APPROXIMATION TYPES DESCRIBING EXTENSIONS OF VALUATIONS TO RATIONAL FUNCTION FIELDS

F.-V. KUHLMANN

Abstract. We introduce the notion of *approximation type* for the partial, and in certain cases the total description of extensions of a given valuation from a field K to the rational function field $K(x)$. To every extension, a unique approximation type of x over K is associated, while x may be the limit of many pseudo Cauchy sequences. Approximation types also provide information in cases where the extensions are not immediate, and we prove that they correspond bijectively to the extensions when K is algebraically closed or lies dense in its algebraic closure.

Contents

Date: November 22, 2021.
The author would like to thank Enric Nart, Giulio Peruginelli and Anna Rzepka for reading the manuscript and providing comments and corrections.

1. Introduction

In this paper we will work with (Krull) valuations on fields and their extensions to rational function fields. As we will show that under certain natural conditions, these extensions are uniquely determined by what we call *approximation types*, it is important to note from the start that we *always identify equivalent valuations*. For basic information on valued fields and for notation, see Section 2.

Take a valued field (K, v_0). It is an important task to describe, analyze and classify all extensions v of the valuation v_0 from K to the rational function field $K(x)$. In order to be able to compute the value of every element of $K(x)$ with respect to v, it suffices to be able to compute the value of all polynomials in x, that is, we only have to deal with the polynomial ring $K[x]$. Indeed, if $f, g \in K[x]$, then necessarily, $v\frac{f}{g} = vf - vg$. We know the values of all elements in K. If in addition we know the value vx, then everything would be easy if for every polynomial

$$f(x) \;=\; \sum_{i=0}^{n} c_i x^i \in K[x] \tag{1.1}$$

the following equation would hold:

$$vf(x) \;=\; \min_{0\le i\le n} v_0 c_i + ivx\,. \tag{1.2}$$

Indeed, we can define valuations on $K(x)$ in this way by choosing vx to be any element in some ordered abelian group which contains vK. If we choose $vx = 0$, we obtain the **Gauß valuation**.

But what if Equation 1.2 does not always hold? Then there are *polynomials of unexpected value*, the value of which is larger than the minimum of the values of its monomials. This observation has led to the theory of *key polynomials*, on which by now an abundant number of articles are available.

Let us give a basic classification of all extensions v of the valuation v_0 of K to $K(x)$. The **rational rank** of an ordered abelian group Γ is $\mathrm{rr}\,\Gamma := \dim_{\mathbb{Q}} \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma$ (note that $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma$ is the **divisible hull** of Γ). As the *Abhyankar Inequality*

$$1 \;=\; \mathrm{trdeg}\,K(x)|K \;\ge\; \mathrm{rr}\,vK(x)/v_0K \;+\; \mathrm{trdeg}\,K(x)v|Kv_0 \tag{1.3}$$

holds by Proposition 2.3, there are the following mutually exclusive cases:

- $(K(x)|K, v)$ is **valuation-algebraic** (case $1 > 0 + 0$):
  $vK(x)/v_0K$ is a torsion group and $K(x)v|Kv_0$ is algebraic,
- $(K(x)|K, v)$ is **value-transcendental** (case $1 = 1 + 0$):
  $vK(x)/v_0K$ has rational rank 1, but $K(x)v|Kv_0$ is algebraic,
- $(K(x)|K, v)$ is **residue-transcendental** (case $1 = 0 + 1$):
  $K(x)v|Kv_0$ has transcendence degree 1, but $vK(x)/v_0K$ is a torsion group.

We will combine the value-transcendental case and the residue-transcendental case by saying that

- $(K(x)|K, v)$ is **valuation-transcendental**:

$vK(x)/v_0K$ has rational rank 1, or $K(x)v|Kv_0$ has transcendence degree 1.

A special case of the valuation-algebraic case is the following:

- $(K(x)|K,v)$ is **immediate**:
  $vK(x) = v_0K$ and $K(x)v = Kv_0$.

