MULTIPLICATIVE VALUED DIFFERENCE FIELDS

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ABSTRACT. The theory of valued difference fields (K, σ, v) depends on how the valuation v interacts with the automorphism σ. Two special cases have already been worked out - the *isometric* case, where $v(\sigma(x)) = v(x)$ for all $x \in K$, has been worked out by Luc Belair, Angus Macintyre and Thomas Scanlon; and the *contractive* case, where $v(\sigma(x)) > nv(x)$ for all $x \in K^{\times}$ with $v(x) > 0$ and $n \in \mathbb{N}$, has been worked out by Salih Azgin. In this paper we deal with a more general version, the *multiplicative* case, where $v(\sigma(x)) = \rho \cdot v(x)$, where ρ (> 0) is interpreted as an element of a real-closed field. We give an axiomatization and prove a relative quantifier elimination theorem for this theory.

1. Introduction

A *valued field* is a structure $\mathcal{K} = (K, \Gamma, k; v, \pi)$, where K is the underlying field, Γ is an ordered abelian group (called the *value group*), and k is a field; $v : K \to \Gamma \cup \{\infty\}$ is the (surjective) *valuation map*, with the *valuation ring* (also called the *ring of integers*) given by $\mathcal{O}_K := \{a \in K : v(a) \geq 0\}$, with a unique maximal ideal given by $\mathfrak{m}_K := \{a \in K : v(a) > 0\}$; and $\pi : \mathcal{O}_K \to k$ is a surjective ring morphism. Then π induces an isomorphism of fields

$$a + \mathfrak{m}_K \mapsto \pi(a) : \mathcal{O}_K/\mathfrak{m}_K \to k,$$

and we identify the residue field $\mathcal{O}_K/\mathfrak{m}_K$ with k via this isomorphism. Accordingly k is called the *residue field*. When K is clear from the context, we denote $\mathcal{O}_K$ and $\mathfrak{m}_K$ by $\mathcal{O}$ and $\mathfrak{m}$ respectively.

A *valued difference field* is a valued field $\mathcal{K}$ as above with a distinguished automorphism (denoted by σ) of the base field K, which also satisfies $\sigma(\mathcal{O}_K) = \mathcal{O}_K$. It then follows that σ induces an automorphism of the residue field:

$$\pi(a) \mapsto \pi(\sigma(a)) : k \to k, \quad a \in \mathcal{O}_K.$$

We denote this automorphism by $\bar{\sigma}$; and k equipped with $\bar{\sigma}$ is called the *residue difference field* of $\mathcal{K}$. Likewise, σ induces an automorphism of the value group as well:

$$\gamma \mapsto \sigma(\gamma) := v(\sigma(a)), \quad \text{where } \gamma = v(a).$$

We denote this automorphism also by σ, and construe the value group as an ordered abelian group equipped with this special automorphism. Such a structure is called a *valued difference group*.

Depending on how the automorphism interacts with the valuation, we get different structures and hence different theories. For example, σ is called *isometric* if $v(\sigma(x)) = v(x)$ for all $x \in K$; and is called *contractive* if $v(\sigma(x)) > nv(x)$ for all $x \in K^{\times}$ with $v(x) > 0$, and all $n \in \mathbb{N}$. The existence of model companions of both these theories have been worked out in detail [1], [2], [3], [4]. A recent work by Françoise Point [11] on valued ordered difference fields and rings is also somewhat related, although in her case the valued field itself is ordered, which, at least on the face of it, makes this theory quite different from the others mentioned.

No matter how the automorphism interacts with the valuation, if we want any hope of having a model companion of the theory of a valued difference field, we better have a model companion of the theory of the valued difference group at least. Unfortunately, by Kikyo and Shelah's theorem [5], the theory of a structure with the strict order property (e.g., an ordered abelian group) and a distinguished automorphism does not have a model companion. So we need to put some restriction on the automorphism. In the isometric case, σ induces the identity automorphism on the value group; and so in this case, the value group has just the structure of an ordered abelian group, whose model companion is the theory of the ordered divisible abelian groups ($ODAG$). However, in the case when the induced automorphism is not the identity, the model companion (if it exists) should be able to decide how to extend the order between linear difference operators. In particular, for any $L(\gamma) = \sum_{i=0}^{n} a_i \sigma^i(\gamma)$, where $a_i \in \mathbb{Z}$, $a_n \neq 0$ and $\gamma > 0$, the model companion should be able to decide when $L(\gamma) > 0$. In the contractive case, it is easily decided by the following rule:

$$L(\gamma) > 0 \iff a_n > 0.$$

However, in more general cases, the decision criteria are not so simple. For example, it is not known whether the theory of an ordered abelian group Γ with a strictly increasing automorphism ($\sigma(\gamma) > \gamma$ for all $0 < \gamma \in \Gamma$) has a model companion. So we restrict ourselves to a more specific case, where we impose that the induced automorphism σ on the value group should satisfy the following axiom (scheme): for each $a_0, \ldots, a_n \in \mathbb{Z}$ and $L(\gamma) = \sum_{i=0}^{n} a_i \sigma^i(\gamma)$,

$$\Big(\forall \gamma > 0 (L(\gamma) > 0)\Big) \bigvee \Big(\forall \gamma > 0 (L(\gamma) = 0)\Big) \bigvee \Big(\forall \gamma > 0 (L(\gamma) < 0)\Big).$$

This axiom, called Axiom OM (short for *Ordered Module*), induces a quasi-order on the ring $\mathbb{Z}[\sigma]$. Equivalently, as shown in Section 2, we can also represent σ as

$$\sigma(\gamma) = \rho \cdot \gamma$$

for all $\gamma \in \Gamma$, where $\rho > 0$ is interpreted as an element of a real-closed field. For example, $\rho = 2$, or $\rho = \frac{5}{3}$, or $\rho = \sqrt{2}$, or $\rho = \pi$, or $\rho = 3 + \delta$ where δ is an infinitesimal, etc. $\mathbb{Z}[\rho]$ then turns out to be an ordered ring, and Γ is construed as an ordered module over that ordered ring. We call such a Γ a *multiplicative ordered difference abelian group* (henceforth, MODAG). We will show in Section 2 that the theory of such a Γ has a model companion, the theory of divisible multiplicative ordered difference abelian group (henceforth, div-MODAG).

In this paper we are thus interested in dealing with this more general case. We call σ *multiplicative* if σ induces the structure of a MODAG on Γ via the rule

$$v(\sigma(x)) = \rho \cdot v(x) \quad \text{for all } x \in K,$$

where $\rho > 0$ (as interpreted in an ordered ring). The induced automorphism σ on the value group Γ then satisfies $\sigma(\gamma) = \rho \cdot \gamma$ for all $\gamma \in \Gamma$.

Three quick points should be noted here. First, we construe a MODAG Γ as an ordered $\mathbb{Z}[\sigma, \sigma^{-1}]$-module. To be able to extend Γ to a model of div-MODAG, we would then want divisibility by "non-zero" linear difference operators, which typically look like $L = \sum_{l=1}^{n} a_l \sigma^l + \sum_{l=1}^{m} b_l \sigma^{-l}$, with $a_n \neq 0$ or $b_m \neq 0$. Any question of solvability of a system $L(x) = b$ for $b \in \Gamma$, can then easily be transformed to a question involving only σ, by iterating the equation throughout by σ^m. Thus, $(\sum_{l=1}^{n} a_l\sigma^l + \sum_{l=1}^{m} b_l\sigma^{-l})(x) = b$ is solvable if and only if $(\sum_{l=1}^{n} a_l\sigma^{m+l} + \sum_{l=1}^{m} b_l\sigma^{m-l})(x) = \sigma^m(b)$ is solvable. In particular, for all practical purposes we can think of Γ as a $\mathbb{Z}[\sigma]$-module, with the understanding that σ has an inverse.

Secondly, if $\sigma(\gamma) = \rho \cdot \gamma$, then $\sigma^{-1}(\gamma) = \rho^{-1} \cdot \gamma$. In particular, if $0 < \rho \leq 1$, we can shift to σ^{-1}, and instead work with $\rho^{-1} \geq 1$. Thus, without loss of generality, we may assume that $\rho \geq 1$.

And finally, this is a generalization over the isometric and the contractive cases. The case $\rho = 1$ is precisely the isometric case; and the case "$\rho = \infty$", i.e., when all $0 < \gamma \in \Gamma$ satisfy for all $b \in \mathbb{Z}_+$, $\rho \cdot \gamma > b\gamma$, is the contractive case. We can have other finite and infinitesimal values for ρ as well.

Also one more thing needs mention here about the characteristics of the relevant fields. Any automorphism of a field is trivial on the integers. Thus for any $n \in \mathbb{Z}$, we have $\sigma(n) = n$. In particular, this means that for any prime p, if $v(p) > 0$, then $v(p) = v(\sigma(p)) = \rho \cdot v(p)$, which implies $\rho = 1$. Thus the mixed characteristic case does not arise for $\rho > 1$, and the mixed characteristic case for $\rho = 1$ has already been dealt with in [2]. The equi-characteristic p case even without the automorphism is already an enormously difficult problem. So we restrict ourselves only to the equi-characteristic zero case in this paper.

Section 2 deals with the value group and shows that, under additional restriction, the theory of an ordered abelian group with an automorphism has a model companion. Section 3 gives a few basic preliminaries about difference algebra. Section 4 shows that in case ρ is transcendental, difference polynomials satisfy the properties of pseudocontinuity and pseudoconvergence trivially, whereas if ρ is algebraic over the integers, one might have to shift to an equivalent sequence to restore these properties. Section 5 introduces the σ-hensel configuration and gives some properties of σ-henselian valued difference fields. Section 6 shows that for a multiplicative valued difference field $\mathcal{K}$ with linear difference closed residue field, all (σ-algebraically) maximal extensions of $\mathcal{K}$ are unique upto isomorphism over $\mathcal{K}$. Section 7 gives a canonical example of a multiplicative valued difference field and also shows by a counter-example that the assumption of linear difference closed residue field is necessary to get unique maximal extensions. Section 8 shows how to extend isomorphisms of valued difference fields by extending the value groups and the residue fields. Section 9 proves the main theorem (Theorem 9.4) of this paper, which is basically the back-and-forth method for extending partial isomorphisms. Section 10 lists the main consequences of this theorem, namely that the theory of σ-henselian multiplicative valued difference fields admits relative completeness and relative quantifier elimination, relative to the residue-valuation structures (RVs). Finally, section 11 shows that in the presence of a cross-section (in the language), the elementary theory of σ-henselian multiplicative valued difference fields is essentially controlled by the theories of the value groups and the residue fields.

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2. Multiplicative Ordered Difference Abelian Group (MODAG)

We work in the language of ordered groups with a symbol for the automorphism $\mathcal{L}_{OG,\sigma} = \{+, -, 0, <, \sigma\}$.

The $\mathcal{L}_{OG,\sigma}$-theory T_σ of ordered difference abelian groups is axiomatized by the following axioms:

(1) Axioms of Abelian Groups in the language $\{+, -, 0\}$
(2) Axioms of Linear Order in the language $\{<\}$
(3) Axiom about interaction
  - $\forall x \forall y \forall z (x < y \rightarrow x + z < y + z)$
(4) Axioms asserting σ is an $\mathcal{L}$-automorphism, where $\mathcal{L} = \{+, -, 0, <\}$.
  - $\sigma(0) = 0$
  - $\forall x \forall y (\sigma(x + y) = \sigma(x) + \sigma(y))$
  - $\forall x (\sigma(-x) = -\sigma(x))$
  - $\forall x \forall y (x < y \rightarrow \sigma(x) < \sigma(y))$
  - $\forall x \exists y (\sigma(y) = x)$

Remark 2.1. Note that T_σ is an $\forall\exists$-theory in the language $\mathcal{L}_{OG,\sigma}$: the existence of inverse ($\forall x \exists y (\sigma(y) = x)$) is the only $\forall\exists$-axiom. The universal theory of T_σ, $(T_\sigma)_\forall$, is the theory of ordered abelian groups with an injective endomorphism.

Unfortunately the theory of ordered abelian groups has the strict order property. As before, by Kikyo and Shelah's theorem [5], we cannot hope to have a model companion of the theory of ordered difference abelian groups.

However, if we restrict ourselves to a very specific kind of automorphisms, we do actually get a model companion. Each of the intended automorphisms is multiplication by an element of a real-closed field. For example, $\sigma(x) = 2x$, or $\sigma(x) = \sqrt{2}x$, or $\sigma(x) = \delta x$, where δ could be an infinite or infinitesimal element.

The problem is that in general abelian groups such multiplication does not make sense. But since integers embed in any torsion-free abelian group, in particular any ordered abelian group, by imitating what we do for real numbers, we can make sense of such multiplication.

For an abelian group G, multiplication by $m \in \mathbb{N}$ makes sense: $mg := \overbrace{g + \cdots + g}^{m \text{ times}}$. Taking additive inverses, multiplication by integers makes sense: $(-m)g := -(mg)$. If G is torsion-free divisible, then multiplication by rational numbers makes sense: $\frac{m}{n}g = \frac{mg}{n}$ is defined to be the unique $y \in G$ such that $ny = mg$.

Motivation. We carry this idea forward and define cuts in rational numbers to make sense of multiplication by irrationals. Let ρ be an element of a real closed field K. Then, for any $0 < g \in G$, we would like $\rho \cdot g$ to be an element of G such that, for all $r \in \mathbb{Q}$,

$$rg \lesseqgtr \rho \cdot g \iff r \lesseqgtr \rho.$$

Since we are typically interested in preserving the order on G, we also require that $\rho > 0$, because then

$$g_1 \lesseqgtr g_2 \iff \rho \cdot g_1 \lesseqgtr \rho \cdot g_2.$$

Without loss of generality, we also require that $\rho \geq 1$; otherwise, we can work with ρ^{-1} instead. Since ρ is an element of the real closed field K, we can define the cut of ρ in the rationals by

$$cut_{\mathbb{Q}}(\rho) = \{a \in K \ : \ \text{for each } q \in \mathbb{Q},\ q \lesseqgtr a \iff q \lesseqgtr \rho\}.$$

Clearly for all $a \in cut_{\mathbb{Q}}(\rho)$, $\rho \cdot g$ and $a \cdot g$ are order-indistinguishable with respect to the rationals. This is a little bit of a problem because we would typically like to be able to distinguish between $b \cdot g$ and $(b+\epsilon)\cdot g$, where b is an algebraic number, and ϵ is an infinitesimal. This is because if b is algebraic over $\mathbb{Z}$, then b is a root of a polynomial $L(x) = \sum_{i=0}^{n} a_i x^i$, with $a_i \in \mathbb{Z}$ for all $i = 0,\ldots,n$. Then for any $0 \neq g \in G$, we have $L(b)\cdot g = 0$, but $L(b+\epsilon)\cdot g \neq 0$. However, for any $a \in K$ and any polynomial $L(x)$ over $\mathbb{Z}$, we also have $L(a) \in K$. In particular, $L(a) > 0$ or $L(a) = 0$ or $L(a) < 0$. So either $L(a)\cdot g > 0$ for all $0 < g \in G$, or $L(a)\cdot g = 0$ for all $g > 0$, or $L(a)\cdot g < 0$ for all $g > 0$. This is the property we take from this particular setting and apply to the general setting to make the "multiplication" work and define what we call multiplicative ordered difference abelian group (MODAG).

Coming back to the general situation, we have an ordered abelian group G and an automorphism $\sigma : G \to G$. For $i \in \mathbb{N}$, we denote

$$\sigma^i(x) := \overbrace{\sigma(\sigma(\ldots(\sigma}^{i \text{ times}}(x))\ldots)).$$

Definition 2.2. There is a natural map $\Phi : \mathbb{Z}[\sigma] \to \mathrm{End}(G)$, which maps any $L := m_k\sigma^k + m_{k-1}\sigma^{k-1} + \ldots + m_1\sigma + m_0$ (thought of as an element of $\mathbb{Z}[\sigma]$ with the m_i's coming from $\mathbb{Z}$), to an endomorphism $L(\cdot) : G \to G$. Such an L is called a *linear difference operator.*

Due to this action of $\mathbb{Z}[\sigma]$, G has the structure of a $\mathbb{Z}[\sigma]$-module, with the understanding that σ has an inverse. To turn it into an ordered $\mathbb{Z}[\sigma]$-module, we further impose the following condition on σ (motivated from our earlier example with the real closed fields): for each $L \in \mathbb{Z}[\sigma]$,

$$\Big(\forall x > 0\ (L(x) > 0)\Big) \bigvee \Big(\forall x > 0\ (L(x) = 0)\Big) \bigvee \Big(\forall x > 0\ (L(x) < 0)\Big).$$

We call this condition Axiom OM (OM stands for *Ordered Module*). This axiom also makes sense for σ an injective endomorphism.

Axiom OM is consistent with Axioms 1-4 because any ordered abelian group is a model of these axioms with $\sigma(x) = 2x$ for all x, say. Also, with this axiom, $\mathbb{Z}[\sigma]$ becomes a quasi-ordered ring with the order defined as follows: $L_1 \geqq L_2 \iff \forall x > 0\ \Big((L_1 - L_2)(x) \geqq 0\Big)$, and $L_1 > L_2 \iff \forall x > 0\ \Big((L_1 - L_2)(x) > 0\Big)$.

It is easy to see that the relation

$$L_1 \approx L_2 \iff L_1 \geqq L_2 \text{ and } L_2 \geqq L_1 \iff \forall x > 0\ ((L_1 - L_2)(x) = 0)$$

is an equivalence relation. Thus taking a quotient makes sense, and we define

Definition 2.3. $\mathbb{Z}[\rho] := \mathbb{Z}[\sigma]/\approx$, where ρ is the image of σ under this quotient map.

We also define $\mathbb{Q}(\rho)$ to be the fraction field of $\mathbb{Z}[\rho]$.

Remark 2.4. Clearly then $\mathbb{Z}[\rho]$ is an (totally) ordered ring and admits an embedding into a real closed field. So ρ can also be simultaneously thought of as an element of a real closed field.

It is also easy to see that $\mathbb{Z}[\rho] = \mathbb{Z}[\sigma]/\operatorname{Ker}(\Phi)$, where Φ is as defined in Definition 2.2. Note that the kernel of Φ need not be trivial. For example, if $\sigma(x) = 2x$ for all x, then $\sigma - 2 \in \operatorname{Ker}(\Phi)$.

Moreover G is an ordered module over the ordered ring $\mathbb{Z}[\rho]$ with the understanding that ρ has an inverse. So we can denote the automorphism on G equivalently by $\rho\cdot$, i.e. $\sigma(x) = \rho \cdot x$. Axiom OM then is equivalent to: for each $L \in \mathbb{Z}[\rho]$,

$$\Big(\forall x > 0\ (L \cdot x > 0)\Big) \bigvee \Big(\forall x > 0\ (L \cdot x = 0)\Big) \bigvee \Big(\forall x > 0\ (L \cdot x < 0)\Big).$$

Definition 2.5. For any ordered difference abelian group G satisfying Axiom OM, we define the *set of* $\mathbb{Z}[\sigma]$*-positivities of* G as

$$ptp_{\mathbb{Z}[\sigma]}(G) := \{L \in \mathbb{Z}[\sigma] : \forall x \in G\ (x > 0 \implies L(x) > 0)\}.$$

We say that G and G' have the *same* ρ if $ptp_{\mathbb{Z}[\sigma]}(G) = ptp_{\mathbb{Z}[\sigma]}(G')$. We also say G is a MODAG *with a given* ρ if G satisfies a given consistent set of $\mathbb{Z}[\sigma]$-positivities.

