let $\psi\colon X \to X'$. If X' attracts X along ψ and X is principally complete, then $\psi(X) = X'$ and X' is principally complete.*

Proof. By Corollary 1 $\psi(X) = X'$ because every element of X' is an attractor of X.

To prove that X' is principally complete, by (2.1.6) it suffices to show that if λ is a limit ordinal and $(B'_\iota)_{\iota<\lambda}$ is a strictly decreasing family of principal balls of X, then $\bigcap\limits_{\iota<\lambda} B'_\iota \neq \varnothing$. For this purpose it suffices to show that for every $\iota < \lambda$ there exists a principal ball B_ι of X such that $\psi(B_\iota) \subseteq B'_\iota$ and that $(B_\iota)_{\iota<\lambda}$ is a decreasing family of balls of X. If this is shown, by assumption there exists $y \in \bigcap\limits_{\iota<\lambda} B_\iota$. Hence $\psi y \in \psi\big(\bigcap\limits_{\iota<\lambda} B_\iota\big) \subseteq \bigcap\limits_{\iota<\lambda} \psi(B_\iota) \subseteq \bigcap\limits_{\iota<\lambda} B'_\iota$ and the proof would be complete.

Let $B'_\iota = B'(x'_\iota, \bar{z}'_\iota)$ and $B'_{\iota+1} = B'(x'_{\iota+1}, \bar{z}'_{\iota+1})$. Since $B'_\iota \supset B'_{\iota+1}$ then $x'_\iota \notin B'_{\iota+1}$ or $\bar{z}'_\iota \notin B'_{\iota+1}$. By exchanging notation, if necessary, we may assume that $x'_\iota \notin B'_{\iota+1}$, hence $x'_\iota \neq x'_{\iota+1}$.

If $d(x'_{\iota+1}, x'_\iota) < d(x'_\iota, \bar{z}'_\iota)$ let $z'_\iota = \bar{z}'_\iota$. If $d(x'_{\iota+1}, x'_\iota) = d(x'_\iota, \bar{z}'_\iota)$ let $z'_\iota = x'_{\iota+1}$. In both cases $B'_\iota = B'(x'_\iota, z'_\iota)$.

The following fact will be crucial in the proof.

($\star$) Let $x', x'_*, z', z'_* \in X'$ be such that $x' \neq x'_*, x' \neq z', x'_* \neq z'_*$ and if $d(x', x'_*) = d(x', z')$ then $x'_* = z'$. Let $B' = B'(x', z')$, $B'_* = B'(x'_*, z'_*)$ and assume that $B'_* \subset B'$. Let $x \in X$ be such that $\psi x = x'$ and let $B = B(x, \varphi_{z'} x)$. Then we have:

(i) There exists $x_* \in B$ such that $\psi x_* = x'_*$.

(ii) $B_* = B(x_*, \varphi_{z'_*} x_*) \subseteq B$.

(iii) $\psi(B) \subseteq B'$ and $\psi(B_*) \subseteq B'_*$.

Proof of ($\star$).(i): By Corollary 3 there exists $x_* \in B(x, \varphi_{x'_*} x)$ such that $\psi x_* = x'_*$. From $x'_* \in B'_* \subset B'$ then $d(x', x'_*) \leq d(x', z')$. If $d(x', x'_*) = d(x', z')$ then $x'_* = z'$, so $x_* \in B(x, \varphi_{z'} x) = B$. If $d(x', x'_*) < d(x', z')$ from $x \in B(x, \varphi_{x'_*} x) \cap B$, by (At4) $d(x, \varphi_{x'_*} x) < d(x, \varphi_{z'} x)$. Hence $x_* \in B(x, \varphi_{x'_*} x) \subset B$.

(ii): We have $d(x'_*, z'_*) < d(x', z')$ and $x_* \in B_* \cap B$. By (At4), $d(x_*, \varphi_{z'_*} x_*) < d(x, \varphi_{z'} x)$. Hence $B_* \subseteq B$.

(iii): By (At2) $\psi(B) \subseteq B'$ and $\psi(B_*) \subseteq B'_*$.

This concludes the proof of ($\star$).