For more details on this notion, see Section 2.2.

Remark 1.1. It was observed by several authors that an immediate extension of v_0 from K to $K(x)$ can be represented as a limit of an infinite sequence of residue-transcendental extensions; see e.g. [1]. The approach is particularly important because residue-transcendental extensions behave better than other types of extensions: the corresponding extensions of value group and residue field are finitely generated (see Corollary 2.7 in [10]), and they do not generate a defect that is not already present in (K,v_0): see the Generalized Stability Theorem in [13] and its applications in [9, 12]. #

We denote by $\tilde{K}$ the algebraic closure of K. We will assume throughout that v is extended to $\widetilde{K(x)}$; this also induces an extension of v from K to $\tilde{K}$. In this way, we associate to $(K(x)|K,v)$ the extension $(\tilde{K}(x)|\tilde{K},v)$. Note that the value group of an algebraically closed valued field is divisible, and its residue field is also algebraically closed. Further, $v\tilde{K}/v_0K$ is a torsion group, and $\tilde{K}v|Kv_0$ is algebraic.

If K is algebraically closed, then v_0K is divisible and Kv_0 is algebraically closed. In this case, for any extension $(K(x)|K,v)$ there are only the following mutually exclusive cases:

$(K(x)|K,v)$ is immediate: $vK(x) = v_0K$ and $K(x)v = Kv_0$,
$(K(x)|K,v)$ is value-transcendental: $\mathrm{rr}\, vK(x)/v_0K = 1$, but $K(x)v = Kv_0$,
$(K(x)|K,v)$ is residue-transcendental: $\mathrm{trdeg}\, K(x)v|Kv_0 = 1$, but $vK(x) = v_0K$.

As a consequence, if (K,v_0) is an arbitrary valued field and we have an arbitrary extension v of v_0 to $K(x)$, then

- $(K(x)|K,v)$ is valuation-algebraic if and only if
  $(\tilde{K}(x)|\tilde{K},v)$ is immediate,
- $(K(x)|K,v)$ is value-transcendental if and only if
  $(\tilde{K}(x)|\tilde{K},v)$ is value-transcendental,
- $(K(x)|K,v)$ is residue-transcendental if and only if
  $(\tilde{K}(x)|\tilde{K},v)$ is residue-transcendental.

Ostrowski in [15] and Kaplansky in [8] gave us a powerful tool for the analysis and the construction of immediate extensions: *pseudo Cauchy sequences* (also called *pseudo convergent sequences*). As is the case for Cauchy sequences, an element in a valued field extension that is a limit of a pseudo Cauchy sequence will in general be a limit of many different pseudo Cauchy sequences. It is therefore desirable to have a unique object that can readily be assigned to an element in a valued field extension and that contains all information that is contained in pseudo Cauchy sequences, and possibly more. We will define such objects, called *approximation types*, in Section 3. They are nests of ultrametric balls; for basic definitions and properties of ultrametric balls, see Section 2.6. In a given extension $(K(x)|K,v)$, the approximation type of x over K is obtained by intersecting all ultrametric balls with center x in $(K(x),v)$ with K.

In Section 3.2 we define *immediate approximation types*, and in Section 3.4 we describe how to obtain them from pseudo Cauchy sequences, and vice versa. In Section 3.3 we prove the analogues of Kaplansky's basic theorems for pseudo Cauchy sequences in the language of approximation types. Immediate approximation types were introduced and extensively used in [14].

Pseudo Cauchy sequences and immediate approximation types are suitable tools to describe immediate extensions $(K(x)|K,v)$, but they do not carry enough information to describe extensions that are not immediate. For that reason, the new notions of *pseudo monotone sequences* and *pseudo divergent sequences* were introduced and used in [4, 17]. However, like pseudo Cauchy sequences they are not unique objects associated with the valuation v and the element x. Moreover, they appear to be somewhat unnatural constructs for capturing the necessary information. Therefore, we propose to use the more uniform approximation types in place of all of these sequences. Still it should be noted that like these sequences, the approximation types cannot describe all extensions when K is not algebraically closed. Nevertheless, we will define in Sections 5.1 and 5.2 important subclasses of extensions which can be fully described.