Definition 2.6. An ordered difference abelian group is called *multiplicative* if it satisfies Axiom OM. The theory of such structures (also called MODAG) is axiomatized by Axioms 1-4 and Axiom OM. Note that this theory is also an $\forall\exists$-theory.

MODAG$_\rho$ denotes the theory of the class of all MODAGs with a given ρ.

Definition 2.7. If there is a non-zero $L \in \mathbb{Z}[\sigma]$ such that $\forall x > 0\ (L(x) = 0)$, we say ρ *satisfies* L and ρ is *algebraic* (over the integers); otherwise ρ is *transcendental*. If ρ is algebraic, there is a minimal (degree) polynomial that ρ satisfies.

Definition 2.8. A MODAG G is called *divisible* (or *linear difference closed*) if for any $0 \not\approx L \in \mathbb{Z}[\sigma]$ and $b \in G$, the equation $L(x) = b$ has a solution in G.

Definition 2.9 (Language for MODAG)**.** We study MODAG in the language of ordered abelian groups together with a symbol for the automorphism, $\mathcal{L}_{OG,\sigma} := \{+, -, 0, <, \sigma\}$.

Definition 2.10. Let div-MODAG be the $\mathcal{L}_{OG,\sigma}$-theory of non-trivial divisible multiplicative ordered difference abelian groups. This theory is axiomatized by the above axioms along with

$$\exists x (x \neq 0)$$

and the following additional infinite list of axioms: for each $L \in \mathbb{Z}[\sigma]$,

$$\Big(\forall x\ (L(x) = 0)\Big) \vee \Big(\forall y \exists x\ (L(x) = y)\Big),$$

i.e., all non-zero linear difference operators are surjective. Thus, div-MODAG is an $\forall\exists$-theory. Similarly as above, we denote by div-MODAG$_\rho$ the theory of the class of all div-MODAGs with a same ρ.

We now show that div-MODAG is the model companion of MODAG. By abuse of terminology, we refer to both the theory and any model of the theory as MODAG (respectively div-MODAG).

Remark 2.11. It might already be clear from the definitions above that for a given ρ, div-MODAG$_\rho$ is basically the theory of non-trivial ordered vector spaces over the ordered field $\mathbb{Q}(\rho)$ and then quantifier elimination actually follows from [12]. However, here we are doing things a little differently. Instead of proving the result for a particular ρ, we prove it uniformly across all ρ using Axiom OM.

Lemma 2.12. MODAG *and* div-MODAG *are co-theories.*

Proof. We will prove something stronger: for a fixed ρ, MODAG_ρ and div-MODAG_ρ are co-theories. Any model of div-MODAG_ρ is trivially a model of MODAG_ρ. So it remains to show that we can embed any model of MODAG_ρ into a model of div-MODAG_ρ.

We will actually show something even stronger. Let (G, σ) be an ordered abelian group with an injective endomorphism that is multiplicative with a given ρ. Then we can extend it to a "smallest" model (H, σ) of div-MODAG_ρ, which we call the (multiplicative) divisible hull of G.

If G is trivial, we can embed it into $\mathbb{Q}(\rho)$. If G is non-trivial, G becomes an ordered, and hence torsion-free, $\mathbb{Z}[\rho]$-module, and thus, by a classical result (see [16, Chapter II, §2.2, Proposition 3]), embeds in its module of fractions $\mathfrak{F}$ over $\mathbb{Q}(\rho)$. Clearly, $\mathfrak{F}$ is a model of div-MODAG_ρ. This embedding ι also has the universal property that any other embedding of G into a model of div-MODAG_ρ factors through ι. Thus, $\mathfrak{F}$ is the smallest model of div-MODAG_ρ in which G embeds. □

Remark 2.13. Since div-$\mathrm{MODAG}_\forall$ is the theory of ordered abelian groups with a multiplicative injective endomorphism, the above proof shows that div-MODAG_ρ has algebraically prime models, namely the (multiplicative) divisible hull.

Lemma 2.14. div-MODAG_ρ *has quantifier elimination.*

Proof. We have already shown that div-MODAG_ρ has algebraically prime models. All we need to show now is that div-MODAG_ρ is simply closed.

So suppose $G \subseteq H$ are two models of div-MODAG_ρ. We want to show $G \prec_s H$.

Suppose $\varphi(v, \bar{w})$ is a quantifier-free formula, $\bar{g} \in G$ and for some $h \in H$, $H \models \varphi(h, \bar{g})$. It suffices to consider the case where φ is a conjunction of atomic and negated atomic formulas. If $\theta(v, \bar{w})$ is atomic, then θ is equivalent to $\sum_{i=1}^{n} L_i(w_i) + L(v) = 0$ or $\sum_{i=1}^{n'} L'_i(w_i) + L'(v) > 0$ for some $L, L', L_i, L'_i \in \mathbb{Z}[\sigma]$. In particular, there is an element $a \in G$ such that $\theta(v, \bar{g})$ is of the form $L(v) = a$ or $L(v) > a$. Also note that $L(v) \neq a$ is equivalent to $L(v) > a$ or $-L(v) > a$. So we may assume that

$$\varphi(v, \bar{g}) \leftrightarrow \bigwedge L_i(v) = a_i \wedge \bigwedge L'_i(v) > b_i,$$

where $a_i, b_i \in G$ and $L_i, L'_i \in \mathbb{Z}[\sigma]$. We may also assume that $L_i \not\approx 0$ because either the corresponding a_i is zero, in which case the equation is trivially true for all v, or the corresponding a_i is non-zero, in which case the equation is inconsistent.

If there is actually a conjunct $L_i(v) = a_i$ with $0 \not\approx L_i$, then we must have $h \in G$ because G is divisible and a non-zero linear equation has a unique solution. So suppose $\varphi(v, \bar{g}) = \bigwedge L'_i(v) > b_i$. Let $h_1 = \min\{[(b_i, L'_i)] : L'_i < 0\}$ and $h_2 = \max\{[(b_i, L'_i)] : L'_i > 0\}$. Since $H \models \varphi(h, \bar{g})$, we have $h_2 < h < h_1$. In particular, $h_2 < h_1$. Now since $G \models$ div-MODAG_ρ, G is densely ordered because if $g < h$, then $g < \dfrac{g+h}{2} < h$. So there is $d \in G$ such that $h_2 < d < h_1$, and then $G \models \varphi(d, \bar{g})$. Thus, $G \prec_s H$. □

Thus, div-MODAG_ρ eliminates quantifiers. In particular, div-MODAG_ρ is model complete. Moreover, for a fixed ρ, $\mathbb{Q}(\rho)$ with the induced ordering is a prime model of div-MODAG_ρ. In particular, div-MODAG_ρ is complete. Note that div-MODAG is not complete; its completions are given by div-MODAG_ρ by fixing a (consistent) set of $\mathbb{Z}[\sigma]$-positivities. Finally we have,

Theorem 2.15. div-MODAG *is the model companion of* MODAG.

Proof. By Lemma 2.12, MODAG and div-MODAG are co-theories. All we need to show now is that div-MODAG is model complete.

So let $G \subseteq H$ be two models of div-MODAG. Want to show that $G \prec H$.

Since $G \subseteq H$ and both are non-trivial, in particular they have the same set of $\mathbb{Z}[\sigma]$-positivities. Thus, for some fixed ρ, we have $G, H \models$ div-MODAG$_\rho$. But div-MODAG$_\rho$ is model complete.

Hence, $G \prec H$. □

3. Preliminaries

Let $\mathcal{K} \prec \mathcal{K}'$ be an extension of valued difference fields. For any $a \in \mathcal{K}'$, $K\langle a\rangle$ denotes the smallest difference subfield of $\mathcal{K}'$ containing K and a. The underlying field of $K\langle a\rangle$ is $K(\sigma^i(a) : i \in \mathbb{Z})$. In literature a difference field generally means a field with an endomorphism. For our case, a difference field always means a field with an automorphism. So "the smallest difference subfield" in our context actually means the smallest inversive difference subfield.

For any $(n+1)$-variable polynomial $P(X_0, \ldots, X_n) \in K[X_0, \ldots, X_n]$, we define a corresponding 1-variable σ-polynomial $f(x) = P(x, \sigma(x), \sigma^2(x), \ldots, \sigma^n(x))$. We define the *degree* of f to be the total degree of P; and the *order of* f to be the largest integer $0 \leq d \leq n$ such that the coefficient of $\sigma^d(x)$ in $f(x)$ is non-zero. If $f \in K$, then order$(f) := -\infty$. Finally we define the complexity of f as

$$\text{complexity}(f) := (d, \deg_{x_d} f, \deg f) \in (\mathbb{N} \cup \{-\infty\})^3,$$

where complexity$(0) := (-\infty, -\infty, -\infty)$ and for $f \in K, f \neq 0$, complexity$(f) := (-\infty, 0, 0)$. We order complexities lexicographically.

Let $\mathbf{x} = (x_0, \ldots, x_n)$, $\mathbf{y} = (y_0, \ldots, y_n)$ be tuples of indeterminates and $\boldsymbol{a} = (a_0, \ldots, a_n)$ be a tuple of elements from some field. Let $\boldsymbol{I} = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ be a multi-index. We define the *length* of $\boldsymbol{I}$ as $|\boldsymbol{I}| := i_0 + \cdots + i_n$ and $\boldsymbol{a}^{\boldsymbol{I}} := a_0^{i_0} \cdots a_n^{i_n}$. For any element ρ of any ring, we define the ρ-length of $\boldsymbol{I}$ as $|\boldsymbol{I}|_\rho := i_0\rho^0 + i_1\rho^1 + \cdots + i_n\rho^n$. Then $|\boldsymbol{I}| \in \mathbb{N}$ and $|\boldsymbol{I}|_\rho$ is an element of that ring. For any polynomial $P(\mathbf{x})$ over K, we have a unique Taylor expansion in $K[\mathbf{x}, \mathbf{y}]$:

$$P(\mathbf{x} + \mathbf{y}) = \sum_{\boldsymbol{I}} P_{(\boldsymbol{I})}(\mathbf{x}) \cdot \mathbf{y}^{\boldsymbol{I}},$$

where the sum is over all $\boldsymbol{I} = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$, each $P_{(\boldsymbol{I})}(x) \in K[\mathbf{x}]$, with $P_{(\boldsymbol{I})} = 0$ for $|\boldsymbol{I}| > \deg(P)$, and $\mathbf{y}^{\boldsymbol{I}} := y_0^{i_0} \cdots y_n^{i_n}$. Thus $\boldsymbol{I}! P_{(\boldsymbol{I})} = \partial_{\boldsymbol{I}} P$ where $\partial_{\boldsymbol{I}}$ is the operator $(\partial/\partial x_0)^{i_0} \cdots (\partial/\partial x_n)^{i_n}$ on $K[\mathbf{x}]$, and $\boldsymbol{I}! := i_0! \cdots i_n!$. We construe $\mathbb{N}^{n+1}$ as a monoid under $+$ (componentwise addition), and let $\leq$ be the (partial) product ordering on $\mathbb{N}^{n+1}$ induced by the natural order on $\mathbb{N}$. Define for $\boldsymbol{I} \leq \boldsymbol{J} \in \mathbb{N}^{n+1}$,

$$\binom{\boldsymbol{J}}{\boldsymbol{I}} := \binom{j_0}{i_0} \cdots \binom{j_n}{i_n}.$$

Then it is easy to check that for $\boldsymbol{I}, \boldsymbol{J} \in \mathbb{N}^{n+1}$,

$$(f_{(\boldsymbol{I})})_{(\boldsymbol{J})} = \binom{\boldsymbol{I} + \boldsymbol{J}}{\boldsymbol{I}} f_{(\boldsymbol{I} + \boldsymbol{J})}.$$

Let x be an indeterminate. When n is clear from the context, we set $\boldsymbol{\sigma}(x) := (x, \sigma(x), \ldots, \sigma^n(x))$, and also $\boldsymbol{\sigma}(a) = (a, \sigma(a), \ldots, \sigma^n(a))$ for $a \in K$. Then for $P \in K[x_0, \ldots, x_n]$ as above and $f(x) = P(\boldsymbol{\sigma}(x))$, we have

$$\begin{aligned} f(x+y) &= P(\boldsymbol{\sigma}(x+y)) = P(\boldsymbol{\sigma}(x) + \boldsymbol{\sigma}(y)) \\ &= \sum_{\boldsymbol{I}} P_{(\boldsymbol{I})}(\boldsymbol{\sigma}(x)) \cdot \boldsymbol{\sigma}(y)^{\boldsymbol{I}} = \sum_{\boldsymbol{I}} f_{(\boldsymbol{I})}(x) \cdot \boldsymbol{\sigma}(y)^{\boldsymbol{I}}, \end{aligned}$$

where $f_{(\boldsymbol{I})}(x) := P_{(\boldsymbol{I})}(\boldsymbol{\sigma}(x))$.

A *pseudo-convergent sequence* (henceforth, *pc-sequence*) from K is a limit ordinal indexed sequence $\{a_\eta\}_{\eta<\lambda}$ of elements of K such that for some index η_0,

$$\eta'' > \eta' > \eta \geq \eta_0 \implies v(a_{\eta''} - a_{\eta'}) > v(a_{\eta'} - a_\eta).$$

An element $a \in K$ is called a *pseudo-limit* of a limit ordinal indexed sequence $\{a_\eta\}$ from K (denoted $a_\eta \rightsquigarrow a$) if there is some index η_0 such that

$$\eta' > \eta \geq \eta_0 \implies v(a - a_{\eta'}) > v(a - a_\eta).$$

Such a sequence is necessarily a pc-sequence in K. For a pc-sequence $\{a_\eta\}$ as above, let $\gamma_\eta := v(a_{\eta'} - a_\eta)$ for $\eta' > \eta \geq \eta_0$; note that this depends only on η. Then $\{\gamma_\eta\}_{\eta \geq \eta_0}$ is strictly increasing. The *breadth* of $\{a_\eta\}$ is defined as the set

$$\{y \in K : v(y) > \gamma_\eta \text{ for all } \eta \geq \eta_0\}.$$

Two pc-sequences $\{a_\eta\}$ and $\{b_\eta\}$ from K are said to be *equivalent* if they have the same pseudo-limits in all valued field extensions of $\mathcal{K}$. Equivalently, by [6, Lemma 3], $\{a_\eta\}$ and $\{b_\eta\}$ are equivalent iff they have the same breadth and a common pseudo-limit in some extension of $\mathcal{K}$.

4. Pseudoconvergence and Pseudocontinuity

Definition 4.1. Let $\mathcal{K} = (K, \Gamma, k; v, \pi, \sigma)$ be a valued difference field. The automorphism σ is called *multiplicative* if the induced structure on Γ is that of a MODAG with $\rho \geq 1$.

Then $\mathcal{K}$ is called a *multiplicative valued difference field*.

All we need in the above definition is $\rho > 0$. However, if $\rho \leq 1$, we can switch to σ^{-1} and then we will have $\rho \geq 1$. As noted in the introduction, we restrict ourselves only to the equi-characteristic zero case.

We are interested in proving an Ax-Kochen-Ershov type theorem for (and hence, finding the model companion of) the theory of multiplicative valued difference fields. We denote the induced automorphisms on the residue field k and the value group Γ by $\bar{\sigma}$ and $\rho\cdot$ respectively. Our main axiom is

Axiom 1. Γ is a MODAG with $\rho \geq 1$, and $v(\sigma(x)) = \rho \cdot v(x) \quad \forall x \in K$.

From now on, we assume that all our valued difference fields and valued difference field extensions satisfy Axiom 1.

Our first goal is to prove pseudo-continuity. It follows from [6] that if $\{a_\eta\}$ is a pc-sequence from K, and $a_\eta \rightsquigarrow a$ with $a \in K$, then for any ordinary non-constant polynomial $P(x) \in K[x]$, we have $P(a_\eta) \rightsquigarrow P(a)$. Unfortunately, this is not true in general for non-constant σ-polynomials over valued difference fields. As it turns out, this is true when ρ is transcendental over the integers (which includes the contractive case "$\rho = \infty$"), but not true if ρ is algebraic (which includes the isometric case $\rho = 1$). Fortunately, in the algebraic case, we can remedy the situation by resorting to equivalent pc-sequences. We will follow the treatment of [2], [3] with appropriate modifications. We will need the following basic lemma.

Lemma 4.2. *Let $\{\gamma_\eta\}$ be an increasing sequence of elements in a* MODAG Γ. *Let $A = \{|I_i|_\rho : I_i \in \mathbb{Z}^{n+1}, i = 1, \ldots, l\}$ be a finite set with $|A| = m$, and for $i = 1, \ldots, m$, let $c_i + n_i \cdot x$, $c_i \in \Gamma$, $n_i \in A$, be linear functions of x with distinct n_i. Then there is a μ, and an enumeration $i_1, i_2, \ldots, i_m$ of $\{1, \ldots, m\}$ such that for $\eta > \mu$, $c_{i_1} + n_{i_1} \cdot \gamma_\eta < c_{i_2} + n_{i_2} \cdot \gamma_\eta < \cdots < c_{i_m} + n_{i_m} \cdot \gamma_\eta$.*

Proof. Since Γ is a MODAG, there is a linear order amongst the n_i's. Suppose $n_i \neq n_j \in A$. WMA $n_i < n_j$. Then either $c_j + n_j \cdot \gamma_\eta < c_i + n_i \cdot \gamma_\eta$ for all η, or for some η_{ij}, $c_i + n_i \cdot \gamma_{\eta_{ij}} \leq c_j + n_j \cdot \gamma_{\eta_{ij}}$. But in the later case, for all $\eta > \eta_{ij}$, we have $c_i + n_i \cdot \gamma_\eta < c_j + n_j \cdot \gamma_\eta$, as $n_i < n_j$ and $\{\gamma_\eta\}$ is increasing. Since A is a finite set, the set of all such η_{ij}'s is also finite, and hence taking μ to be the maximum of those η_{ij}'s, we have our result. ☐

<u>Basic Calculation.</u>

Suppose $\mathcal{K}$ is a multiplicative valued difference field. Let $\{a_\eta\}$ be a pc-sequence from K with a pseudo-limit a in some extension. Let $P(x)$ be a non-constant σ-polynomial over K of order $\leq n$.

Case I. ρ is transcendental.