Let $\mathbb{F}$ denote the set of decreasing chains $\mathcal{C} = (B_\iota)_{\iota<\kappa}$ of balls $B_\iota = B(x_\iota, \varphi_{z'_\iota} x_\iota)$ such that $\psi x_\iota = x'_\iota$ for all $\iota < \kappa$ and $\kappa \leq \lambda$. $\mathbb{F}$ is not empty: Indeed, $B'_0 = B'(x'_0, z'_0)$ is a ball in X'. Since x'_0 is an attractor of X along ψ, there exists an approximator $\varphi_{x'_0}$, and from X principally complete, it follows by the Attractor Theorem that there exists $x_0 \in X$ such that $\psi x_0 = x'_0$. Since $x'_0 \neq z'_0$ then $x_0 \neq \varphi_{z'_0} x_0$. Let $B_0 = B(x_0, \varphi_{z'_0} x_0)$. The family with $\kappa = 1$, consisting only of B_0, belongs to $\mathbb{F}$.

The set $\mathbb{F}$ is ordered as follows. Let $\mathcal{C} = (B_\iota)_{\iota<\kappa}$ with $B_\iota = B(x_\iota, \varphi_{z'_\iota} x_\iota)$ for all $\iota < \kappa$. Let $\widetilde{\mathcal{C}} = (\widetilde{B}_\iota)_{\iota<\widetilde{\kappa}}$ with $\widetilde{B}_\iota = B(\widetilde{x}_\iota, \varphi_{z'_\iota} \widetilde{x}_\iota)$ for all $\iota < \widetilde{\kappa}$. We define $\mathcal{C} \leq \widetilde{\mathcal{C}}$ when $\kappa \leq \widetilde{\kappa}$ and for every $\iota < \kappa$ we have $x_\iota = \widetilde{x}_\iota$. With this order, $\mathbb{F}$ is inductive and by Zorn's Lemma there exists a maximal $\mathcal{C} = (B_\iota)_{\iota<\kappa}$ in $\mathbb{F}$. We need to show that $\kappa = \lambda$ and we assume that $\kappa < \lambda$ to derive a contradiction.

Case 1: $\kappa = \mu + 1$.

Let $B'_\mu = B'(x'_\mu, z'_\mu)$, $B'_\kappa = B'(x'_\kappa, z'_\kappa)$, $B_\mu = B(x_\mu, \varphi_{z'_\mu} x_\mu)$ where $\psi\, x_\mu = x'_\mu$. By ($\star$) there exists $x_\kappa \in B_\mu$ such that $\psi x_\kappa = x'_\kappa$ and $B_\kappa = B(x_\kappa, \varphi_{z'_\kappa} x_\kappa) \subseteq B_\mu$. Let $\widetilde{\mathcal{C}} = (\widetilde{B}_\iota)_{\iota<\kappa+1}$ where $\widetilde{B}_\iota = B_\iota$ for all $\iota < \kappa$ and $\widetilde{B}_\kappa = B_\kappa$ as defined above. Then $\mathcal{C} < \widetilde{\mathcal{C}}$. But this is contrary to the maximality of $\mathcal{C}$.

Case 2: κ is a limit ordinal.