We fix a valuation v_0 on K and denote by the letter[1] $\mathcal{V}$ the set of all extensions of v_0 to $K(x)$. To every $v \in \mathcal{V}$ we then associate the approximation type of x with respect to v, which we denote by $\mathrm{appr}_v(x,K)$. By $\mathcal{A}$ we denote the set of all non-trivial (abstract) approximation types over (K,v_0). We will prove:

Theorem 1.2. *Let $K(x)$ be the rational function field in one variable over K. Then for every non-trivial approximation type $\mathbf{A}$ over (K,v_0) there is an extension v of v_0 to $K(x)$ such that $\mathrm{appr}_v(x,K)=\mathbf{A}$. In other words, the function*

$$\mathcal{V} \longrightarrow \mathcal{A}\,, \qquad v \mapsto \mathrm{appr}_v(x,K) \tag{1.4}$$

is surjective.

Theorem 1.3. *Let $K(x)$ be the rational function field in one variable over K. Assume that K is algebraically closed or that (K,v_0) lies dense in its algebraic closure. Then the valuations $v \in \mathcal{V}$ are fully characterized by the approximation types $\mathrm{appr}_v(x,K)$. In particular, the function (1.4) is a bijection.*

These theorems will be proved in Sections 4.1 and 5.3, where we will also make precise what we mean by "fully characterized". The classification of extensions

[1] Note that $\mathcal{V}$ is the calligraphic version of the capital letter V, and certainly *not* a capital greek letter nu, as the capital version of ν in the Greek alphabet is N. Addressing $\mathcal{V}$ as "capital nu" would be something capitally new.

we introduced above is clearly reflected in the approximation types, together with related information necessary to fully describe the valuations.

Theorem 1.3 is a special case of a more general theorem. Using the notion of *pure extension* which we introduced in [10], we show in Theorem 5.8 that the statement of Theorem 1.3 holds for all pure extensions, provided that vK is divisible. The surjectivity of the function remains valid when we restrict to a natural subset of the set of all approximation types. We obtain Theorem 1.3 for the case of algebraically closed K from Proposition 5.2 which states that if K is algebraically closed, then every extension $v \in \mathcal{V}$ is pure.

In order to cover the case where (K, v_0) lies dense in its algebraic closure for some and hence for all extensions of v_0 to $\tilde{K}$ (see Lemma 5.5), we generalize the notion *pure extension* to what we call *almost pure extension*. In Proposition 5.6 we will then show that in this case every extension $v \in \mathcal{V}$ is almost pure. As we will show that Theorem 5.8 also holds for almost pure extensions, Theorem 1.3 will follow from it.

Two important cases where (K, v_0) lies dense in its algebraic closure are:

1) The rank of (K, v_0) is 1, i.e., v_0K is archimedean ordered (hence order embeddable in $\mathbb{R}$), and its henselization is algebraically closed. The latter occurs when v_0K is divisible and Kv_0 is algebraically closed of characteristic 0 (see [19, 32.21]).

2) The field K is separable-algebraically closed and v_0 is non-trivial (see [11, Theorem 1.11]).

An alternative approach to the proof of Theorem 1.2 can be given by use of model theory. Under the minimal necessary additional assumptions we show in Theorem 4.5 that every non-trivial approximation type $\mathbf{A}$ over (K, v_0) can be realized in some elementary extension (K^*, v) of (K, v_0). For the proof we show that under the additional assumptions, the approximation types are subsets of suitable 1-types over (K, v_0). This in fact explains the choice of the name "approximation type".

Note that for simplicity we will often write "v" in place of "v_0" for the valuation on K when it is not necessary to distinguish it from its extensions.