Let $\gamma_\eta = v(a_\eta - a)$. Then for each η we have,

$$P(a_\eta) - P(a) \quad = \sum_{\substack{\boldsymbol{L} \in \mathbb{N}^{n+1} \\ 1 \leq |\boldsymbol{L}| \leq \deg(P)}} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(a_\eta - a)^{\boldsymbol{L}} =: \sum_{\substack{\boldsymbol{L} \in \mathbb{N}^{n+1} \\ 1 \leq |\boldsymbol{L}| \leq \deg(P)}} Q_{\boldsymbol{L}}(\eta)$$

To calculate $v(P(a_\eta) - P(a))$, we need to calculate the valuation of each summand $Q_{\boldsymbol{L}}(\eta)$. We claim that there is a unique $\boldsymbol{L}$ for which the valuation of $Q_{\boldsymbol{L}}(\eta)$ is minimum eventually. Suppose not. Note that the valuation of $Q_{\boldsymbol{L}}(\eta)$

$$v(Q_{\boldsymbol{L}}(\eta)) = v(P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(a_\eta - a)^{\boldsymbol{L}}) = v(P_{(\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma_\eta$$

is a linear function in γ_η. Thus, by Lemma 4.2, the only way there isn't a unique $\boldsymbol{L}$ with the valuation of $Q_{\boldsymbol{L}}(\eta)$ minimum eventually is if there are $\boldsymbol{L} \neq \boldsymbol{L'}$ with $|\boldsymbol{L}|_\rho = |\boldsymbol{L'}|_\rho$. But then,

$$\begin{aligned} |\boldsymbol{L}|_\rho = |\boldsymbol{L'}|_\rho \quad &\Longrightarrow \quad |\boldsymbol{L} - \boldsymbol{L'}|_\rho = 0 \\ &\Longrightarrow \quad (l_0 - l'_0)\rho^0 + (l_1 - l'_1)\rho^1 + \cdots + (l_n - l'_n)\rho^n = 0 \end{aligned}$$

which implies that ρ is algebraic over $\mathbb{Z}$, a contradiction. Hence, the claim holds. In particular, there is a unique $\boldsymbol{L}_0$ such that eventually (in η),

$$v(P(a_\eta) - P(a)) = v(P_{(\boldsymbol{L}_0)}(a)) + |\boldsymbol{L_0}|_\rho \cdot \gamma_\eta,$$

which is strictly increasing. Hence, $P(a_\eta) \rightsquigarrow P(a)$.

Note that if $\rho = \infty$, then ρ is transcendental over $\mathbb{Z}$. Hence, the contractive case is included in Case I.

Case II. ρ is algebraic.

Since ρ satisfies some algebraic equation over the integers, there can be accidental cancelations and we might have $v(Q_{\boldsymbol{L}}(\eta)) = v(Q_{\boldsymbol{L'}}(\eta))$ for infinitely many η and

$\boldsymbol{L} \neq \boldsymbol{L'}$, and the above proof fails. To remedy this, we construct an equivalent pc-sequence $\{b_\eta\}$ such that $P(b_\eta) \rightsquigarrow P(a)$.

Put $\gamma_\eta := v(a_\eta - a)$; then $\{\gamma_\eta\}$ is eventually strictly increasing. Since v is surjective, choose $\theta_\eta \in K$ such that $v(\theta_\eta) = \gamma_\eta$. Set $b_\eta := a_\eta + \mu_\eta\theta_\eta$, where we demand that $\mu_\eta \in K$ and $v(\mu_\eta) = 0$. Define d_η by $a_\eta - a = \theta_\eta d_\eta$. So $v(d_\eta) = 0$ and d_η depends on the choice of θ_η. Since a is normally not in K, d_η won't normally be in K either. Then,

$$\begin{aligned} b_\eta - a &= b_\eta - a_\eta + a_\eta - a \\ &= \theta_\eta(\mu_\eta + d_\eta). \end{aligned}$$

We impose $v(\mu_\eta + d_\eta) = 0$. This ensures $b_\eta \rightsquigarrow a$, and that $\{a_\eta\}$ and $\{b_\eta\}$ have the same breadth; so they are equivalent. Let $A := \{|\boldsymbol{L}|_\rho : \boldsymbol{L} \in \mathbb{N}^{n+1}$ and $1 \leq |\boldsymbol{L}| \leq \deg(P)\}$. Now,

$$\begin{aligned} P(b_\eta) - P(a) &= \sum_{|\boldsymbol{L}|_\rho \,\in\, A} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(b_\eta - a)^{\boldsymbol{L}} \\ &= \sum_{m \in A} \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(b_\eta - a)^{\boldsymbol{L}} \\ &= \sum_{m \in A} \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(\theta_\eta(\mu_\eta + d_\eta))^{\boldsymbol{L}} \\ &= \sum_{m \in A} \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(\theta_\eta)^{\boldsymbol{L}} \cdot \boldsymbol{\sigma}(\mu_\eta + d_\eta)^{\boldsymbol{L}} \\ &= \sum_{m \in A} P_{m,\eta}(\mu_\eta + d_\eta) \end{aligned}$$

where $P_{m,\eta}$ is the σ-polynomial over $K\langle a \rangle$ given by

$$P_{m,\eta}(x) = \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(\theta_\eta)^{\boldsymbol{L}} \cdot \boldsymbol{\sigma}(x)^{\boldsymbol{L}}.$$

Since $P \notin K$, there is an $m \in A$ such that $P_{m,\eta} \neq 0$. For such m, pick $\boldsymbol{L} = \boldsymbol{L}(m)$ with $|\boldsymbol{L}|_\rho = m$ for which $v(P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(\theta_\eta)^{\boldsymbol{L}})$ is minimal, so

$$P_{m,\eta}(x) = P_{(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(\theta_\eta)^{\boldsymbol{L}} \cdot p_{m,\eta}(\boldsymbol{\sigma}(x)),$$

where $p_{m,\eta}(x_0, \ldots, x_n)$ has its coefficients in the valuation ring of $K\langle a \rangle$, with one of its coefficients equal to 1. Then

$$v(P_{m,\eta}(\mu_\eta + d_\eta)) = v(P_{(\boldsymbol{L})}(a)) + m \cdot \gamma_\eta + v(p_{m,\eta}(\boldsymbol{\sigma}(\mu_\eta + d_\eta))).$$

This calculation suggests a new constraint on $\{\mu_\eta\}$, namely that for each $m \in A$ with $P_{m,\eta} \neq 0$,

$$v(p_{m,\eta}(\boldsymbol{\sigma}(\mu_\eta + d_\eta))) = 0 \quad \text{(eventually in } \eta).$$

Assume this constraint is met. Then Lemma 4.2 yields a fixed $m_0 \in A$ such that if $m \in A$ and $m \neq m_0$, then eventually in η,

$$v(P_{m_0,\eta}(\mu_\eta + d_\eta)) < v(P_{m,\eta}(\mu_\eta + d_\eta))$$

For this m_0 we have, eventually in η,

$$v(P(b_\eta) - P(a)) = v(P_{(\boldsymbol{L})}(a)) + m_0 \cdot \gamma_\eta, \quad \boldsymbol{L} = \boldsymbol{L}(m_0),$$

which is increasing. So $P(b_\eta) \rightsquigarrow P(a)$, as desired.

To have $\{\mu_\eta\}$ satisfy all constraints, we introduce an axiom (scheme) about $\mathcal{K}$ which involves only the residue field k of $\mathcal{K}$:

Axiom 2. For each integer $d > 0$ there is $y \in k$ such that $\bar{\sigma}^d(y) \neq y$.

By [14, p. 201], this axiom implies that there are no residual $\bar{\sigma}$-identities at all, that is, for every non-zero $f \in k[x_0, \ldots, x_n]$, there is a $y \in k$ with $f(\bar{\boldsymbol{\sigma}}(y)) \neq 0$ (and thus the set $\{y \in k : f(\bar{\boldsymbol{\sigma}}(y)) \neq 0\}$ is infinite). Now note that the p_m's are over $K\langle a\rangle$, and we need $\bar{\mu}_\eta \in k$. The following lemma will take care of this.

Lemma 4.3. *Let $k \subseteq k'$ be a field extension, and $p(x_0, \ldots, x_n)$ a non-zero polynomial over k'. Then there is a non-zero polynomial $f(x_0, \ldots, x_n)$ over k such that whenever $y_0, \ldots, y_n \in k$ and $f(y_0, \ldots, y_n) \neq 0$, then $p(y_0, \ldots, y_n) \neq 0$.*

Proof. Using a basis $b_1, \ldots, b_m$ of the k-vector subspace of k' generated by the coefficients of p, we have $p = b_1 f_1 + \cdots + b_m f_m$, with $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$. Let f be one of the f_i's. Then f has the required property. □

Consider an $m \in A$ with non-zero $P_{m,\eta}$, and define

$$q_{m,\eta}(x_0, \ldots, x_n) := p_{m,\eta}(x_0 + d_\eta, \ldots, x_n + \sigma^n(d_\eta)).$$

Then the reduced polynomial

$$\bar{q}_{m,\eta}(x_0, \ldots, x_n) := \bar{p}_m(x_0 + \bar{d}_\eta, \ldots, x_n + \bar{\sigma}^n(\bar{d}_\eta))$$

is also non-zero for each η. By Lemma 4.3, we can pick a non-zero polynomial $f_\eta(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]$ such that if $y \in \mathcal{O}_K$ and $f_\eta(\bar{\boldsymbol{\sigma}}(\bar{y})) \neq 0$, then $\bar{q}_{m,\eta}(\bar{\boldsymbol{\sigma}}(\bar{y})) \neq 0$ for each $m \in A$ with $P_{m,\eta} \neq 0$.

Conclusion: if for each η the element $\mu_\eta \in \mathcal{O}_K$ satisfies $\bar{\mu}_\eta \neq 0, \bar{\mu}_\eta + \bar{d}_\eta \neq 0$, and $f_\eta(\bar{\boldsymbol{\sigma}}(\bar{\mu}_\eta)) \neq 0$, then all constraints on $\{\mu_\eta\}$ are met.

Axiom 2 allows us to meet these constraints, even if instead of a single $P(x)$ of order $\leq n$ we have finitely many non-constant σ-polynomials $Q(x)$ of order $\leq n$ and we have to meet simultaneously the constraints coming from each of those Q's. This leads to:

Theorem 4.4. *Suppose $\mathcal{K}$ satisfies Axiom 2. Suppose $\{a_\eta\}$ in K is a pc-sequence and $a_\eta \rightsquigarrow a$ in an extension with $\gamma_\eta := v(a - a_\eta)$. Let Σ be a finite set of σ-polynomials $P(x)$ over K.*

- *If ρ is transcendental, then $P(a_\eta) \rightsquigarrow P(a)$, for all non-constant $P \in K[x]$; more specifically there is a unique $\boldsymbol{L}_0 = \boldsymbol{L}_0(P)$ such that for all $\boldsymbol{I} \neq \boldsymbol{L}_0$, eventually*

$$v(P(a_\eta) - P(a)) = v(P_{(\boldsymbol{L}_0)}(a)) + |\boldsymbol{L}_0|_\rho \cdot \gamma_\eta < v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta.$$

- *If ρ is algebraic, then there is a pc-sequence $\{b_\eta\}$ from K, equivalent to $\{a_\eta\}$, such that $P(b_\eta) \rightsquigarrow P(a)$ for each non-constant $P \in \Sigma$; more specifically there is a unique $m_0 = m_0(P)$ such that for all $\boldsymbol{I}$ with $|\boldsymbol{I}|_\rho \neq m_0$, eventually*

$$v(P(b_\eta) - P(a)) = \min_{|\boldsymbol{L}_0|_\rho = m_0} v(P_{(\boldsymbol{L}_0)}(a)) + |\boldsymbol{L}_0|_\rho \cdot \gamma_\eta < v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta.$$

Refinement of the Basic Calculation. The following improvement of the basic calculation will be needed later on.

Theorem 4.5. *Suppose $\mathcal{K}$ satisfies Axiom 2 and ρ is algebraic. Let $\{a_\eta\}$ be a pc-sequence from K and let $a_\eta \rightsquigarrow a$ in some extension. Let $P(x)$ be a σ-polynomial over K such that* *(i)* $P(a_\eta) \rightsquigarrow 0$,

(ii) $P_{(\boldsymbol{L})}(b_\eta) \not\rightsquigarrow 0$, *whenever* $|\boldsymbol{L}| \geq 1$ *and* $\{b_\eta\}$ *is a pc-sequence in* K *equivalent to* $\{a_\eta\}$.

Let Σ *be a finite set of* σ*-polynomials* $Q(x)$ *over* K*. Then there is a pc-sequence* $\{b_\eta\}$ *in* K*, equivalent to* $\{a_\eta\}$*, such that* $P(b_\eta) \rightsquigarrow 0$*, and* $Q(b_\eta) \rightsquigarrow Q(a)$ *for all non-constant* Q *in* Σ.

Proof. By augmenting Σ, we can assume $P_{(\boldsymbol{L})} \in \Sigma$ for all $\boldsymbol{L}$. Let n be such that all $Q \in \Sigma$ have order $\leq n$. Let $\{\theta_\eta\}$ and $\{d_\eta\}$ be as before. By following the proof in the basic calculation and using Axiom 2, we get non-zero polynomials $f_\eta \in k[x_0, \ldots, x_n]$ and a sequence $\{\mu_\eta\}$ satisfying the constraints

$$\mu_\eta \in \mathcal{O}, \quad \bar{\mu}_\eta \neq 0, \quad \bar{\mu}_\eta + \bar{d}_\eta \neq 0, \quad f_\eta(\bar{\boldsymbol{\sigma}}(\bar{\mu}_\eta)) \neq 0,$$

such that, by setting $b_\eta := a_\eta + \theta_\eta \mu_\eta$, we have

$$Q(b_\eta) \rightsquigarrow Q(a) \qquad \text{for each non-constant } Q \in \Sigma.$$

We would like to constrain $\{\mu_\eta\}$ further so that we also have $P(b_\eta) \rightsquigarrow 0$. Letting $A := \{|\boldsymbol{L}|_\rho : \boldsymbol{L} \in \mathbb{N}^{n+1} \text{ and } 1 \leq |\boldsymbol{L}| \leq \deg(P)\}$, we have

$$\begin{aligned} P(a_\eta) &= P(b_\eta - \theta_\eta \mu_\eta) \\ &= P(b_\eta) + \sum_{m \in A} \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(b_\eta) \cdot \boldsymbol{\sigma}(-\theta_\eta \mu_\eta)^{\boldsymbol{L}} \\ &= P(b_\eta) + \sum_{m \in A} \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(b_\eta) \cdot \boldsymbol{\sigma}(-\theta_\eta)^{\boldsymbol{L}} \cdot \boldsymbol{\sigma}(\mu_\eta)^{\boldsymbol{L}} \\ &= P(b_\eta) + \sum_{m \in A} Q_{m,\eta}(\mu_\eta) \end{aligned}$$

where $Q_{m,\eta}$ is the σ-polynomial over K given by

$$Q_{m,\eta}(x) = \sum_{|\boldsymbol{L}|_\rho = m} P_{(\boldsymbol{L})}(b_\eta) \cdot \boldsymbol{\sigma}(-\theta_\eta)^{\boldsymbol{L}} \cdot \boldsymbol{\sigma}(x)^{\boldsymbol{L}}.$$

Now, for $|\boldsymbol{L}| \geq 1$, $P_{(\boldsymbol{L})}(b_\eta) \not\rightsquigarrow 0$, and, provided $P_{(\boldsymbol{L})} \notin K$, $P_{(\boldsymbol{L})}(b_\eta) \rightsquigarrow P_{(\boldsymbol{L})}(a)$. Hence, $v(P_{(\boldsymbol{L})}(b_\eta))$ settles down eventually. Let $\gamma_{\boldsymbol{L}}$ be this eventual value. For each $m \in A$ such that $Q_{m,\eta} \neq 0$, let $\boldsymbol{L} = \boldsymbol{L}(m)$ be such that $P_{(\boldsymbol{L})}(b_\eta) \cdot \boldsymbol{\sigma}(-\theta_\eta)^{\boldsymbol{L}}$ has minimal valuation. Then, for such $Q_{m,\eta}$, we can write (eventually in η),

$$Q_{m,\eta}(x) = c_{m,\eta} \cdot q_{m,\eta}(\boldsymbol{\sigma}(x)),$$

where $v(c_{m,\eta}) = \gamma_{\boldsymbol{L}} + |\boldsymbol{L}|_\rho \cdot \gamma_\eta$ and $q_{m,\eta}$ is a polynomial over $\mathcal{O}$ with at least one coefficient 1. This suggests another constraint on $\{\mu_\eta\}$, namely, for each $m \in A$ such that $Q_{m,\eta} \neq 0$, $v(q_{m,\eta}(\boldsymbol{\sigma}(\mu_\eta))) = 0$ (eventually in η); equivalently, $\bar{q}_{m,\eta}(\bar{\boldsymbol{\sigma}}(\bar{\mu}_\eta)) \neq 0$. As usual, this constraint can be met by Axiom 2. And then, by Lemma 4.2, we have a unique m_0 such that eventually in η,

$$v\Big(\sum_{m \in A} Q_{m,\eta}(\mu_\eta)\Big) = v(Q_{m_0,\eta}(\mu_\eta)) = \gamma_{\boldsymbol{L}} + m_0 \cdot \gamma_\eta, \qquad \boldsymbol{L} = \boldsymbol{L}(m_0),$$

which is increasing. Now,

$$P(b_\eta) = P(a_\eta) - \sum_{m \in A} Q_{m,\eta}(\mu_\eta).$$

If $v(P(a_\eta)) \neq v(Q_{m_0,\eta}(\mu_\eta))$, we do nothing. However, if $v(P(a_\eta)) = \gamma_{\boldsymbol{L}} + m_0 \cdot \gamma_\eta$, then replacing μ_η by a variable x, consider

$$\begin{aligned} & P(a_\eta) - \textstyle\sum_{m\in A} Q_{m,\eta}(x) \\ = \quad & P(a_\eta)\Big(1 - P(a_\eta)^{-1} \textstyle\sum_{m\in A} Q_{m,\eta}(x)\Big) \\ = \quad & P(a_\eta)\Big(1 - P(a_\eta)^{-1} c_{m_0,\eta} q_{m_0,\eta}(\boldsymbol{\sigma}(x))(1+\epsilon_\eta)\Big) \\ = \quad & P(a_\eta)\Big(1 - c'_{m_0,\eta} q_{m_0,\eta}(\boldsymbol{\sigma}(x)) + \epsilon'_\eta\Big) \end{aligned}$$

where $c'_{m_0,\eta} = P(a_\eta)^{-1} c_{m_0,\eta}$, $v(\epsilon_\eta) > 0$ and $v(\epsilon'_\eta) > 0$. Note that $v(c'_{m_0,\eta}) = 0$ and $q_{m_0,\eta}$ is a polynomial over $\mathcal{O}$ with at least one coefficient 1. So we impose our final constraint on $\{\mu_\eta\}$: for each η such that $v(P(a_\eta)) = \gamma_{\boldsymbol{L}} + m_0 \cdot \gamma_\eta$, we require that

$$1 - \bar{c}'_{m_0,\eta} \bar{q}_{m_0,\eta}(\bar{\boldsymbol{\sigma}}(\bar{\mu}_\eta)) \neq 0.$$

Then we get that eventually

$$v(P(b_\eta)) = \min\{v(P(a_\eta)), v(Q_{m_0,\eta}(\mu_\eta))\},$$

and since both of these are increasing, we have $P(b_\eta) \rightsquigarrow 0$. □

5. Around Newton-Hensel Lemma

For the moment we consider the basic problem of how to start with $a \in K$ and $P(a) \neq 0$, and find $b \in K$ with $v(P(b)) > v(P(a))$.