Since X is principally complete, there exists $y \in \bigcap \mathcal{C} = \bigcap_{\iota<\kappa} B_\iota$. Hence $\psi y \in \bigcap_{\iota<\kappa} \psi(B_\iota) \subseteq \bigcap_{\iota<\kappa} B'_\iota$. Thus for every $\iota < \kappa$ we have $\psi y, x'_\kappa \in B'_{\iota+1}$, hence $d(x'_\kappa, \psi y) \leq d(x'_{\iota+1}, z'_{\iota+1}) < d(x'_\iota, z'_\iota)$. By the assumption $\kappa < \lambda$ we have the ball $B'_\kappa = B'(x'_\kappa, z'_\kappa)$. We shall define $x_\kappa \in X$. If $\psi y = x'_\kappa$ let $x_\kappa = y$. If $\psi y \neq x'_\kappa$ then for every $\iota < \kappa$ we have $y \in B(y, \varphi_{x'_\kappa} y) \cap B_\iota$, hence by (At4) $d(y, \varphi_{x'_\kappa} y) < d(x_\iota, \varphi_{z'_\iota} x_\iota)$, so $B(y, \varphi_{x'_\kappa} y) \subseteq B_\iota$. By Corollary 3 there exists $x_\kappa \in B(y, \varphi_{x'_\kappa} y)$ such that $\psi x_\kappa = x'_\kappa$. So $x_\kappa \in B_\iota$ for all $\iota < \kappa$. We have $x_\kappa \neq \varphi_{z'_\kappa} x_\kappa$ because $\psi x_\kappa = x'_\kappa \neq z'_\kappa$. Let $B_\kappa = B(x_\kappa, \varphi_{z'_\kappa} x_\kappa)$, by (At2) $\psi(B_\kappa) \subseteq B'_\kappa$. Now we show that $B_\kappa \subseteq B_\iota$ for every $\iota < \kappa$. Indeed $d(\psi x_\kappa, z'_\kappa) < d(\psi x_\iota, z'_\iota)$ and $x_\kappa \in B_\kappa \cap B_\iota$, by (At4) $d(x_\kappa, \varphi_{z'_\kappa} x_\kappa) < d(x_\iota, \varphi_{z'_\iota} x_\iota)$, hence $B_\kappa \subseteq B_\iota$ for all $\iota < \kappa$. Let $\widetilde{B}_\iota = B_\iota$ for all $\iota < \kappa$ and $\widetilde{B}_\kappa = B_\kappa$. Then $\widetilde{\mathcal{C}} = (\widetilde{B}_\iota)_{\iota<\kappa+1} \in \mathbb{F}$ and $\mathcal{C} < \widetilde{\mathcal{C}}$, which is contrary to the maximality of $\mathcal{C}$.

We conclude that $\kappa < \lambda$ is impossible. This suffices to prove the theorem. □

Since a solid and principally complete ultrametric space is spherically complete, we deduce from Theorem 7 that if X is principally complete, if X' is solid, if $\psi\colon X \to X'$ and X' attracts X, then X' is spherically complete.

Special case when Γ is totally ordered

Corollary 6. *Let $\psi\colon X \to X'$, assume that Γ is totally ordered and that X is principally complete. Assume also that every $z' \in X'$ has an approximator $\varphi_{z'}$ such that condition (At2) is satisfied for every $x \in X$. Then:*

(1) *X' attracts X along ψ.*
(2) *$\psi(X) = X'$ and X' is principally complete.*

Proof. (1): Since Γ is totally ordered and z' has an approximator $\varphi_{z'}$ satisfying (At2), by Lemma 4 $\varphi_{z'}$ also satisfies (At3), so z' is an attractor of X along ψ. We need to show that condition (At4) is satisfied. Let $t, x \in X$, $y', z' \in X'$ and assume that $d(\psi t, y') < d(\psi x, z')$ and $B(t, \varphi_{y'}\, t) \cap B(x, \varphi_{z'}\, x) \neq \varnothing$. We note that $\psi x \neq z'$, so $x \neq \varphi_{z'}\, x$. To obtain a contradiction, we assume that $d(x, \varphi_{z'}\, x) \leq d(t, \varphi_{y'}\, t)$, then $B(x, \varphi_{z'}\, x) \subseteq B(t, \varphi_{y'}\, t)$. By (At2), $\psi x \in \psi(B(x, \varphi_{z'}\, x)) \subseteq \psi(B(t, \varphi_{y'}\, t)) \subseteq B'(\psi t, y')$. By Corollary 3 there exists $z \in B(x, \varphi_{z'}\, x)$ such that $z' = \psi z \in \psi(B(x, \varphi_{z'}\, x)) \subseteq \psi(B(t, \varphi_{y'}\, t)) \subseteq B'(\psi t, y')$. It follows that $d(\psi x, z') \leq d(\psi t, y')$ and this is a contradiction. So condition (At4) is satisfied, that is, X' attracts X along ψ.

(2): By Theorem 7 $\psi(X) = X'$ and X' is principally complete. □

7. Notes

Part 2 of the Main Theorem, for the special case of a fixed point and for spherically complete ultrametric spaces, was proved in [14]. A special case of the Common Point Theorem was proved in [11]. Lemma 2 was proved in [8].