2. Preliminaries

By a valued field (K, v) we mean a field K endowed with a Krull valuation v. This is a function from K to $\Gamma \cup \{\infty\}$ where Γ is an ordered abelian group and ∞ an element larger than all elements of Γ, which satisfies $v(a) = \infty \Leftrightarrow a = 0$, $v(ab) = v(a) + v(b)$ and $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$. From these laws one deduces that $v(1) = 0$, $v(-a) = v(a)$, $v(a^{-1}) = -v(a)$ and $v(a_1 + \ldots + a_n) \geq \min_{1\leq i\leq n} v(a_i)$ with equality holding if all values $v(a_i)$ are distinct. We will make use of all these facts freely.

The value group $v(K^\times) \subseteq \Gamma$ of (K, v) will be denoted by vK, and its residue field $\{a \in K \mid v(a) \geq 0\}/\{a \in K \mid v(a) > 0\}$ by Kv. The value of an element a will be denoted by va in place of $v(a)$, and its residue by av. By $(L|K, v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. For background on valuation theory, see [6, 7, 18].

Lemma 2.1. *Take any extension $(K(x)|K,v)$. For each $c\in K$, the following assertions are equivalent:*
a) there is some $c'\in K$ such that $v(x-c')>v(x-c)$;
b) there is $d\in K$ such that $vd(x-c)=0$ and $d(x-c)v\in Kv$.
If $d(x-c)v\in Kv$ holds for some $d\in K$ such that $vd(x-c)=0$, then it holds for all such d.

Proof. Assume first that there is some $c'\in K$ such that $v(x-c')>v(x-c)$. Then $v(x-c)=\min\{v(x-c'),v(x-c)\}=v(c-c')\in vK$, so that there is $d\in K$ such that $vd(x-c)=0$. It follows that $v(d(x-c)-d(c'-c))=vd+v(x-c')>vd(x-c)$, whence $d(x-c)v=d(c'-c)v\in Kv$.

Now assume that there is $d\in K$ such that $vd(x-c)=0$ and $d(x-c)v\in Kv$. Pick $b\in K$ such that $vb=0$ and $bv=d(x-c)v$. Then $v(d(x-c)-b)>0$, whence $v(x-c-bd^{-1})>-vd=v(x-c)$, so that $c':=c+bd^{-1}$ satisfies assertion a).

Assume that there is $d\in K$ such that $vd(x-c)=0$ and $d(x-c)v\in Kv$ and take some $d'\in K$ such that $vd'(x-c)=0$. Then $d'(x-c)v=(d'd^{-1}v)(d(x-c)v)\in Kv$ since $vd'd^{-1}=0$. □

2.1. Ordered sets and cuts.

Take any totally ordered set $(T,<)$, an element $\alpha\in T$ and two subsets $D,E\subseteq T$. We write
$\alpha>D$ if $\alpha>\delta$ for all $\delta\in D$,
$\alpha<E$ if $\alpha<\varepsilon$ for all $\varepsilon\in E$,
$D<E$ if $\delta<\varepsilon$ for all $\delta\in D$ and $\varepsilon\in E$,
and similarly for "$\leq$" in place of "$<$". A **cut in** T is a pair (D,E) of subsets of T such that $D<E$ and $D\cup E=T$. If this is the case, then D is an **initial segment** of T, that is, if $\delta\in D$ and $\alpha\leq\delta$, then $\alpha\in D$, and E is a **final segment** of T, that is, if $\varepsilon\in E$ and $\alpha\geq\varepsilon$, then $\alpha\in E$.

If $\alpha\notin T$ is an element in some totally ordered set containing $(T,<)$, then we say that α **realizes the cut** (D,E) if $D<\alpha<E$. The **cut induced by** α **in** T is the pair $(\{\delta\in T\mid\delta<\alpha\},\{\varepsilon\in T\mid\varepsilon>\alpha\})$.