Before we do that, we need a little notation. Let $\mathcal{K} = (K, \sigma, v)$ be a multiplicative valued difference field. As already mentioned, the automorphism σ on K induces an automorphism on the value group Γ, which we also denote by σ, as follows:

$$\gamma \mapsto \sigma(\gamma) := v(\sigma(a)), \qquad \text{where } \gamma = v(a) \text{ for some } a \in K.$$

Then for any multi-index $\boldsymbol{I} = (i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1}$, we have

$$v(\boldsymbol{\sigma}(a)^{\boldsymbol{I}}) = v(a^{i_0}(\sigma(a))^{i_1} \cdots (\sigma^n(a))^{i_n}) = \sum_{j=0}^{n} i_j v(\sigma^j(a)) = \sum_{j=0}^{n} i_j \rho^j \cdot v(a) = |\boldsymbol{I}|_\rho \cdot v(a).$$

If $\boldsymbol{I} = \boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots 0)$ with 1 at the i-th place, we denote $P_{(\boldsymbol{e}_i)}$ by $P_{(i)}$. And then, for $\gamma \in \Gamma$, we have $|\boldsymbol{e}_i|_\rho \cdot \gamma = \rho^i \cdot \gamma$. By abuse of notation, we will often identify $\boldsymbol{e}_i$ with i. For example, we will write $\boldsymbol{J} \neq i$ (for some multi-index $\boldsymbol{J}$) to actually mean $\boldsymbol{J} \neq \boldsymbol{e}_i$. Hopefully, this should be clear from the context.

Let $\mathcal{K}$ be a multiplicative valued difference field. Let $P(x)$ be a σ-polynomial over K of order $\leq n$, and $a \in K$. Let $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{L} \in \mathbb{N}^{n+1}$.

Definition 5.1. We say (P, a) is in σ-hensel configuration if P is not a constant and there is $0 \leq i \leq n$ and $\gamma \in \Gamma$ such that

- (i) $v(P(a)) = v(P_{(i)}(a)) + \rho^i \cdot \gamma \leq v(P_{(j)}(a)) + \rho^j \cdot \gamma$ whenever $0 \leq j \leq n$,
- (ii) $v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J}|_\rho \cdot \gamma < v(P_{(\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma$ whenever $\boldsymbol{0} \neq \boldsymbol{J} < \boldsymbol{L}$ and $P_{(\boldsymbol{J})} \neq 0$.

We say (P, a) is in strict σ-hensel configuration if the inequality in (i) is strict for $j \neq i$.

Remark 5.2. Note that if (P,a) is in (strict) σ-hensel configuration, then $P_{(\boldsymbol{J})}(a) \neq 0$ whenever $\boldsymbol{J} \neq \boldsymbol{0}$ and $P_{(\boldsymbol{J})} \neq 0$, so $P(a) \neq 0$, and therefore γ as above satisfies

$$v(P(a)) = \min_{0 \leq j \leq n} v(P_{(j)}(a)) + \rho^j \cdot \gamma,$$

so is unique, and we set $\gamma(P,a) := \gamma$. If (P,a) is not in σ-hensel configuration, we set $\gamma(P,a) := \infty$. If (P,a) is in strict σ-hensel configuration, then i is unique and we set $i(P,a) := i$.

Remark 5.3. Suppose P is non-constant, $P(a) \neq 0$, $v(P(a)) > 0$ and $v(P_{(\boldsymbol{J})}(a)) = 0$ for all $\boldsymbol{J} \neq \boldsymbol{0}$ with $P_{(\boldsymbol{J})} \neq 0$. Then (P,a) is in σ-hensel configuration with $\gamma(P,a) = v(P(a)) > 0$ and any i with $0 \leq i \leq n$; and for $\rho > 1$, (P,a) is in strict σ-hensel configuration with $\gamma(P,a) = v(P(a)) > 0$ and $i(P,a) = 0$.

Now given (P,a) in (strict) σ-hensel configuration, we aim to find $b \in K$ such that $v(P(b)) > v(P(a))$ and (P,b) is in (strict) σ-hensel configuration. This, however, requires an additional assumption on the residue field k, namely that k should be linear difference-closed. We will justify later on why this assumption is necessary.

Axiom 3_n. If $\alpha_0, \ldots, \alpha_n \in k$ are not all 0, then the equation

$$1 + \alpha_0 x + \alpha_1 \bar{\sigma}(x) + \cdots + \alpha_n \bar{\sigma}^n(x) = 0$$

has a solution in k.

Lemma 5.4. *Suppose $\mathcal{K}$ satisfies Axiom 3_n, and (P,a) is in σ-hensel configuration. Then there is $b \in K$ such that*

(1) $v(b-a) \geq \gamma(P,a), \quad v(P(b)) > v(P(a))$,
(2) *either $P(b) = 0$, or (P,b) is in σ-hensel configuration.*

For any such b, we have $v(b-a) = \gamma(P,a)$ and $\gamma(P,b) > \gamma(P,a)$.

Proof. This is the same proof as [4, Lemma 4.4]. But we include it here for the sake of completeness, and also to set the ground for the next lemma.

Step 1. Let $\gamma = \gamma(P,a)$. Pick $\epsilon \in K$ with $v(\epsilon) = \gamma$. Let $b = a + \epsilon u$, where $u \in K$ is to be determined later; we only impose $v(u) \geq 0$ for now. Consider

$$P(b) = P(a) + \sum_{|\boldsymbol{J}| \geq 1} P_{(\boldsymbol{J})}(a) \cdot \boldsymbol{\sigma}(b-a)^{\boldsymbol{J}}.$$

Therefore, $P(b) = P(a) \cdot (1 + \sum_{|\boldsymbol{J}| \geq 1} c_{\boldsymbol{J}} \cdot \boldsymbol{\sigma}(u)^{\boldsymbol{J}})$, where

$$c_{\boldsymbol{J}} = \frac{P_{(\boldsymbol{J})}(a) \cdot \boldsymbol{\sigma}(\epsilon)^{\boldsymbol{J}}}{P(a)}.$$

From $v(\epsilon) = \gamma$ and the fact that (P,a) is in σ-hensel configuration, we obtain $\min_{0 \leq j \leq n} v(c_j) = 0$ and $v(c_{\boldsymbol{L}}) > 0$ for $|\boldsymbol{L}| > 1$. Then imposing $v(P(b)) > v(P(a))$ forces $\bar{u}$ to be a solution of the equation

$$1 + \sum_{0 \leq j \leq n} \bar{c}_j \cdot \bar{\sigma}^j(x) = 0.$$

By Axiom 3_n, we can take u with this property, and then $v(u) = 0$, so $v(b-a) = \gamma(P,a)$, and $v(P(b)) > v(P(a))$.

Step 2. Assume that $P(b) \neq 0$. It remains to show that then (P, b) is in σ-hensel configuration with $\gamma(P, b) > \gamma$. Let $\boldsymbol{J} \neq \boldsymbol{0}, P_{(\boldsymbol{J})} \neq 0$ and consider

$$P_{(\boldsymbol{J})}(b) = P_{(\boldsymbol{J})}(a) + \sum_{\boldsymbol{L} \neq \boldsymbol{0}} P_{(\boldsymbol{J})(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(b-a)^{\boldsymbol{L}}.$$

Note that $P_{(\boldsymbol{J})}(a) \neq 0$. Since $\mathcal{K}$ is of equi-characteristic zero, $v(P_{(\boldsymbol{J})(\boldsymbol{L})}(a)) = v(P_{(\boldsymbol{J}+\boldsymbol{L})}(a))$. Therefore, for all $\boldsymbol{L} \neq \boldsymbol{0}$,

$$v(P_{(\boldsymbol{J})(\boldsymbol{L})}(a) \cdot \boldsymbol{\sigma}(b-a)^{\boldsymbol{L}}) > v(P_{(\boldsymbol{J})}(a)),$$

hence $v(P_{(\boldsymbol{J})}(b)) = v(P_{(\boldsymbol{J})}(a))$. Since $P(b) \neq 0$, we can pick $\gamma_1 \in \Gamma$ such that

$$v(P(b)) = \min_{0 \leq j \leq n} v(P_{(j)}(a)) + \rho^j \cdot \gamma_1.$$

Then $\gamma < \gamma_1$: Pick $0 \leq i \leq n$ such that $v(P(a)) = v(P_{(i)}(a)) + \rho^i \cdot \gamma$. So

$$\rho^i \cdot \gamma = v(P(a)) - v(P_{(i)}(a)) < v(P(b)) - v(P_{(i)}(a)) \leq \rho^i \cdot \gamma_1.$$

Also for $\boldsymbol{J}, \boldsymbol{L} \neq \boldsymbol{0}$ and $\theta \in \Gamma$ with $\theta > 0$, we have $|\boldsymbol{J}|_\rho \cdot \theta < |\boldsymbol{L}|_\rho \cdot \theta$ for $\boldsymbol{J} < \boldsymbol{L}$ (here we are using the fact that $\rho > 0$ and the fact that $\boldsymbol{J}, \boldsymbol{L}$ are tuples of natural numbers). Thus the inequality

$$v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J}|_\rho \cdot \gamma < v(P_{(\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma$$

together with $\gamma_1 > \gamma$ and $v(P_{(\boldsymbol{I})}(b)) = v(P_{(\boldsymbol{I})}(a))$ for all $\boldsymbol{I}$ yields

$$v(P_{(\boldsymbol{J})}(b)) + |\boldsymbol{J}|_\rho \cdot \gamma_1 < v(P_{(\boldsymbol{L})}(b)) + |\boldsymbol{L}|_\rho \cdot \gamma_1.$$

Hence, (P, b) is in σ-hensel configuration with $\gamma(P, b) = \gamma_1$. □

Lemma 5.5. *Suppose $\mathcal{K}$ satisfies Axiom 3_n and $\rho > 1$, and (P, a) is in σ-hensel configuration. Then there is $c \in K$ such that*

(1) $v(c-a) \geq \gamma(P, a), \quad v(P(c)) > v(P(a))$,
(2) *either $P(c) = 0$, or (P, c) is in strict σ-hensel configuration.*

For any such c, we have $v(c-a) = \gamma(P, a)$, $\gamma(P, c) > \gamma(P, a)$; and if (P, a) was already in strict σ-hensel configuration, then $i(P, c) \leq i(P, a)$.

Proof. Let $\gamma = \gamma(P, a)$ and $i = i(P, a)$ (in case (P, a) is in strict σ-hensel configuration). Since (P, a) is in σ-hensel configuration, by Lemma 5.4, there is $b \in K$ such that $v(b-a) = \gamma(P, a)$, $v(P(b)) > v(P(a))$, $\gamma(P, b) > \gamma(P, a) = \gamma$ and either $P(b) = 0$ or (P, b) is in σ-hensel configuration.

If $P(b) = 0$, let $c := b$ and we are done. So suppose $P(b) \neq 0$. Then, letting $\gamma_1 = \gamma(P, b)$, we have for some $0 \leq j_0 \leq n$,

$$v(P(b)) = v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma_1 \leq v(P_{(j)}(a)) + \rho^j \cdot \gamma_1$$

for all $0 \leq j \leq n$.

If the above inequality is strict for $j \neq j_0$, we are done: Then (P, b) is in strict σ-hensel configuration with $i(P, b) = j_0$ and $\gamma(P, b) = \gamma_1$. Moreover, if (P, a) is already in strict σ-hensel configuration and $i = i(P, a) < j_0$, then $\rho^i \cdot (\gamma_1 - \gamma) \leq \rho^{j_0} \cdot (\gamma_1 - \gamma)$ as $\gamma_1 - \gamma > 0$ and $\rho \geq 1$, and we have

$$\begin{aligned} v(P_{(i)}(a)) + \rho^i \cdot \gamma &< v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma \\ \implies v(P_{(i)}(a)) + \rho^i \cdot \gamma + \rho^i \cdot (\gamma_1 - \gamma) &< v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma + \rho^{j_0} \cdot (\gamma_1 - \gamma) \\ \implies v(P_{(i)}(a)) + \rho^i \cdot \gamma_1 &< v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma_1, \end{aligned}$$

which is a contradiction. Thus $j_0 \leq i$. Hence, letting $c := b$ works.

However, if there is no such unique j_0, then it means there are $0 \leq j_0 < j_1 < \cdots < j_m \leq n$ such that

$$v(P(b)) = v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma_1 = v(P_{(j_1)}(a)) + \rho^{j_1} \cdot \gamma_1 = \cdots = v(P_{(j_m)}(a)) + \rho^{j_m} \cdot \gamma_1.$$

Since (P, b) is in σ-hensel configuration, we can find $b' \in K$ such that $v(P(b')) > v(P(b)) > v(P(a)), \gamma(P, b') > \gamma(P, b) > \gamma(P, a)$, $v(b' - b) = \gamma(P, b)$ and either $P(b') = 0$ or (P, b') is in σ-hensel configuration. It follows that

$$\begin{aligned} v(b'-a) &= v(b' - b + b - a) \\ &\geq \min\{v(b'-b), v(b-a)\} \\ &= \min\{\gamma(P,b), \gamma(P,a)\} \\ \implies v(b'-a) &= \gamma(P,a) \qquad \text{since, } \gamma(P,a) < \gamma(P,b). \end{aligned}$$

If $P(b') = 0$, we are done. So suppose $P(b') \neq 0$. Let $\gamma_2 = \gamma(P, b')$. Since $\gamma_2 - \gamma_1 > 0$ and $\rho > 1$ (this is where we crucially use this hypothesis), we have

$$\rho^{j_0} \cdot (\gamma_2 - \gamma_1) < \rho^{j_1} \cdot (\gamma_2 - \gamma_1) < \cdots < \rho^{j_m} \cdot (\gamma_2 - \gamma_1).$$

But then by doing the same trick as in the previous paragraph, we obtain

$$v(P_{(j_0)}(a)) + \rho^{j_0} \cdot \gamma_2 < v(P_{(j_1)}(a)) + \rho^{j_1} \cdot \gamma_2 < \cdots < v(P_{(j_m)}(a)) + \rho^{j_m} \cdot \gamma_2.$$

Thus, we have succeeded in finding a better approximation b' than b in the sense that (P, b') is in σ-hensel configuration with its minimal valuation occurring at a possibly lower index than that of (P, b). And these inequalities will remain as we pass to higher γ. Since $i(P, a)$ is finite, there are only finitely many possibilities for this index to go down. So by repeating this step finitely many times, we end up at our required c with $v(c - a) = \gamma(P, a)$ such that either $P(c) = 0$ or (P, c) is in strict σ-hensel configuration with $\gamma(P, c) > \gamma(P, a)$ and $i(P, c) \leq i(P, a)$. □

Lemma 5.6. *Suppose $\mathcal{K}$ satisfies Axiom 3_n, and (P, a) is in σ-hensel configuration. Suppose also there is no $b \in K$ such that $P(b) = 0$ and $v(b - a) = \gamma(P, a)$. Then there is a pc-sequence $\{a_\eta\}$ in K with the following properties:*

1. *$a_0 = a$ and $\{a_\eta\}$ has no pseudolimit in K;*
2. *$\{v(P(a_\eta))\}$ is strictly increasing, and thus $P(a_\eta) \rightsquigarrow 0$;*
3. *$v(a_{\eta'} - a_\eta) = \gamma(P, a_\eta)$ whenever $\eta < \eta'$;*
4. *(P, a_η) is in σ-hensel configuration with $\gamma(P, a_\eta) < \gamma(P, a_{\eta'})$ for $\eta < \eta'$;*
5. *for any extension $\mathcal{K}'$ of $\mathcal{K}$ and $b, c \in K'$ such that $a_\eta \rightsquigarrow b$ (and hence, (P, b) is in σ-hensel configuration) and $v(c - b) \geq \gamma(P, b)$, we have $a_\eta \rightsquigarrow c$.*

Proof. We will build the sequence by transfinite recursion. Start with $a_0 := a$. Suppose for some ordinal $\lambda > 0$, we have built the sequence $\{a_\eta\}_{\eta<\lambda}$ such that

(i) (P, a_η) is in σ-hensel configuration, for all $\eta < \lambda$,
(ii) $v(a_{\eta'} - a_\eta) = \gamma(P, a_\eta)$ whenever $\eta < \eta' < \lambda$,
(iii) $v(P(a_{\eta'})) > v(P(a_\eta))$ and $\gamma(P, a_{\eta'}) > \gamma(P, a_\eta)$ whenever $\eta < \eta' < \lambda$.

Now we will have to deal with the inductive case. If λ is a successor ordinal, say $\lambda = \mu + 1$, then by Lemma 5.4, there is $a_\lambda \in K$ such that $v(a_\lambda - a_\mu) = \gamma(P, a_\mu)$, $v(P(a_\lambda)) > v(P(a_\mu))$ and $\gamma(P, a_\lambda) > \gamma(P, a_\mu)$. Then the extended sequence $\{a_\eta\}_{\eta<\lambda+1}$ has the above properties with $\lambda + 1$ instead of λ.

Suppose λ is a limit ordinal. Then $\{a_\eta\}$ is a pc-sequence and $P(a_\eta) \rightsquigarrow 0$. If $\{a_\eta\}$ has no pseudolimit in K, we are done. Otherwise, let $a_\lambda \in K$ be a pseudolimit of

$\{a_\eta\}$. Then $v(a_\lambda - a_\eta) = v(a_{\eta+1} - a_\eta) = \gamma(P, a_\eta)$; also, for any $\eta < \lambda$,

$$P(a_\lambda) = P(a_\eta) + \sum_{|\boldsymbol{I}|\geq 1} P_{(\boldsymbol{I})}(a_\eta) \cdot \boldsymbol{\sigma}(a_\lambda - a_\eta)^{\boldsymbol{I}};$$

since $P(a_\eta)$ has the minimal valuation of all the summands, we have $v(P(a_\lambda)) \geq v(P(a_\eta))$ for all $\eta < \lambda$. Since $\{v(P(a_\eta))\}_{\eta<\lambda}$ is increasing by inductive hypothesis, we get $v(P(a_\lambda)) > v(P(a_\eta))$ for all $\eta < \lambda$. And then by Step 2 of Lemma 5.4, it follows that (P, a_λ) is in σ-hensel configuration with $\gamma(P, a_\lambda) > \gamma(P, a_\eta)$ for all $\eta < \lambda$. Thus the extended sequence $\{a_\eta\}_{\eta<\lambda+1}$ satisfies all the above properties with $\lambda + 1$ instead of λ. Eventually we will have a sequence cofinal in K, and hence the building process must come to a stop, yielding a pc-sequence satisfying $(1), (2), (3)$ and (4).