Part 1 of the combined Fixed Point and Stability Theorem was proved for spherically complete spaces in [10]. The notion of an attractor in this paper, which is weaker than in [14], allows to show that every element of the image $\psi(X)$ is an attractor of X along ψ. Theorem 6 was proved by Kuhlmann [3] under the additional assumption that the sets of values of the distance of X and X' are totally ordered. In the general form, the content of §6 has not appeared earlier in the literature. However, when Γ and Γ' are totally ordered the result of Theorem 7 was already proved by Kuhlmann and circulated in preprints, dated 1997, 1999, 2002, now accepted for publication [4].

References

[1] Hitzler, P. and Seda, A., Multivalued mappings, fixed point theorems and disjunctive databases, in: Third Irish Workshop on Formal Methods in Computing, British Computer Society, 1999, 18 pages.

[2] Hitzler, P. and Seda, A., The fixed-point theorems of Priess-Crampe and Ribenboim in logic programming. Valuation theory and its applications, Vol.I (Saskatoon,SK,1999), 219-235, Fields Inst. Commun. 32, AMS, Providence, RI,2002.

[3] Kuhlmann, F.-V., A theorem about mappings on spherically complete ultrametric spaces with applications. Séminaire sur les Structures Ordonnées, Paris, 1999, nº 72.

[4] Kuhlmann, F.-V., Maps on ultrametric spaces, Hensel's Lemma, and differential equations over valued fields, accepted for publication in Comm. Alg..

[5] Monna, A.F., *Analyse Non-Archimédienne*, 1970, Springer-Verlag, Berlin-Heidelberg.

[6] Narkiewicz, W., *Polynomial mappings.* Lecture Notes in Mathematics, #1600, 1995, Springer-Verlag, New York.

[7] Ochsenius, H., Schikkof, W., Norm Hilbert spaces over Krull valued fields, Indag. Math., N.S. 17 (2006), 65-84.

[8] Priess-Crampe, S., Der Banachsche Fixpunktsatz für ultrametrische Räume, Result. Math. 18 (1990), 178-186.

[9] Priess-Crampe, S., Remarks on some theorems of functional analysis, Contemporary Mathematics 384 (2005), 235-246.

[10] Priess-Crampe, S., Fixed points and stable balls in ultrametric spaces, Result. Math. 43 (2003), 163-167.

[11] Priess-Crampe, S. and Ribenboim, P., The common point theorem for ultrametric spaces, Geom. Ded. 72 (1998), 105-110.

[12] Priess-Crampe, S. and Ribenboim, P., Logic programming and ultrametric spaces, Rend. Mat. Roma, ser. VII, 10 (1999), 155-176.

[13] Priess-Crampe, S. and Ribenboim, P., Ultrametric spaces and logic programming, J. Logic Progr. 42 (2000), 59-70.

[14] Priess-Crampe, S.and Ribenboim, P., Fixed point and attractor theorems for ultrametric spaces, Forum Math. 12 (2000), 53-64.

[15] Priess-Crampe, S. and Ribenboim, P., Systems of differential equations over valued fields, Contemp. Math. 319 (2003), 299-318.

[16] Priess-Crampe, S. and Ribenboim, P., Differential equations over valued fields (and more), J. Reine Angew. Math. 576 (2004), 123-147.

[17] Ribenboim, P., Fermat's equation for matrices or quaternions over q-adic fields, Acta Arith. 113 (2004), 241-250.

[18] Rosenlicht, M., Hardy fields, J. Math. Anal. Appl. 93 (1983), 297–311.

[19] Rosenlicht, M., The rank of a Hardy field, Trans. A.M.S. 280 (1983), 659-671.

[20] Schörner, E., Zur Konstruktion der Hahn-Ternärkörper als Ternärkörper formaler Potenzreihen, Geom. Dedicata 80 (2000), 157-171.

[21] Silverman, J.H., *The Arithmetic of Dynamical Systems*, 2007, Springer-Verlag, New York.

[22] van der Hoeven, J., *Transseries and Real Differential Algebra.* Lecture Notes in Mathematics, #1888, 2006, Springer-Verlag, Berlin.

[23] van Rooij, A.C.M., *Non-Archimedean Functional Analysis*, 1978, Marcel Dekker, New York.

Mathematisches Institut der Universität München, Theresienstr.39, 80333 München
E-mail address: priess@math.lmu.de

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L3N6,Canada
E-mail address: mathstat@mast.queensu.ca