2.2. Immediate extensions.

An extension $(L|K,v)$ is called **immediate** if the canonical embeddings $vK\hookrightarrow vL$ and $Kv\hookrightarrow Lv$ are both onto. Instead, we will also say "if (K,v) and (L,v) have the same value group and the same residue field" or just "if $vL=vK$ and $Lv=Kv$" (recall that we are identifying equivalent valuations and places, so we may view vK as a subgroup of vL and Kv as a subfield of Lv). However, the reader should note that this is less precise and can be misunderstood. For instance, if $vK\simeq\mathbb{Z}$ and $L|K$ is finite, then still, $vK\simeq\mathbb{Z}$ even if the embedding of vK in vL is not onto.

Important examples of immediate extensions are henselizations and completions of valued fields.

Lemma 2.2. *An extension* $(L|K,v)$ *is immediate if and only if for every* $x \in L \setminus K$ *there is* $a \in K$ *such that* $v(x-a) > vx$.

Proof. "$\Rightarrow$": Assume that $(L|K,v)$ is immediate and take $x \in L \setminus K$. Then $x \neq 0$ and therefore, $vx \in vL = vK$. Thus there is $d \in K$ such that $vdx = 0$ and $dxv \in Lv = Kv$. Now we apply Lemma 2.1 with $c = 0$ to obtain $c' \in K$ such that $v(x-c') > vx$. Thus $a := c'$ is the required element.

"$\Leftarrow$": Take $\alpha \in vL$ and $x \in L$ such that $vx = \alpha$. If there is $a \in K$ such that $v(x-a) > vx$, then $\alpha = vx = va \in vK$. Now let $\zeta \in Lv$ and $x \in L$ be such that $xv = \zeta$. If there is $a \in K$ such that $v(x-a) > vx = 0$, then $\zeta = xv = av \in Kv$. □

The assertion of this lemma is an important observation, as it allows us to generalize the definition of "immediate extension" to more general valued structures for which the invariants associated to them can be much more complicated than the pair of value group and residue field. One of such cases occurs when we consider a valued field extension $(L|K,v)$ and a K-vector space $V \subseteq L$. When equipped with the restriction of the valuation v of L, we call (V,v) a **valued** (K,v)**-vector space**. If $V \subseteq V' \subseteq L$, then we call $(V'|V,v)$ an **immediate extension of valued vector spaces** if for every $a \in V'$ there is $b \in V$ such that $v(a-b) > va$.

2.3. Algebraic valuation independence.

The following proposition has turned out to be amazingly universal in many different applications of valuation theory. It plays an important role for example in algebraic geometry as well as in the model theory of valued fields, in real algebraic geometry, or in the structure theory of exponential Hardy fields (= nonarchimedean ordered fields which encode the asymptotic behaviour of real-valued functions including exp and log). For more details and the easy but technical proof of the proposition, see [3, Chapter VI, §10.3, Theorem 1].

Proposition 2.3. *Let* $(L|K,v)$ *be an extension of valued fields. Take elements* $x_i, y_j \in L$, $i \in I$, $j \in J$, *such that the values* vx_i, $i \in I$, *are rationally independent over* vK, *and the residues* $y_j v$, $j \in J$, *are algebraically independent over* Kv. *Then the elements* x_i, y_j, $i \in I$, $j \in J$, *are algebraically independent over* K.

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J] \tag{2.1}$$

in such a way that whenever $k \neq \ell$, *then there is some* i *s.t.* $\mu_{k,i} \neq \mu_{\ell,i}$ *or some* j *s.t.* $\nu_{k,j} \neq \nu_{\ell,j}$, *then*

$$vf \;=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k vc_k + \sum_{i\in I} \mu_{k,i} vx_i\,. \tag{2.2}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i \in I} \mathbb{Z} v x_i \\ K(x_i, y_j \mid i \in I, j \in J)v &= Kv\,(y_j v \mid j \in J)\,. \end{aligned}$$

Moreover, the valuation v and the residue map on $K(x_i, y_j \mid i \in I, j \in J)$ are uniquely determined by their restriction to K, the values vx_i and the residues $y_j v$.