Now $a_\eta \rightsquigarrow b$. Thus $v(b - a_\eta) = v(a_{\eta+1} - a_\eta) = \gamma(P, a_\eta)$ for all η, and (P, b) is in σ-hensel configuration with $\gamma(P, b) > \gamma(P, a_\eta)$ for all η. In particular,

$$\begin{aligned} v(c - a_\eta) &= v(c - b + b - a_\eta) \\ &\geq \min\{v(c - b), v(b - a_\eta)\} \\ &\geq \min\{\gamma(P, b), \gamma(P, a_\eta)\} \\ \implies v(c - a_\eta) &= \gamma(P, a_\eta) \end{aligned}$$

Since $\{\gamma(P, a_\eta)\}$ is increasing, we have $a_\eta \rightsquigarrow c$. □

It follows similarly (with ideas from the proof of Lemma 5.5) that

Lemma 5.7. *Suppose $\mathcal{K}$ satisfies Axiom 3_n, $\rho > 1$ and (P, a) is in strict σ-hensel configuration. Suppose also there is no $b \in K$ such that $P(b) = 0$ and $v(b - a) = \gamma(P, a)$. Then there is a pc-sequence $\{a_\eta\}$ in K with the following properties:*

1. *$a_0 = a$ and $\{a_\eta\}$ has no pseudolimit in K;*
2. *$\{v(P(a_\eta))\}$ is strictly increasing, and thus $P(a_\eta) \rightsquigarrow 0$;*
3. *$v(a_{\eta'} - a_\eta) = \gamma(P, a_\eta)$ whenever $\eta < \eta'$;*
4. *(P, a_η) is in strict σ-hensel configuration with $\gamma(P, a_\eta) < \gamma(P, a_{\eta'})$ and $i(P, a_{\eta'}) \leq i(P, a_\eta)$ for $\eta < \eta'$;*
5. *for any extension $\mathcal{K}'$ of $\mathcal{K}$ and $b, c \in K'$ such that $a_\eta \rightsquigarrow b$ (and hence, (P, b) is in σ-hensel configuration) and $v(c - b) \geq \gamma(P, b)$, we have $a_\eta \rightsquigarrow c$.*

Definition 5.8. A multiplicative valued difference field $\mathcal{K}$ is called (strict) σ-henselian if for all (P, a) in (strict) σ-hensel configuration there is $b \in K$ such that $v(b - a) = \gamma(P, a)$ and $P(b) = 0$.

By **Axiom 3** we mean the set $\{\text{Axiom } 3_n : n = 0, 1, 2, \ldots\}$. So Axiom 3 is really an axiom scheme and $\mathcal{K}$ satisfies Axiom 3 if and only if k is linear difference closed.

Corollary 5.9. *If $\mathcal{K}$ is maximally complete as a valued field and satisfies Axiom 3, then $\mathcal{K}$ is σ-henselian (strict σ-henselian if $\rho > 1$). In particular, if $\mathcal{K}$ is complete with discrete valuation and satisfies Axiom 3, then $\mathcal{K}$ is σ-henselian (strict σ-henselian if $\rho > 1$).*

Lemma 5.10. (1) *If $\mathcal{K}$ is σ-henselian, then $\mathcal{K}$ satisfies Axiom 3.*
(2) *If $\mathcal{K}$ satisfies Axiom 3, then $\mathcal{K}$ satisfies Axiom 2.*

Proof. (1) Assume that $\mathcal{K}$ is σ-henselian and let $Q(x) = 1 + \alpha_0 x + \alpha_1 \bar{\sigma}(x) + \cdots + \alpha_n \bar{\sigma}^n(x) \in k\langle x\rangle$ such that not all α_i's are zero. We want to find $b \in k$ such that $Q(b) = 0$.

Let $P(a) = 1 + a_0x + a_1\sigma(x) + \cdots + a_n\sigma^n(x)$, where for all i, $a_i \in K$, $a_i = 0$ if $\alpha_i = 0$, and $v(a_i) = 0$ with $\bar{a}_i = \alpha_i$ if $\alpha_i \neq 0$. It is easy to see that $(P, 0)$ is in σ-hensel configuration with $\gamma(P, 0) = 0$. By σ-henselianity, there is $a \in K$ such that $v(a) = 0$ and $P(a) = 0$. Set $b := \bar{a}$.

(2) For $\mathcal{K}$ to satisfy Axiom 2, we need for each $d \in \mathbb{Z}_+$, an element $a \in k$ such that $\bar{\sigma}^d(a) \neq a$. Consider the linear difference polynomial $P_d(x) = \bar{\sigma}^d(x) - x + 1$ over k. Since $\mathcal{K}$ satisfies Axiom 3, there is $a \in k$ such that $P_d(a) = 0$, i.e., $\bar{\sigma}^d(a) = a - 1$. In particular, $\bar{\sigma}^d(a) \neq a$. □

Remark 5.11. (1) If $\Gamma = \{0\}$, then $\mathcal{K}$ is σ-henselian.

(2) If $\Gamma \neq \{0\}$ and $\mathcal{K}$ is σ-henselian, then $\mathcal{K}$ satisfies Axiom 3 by Lemma 5.10. In particular, $\bar{\sigma}^n \neq id_k$ for all $n \geq 1$. Thus, $\mathcal{K}$ satisfies Axiom 2 as well.

(3) If $\rho > 1$ and $\mathcal{K}$ satisfies Axiom 3, then $\mathcal{K}$ is σ-henselian iff $\mathcal{K}$ is strict σ-henselian: the "only-if" direction is trivial, and the "if" direction follows from Lemma 5.5.

Definition 5.12. We say $\{a_\eta\}$ is of σ-algebraic type over K if $P(b_\eta) \rightsquigarrow 0$ for some σ-polynomial $P(x)$ over K and an equivalent pc-sequence $\{b_\eta\}$ in K. Otherwise, we say $\{a_\eta\}$ is of σ-transcendental type.

If $\{a_\eta\}$ is of σ-algebraic type over K, then a minimal σ-polynomial of $\{a_\eta\}$ over K is a σ-polynomial $P(x)$ over K with the following properties:

(i) $P(b_\eta) \rightsquigarrow 0$ for some pc-sequence $\{b_\eta\}$ in K equivalent to $\{a_\eta\}$;

(ii) $Q(b_\eta) \not\rightsquigarrow 0$ whenever $Q(x)$ is σ-polynomial over K of lower complexity than $P(x)$ and $\{b_\eta\}$ is a pc-sequence in K equivalent to $\{a_\eta\}$.

Lemma 5.13. *Suppose $\mathcal{K}$ satisfies Axiom 2. Let $\{a_\eta\}$ from K be a pc-sequence of σ-algebraic type over K with minimal σ-polynomial $P(x)$ over K, and with pseudolimit a in some extension. Let Σ be a finite set of σ-polynomials $Q(x)$ over K. Then there is a pc-sequence $\{b_\eta\}$ in K, equivalent to $\{a_\eta\}$, such that, with $\gamma_\eta := v(a - a_\eta)$:*

(I) *$v(a - b_\eta) = \gamma_\eta$, eventually, and $P(b_\eta) \rightsquigarrow 0$;*

(II) *if $Q \in \Sigma$ and $Q \notin K$, then $Q(b_\eta) \rightsquigarrow Q(a)$;*

(III) *(P, b_η) is in σ-hensel configuration with $\gamma(P, b_\eta) = \gamma_\eta$, eventually;*

(IV) *if $P(a) \neq 0$, then (P, a) is in σ-hensel configuration with $\gamma(P, a) > \gamma_\eta$ eventually.*

If $\rho > 1$, then (P, b_η) is actually in strict σ-hensel configuration. Also there is some a', pseudolimit of $\{a_\eta\}$, such that $(I), (II)$ and (IV) hold with a replaced by a', and either $P(a') = 0$ or (P, a') is in strict σ-hensel configuration with $\gamma(P, a') > \gamma_\eta$ eventually.

Proof. Let P have order n. Let us augment Σ with all $P_{(\boldsymbol{I})}$ for $1 \leq |\boldsymbol{I}| \leq \deg(P)$. In the rest of the proof, all multi-indices range over $\mathbb{N}^{n+1}$. Also since P is a minimal polynomial of $\{a_\eta\}$, there is an equivalent sequence $\{c_\eta\}$ such that $P(c_\eta) \rightsquigarrow 0$.

Now if ρ is transcendental, then by Theorem 4.4, $Q(c_\eta) \rightsquigarrow Q(a)$ for all $Q \in \Sigma$ and $Q \notin K$. Let $b_\eta := c_\eta$. Thus, $\{b_\eta\}$ satisfies (I) and (II). And if ρ is algebraic, then by Theorem 4.5, there is a pc-sequence $\{b_\eta\}$, equivalent to $\{c_\eta\}$ (and hence to $\{a_\eta\}$), such that (I) and (II) hold. Theorem 4.4 also shows that in the transcendental case, there is a unique $\boldsymbol{L}_0$ such that eventually for all $\boldsymbol{I} \neq \boldsymbol{L}_0$,

$$v(P(b_\eta) - P(a)) = v(P_{(\boldsymbol{L}_0)}(a)) + |\boldsymbol{L}_0|_\rho \cdot \gamma_\eta < v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta,$$

and in the algebraic case there is a unique m_0 such that eventually for all $\boldsymbol{I}$ with $|\boldsymbol{I}|_\rho \neq m_0$,

$$v(P(b_\eta) - P(a)) = \min_{|\boldsymbol{L}_0|_\rho = m_0} v(P_{(\boldsymbol{L}_0)}(a)) + m_0 \cdot \gamma_\eta < v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta. \tag{1}$$

We will show that in either case $|\boldsymbol{L}_0| = 1$. Since for $\rho > 1$, there is a unique $\boldsymbol{L}_0$ such that $|\boldsymbol{L}_0|_\rho = m_0$ and $|\boldsymbol{L}_0| = 1$, this gives us that for $\rho > 1$ (both algebraic and transcendental), there is a unique $\boldsymbol{L}_0$ such that eventually for all $\boldsymbol{I} \neq \boldsymbol{L}_0$,

$$v(P(b_\eta) - P(a)) = v(P_{(\boldsymbol{L}_0)}(a)) + |\boldsymbol{L}_0|_\rho \cdot \gamma_\eta < v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta. \tag{2}$$

This actually gives the strict σ-hensel configuration of (P, b_η) for $\rho > 1$.

For any $\boldsymbol{I}$ such that $P_{(\boldsymbol{I})} \neq 0$, we claim that if $\boldsymbol{I} < \boldsymbol{J}$, then

$$v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta < v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J}|_\rho \cdot \gamma_\eta$$

eventually: Theorem 4.4 with $\Sigma = \{P, P_{(\boldsymbol{I})}\}$ shows that we can arrange that our sequence $\{b_\eta\}$ also satisfies

$$v(P_{(\boldsymbol{I})}(b_\eta) - P_{(\boldsymbol{I})}(a)) \leq v(P_{(\boldsymbol{I})(\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma_\eta,$$

eventually for all $\boldsymbol{L}$ with $|\boldsymbol{L}| \geq 1$. Since $v(P_{(\boldsymbol{I})}(b_\eta)) = v(P_{(\boldsymbol{I})}(a))$ eventually (as P is a minimal polynomial for $\{a_\eta\}$), this yields

$$v(P_{(\boldsymbol{I})}(a)) \leq v(P_{(\boldsymbol{I})(\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma_\eta = v(P_{(\boldsymbol{I}+\boldsymbol{L})}(a)) + |\boldsymbol{L}|_\rho \cdot \gamma_\eta.$$

For $\boldsymbol{L}$ with $\boldsymbol{I} + \boldsymbol{L} = \boldsymbol{J}$, this yields

$$v(P_{(\boldsymbol{I})}(a)) \leq v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J} - \boldsymbol{I}|_\rho \cdot \gamma_\eta.$$

As $\{\gamma_\eta\}$ is increasing, we have eventually in η,

$$v(P_{(\boldsymbol{I})}(a)) < v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J} - \boldsymbol{I}|_\rho \cdot \gamma_\eta.$$

Since eventually $v(P_{(\boldsymbol{I})}(b_\eta)) = v(P_{(\boldsymbol{I})}(a))$, we have

$$\begin{aligned} v(P_{(\boldsymbol{I})}(b_\eta)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta &< v(P_{(\boldsymbol{J})}(b_\eta)) + |\boldsymbol{J}|_\rho \cdot \gamma_\eta, \quad \text{and} \\ v(P_{(\boldsymbol{I})}(a)) + |\boldsymbol{I}|_\rho \cdot \gamma_\eta &< v(P_{(\boldsymbol{J})}(a)) + |\boldsymbol{J}|_\rho \cdot \gamma_\eta \end{aligned}$$

It follows that $|\boldsymbol{L_0}| = 1$ (for $\rho = 1$, this means $m_0 = 1$). In particular, we have established (1) with $m_0 = 1$ for $\rho = 1$, and (2) for $\rho > 1$. Since $P(b_\eta) \rightsquigarrow 0$, this yields $v(P(a)) > v(P(b_\eta))$ eventually, i.e., $v(P(b_\eta) - P(a)) = v(P(b_\eta))$. It follows from this and (1) that (P, b_η) is in σ-hensel configuration eventually with $\gamma(P, b_\eta) = \gamma_\eta$; and it follows from (2) that for $\rho > 1$, (P, b_η) is in strict σ-hensel configuration.

Finally by Step 2 of Lemma 5.4, it follows that if $P(a) \neq 0$, then (P, a) is also in σ-hensel configuration with $\gamma(P, a) > \gamma_\eta$ eventually. And for $\rho > 1$, if (P, a) is already in strict σ-hensel configuration, we are done. Otherwise follow the proof of Lemma 5.5 to find the required a'. □

6. Immediate Extensions

Throughout this section, $\mathcal{K} = (K, \Gamma, k; v, \pi)$ is a multiplicative valued difference field satisfying Axiom 2. Note that then any immediate extension of $\mathcal{K}$ also satisfies Axiom 2. We state here a few basic facts on immediate extensions.

Lemma 6.1. *Let $\{a_\eta\}$ from K be a pc-sequence of σ-transcendental type over K. Then $\mathcal{K}$ has a proper immediate extension $(K\langle a\rangle, \Gamma, k; v_a, \pi_a)$ such that:*

(1) *a is σ-transcendental over K and $a_\eta \rightsquigarrow a$;*
(2) *for any extension $(K_1, \Gamma_1, k_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $a_\eta \rightsquigarrow b$, there is a unique embedding*
$$(K\langle a\rangle, \Gamma, k; v_a, \pi_a) \longrightarrow (K_1, \Gamma_1, k_1; v_1, \pi_1)$$
over K that sends a to b.

Proof. See [3, Lemma 5.2]. (All that is needed in the proof is the pseudo-continuity of the σ-polynomials (upto equivalent sequences). So the same proof works here.) □

As a consequence of both (1) and (2) of Lemma 6.1, we have:

Corollary 6.2. *Let a from some extension of $\mathcal{K}$ be σ-algebraic over K and let $\{a_\eta\}$ be a pc-sequence in K such that $a_\eta \rightsquigarrow a$. Then $\{a_\eta\}$ is of σ-algebraic type over K.*

Lemma 6.3. *Let $\{a_\eta\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudolimit in K. Let $P(x)$ be a minimal σ-polynomial of $\{a_\eta\}$ over K. Then $\mathcal{K}$ has a proper immediate extension $(K\langle a\rangle, \Gamma, k; v_a, \pi_a)$ such that:*

(1) *$P(a) = 0$ and $a_\eta \rightsquigarrow a$;*
(2) *for any extension $(K_1, \Gamma_1, k_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $P(b) = 0$ and $a_\eta \rightsquigarrow b$, there is a unique embedding*
$$(K\langle a\rangle, \Gamma, k; v_a, \pi_a) \longrightarrow (K_1, \Gamma_1, k_1; v_1, \pi_1)$$
over K that sends a to b.

Proof. See [3, Lemma 5.3]. □

Definition 6.4. $\mathcal{K}$ is said to be σ-algebraically maximal if it has no proper immediate σ-algebraic extension; and $\mathcal{K}$ is said to be maximal if it has no proper immediate extension.

Corollary 6.5. (1) *$\mathcal{K}$ is σ-algebraically maximal if and only if each pc-sequence in K of σ-algebraic type over K has a pseudolimit in K;*
(2) *if $\mathcal{K}$ satisfies Axiom 3 and is σ-algebraically maximal, then $\mathcal{K}$ is σ-henselian.*

Proof. (1) The "only if" direction follows from Lemma 6.3. For the "if" direction, suppose for a contradiction that $\mathcal{K}_1 := (K_1, \Gamma, k; v_1, \pi_1)$ is a proper immediate σ-algebraic extension of $\mathcal{K}$. Since the extension is proper, there is $a \in K_1 \setminus K$. Since the extension is immediate, we can find a pc-sequence $\{a_\eta\}$ from K such that $a_\eta \rightsquigarrow a$. Since the extension is σ-algebraic, a is σ-algebraic over K. Then by Corollary 6.2, $\{a_\eta\}$ is of σ-algebraic type over K. So by assumption, there is $b \in K$ such that $a_\eta \rightsquigarrow b$. But then by part (2) of Lemma 6.3, we have
$$(K\langle a\rangle, \Gamma, k; v_a, \pi_a) \cong (K\langle b\rangle, \Gamma, k; v_b, \pi_b) \cong (K, \Gamma, k; v, \pi),$$
i.e., $a \in K$, a contradiction.

(2) Let $P(x)$ be a σ-polynomial over K of order $\leq n$, and $a \in K$ be such that (P, a) is in σ-hensel configuration. If there is no $b \in K$ such that $v(b-a) = \gamma(P, a)$ and $P(b) = 0$, then by Lemma 5.6, there is a σ-algebraic pc-sequence $\{a_\eta\}$ in K such that $\{a_\eta\}$ has no pseudolimit in K. But then by part (1) of this corollary, K is not σ-algebraically maximal, a contradiction. □

It is clear that $\mathcal{K}$ has σ-algebraically maximal immediate σ-algebraic extensions, and also maximal immediate extensions. We will show that, provided $\mathcal{K}$ satisfies Axiom 3, both kinds of extensions are unique up to isomorphism.

Lemma 6.6. *Let $\mathcal{K}'$ be a σ-algebraically maximal extension of $\mathcal{K}$ satisfying Axiom 3. Let $\{a_\eta\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudolimit in K, and with minimal σ-polynomial $P(x)$ over K. Then there exists $b \in K'$ such that $a_\eta \rightsquigarrow b$ and $P(b) = 0$.*

Proof. By Corollary 6.5 (1), there exists $a \in K'$ such that $a_\eta \rightsquigarrow a$. If $P(a) = 0$, we are done. So let us assume $P(a) \neq 0$. Then by Lemma 5.13 (IV), (P, a) is in σ-hensel configuration with $\gamma(P, a) > v(a - a_\eta)$ eventually. Since $\mathcal{K}'$ satisfies Axiom 3, by Corollary 6.5 (2), there is $b \in K'$ such that $v(b-a) = \gamma(P, a)$ and $P(b) = 0$. Finally $v(b-a_\eta) = v(b-a+a-a_\eta) = v(a-a_\eta)$, since $v(b-a) = \gamma(P, a) > v(a-a_\eta)$. Thus, $a_\eta \rightsquigarrow b$. □

Together with Lemmas 6.1 and 6.3, this yields:

Theorem 6.7.
(1) *Suppose $\mathcal{K}'$ is a proper immediate σ-henselian extension of $\mathcal{K}$, and let $a \in K' \setminus K$. Let $\mathcal{K}_1$ be a σ-henselian extension of $\mathcal{K}$ satisfying Axiom 2, such that every pc-sequence from $\mathcal{K}_1$ of length at most card(Γ) has a pseudolimit in $\mathcal{K}_1$. Then $\mathcal{K}\langle a\rangle$ embeds in $\mathcal{K}_1$ over $\mathcal{K}$.*
(2) *Suppose $\mathcal{K}'$ is a proper immediate σ-henselian σ-algebraic extension of $\mathcal{K}$, and let $a \in K' \setminus K$. Let $\mathcal{K}_1$ be a σ-henselian extension of $\mathcal{K}$ satisfying Axiom 2, such that every pc-sequence of σ-algebraic type over $\mathcal{K}_1$ and of length at most card(Γ) has a pseudolimit in $\mathcal{K}_1$. Then $\mathcal{K}\langle a\rangle$ embeds in $\mathcal{K}_1$ over $\mathcal{K}$.*

Proof. (1) By [6, Theorem 1], there is a pc-sequence $\{a_\eta\}$ from K such that $a_\eta \rightsquigarrow a$ and $\{a_\eta\}$ has no pseudolimit in K. By assumption, there is $b \in K_1$ such that $a_\eta \rightsquigarrow b$.