Conversely, if (K, v) is any valued field, the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K, and we assign to the vx_i any values in an ordered group extension of vK which are rationally independent, then (2.2) defines a valuation on $K(x_i, y_j \mid i \in I, j \in J)$, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv. □

The proof of the following corollary is straightforward:

Corollary 2.4. *Take a valued field (K, v_0) and an element z transcendental over K. Take an element α either in $v_0 K$, or in some ordered abelian group containing $v_0 K$. In the latter case, assume that α is not a torsion element modulo $v_0 K$. For every polynomial $f(X) = \sum_{i=0}^{n} c_i X^i \in K[X]$, define*

$$vf(z) := \min_i v_0 c_i + i\alpha\,,$$

and extend v to $K(z)$ in the canonical way. Then v is a valuation on $K(z)$.

If $\alpha \in v_0 K$ and $d \in K$ is such that $v_0 d = -vz$, then dzv is transcendental over Kv_0, $K(z)v = Kv_0(dzv)$ and $vK(z) = v_0 K$. If $\alpha \notin v_0 K$, then $vK(z) = v_0 K \oplus \mathbb{Z}\alpha$ and $K(z)v = Kv_0$.

We need to know when the construction described in this corollary still yields the same valuation even when z and α are changed. Although we have already announced that we identify equivalent valuations, at this point we will be a bit more precise. If v_1 and v_2 are extensions of a valuation v_0 from K to some extension field L, then we will say that v_1 and v_2 are **equivalent over** v_0 if there is an order preserving isomorphism ρ from $v_1 L$ to $v_2 L$ which is the identity on $v_0 K$ such that $v_2 = \rho \circ v_1$.

Lemma 2.5. *Assume that Γ is an ordered abelian group and $\alpha \notin \Gamma$ is an element in some ordered abelian group containing Γ. Then the ordering on $\Gamma \oplus \mathbb{Z}\alpha$ is uniquely determined by the cut that α induces in Γ, provided that*
a) Γ is divisible, or
b) $\alpha > \Gamma$ (in which case the induced cut is $(\Gamma, \emptyset)$), or
c) $\alpha < \Gamma$ (in which case the induced cut is $(\emptyset, \Gamma)$).

Proof. We have to show that it can be deduced from the cut (D, E) induced by α in Γ whether any given element $\gamma + n\alpha \in \Gamma \oplus \mathbb{Z}\alpha$ with $\gamma \in \Gamma$ and $n \in \mathbb{Z} \setminus \{0\}$ is positive or not.

Assume first that Γ is divisible, so that $\frac{\gamma}{n} \in \Gamma$. If n is positive, then $\gamma + n\alpha > 0 \Leftrightarrow \alpha > -\frac{\gamma}{n} \Leftrightarrow -\frac{\gamma}{n} \in D$. If n is negative, then $\gamma + n\alpha > 0 \Leftrightarrow \alpha < -\frac{\gamma}{n} \Leftrightarrow -\frac{\gamma}{n} \in E$.

If $\alpha > \Gamma$, then $\gamma + n\alpha > 0 \Leftrightarrow n > 0$. If $\alpha < \Gamma$, then $\gamma + n\alpha > 0 \Leftrightarrow n < 0$. □

Lemma 2.6. *Take a valued field (K, v_0) and an element z transcendental over K. Pick any α_1 and α_2 in some ordered abelian groups containing v_0K. For $i = 1, 2$, assume that α_i is not a torsion element modulo v_0K if $\alpha_i \notin v_0K$ and extend v_0 to $K(z)$ by using Corollary 2.4, assigning the value α_i to z. Then the following assertions hold:*

1) The valuations v_1 and v_2 can only be equivalent if either both $\alpha_1 \notin v_0K$ and $\alpha_2 \notin v_0K$ or both $\alpha_1 \in v_0K$ and $\alpha_2 \in v_0K$.

2) Assume that $\alpha_1 \notin v_0K$ and $\alpha_2 \notin v_0K$. If v_1 and v_2 are equivalent over v_0, then α_1 and α_2 realize the same cut (D, E) in v_0K. The converse holds if v_0K is divisible, or $D = \emptyset$, or $E = \emptyset$.

Proof. 1): If one of α_1, α_2 lies in v_0K and the other does not, then by Proposition 2.3, one of $v_1K(z)$, $v_2K(z)$ is equal to v_0K and the other is not, so v_1 and v_2 cannot be equivalent.

2): Assume that $\alpha_1 \notin v_0K$ and $\alpha_2 \notin v_0K$. Further, assume first that v_1 and v_2 are equivalent over v_0, that is, there is an order preserving isomorphism ρ from $v_1K(z)$ to $v_2K(z)$ fixing v_0K such that $v_2 = \rho \circ v_1$. Thus $\alpha_2 = v_2z = \rho\alpha_1 \in v_2K(z)$. Since ρ is order preserving and fixes v_0K, this shows that α_1 and α_2 realize the same cut in v_0K.

For the converse, assume that α_1 and α_2 realize the same cut (D, E) in v_0K. First, assume that v_0K is divisible. This implies that $\alpha_1 \notin v_0K$ and $\alpha_2 \notin v_0K$ are not torsion elements modulo v_0K. Thus sending α_1 to α_2 induces an isomorphism ρ from $v_0K \oplus \mathbb{Z}\alpha_1$ to $v_0K \oplus \mathbb{Z}\alpha_2$ which leaves v_0K fixed. By Lemma 2.5, there is a unique ordering on $v_0K \oplus \mathbb{Z}\alpha_1$ determined by the cut (D, E). Through ρ it induces on $v_0K \oplus \mathbb{Z}\alpha_2$ an ordering, which again by Lemma 2.5 must coincide with the ordering determined by (D, E). This shows that ρ is order preserving. By Corollary 2.4, the extension of v_0 to $K(z)$ is uniquely determined by the value $\alpha_2 = \rho\alpha_1$ assigned to the element z, so $\rho \circ v_1 = v_2$, showing that v_1 and v_2 are equivalent over v_0.

Now assume that $D = \emptyset$ (the case of $E = \emptyset$ is treated analogously). Then again, $\alpha_1 \notin v_0K$ and $\alpha_2 \notin v_0K$ are not torsion elements modulo v_0K, and the proof proceeds as before. □

Remark 2.7. More generally, the converse in part 2) of the lemma always holds when the archimedean class of any element realizing the cut is not equal to the archimedean class of any element in Γ. This happens if and only if there is a convex subgroup Δ of Γ which is cofinal in D or coinitial in E. #

2.4. **The sets** $v(x-K)$**.**

Take an extension $(L|K,v)$ and an element $x\in L$. In this section we will investigate the set

$$v(x-K) \;:=\; \{v(x-c)\mid c\in K\} \;\subseteq\; vL\cup\{\infty\}\,.$$

The following facts were proved in [2, Lemma 3.1]:

Lemma 2.8. *Take a valued field extension $(L|K,v)$ and $x\in L$.*

1) The set $v(x-K)\cap vK$ is an initial segment of vK.

2) The set $v(x-K)\setminus vK$ has at most one element.

3) If $\alpha\in v(x-K)\setminus vK$, then $\alpha=\max v(x-K)$ and

$$v(x-K)\cap vK=\{\gamma\in vK\mid \gamma<\alpha\}\,,$$

which is the lower cut set of the cut induced by α in vK.

4) For every $c\in K$, $\{\gamma\in v(x-K)\mid \gamma<v(x-c)\}$ is a subset of vK and thus an initial segment of vK.

5) We have $v(x-c)=\max v(x-K)$ if and only if $v(x-c)\notin vK$ or $(d(x-c))v\notin Kv$ for every $d\in K$ such that $v(d(x-c))=0$.

Proof. Take any $c\in K$ and $\gamma\in vK$ such that $\gamma<v(x-c)$. Pick $c_\gamma\in K$ such that $vc_\gamma=\gamma$. Then $c+c_\gamma\in K$ and $v(x-(c+c_\gamma))=\min\{v(x-c),vc_\gamma\}=vc_\gamma=\gamma$, so $\gamma\in v(x-K)\cap vK$. This proves part 1) and the inclusion "$\supseteq$" in the second assertion of part 3).