If $\{a_\eta\}$ is of σ-transcendental type, then by Corollary 6.2, both a and b must be σ-transcendental over K. Now apply Lemma 6.1.

If $\{a_\eta\}$ is of σ-algebraic type, let $P(x)$ be a minimal polynomial for $\{a_\eta\}$. By Theorem 5.13, we get an equivalent pc-sequence $\{b_\eta\}$ from K with $b_\eta \rightsquigarrow a$, such that $P(b_\eta) \rightsquigarrow 0$, $P(b_\eta) \rightsquigarrow P(a)$, $P_{(\boldsymbol{L})}(b_\eta) \rightsquigarrow P_{(\boldsymbol{L})}(a)$ (but not to 0) for all $|\boldsymbol{L}| \geq 1$, (P, b_η) is in σ-hensel configuration eventually, and either $P(a) = 0$, or (P, a) is also in σ-hensel configuration with $\gamma(P, a) > \gamma(P, b_\eta)$ eventually. If $P(a) = 0$, we are done. Otherwise, since $\mathcal{K}'$ is σ-henselian, we have $a' \in K'$ such that $P(a') = 0$ and $v(a' - a) = \gamma(P, a)$. Since $\gamma(P, a) > \gamma(P, b_\eta)$ eventually, we have $b_\eta \rightsquigarrow a'$. Thus, in either case, we have $a' \in K'$ such that $P(a') = 0$ and $b_\eta \rightsquigarrow a'$. Similarly, we get $b' \in K_1$ such that $b_\eta \rightsquigarrow b'$ and $P(b') = 0$.

By Lemma 6.3, $K\langle a'\rangle$ is isomorphic to $K\langle b'\rangle$ as multiplicative valued fields over K, with a' mapped to b'.

Now, $\mathcal{K}'$ is immediate over $K\langle a'\rangle$. If a is not already in $K\langle a'\rangle$, then we may repeat the argument and conclude by a standard maximality argument.

(2) By Corollary 6.2, there is a pc-sequence $\{a_\eta\}$ from K of σ-algebraic type pseudoconverging to a, with no pseudolimit in K. Then the calculation in (1) works, since every extension or pc-sequence considered will be of σ-algebraic type. □

Corollary 6.8. *Suppose $\mathcal{K}$ satisfies Axiom 3. Then all its maximal immediate extensions are isomorphic over $\mathcal{K}$, and all its σ-algebraically maximal immediate σ-algebraic extensions are isomorphic over $\mathcal{K}$.*

Proof. We have already noticed the existence of both kinds of maximal immediate extensions. By Corollary 6.5 (2), they are also σ-henselian. The desired uniqueness then follows by a standard maximality argument using Theorem 6.7 (1) and (2). □

We now state minor variants of the last two results using the notion of saturation from model theory.

Lemma 6.9. *Let $\mathcal{K}'$ be a $|\Gamma|^+$-saturated σ-henselian extension of $\mathcal{K}$. Let $\{a_\eta\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudolimit in K, and with minimal σ-polynomial $P(x)$ over K. Then there exists $b \in K'$ such that $a_\eta \rightsquigarrow b$ and $P(b) = 0$.*

Proof. By the saturation assumption, there is a pseudolimit $a \in \mathcal{K}'$ of $\{a_\eta\}$. If $P(a) = 0$, we are done. So let's assume $P(a) \neq 0$. But then by Lemma 5.13 (IV), (P, a) is in σ-hensel configuration with $\gamma(P, a) > v(a - a_\eta)$ eventually. Since $\mathcal{K}'$ is σ-henselian, there is $b \in K'$ such that $v(b - a) = \gamma(P, a)$ and $P(b) = 0$. Finally, $a_\eta \rightsquigarrow b$, since $v(b - a) = \gamma(P, a) > v(a - a_\eta)$. □

In combination with Lemmas 6.1 and 6.3, this yields:

Corollary 6.10. *Suppose that $\mathcal{K}$ satisfies Axiom 3 and $\mathcal{K}'$ is a $|\Gamma|^+$-saturated σ-henselian extension of $\mathcal{K}$. Let $\mathcal{K}^*$ be a maximal immediate extension of $\mathcal{K}$. Then $\mathcal{K}^*$ can be embedded in $\mathcal{K}'$ over $\mathcal{K}$.*

7. Example and Counter-example

We will now show the consistency of our axioms by building models for our theory. The canonical models we have in mind are the generalized power series fields $k((t^\Gamma))$, also known as the Hahn series. Given any difference field k of characteristic zero with automorphism $\bar{\sigma}$, and any MODAG Γ with automorphism $\sigma(\gamma) = \rho \cdot \gamma$, we form the multiplicative difference valued field $k((t^\Gamma))$ as follows.

As a set, $k((t^\Gamma)) := \{f : \Gamma \to k \mid \operatorname{supp}(f) := \{x \in \Gamma : f(x) \neq 0\}$ is well-ordered in the ordering induced by $\Gamma\}$. An element $f \in k((t^\Gamma))$ is thought of as a formal power series $f \leftrightarrow \sum_{\gamma\in\Gamma} f(\gamma)t^\gamma$. Addition, multiplication and valuation on $k((\Gamma))$ are defined the usual way. And, we define the automorphism on $k((\Gamma))$ as follows:

$$\sigma(f) \quad := \quad \sum_{\gamma\in\Gamma} \bar{\sigma}(f(\gamma))t^{\rho\cdot\gamma}$$

If we choose $\rho \geq 1$, $k((t^\Gamma))$ satisfies Axiom 1. Also if we impose that $\bar{\sigma}$ is a linear difference closed automorphism on k, then $k((t^\Gamma))$ satisfies Axiom 2 and Axiom 3 as well. Moreover, using the fact that $k((t^\Gamma))$ is maximally complete (see [7]), it follows from Corollary 5.9 that $k((t^\Gamma))$ is σ-henselian for $\rho \geq 1$, and strict σ-henselian for $\rho > 1$. Also note that the residue field of $k((t^\Gamma))$ is k, and the value group is Γ.

Now we will justify why we need Axiom 3 (at least for the case $\rho > 1$). We will provide an example that shows why Axiom 3 cannot be dropped. This example is adapted from [4, Example 5.11], which is in turn adapted from [6, Section 5].

Example 7.1. Let ρ be any element of a real-closed field and $\rho > 1$. Let Γ be the MODAG generated by ρ over $\mathbb{Z}$. Thus we construe Γ as the ordered difference group $\mathbb{Z}[\rho, \rho^{-1}]$ with the order induced by the cut of ρ. Let k be any field of characteristic zero, construed as a difference field equipped with its identity automorphism. And let $\mathcal{K}$ be the multiplicative valued difference field $(k((t^\Gamma)), \Gamma, k; v, \pi)$.

For each n, let $\Gamma_n := \rho^{-n}\mathbb{Z}[\rho]$ and let $\mathcal{K}_n$ be the multiplicative valued difference field $k((t^{\Gamma_n}), \Gamma_n, k; v, \pi)$. Let

$$\mathcal{K}_\infty := \Big(\bigcup_n k((t^{\Gamma_n})), \Gamma, k; v, \pi\Big).$$

Then $\mathcal{K}_\infty$ equipped with the restriction of σ, is a valued difference subfield of $\mathcal{K}$ and σ is multiplicative. Let us define a sequence $\{a_n\}$ as follows:

$$a_n = \sum_{i=1}^{n} t^{-\rho^{-i}}.$$

We claim that $\{a_n\}$ is a pc-sequence : Note that since $\rho > 1$, we have for $i < j \in \mathbb{N}$, $v(t^{-\rho^{-i}}) = -\rho^{-i} < -\rho^{-j} = v(t^{-\rho^{-j}})$. Hence, $v(a_{n+1} - a_n) = v(t^{-\rho^{-(n+1)}}) = -\rho^{-(n+1)}$, which is increasing as $n \to \infty$.

Also it is clear that $\{a_n\}$ has no pseudolimit in $\mathcal{K}_\infty$. Moreover, since $\sigma(t^{-\rho^{-i}}) = t^{-\rho^{-i+1}}$, we have for

$$P(x) = \sigma(x) - x - t^{-1},$$

$P(a_n) = t^{-\rho^{-n}} \rightsquigarrow 0$, and hence $\{a_n\}$ is of σ-algebraic type over $\mathcal{K}_\infty$. Now $\mathcal{K}_\infty$ is a union of henselian valued fields, and hence is henselian. Moreover it is of characteristic zero. Hence $\mathcal{K}_\infty$ is algebraically maximal. Therefore, $P(x)$ is a minimal σ-polynomial of $\{a_n\}$ over $\mathcal{K}_\infty$. Also,

$$P(a_n) + 1 \rightsquigarrow 0,$$

and so $P(x)+1$ is a minimal σ-polynomial of $\{a_n\}$ over $\mathcal{K}_\infty$ as well. By Lemma 6.3, there are immediate extensions $\mathcal{K}_\infty\langle a\rangle, \mathcal{K}_\infty\langle a'\rangle$ of $\mathcal{K}_\infty$ such that $a_n \rightsquigarrow a$, $P(a) = 0$, and $a_n \rightsquigarrow a'$, $P(a') + 1 = 0$. Let $\mathcal{L}_1, \mathcal{L}_2$ be σ-algebraically maximal, immediate, σ-algebraic extensions of $\mathcal{K}_\infty\langle a\rangle, \mathcal{K}_\infty\langle a'\rangle$ respectively.

Now we claim that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not isomorphic over $\mathcal{K}_\infty$. Suppose for a contradiction that they are isomorphic. Then there is $b \in \mathcal{L}_1$ such that $P(b)+1 = 0$. Since $P(a) = 0$ we have

$$Q(a, b) := \sigma(b - a) - (b - a) + 1 = 0.$$

We claim that this is only possible when $v(b - a) = 0$: if $v(b - a) > 0$, then since $\rho > 1$, we have $v(\sigma(b - a)) > v(b - a) > 0 = v(1)$. Hence, $v(Q(a, b)) = v(1) = 0$, and thus $Q(a, b) \neq 0$, a contradiction; similarly, if $v(b - a) < 0$, then $v(\sigma(b - a)) < v(b - a) < 0 = v(1)$, in which case again $v(Q(a, b)) = 0$, a contradiction. Thus, $v(b - a) = 0$.

But then, $\overline{b - a} \in k$ and $\overline{b - a}$ is a solution of

$$\bar{\sigma}(x) - x + 1 = 0,$$

which is impossible since $\bar{\sigma} = id$, contradiction.

Here we considered a particular instance of failure of Axiom 3; namely, when $\bar{\sigma}$ is the identity, the above $\bar{\sigma}$-linear equation does not have a solution in k. However, one can produce a similar construction for any non-degenerate inhomogeneous $\bar{\sigma}$-linear equation which does not have a solution in k.

8. Extending Residue Field and Value Group

Let $\mathcal{K} = (K, \Gamma, k; v, \pi)$ and $\mathcal{K}' = (K', \Gamma', k'; v', \pi')$ be two multiplicative valued difference fields with $ptp_{\mathbb{Z}[\rho]}(\Gamma) = ptp_{\mathbb{Z}[\rho]}(\Gamma')$. Let $\mathcal{O}$ and $\mathcal{O}'$ be their respective ring of integers, and let σ denote both their difference operators. Let $\mathcal{E} = (E, \Gamma_E, k_E; v, \pi)$ be a common multiplicative valued difference subfield of both $\mathcal{K}$ and $\mathcal{K}'$, that is, $\mathcal{E} \leq \mathcal{K}, \mathcal{E} \leq \mathcal{K}'$. Then we have:

Lemma 8.1. *Let $a \in \mathcal{O}$ and assume $\alpha = \pi(a)$ is $\bar{\sigma}$-transcendental over k_E. Then*

- *$v(P(a)) = \min_{\boldsymbol{L}}\{v(b_{\boldsymbol{L}})\}$ for each σ-polynomial $P(x) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ over E;*
- *$v(E\langle a\rangle^\times) = v(E^\times) = \Gamma_E$, and $\mathcal{E}\langle a\rangle$ has residue field $k_E\langle\alpha\rangle$;*
- *if $b \in \mathcal{O}'$ is such that $\beta = \pi(b)$ is $\bar{\sigma}$-transcendental over k_E, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ over $\mathcal{E}$ sending a to b.*

Proof. See [3, Lemma 2.5]. □

Lemma 8.2. *Let $P(x)$ be a non-constant σ-polynomial over the valuation ring of E whose reduction $\bar{P}(x)$ has the same complexity as $P(x)$. Let $a \in \mathcal{O}, b \in \mathcal{O}'$, and assume that $P(a) = 0, P(b) = 0$, and that $\bar{P}(x)$ is a minimal $\bar{\sigma}$-polynomial of $\alpha := \pi(a)$ and of $\beta := \pi'(b)$ over k_E. Then*

- *$\mathcal{E}\langle a\rangle$ has value group $v(E^\times) = \Gamma_E$ and residue field $k_E\langle\alpha\rangle$;*
- *if there is a difference field isomorphism $k_E\langle\alpha\rangle \to k_E\langle\beta\rangle$ over k_E sending α to β, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ over $\mathcal{E}$ sending a to b.*

Proof. See [3, Lemma 2.6]. □

Now we will show how to extend the value group. Recall that Γ is a model of MODAG. Before stating the results, we need a couple of definitions.

Definition 8.3. For a given σ-polynomial $P(x) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ over K and $a \in K$, we say a is generic for P if $v(P(a)) = \min\{v(b_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot v(a)\}$.

Definition 8.4. An element $a \in K$ (or K') is said to be generic over $\mathcal{E}$ if a is generic for all σ-polynomials $P(x) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ over E.

Lemma 8.5. *Assume $\mathcal{K}$ satisfies Axiom 2. Then, for each $\gamma \in \Gamma$ and $P(x) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ over K, there is $a \in K$ such that $v(a) = \gamma$ and a is generic for P.*

Proof. Let $c \in K$ be such that $v(c) = \gamma$. If c is already generic for P, set $a := c$ and we are done. Otherwise, pick $\epsilon \in K$ such that $v(\epsilon) = 0$ (we will decide later how to choose ϵ) and set $a := c\epsilon$. Note that $v(a) = v(c) = \gamma$. Then, $P(a) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(c)^{\boldsymbol{L}} \boldsymbol{\sigma}(\epsilon)^{\boldsymbol{L}}$. Choosing $d \in K^\times$ such that $v(d) = \min\{v(b_{\boldsymbol{L}} \boldsymbol{\sigma}(c)^{\boldsymbol{L}})\} = \min\{v(b_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot \gamma\}$, we can write $P(a) = dQ_P(\epsilon)$, where $Q_P(\epsilon)$ is over the ring of integers of K. Consider $\overline{Q_P}(\bar{\epsilon})$, a $\bar{\sigma}$-polynomial over k; choose $\bar{\epsilon} \in k$ such that $\overline{Q_P}(\bar{\epsilon}) \neq 0$, which is possible since $\mathcal{K}$ satisfies Axiom 2. Let $\epsilon \in K$ be such that $\pi(\epsilon) = \bar{\epsilon}$. Then $v(Q_P(\epsilon)) = 0$, and hence $v(P(a)) = v(d) = \min\{v(b_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot \gamma\}$. Thus, a is generic for P and $v(a) = \gamma$. □

Remark 8.6. It is clear from the proof of Lemma 8.5 that if we replace $P(x)$ by a finite set of σ-polynomials $\{P_1(x), \ldots, P_m(x)\}$ of possibly different orders, then by choosing $\bar{\epsilon} \in k$ such that it doesn't solve any of the related m equations $\overline{Q_{P_i}}(x) = 0$ over k (which is again possible to do as $\mathcal{K}$ satisfies Axiom 2), we can find $a \in K$ such that a is generic for $\{P_1, \ldots, P_m\}$.

Definition 8.7. Let $P(x) = \sum b_{\boldsymbol{L}} \boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ be a σ-polynomial over K and $a \in K$. Write $P(ax) = dQ_P(x)$, where $d \in K$ is such that $v(d) = \min\{v(b_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot v(a)\}$. Then $Q_P(x)$ is a σ-polynomial over $\mathcal{O}_K$, and thus $\overline{Q_P}(x)$ is a $\bar{\sigma}$-polynomial over k. We say $\overline{Q_P}(x)$ is a k-$\bar{\sigma}$-polynomial corresponding to (P, a).

Lemma 8.8. *Let $\gamma \in \Gamma \setminus \Gamma_E$. Let κ be an infinite cardinal such that $|k_E| \leq \kappa$. Assume $\mathcal{K}, \mathcal{K}'$ satisfy Axiom 2 and are κ^+-saturated. Then*

(i) *there is $a \in K$, generic over $\mathcal{E}$, with $v(a) = \gamma$;*
(ii) *$E\langle a \rangle$ has value group $\Gamma_E\langle\gamma\rangle$, and residue field $k_{E\langle a \rangle}$ of size $\leq \kappa$;*
(iii) *if $\gamma' \in \Gamma'$ is such that there is a valued difference group isomorphism $\Gamma_E\langle\gamma\rangle \to \Gamma_E\langle\gamma'\rangle$ over Γ_E (in the language of* MODAG*), and $a' \in K'$ is such that a' is generic over $\mathcal{E}$ with $v(a') = \gamma'$, then there is a valued difference field isomorphism $\mathcal{E}\langle a \rangle \to \mathcal{E}\langle a' \rangle$ over $\mathcal{E}$ sending a to a'.*

Proof. (i) Fix $c \in K$ such that $v(c) = \gamma$. For each σ-polynomial $P(x)$ over E, let $\overline{Q_P}(x)$ be a k-$\bar{\sigma}$-polynomial corresponding to (P, c), and define

$$\varphi_P(y) := \overline{Q_P}(y) \neq 0$$

i.e. $\varphi_P(y)$ is the first-order formula with parameters from k saying "y is not a root of $\overline{Q_P}$". Let

$$p(y) \quad := \quad \{\varphi_P(y) \mid P \text{ is a } \sigma\text{-polynomial over } E\}.$$

By Axiom 2, $p(y)$ is finitely consistent, and hence consistent. So it is a type over E. Moreover by cardinality considerations, $|p(y)| \leq \kappa^{<\omega} = \kappa$ (since κ is infinite). Since $\mathcal{K}$ is κ^+-saturated, $p(y)$ is realized in $\mathcal{K}$. In particular, there is $y \in k$ such that y is not a root of any $\overline{Q_P}$. Choosing $\epsilon \in \mathcal{O}$ with $\pi(\epsilon) = y$ and setting $a := c\epsilon$, we then have that $v(a) = \gamma$ and a is generic for all σ-polynomials $P(x)$ over E, i.e., a is generic over $\mathcal{E}$.

(ii) Since a is generic over $\mathcal{E}$, it is clear that $v(E\langle a \rangle^\times) = \Gamma_E\langle\gamma\rangle$, which clearly has size at most κ. Moreover, since each element of the residue field comes from an element of valuation zero, the size of $k_{E\langle a \rangle}$ is at most the size of the set $\{P(x) \mid P \text{ is a } \sigma\text{-polynomial over } E \text{ and } v(P(a)) = 0\}$, which, again by cardinality considerations, is at the most κ. Thus $|k_{E\langle a \rangle}| \leq \kappa$.

(iii) Finally notice that if b is generic over $\mathcal{E}$, then $v(P(b)) \neq \infty$ for any σ-polynomial $P(x)$ over E; i.e., $P(b) \neq 0$. So b is σ-transcendental over E. In particular, a and a' are σ-transcendental over E. Thus there is a difference field isomorphism $\psi : E\langle a \rangle \to E\langle a' \rangle$ over E sending a to a'. But, since $v(a) = \gamma$, $v(a') = \gamma'$, $\Gamma_E\langle\gamma\rangle \cong \Gamma_E\langle\gamma'\rangle$ over Γ_E, and a and a' are both generic over $\mathcal{E}$, the valuations are already determined and matched up on both sides, i.e., ψ is actually a valued difference field isomorphism. $\square$

9. Embedding Theorem

To prove completeness and relative quantifier elimination of the theory of multiplicative valued difference fields, we use a standard back and forth test, see [13, Proposition 4.3.28]. To that end we would like to know when can we extend isomorphism between "small" substructures, and the main theorem of this section, Theorem 9.4, gives an answer to that question.

For the moment we will work in a 4-sorted language $\mathcal{L}_{4vdf}$, where we have our usual 3 sorts for the valued field K, the value group Γ and the residue field k, and we add to it a fourth sort called the residue-valuation structure RV. This represents the language of the leading terms introduced in [8], and explained further in [9] and [10]. We replace the symbol for the induced automorphism $\tilde{\sigma}$ on the value group Γ by $\rho\cdot$. We could have just worked with a 2-sorted language with K and RV (we call this the leading term language). But the 2-sorted language is interpretable in and also interprets the 4-sorted language. So to make things more transparent we stick to the 4-sorted language. Before we proceed further, we would like to recall some preliminaries about the RVs. Recall that we are always dealing with the equi-characteristic zero case.

Preliminaries

Let $\mathcal{K} = (K, \Gamma, k; v, \pi)$ be a multiplicative valued difference field. Let $\mathcal{O}$ be the ring of integers, and $\mathfrak{m}$ be its maximal ideal. Let $K^\times$, $\mathcal{O}^\times$ and $k^\times$ denote the set of units of K, $\mathcal{O}$ and k respectively. It is easy to see that $(1+\mathfrak{m})$ is a subgroup of $K^\times$ under multiplication. We denote the factor group as $\mathrm{RV} := K^\times/1+\mathfrak{m}$ and the natural quotient map as $\mathrm{rv} : K^\times \to \mathrm{RV}$. To extend the map to whole of K, we introduce a new symbol "∞" (as we do with value groups) and define $\mathrm{rv}(0) = \infty$. Though RV is defined merely as a group, it inherits much more structure from $\mathcal{K}$.

To start with, since the valuation v on $\mathcal{K}$ is given by the exact sequence

$$1 \longrightarrow \mathcal{O}^\times \xrightarrow{\iota} K^\times \xrightarrow{v} \Gamma \longrightarrow 0$$

and since $1+\mathfrak{m} \leq \mathcal{O}^\times$, the valuation descends to RV giving the following exact sequence

$$1 \longrightarrow k^\times \xrightarrow{\iota} \mathrm{RV} \xrightarrow{v_{\mathrm{rv}}} \Gamma \longrightarrow 0$$

(note that $\mathcal{O}^\times/1+\mathfrak{m} = (\mathcal{O}/\mathfrak{m})^\times = k^\times$). In fact, it follows straight from the definitions that,

Lemma 9.1. *For all non-zero $x, y \in K$, the following are equivalent:*

(1) $\mathrm{rv}(x) = \mathrm{rv}(y)$
(2) $v(x-y) > v(y)$
(3) $\pi(x/y) = 1$

Proof. See [10, Proposition 1.3.3].

Also note that if $x, y \in \mathcal{O}$, then the last condition is equivalent to saying $\pi(x) = \pi(y)$. And both (2) and (3) imply that $v(x) = v(y)$. □

Now $\sigma(\mathcal{O}) = \mathcal{O}$ implies $\sigma(\mathfrak{m}) = \mathfrak{m}$: $v(x) > 0 \iff v(x^{-1}) < 0 \iff v(\sigma(x^{-1})) < 0 \iff -v(\sigma(x)) < 0 \iff v(\sigma(x)) > 0$. Thus, the difference operator on K induces a difference operator (which we call by σ_{rv}) on RV as follows:

$$\sigma_{\mathrm{rv}}(\mathrm{rv}(x)) := \mathrm{rv}(\sigma(x)).$$

It trivially follows that the induced σ is also multiplicative with the same ρ.

RV also inherits an image of addition from K via the relation

$$\oplus(x^1_{\mathrm{rv}},\dots,x^n_{\mathrm{rv}},z_{\mathrm{rv}}) \;=\; \exists x^1,\dots,x^n,z\in K\ (x^1_{\mathrm{rv}}=\mathrm{rv}(x^1)\ \wedge\ \cdots\ \wedge\ x^n_{\mathrm{rv}}=\mathrm{rv}(x^n)$$
$$\wedge\ z_{\mathrm{rv}}=\mathrm{rv}(z)\ \wedge\ x^1+\cdots+x^n=z).$$

The sum $x^1_{\mathrm{rv}}+\cdots+x^n_{\mathrm{rv}}$ is said to be well-defined (and $=z_{\mathrm{rv}}$) if there is exactly one z_{rv} such that $\oplus(x^1_{\mathrm{rv}},\dots,x^n_{\mathrm{rv}},z_{\mathrm{rv}})$ holds. Unfortunately this is not always the case. Fortunately, the sum is well-defined when it is "expected" to be. Formally,

Lemma 9.2. $\mathrm{rv}(x_1)+\cdots+\mathrm{rv}(x_n)$ *is well-defined* (*and is equal to* $\mathrm{rv}(x_1+\cdots+x_n)$) *if and only if* $v(x_1+\cdots+x_n)=\min\{v(x_1),\dots,v(x_n)\}$.

Proof. See [10, Proposition 1.3.6, 1.3.7 and 1.3.8]. □

Thus, we construe RV as a structure in the language $\mathcal{L}_{\mathrm{rv},\sigma_{\mathrm{rv}}}:=\{\cdot,^{-1},\oplus,1,v,\sigma_{\mathrm{rv}}\}$. And finally, we have

Proposition 9.3. Γ *and* k *are interpretable in* RV.

Proof. See [10, Proposition 3.1.4]. Note that the proof there is done for valued fields. For our case, the difference operator on Γ is interpreted in terms of the difference operator on RV as $\rho\cdot v_{\mathrm{rv}}(x)=v_{\mathrm{rv}}(\sigma_{\mathrm{rv}}(x))$. And since the nonzero elements $\bar{x}$ of the residue field are in bijection with $x\in\mathrm{RV}$ such that $v_{\mathrm{rv}}(x)=0$, $\bar{\sigma}(\bar{x})$ is interpreted in the obvious way as $\sigma_{\mathrm{rv}}(x)$. □

Now we describe the embeddings. Let $\mathcal{K}=(K,\Gamma,k,\mathrm{RV};v,\pi,v_{\mathrm{rv}},\iota,\mathrm{rv})$ and $\mathcal{K}'=(K',\Gamma',k',\mathrm{RV}';v',\pi',v'_{\mathrm{rv}},\iota',\mathrm{rv}')$ be two σ-henselian multiplicative valued difference fields of equal characteristic zero, satisfying Axiom 1 with $ptp_{\mathbb{Z}[\rho]}(\Gamma)=ptp_{\mathbb{Z}[\rho]}(\Gamma')$. By Lemma 5.10, $\mathcal{K}$ and $\mathcal{K}'$ satisfy Axiom 2 and Axiom 3. We denote the difference operator of both $\mathcal{K}$ and $\mathcal{K}'$ by σ, and their rings of integers by $\mathcal{O}$ and $\mathcal{O}'$ respectively. Let $\mathcal{E}=(E,\Gamma_E,k_E,\mathrm{RV}_E;v,\pi,v_{\mathrm{rv}},\iota,\mathrm{rv})$ and $\mathcal{E}'=(E',\Gamma'_{E'},k'_{E'},\mathrm{RV}'_{E'};v',\pi',v'_{\mathrm{rv}},\iota',\mathrm{rv}')$ be valued difference subfields of $\mathcal{K}$ and $\mathcal{K}'$ respectively. We say a bijection $\psi:E\to E'$ is an *admissible isomorphism* if it has the following properties:

(1) ψ is an isomorphism of multiplicative valued difference fields;
(2) the induced isomorphism $\psi_{\mathrm{rv}}:\mathrm{RV}_E\to\mathrm{RV}'_{E'}$ of RVs is *elementary*, i.e., for all formulas $\varphi(x_1,\dots,x_n)$ in $\mathcal{L}_{\mathrm{rv},\sigma_{\mathrm{rv}}}$, and $\xi_1,\dots,\xi_n\in\mathrm{RV}_E$,
$$\mathrm{RV}\models\varphi(\xi_1,\dots,\xi_n)\iff\mathrm{RV}'\models\varphi(\psi_{\mathrm{rv}}(\xi_1),\dots,\psi_{\mathrm{rv}}(\xi_n));$$
(3) the induced isomorphism $\psi_r:k_E\to k'_{E'}$ of difference fields is *elementary*, i.e., for all formulas $\varphi(x_1,\dots,x_n)$ in the language $\mathcal{L}_{R,\sigma}$ of difference rings, and $\alpha_1,\dots,\alpha_n\in k_E$,
$$k\models\varphi(\alpha_1,\dots,\alpha_n)\iff k'\models\varphi(\psi_r(\alpha_1),\dots,\psi_r(\alpha_n));$$
(4) the induced isomorphism $\psi_v:\Gamma_E\to\Gamma'_{E'}$ of MODAGs is elementary, i.e, for all formulas $\varphi(x_1,\dots,x_n)$ in $\mathcal{L}_{OG,\rho\cdot}$, and $\gamma_1,\dots,\gamma_n\in\Gamma_E$,
$$\Gamma\models\varphi(\gamma_1,\dots,\gamma_n)\iff\Gamma'\models\varphi(\psi_v(\gamma_1),\dots,\psi_v(\gamma_n)).$$

(Note that it is enough to maintain (1) and (2) above, since (3) and (4) are consequences of (2) because of Proposition 9.3).

Our main goal is to be able to extend such admissible isomorphisms. For this we need certain degree of saturation on $\mathcal{K}$ and $\mathcal{K}'$. Fix an infinite cardinal κ and let us assume that $\mathcal{K}$ and $\mathcal{K}'$ are κ^+-saturated. We then say a substructure $\mathcal{E} = (E, \Gamma_E, k_E, \mathrm{RV}_E; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ of $\mathcal{K}$ (respectively of $\mathcal{K}'$) is *small* if $|\Gamma_E|, |k_E| \leq \kappa$. While extending the isomorphism, we do it in steps and at each step we typically extend the isomorphism from some E to $E\langle a\rangle$, which is obviously small if E is; and then reiterate the process κ many times, which again preserves smallness. Eventually we reiterate this process countably many times and take union of an increasing sequence of countably many small fields, which also preserves smallness. Having said all that, we now state the Embedding Theorem.

Theorem 9.4 (Embedding Theorem)**.** *Suppose $\mathcal{K}, \mathcal{K}', \mathcal{E}, \mathcal{E}'$ are as above with $\mathcal{K}, \mathcal{K}'$ κ^+-saturated and $\mathcal{E}, \mathcal{E}'$ small. Assume $\psi : E \to E'$ is an admissible isomorphism and let $a \in K$. Then there exist $b \in K'$ and an admissible isomorphism $\psi' : E\langle a\rangle \cong E'\langle b\rangle$ extending ψ with $\psi(a) = b$.*

Proof. Note that the theorem is obvious if $\Gamma = \{0\}$. So let us assume that $\Gamma \neq \{0\}$. Also without loss of generality, we may assume $a \in \mathcal{O}_K$. We will extend the isomorphism in steps. Note that we have three cases to consider:

Case 1. There exists $c \in E\langle a\rangle$ such that $\pi(c) \in k \setminus k_E$;
Case 2. There exists $c \in E\langle a\rangle$ such that $v(c) \in \Gamma \setminus \Gamma_E$;
Case 3. For all $c \in E\langle a\rangle, \pi(c) \in k_E$ and $v(c) \in \Gamma_E$.

Step I: Extending the residue field

Let $c \in E\langle a\rangle$ be such that $\alpha := \pi(c) \notin k_E$. Since $k^\times \hookrightarrow \mathrm{RV}$, $\alpha \in \mathrm{RV}$. By saturation of $\mathcal{K}'$, we can find $\alpha' \in \mathrm{RV}'$ and an $\mathcal{L}_{\mathrm{rv}}$-isomorphism $\mathrm{RV}_E\langle\alpha\rangle \cong \mathrm{RV}'_{E'}\langle\alpha'\rangle$ extending ψ_{rv} and sending $r \mapsto r'$ that is elementary as a partial map between RV and RV$'$. Note that then $\alpha' \in k'$. Now we have two cases to consider.

Subcase I. α (respectively, α') is $\bar{\sigma}$-transcendental over k_E (respectively, $k'_{E'}$). In that case, pick $d \in \mathcal{O}$ and $d' \in \mathcal{O}'$ such that $\pi(d) = \alpha$ and $\pi(d') = \alpha'$. Then by Lemma 8.1, there is an admissible isomorphism $\mathcal{E}\langle d\rangle \cong \mathcal{E}'\langle d'\rangle$ extending ψ with small domain $(E\langle d\rangle, \Gamma_E, k_E\langle\alpha\rangle)$ and sending d to d'.

Subcase II. α is $\bar{\sigma}$-algebraic over k_E. Let $P(x)$ be a σ-polynomial over $\mathcal{O}_{\mathcal{E}}$ such that $\bar{P}(x)$ is a minimal $\bar{\sigma}$-polynomial of α. Pick $d \in \mathcal{O}$ such that $\pi(d) = \alpha$. If $P(d) \neq 0$ already, then (P, d) is in σ-hensel configuration with $\gamma(P, d) > 0$. Since $\mathcal{K}$ is σ-henselian, there is $e \in \mathcal{O}$ such that $P(e) = 0$ and $\pi(e) = \pi(d) = \alpha$. Likewise, there is $e' \in \mathcal{O}'$ such that $P^\psi(e') = 0$ and $\pi(e') = \alpha'$, where P^ψ is the difference polynomial over E' corresponding to P under ψ. Then by Lemma 8.2, there is an admissible isomorphism $\mathcal{E}\langle e\rangle \cong \mathcal{E}'\langle e'\rangle$ extending ψ with small domain $(E\langle e\rangle, \Gamma_E, k_E\langle\alpha\rangle)$ and sending e to e'.

Note that in either case, we have been able to extend the admissible isomorphism to a small domain that includes α. Since E is small, so is $E\langle a\rangle$, i.e., $|k_{E\langle a\rangle}| \leq \kappa$. Thus, by repeating Step I κ many times, we are able to extend the admissible isomorphism to a small domain $\mathcal{E}_1$ such that for all $c \in E\langle a\rangle$ with $\pi(c) \notin k_E$, we have $\pi(c) \in k_{E_1}$. Continuing this countably many times, we are able to build an increasing sequence of small domains $\mathcal{E} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_i \subset \cdots$ such that for each $c \in E_i\langle a\rangle$ with $\pi(c) \notin k_{E_i}$, we have $\pi(c) \in k_{E_{i+1}}$. Taking the union of these countably many small domains, we get a small domain, which we still call $\mathcal{E}$, such that ψ extends to an admissible isomorphism with domain E and for all $c \in E\langle a\rangle$, we have $\pi(c) \in k_E$, i.e., we are not in Case 1 anymore.

Step II: Extending the value group

Let $c \in E\langle a\rangle$ be such that $\gamma := v(c) \notin \Gamma_E$. Let $b \in K$ be generic over $\mathcal{E}$ with $v(b) = \gamma$. Let $r := \mathrm{rv}(b)$. By saturation of $\mathcal{K}'$, find $r' \in \mathrm{RV}'$ and an $\mathcal{L}_{\mathrm{rv}}$-isomorphism $\mathrm{RV}_E\langle r\rangle \cong \mathrm{RV}'_{E'}\langle r'\rangle$ extending ψ_{rv}, sending $r \mapsto r'$, that is elementary as a partial map between RV and RV$'$. Let $b' \in K'$ be such that $\mathrm{rv}'(b') = r'$.

We claim that b' is generic over $\mathcal{E}'$: for any $P(x) = \sum b_{\boldsymbol{L}}\boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ with $b_{\boldsymbol{L}} \in E'$, let $P^{\psi^{-1}}(x) = \sum a_{\boldsymbol{L}}\boldsymbol{\sigma}(x)^{\boldsymbol{L}}$ be the corresponding σ-polynomial over E with $a_{\boldsymbol{L}} \in E$ and $\psi(a_{\boldsymbol{L}}) = b_{\boldsymbol{L}}$. Since b is generic over $\mathcal{E}$, we have $v(P^{\psi^{-1}}(b)) = \min\{v(a_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot \gamma\}$, and hence by Lemma 9.2, we have

$$\mathrm{rv}(P^{\psi^{-1}}(b)) = \sum \mathrm{rv}(a_{\boldsymbol{L}}\boldsymbol{\sigma}(b)^{\boldsymbol{L}}) = \sum \mathrm{rv}(a_{\boldsymbol{L}})\boldsymbol{\sigma}(\mathrm{rv}(b))^{\boldsymbol{L}} = \sum \mathrm{rv}(a_{\boldsymbol{L}})\boldsymbol{\sigma}(r)^{\boldsymbol{L}}.$$

Then,

$$\begin{aligned} \mathrm{rv}'(P(b')) &= \psi_{\mathrm{rv}}(\mathrm{rv}(P^{\psi^{-1}}(b))) = \psi_{\mathrm{rv}}\Big(\sum \mathrm{rv}(a_{\boldsymbol{L}})\boldsymbol{\sigma}(r)^{\boldsymbol{L}}\Big) \\ &= \sum \mathrm{rv}'(b_{\boldsymbol{L}})\boldsymbol{\sigma}(r')^{\boldsymbol{L}} = \sum \mathrm{rv}'(b_{\boldsymbol{L}}\boldsymbol{\sigma}(b')^{\boldsymbol{L}}), \end{aligned}$$

and hence by Lemma 9.2 again, we have $v(P(b')) = \min\{v(b_{\boldsymbol{L}}) + |\boldsymbol{L}|_\rho \cdot v(b')\}$, i.e., b' is generic over $\mathcal{E}'$. Then by Lemma 8.8, since $\mathcal{K}$ satisfies Axiom 2, there is an admissible isomorphism from $\mathcal{E}\langle b\rangle \xrightarrow{\cong} \mathcal{E}'\langle b'\rangle$ extending ψ and sending $b \mapsto b'$.

Thus we have been able to extend the admissible isomorphism to a small domain that includes γ. Since E is small, so is $E\langle a\rangle$, i.e., $|\Gamma_{E\langle a\rangle}| \leq \kappa$. Thus, by repeating Step II κ many times, we are able to extend the admissible isomorphism to a small domain $\mathcal{E}_1$ such that for all $c \in E\langle a\rangle$ with $v(c) \notin \Gamma_E$, we have $v(c) \in \Gamma_{E_1}$. Continuing this countably many times, we are able to build an increasing sequence of small domains $\mathcal{E} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_i \subset \cdots$ such that for each $c \in E_i\langle a\rangle$ with $v(c) \notin \Gamma_{E_i}$, we have $v(c) \in \Gamma_{E_{i+1}}$. Taking the union of these countably many small domains, we get a small domain, which we still call $\mathcal{E}$, such that ψ extends to an admissible isomorphism with domain E and for all $c \in E\langle a\rangle$, we have $v(c) \in \Gamma_E$, i.e., we are not in Case 2 anymore.

Step III: Immediate Extension

After doing Steps I and II, we are reduced to the case when $E\langle a\rangle$ is an immediate extension of $\mathcal{E}$ where both fields are equipped with the valuation induced by $\mathcal{K}$. Let $\mathcal{E}\langle a\rangle$ be the valued difference subfield of $\mathcal{K}$ that has $E\langle a\rangle$ as the underlying field. In this situation, we would like to extend the admissible isomorphism, not just to $\mathcal{E}\langle a\rangle$, but to a maximal immediate extension of $\mathcal{E}\langle a\rangle$ and use Corollary 6.10. However, for that we need $\mathcal{E}$ to satisfy Axiom 2 and Axiom 3. Since Axiom 3 implies Axiom 2 by Lemma 5.10, it is enough to extend $\mathcal{E}$ such that it satisfies Axiom 3. Recall that $\mathcal{K}$ satisfies Axiom 3. Now to make $\mathcal{E}$ satisfy Axiom 3, for each linear $\bar{\sigma}$-polynomial $P(x)$ over k_E, if there is already no solution to $P(x)$ in k_E, find a solution $\alpha \in k$ and follow Step I. Since there are at most κ many such polynomials, we end up in a small domain. Thus, after doing all these, we can assume $\mathcal{E}$ satisfies Axiom 2 and Axiom 3. Let $\mathcal{E}^*$ be a maximal immediate valued difference field extension of $\mathcal{E}\langle a\rangle$. Then $\mathcal{E}^*$ is a maximal immediate extension of $\mathcal{E}$ as well. Similarly let $\mathcal{E}'^*$ be a maximal immediate extension of $\mathcal{E}'$. Since such extensions are unique by Corollary 6.8, and by Corollary 6.10 they can be embedded in $\mathcal{K}$ (respectively $\mathcal{K}'$) over $\mathcal{E}$ (respectively $\mathcal{E}'$) by saturatedness of $\mathcal{K}$ (respectively $\mathcal{K}'$), we have that ψ extends to a valued difference field isomorphism $\mathcal{E}^* \cong \mathcal{E}'^*$. Since $k_{E^*} = k_E$ and $\Gamma_{E^*} = \Gamma_E$, it follows by Snake Lemma on the following diagram that $\mathrm{RV}_{E^*} = \mathrm{RV}_E$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k_E^\times & \longrightarrow & \mathrm{RV}_E & \longrightarrow & \Gamma_E & \longrightarrow & 0\ . \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & k_{E^*}^\times & \longrightarrow & \mathrm{RV}_{E^*} & \longrightarrow & \Gamma_{E^*} & \longrightarrow & 0 \end{array}$$

Thus, the isomorphism is actually admissible. It remains to note that a is in the underlying difference field of $\mathcal{E}^*$. □

Remark 9.5. It follows from the above proof that we can impose more structure on the RV and still have the Embedding Theorem go through. We just need to re-define the admissible maps accordingly.

10. Completeness and Quantifier Elimination Relative to RV

We now state some model-theoretic consequences of Theorem 9.4. We use '$\equiv$' to denote the relation of elementary equivalence, and $\preccurlyeq$ to denote the relation of elementary submodel. Recall that we are working in the 4-sorted language $\mathcal{L}_{4vdf}$ with sorts K for the valued field, Γ for the value group, k for the residue field and RV for the residue-valuation structure. The language also has a function symbol σ going from the field sort to itself. Let $\mathcal{K} = (K, \Gamma, k, \mathrm{RV}; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ and $\mathcal{K}' = (K', \Gamma', k', \mathrm{RV}'; v', \pi', v'_{\mathrm{rv}}, \iota', \mathrm{rv}')$ be two σ-henselian multiplicative valued difference fields (in the 4-sorted language) of equi-characteristic zero satisfying Axiom 1 with $ptp_{\mathbb{Z}[\rho]}(\Gamma) = ptp_{\mathbb{Z}[\rho]}(\Gamma')$.

Theorem 10.1. *$\mathcal{K} \equiv_{\mathcal{L}_{4vdf}} \mathcal{K}'$ if and only if* $\mathrm{RV} \equiv_{\mathcal{L}_{\mathrm{rv},\sigma_{\mathrm{rv}}}} \mathrm{RV}'$.

Proof. The "only if" direction is obvious. For the converse, note that $(\mathbb{Q}, \{0\}, \mathbb{Q}, \mathbb{Q}; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$, with $v(q) = 0$, $\pi(q) = q$, $\mathrm{rv}(q) = q$, $v_{\mathrm{rv}}(q) = 0$ and $\iota(q) = q$ for all $0 \neq q \in \mathbb{Q}$, is a substructure of both $\mathcal{K}$ and $\mathcal{K}'$, and thus the identity map between these two substructures is an admissible isomorphism. Now apply Theorem 9.4. □

Theorem 10.2. *Let $\mathcal{E} = (E, \Gamma_E, k_E, \mathrm{RV}_E; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ be a σ-henselian multiplicative valued difference subfield of $\mathcal{K}$, satisfying Axiom 1, such that $\mathrm{RV}_E \preccurlyeq_{\mathcal{L}_{\mathrm{rv},\sigma_{\mathrm{rv}}}} \mathrm{RV}$. Then $\mathcal{E} \preccurlyeq_{\mathcal{L}_{4vdf}} \mathcal{K}$.*

Proof. Take an elementary extension $\mathcal{K}'$ of $\mathcal{E}$. Then $\mathcal{K}'$ satisfies Axiom 1, and is also σ-henselian. Moreover $(E, \Gamma_E, k_E, \mathrm{RV}_E; \cdots)$ is a substructure of both $\mathcal{K}$ and $\mathcal{K}'$, and hence the identity map is an admissible isomorphism. Hence, by Theorem 9.4, we have $\mathcal{K} \equiv_{\mathcal{E}} \mathcal{K}'$. Since $\mathcal{E} \preccurlyeq_{\mathcal{L}_{4vdf}} \mathcal{K}'$, this gives $\mathcal{E} \preccurlyeq_{\mathcal{L}_{4vdf}} \mathcal{K}$. □

Corollary 10.3. *$\mathcal{K}$ is decidable if and only if* RV *is decidable.*

The proofs of these results use only weak forms of the Embedding Theorem, but now we turn to a result that uses its full strength: a relative elimination of quantifiers for the theory of σ-henselian multiplicative valued difference fields of equi-characteristic 0 that satisfy Axiom 1.

Theorem 10.4. *Let T be the $\mathcal{L}_{4vdf}$-theory of σ-henselian multiplicative valued difference fields of equi-characteristic zero satisfying Axiom 1, and $\phi(x)$ be an $\mathcal{L}_{4vdf}$-formula. Then there is an $\mathcal{L}_{4vdf}$-formula $\varphi(x)$ in which all occurrences of field* *variables are free, such that*

$$T \vdash \phi(x) \leftrightarrow \varphi(x).$$

Proof. Let φ range over $\mathcal{L}_{4vdf}$-formulas in which all occurrences of field variables are free. For a model $\mathcal{K} = (K, \Gamma, k, \mathrm{RV}; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ of T and $a \in K^l$, $\gamma \in \Gamma^m$, $\alpha \in k^n$ and $r \in \mathrm{RV}^s$, let

$$\mathrm{fqftp}_{\mathcal{K}}(a, \gamma, \alpha, r) := \{\varphi : \mathcal{K} \models \varphi(a, \gamma, \alpha, r)\}.$$

Let $\mathcal{K}, \mathcal{K}'$ be models of T and suppose

$$(a, \gamma, \alpha, r) \in K^l \times \Gamma^m \times k^n \times \mathrm{RV}^s, \qquad (a', \gamma', \alpha', r') \in K'^l \times \Gamma'^m \times k'^n \times \mathrm{RV}'^s$$

are such that $\mathrm{fqftp}_{\mathcal{K}}(a, \gamma, \alpha, r) = \mathrm{fqftp}_{\mathcal{K}'}(a', \gamma', \alpha', r')$. It suffices to show that

$$tp_{\mathcal{K}}(a, \gamma, \alpha, r) = tp_{\mathcal{K}'}(a', \gamma', \alpha', r').$$

Let $\mathcal{E}$ (respectively $\mathcal{E}'$) be the multiplicative valued difference subfield of $\mathcal{K}$ (respectively $\mathcal{K}'$) generated by a, γ, α and r (respectively a', γ', α' and r'). Then there is an admissible isomorphism $\mathcal{E} \to \mathcal{E}'$ that maps $a \to a'$, $\gamma \to \gamma'$, $\alpha \to \alpha'$ and $r \to r'$. Now apply Theorem 9.4. □

11. Completeness and Quantifier Elimination Relative to (k, Γ)

Although the leading term language is already interpretable in the language of pure valued fields and is therefore closer to the basic language, we would now like to move to the 3-sorted language $\mathcal{L}_{3vdf}$, with a sort for the valued field K, a sort for the value group Γ, and a sort for the residue field k. It is well-known that in the presence of a "cross-section", the four-sorted structure $(K, \Gamma, k, \mathrm{RV})$ is interpretable in the three-sorted structure (K, Γ, k). As a result any admissible isomorphism, as defined in the section on Embedding Theorem, boils down to one that satisfies properties (1), (3) and (4) only, because in the presence of a cross-section, (2) follows from (3) and (4). What that effectively means is that now we have completeness relative to the value group and the residue field. Let us now make all these explicit.

Let $\mathcal{K} = (K, \Gamma, k, \mathrm{RV}; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ be a multiplicative valued difference field. We construe RV as a left $\mathbb{Z}[\sigma]$-module (w.r.t. multiplication) under the action

$$\Big(\sum_{j=0}^{n} i_j \sigma^j\Big) a = \boldsymbol{\sigma}_{\mathbf{rv}}(a)^{\boldsymbol{I}},$$

where $\boldsymbol{I} = (i_0, \ldots, i_n)$ (we will freely switch between these two notations and the corresponding $\boldsymbol{I}$ or the i_j's will be clear from the context); similarly we construe Γ also as a left $\mathbb{Z}[\sigma]$-module (w.r.t. addition) under the action

$$\Big(\sum_{j=0}^{n} i_j \sigma^j\Big) \gamma = \sum_{j=0}^{n} i_j \rho^j \cdot \gamma.$$

With these actions in place, we make the following

Definition 11.1. A *cross-section* $s : \Gamma \to \mathrm{RV}$ on $\mathcal{K}$ is a group homomorphism such that for all $\gamma \in \Gamma$ and $\tau = \sum_{j=0}^{n} i_j \sigma^j \in \mathbb{Z}[\sigma]$, we have $v_{\mathrm{rv}}(s(\gamma)) = \gamma$, and

$$s((\tau)\gamma) = (\tau)s(\gamma).$$

Example 11.2. For Hahn difference fields $k((t^\Gamma))$, the map given by $s(\gamma) = t^\gamma$ is a cross-section.

Since $v(\boldsymbol{\sigma}(a)^{\boldsymbol{I}}) = \sum_{j=0}^{n} i_j \rho^j \cdot v(a)$, we have an exact sequence of $\mathbb{Z}[\sigma]$-modules

$$1 \longrightarrow k^\times \xrightarrow{\ \iota\ } \mathrm{RV} \xrightarrow{\ v_{\mathrm{rv}}\ } \Gamma \longrightarrow 0\ ,$$

where $k^\times$ is the set of non-zero elements of k. Clearly then, existence of a cross-section on $\mathcal{K}$ corresponds to this exact sequence being a split sequence.

Unfortunately, cross sections do not always exist. To make this happen, we need to impose additional conditions on the residue field or on the value group or on the valued field itself. For example, one could demand that Γ be a flat $\mathbb{Z}[\sigma]$-module, or the residue field be closed under taking roots with respect to $\mathbb{Q}(\sigma)$-monomials, or that $\mathcal{O}^\times$ be pure in $K^\times$, etc. This is a future research direction.

In any case, once we have the cross-section in place, we have the following result.

Proposition 11.3. *Suppose $\mathcal{K}$ has a cross-section $s : \Gamma \to \mathrm{RV}$. Then* RV *is interpretable in the two-sorted structure* (Γ, k) *with the first sort in the language of* MODAG *and the second in the language of difference fields.*

Proof. Let $S = (\Gamma \times k^\times) \cup \{(0,0)\}$. Note that S is a definable subset of $\Gamma \times k$ (in particular, the second co-ordinate is zero only when the first is too). Define $f : S \to \mathrm{RV} \cup \{\infty\}$ by

$$f((\gamma, a)) = \begin{cases} s(\gamma)a & \text{if } a \neq 0 \\ \infty & \text{if } (\gamma, a) = (0,0) \end{cases}$$

Now it follows from [10, Proposition 3.1.6], that f is a bijection, and that the inverse images of multiplication and $\oplus$ on RV are definable in S. Moreover, if $a \neq 0$, then $v_{\mathrm{rv}}(s(\gamma)a) = v_{\mathrm{rv}}(s(\gamma)) + v_{\mathrm{rv}}(a) = \gamma + 0 = \gamma$, and if $a = 0$, then $v(\infty) = \infty$. Thus the inverse image of the valuation map is $\{\langle(\gamma, a), \gamma\rangle\} \cup \{\langle(0,0), \infty\rangle\}$. Finally, since $\sigma_{\mathrm{rv}}(s(\gamma)a) = s(\rho \cdot \gamma)\bar{\sigma}(a)$, the inverse image of the difference operator on RV is given by $\{\langle(\gamma, a), (\rho \cdot \gamma, \bar{\sigma}(a))\rangle\}$. Hence the result follows. □

As an immediate corollary of Proposition 11.7, we have

Corollary 11.4. *If $\mathcal{K} = (K, \Gamma, k, \mathrm{RV}; v, \pi, v_{\mathrm{rv}}, \iota, \mathrm{rv})$ and $\mathcal{K}' = (K', \Gamma', k', \mathrm{RV}'; v', \pi', v'_{\mathrm{rv}}, \iota', \mathrm{rv}')$ are two multiplicative valued difference fields satisfying Axiom 1 with $ptp_{\mathbb{Z}[\rho]}(\Gamma) = ptp_{\mathbb{Z}[\rho]}(\Gamma')$, have a cross-section, and $\Gamma \equiv_{\mathcal{L}_{OG,\rho\cdot}} \Gamma'$ (as* MODAG*s) and $k \equiv_{\mathcal{L}_{R,\sigma}} k'$ (as difference fields), then $\mathrm{RV} \equiv_{\mathcal{L}_{\mathrm{rv},\sigma_{\mathrm{rv}}}} \mathrm{RV}'$.*

This allows us to work in the 3-sorted language $\mathcal{L}_{3vdfs}$ (eliminating the need for the RV sort), where we have a symbol s for the cross-section. Combining Corollary 11.8 with Theorems 10.1, 10.2 and 10.4 and Corollary 10.3, we then have the following nice results. Let $\mathcal{K} = (K, \Gamma, k; v, \pi, s)$ and $\mathcal{K}' = (K', \Gamma', k'; v', \pi', s')$ be two σ-henselian multiplicative valued difference fields, satisfying Axiom 1 with $ptp_{\mathbb{Z}[\rho]}(\Gamma) = ptp_{\mathbb{Z}[\rho]}(\Gamma')$, of equi-characteristic zero, and having a cross-section. Then,

Theorem 11.5. *$\mathcal{K} \equiv_{\mathcal{L}_{3vdfs}} \mathcal{K}'$ (as multiplicative valued difference fields with a cross-section) if and only if $\Gamma \equiv_{\mathcal{L}_{OG,\rho\cdot}} \Gamma'$ (as* MODAG*s) and $k \equiv_{\mathcal{L}_{R,\sigma}} k'$ (as difference fields).*

Theorem 11.6. *Let $\mathcal{E} = (E, \Gamma_E, k_E; v, \pi, s)$ be a σ-henselian multiplicative valued difference subfield of $\mathcal{K}$, satisfying Axiom 1 and having a cross-section, such that $\Gamma_E \preccurlyeq_{\mathcal{L}_{OG,\rho\cdot}} \Gamma$ (as* MODAG*s) and $k_E \preccurlyeq_{\mathcal{L}_{R,\sigma}} k$ (as difference fields). Then $\mathcal{E} \preccurlyeq_{\mathcal{L}_{3vdfs}} \mathcal{K}$ (as multiplicative valued difference fields with a cross-section).*

Theorem 11.7. *Theory of $\mathcal{K}$ in the language $\mathcal{L}_{3vdfs}$ is decidable if and only if theories of Γ and k are decidable.*

Since we have expanded the language to contain a symbol for the cross-section, we also have the following relative quantifier elimination result.

Theorem 11.8. *Let T be the $\mathcal{L}_{3vdfs}$-theory of σ-henselian multiplicative valued difference fields of equi-characteristic zero satisfying Axiom 1 and having a cross-section, and $\phi(x)$ be an $\mathcal{L}_{3vdfs}$-formula. Then there is an $\mathcal{L}_{3vdfs}$-formula $\varphi(x)$ in which all occurrences of field variables are free, such that*

$$T \vdash \phi(x) \leftrightarrow \varphi(x).$$

And finally,

Theorem 11.9. *In equi-characteristic zero, the model companion of the theory of multiplicative valued difference fields satisfying Axiom 1 and Axiom 3 and having a cross-section is the theory of σ-henselian multiplicative valued difference fields satisfying Axiom 1 and having a cross-section, where the value group is a model of* div-MODAG *and the residue field is a model of $ACFA$.*

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