Further, take $c_1,c_2\in K$ such that $v(x-c_1)<v(x-c_2)$. Then

$$v(x-c_1)\;=\;\min\{v(x-c_1),v(x-c_2)\}\;=\;v(c_2-c_1)\;\in\;vK\;,$$

so the assertion of part 2) as well as the first assertion of part 3) and the inclusion "$\subseteq$" in its second assertion must hold.

4): If $v(x-c)\in vK$, then the assertion follows from part 1), and otherwise from part 3).

5): This is the contrapositive of Lemma 2.1. □

Lemma 2.9. *Take a valued field extension $(L|K,v)$ and elements $x,y\in L$.*

1) If $v(x-K)$ has no maximal element, then it is an initial segment of vK.

2) If $(K(x)|K,v)$ is immediate, then $v(x-K)$ has no maximal element.

3) If for all $x\in L$, $v(x-K)$ has no maximal element, then the extension $(L|K,v)$ is immediate.

4) If $v(x-y)>v(x-K)$, then $v(x-c)=v(y-c)$ for all $c\in K$, and $v(x-K)=v(y-K)$.

5) If $v(x-K)$ has no maximal element, then the following are equivalent:

a) $v(x-y)>v(x-K)$,

b) $v(x-y)\geq v(x-K)$,

c) $v(x-c)=v(y-c)$ for all $c\in K$.

6) Take extensions $(L|K,v)$ and $(L(x)|L,v)$, and assume that $vL = vK$. If $v(x-K) \neq v(x-L)$, then for some $y \in L$, $v(x-y) > v(x-K)$ and $v(x-K) = v(y-K)$.

Proof. 1): If it has no maximal element, then $v(x-K)$ is the union of the sets $\{\gamma \in v(x-K) \mid \gamma < v(x-c)\}$ where c runs through all elements of K. By part 4) of Lemma 2.8 all of these sets are initial segments of vK, hence so is their union.

2): This follows from part 5) of Lemma 2.8.

3): This follows from Lemma 2.2.

4): If $c \in K$, then by assumption, $v(x-y) > v(x-c)$, which implies that $v(y-c) = \min\{v(x-y), v(x-c)\} = v(x-c)$. As this holds for all $c \in K$, we also obtain that $v(x-K) = v(y-K)$.

5): Assume that $v(x-K)$ has no maximal element. Then assertions a) and b) are trivially equivalent. The implication a)⇒c) is part 4) of our lemma. Now suppose that assertion b) does not hold, and pick some $c \in K$ such that $v(x-c) > v(x-y)$. It follows that

$$v(y-c) \;=\; \min\{v(x-y), v(x-c)\} \;=\; v(x-y) \;\neq\; v(x-c)\,,$$

so assertion c) does not hold.

6): Since $K \subseteq L$, we have that $v(x-K) \subseteq v(x-L)$. Assume that $v(x-K) \neq v(x-L)$. Then there exists $y \in L$ such that $v(x-y) \notin v(x-K)$. Suppose that there is some $c \in K$ such that $v(x-c) > v(x-y)$. Then $v(x-y) = \min\{v(x-y), v(x-c)\} = v(y-c) \in vL = vK$; but then $v(x-y) \in v(x-K)$ by part 4) of Lemma 2.8, a contradiction. This proves that $v(x-y) > v(x-K)$. Hence by part 4) of the present lemma, $v(x-K) = v(y-K)$. □

Note that the converse of part 2) of this lemma does in general not hold.

2.5. **Pseudo Cauchy sequences.**

Take a valued field (K,v) and a sequence $(c_\nu)_{\nu<\lambda}$ of elements in K, indexed by ordinals $\nu < \lambda$ where λ is a limit ordinal. It is called a **pseudo Cauchy sequence** (or **pseudo convergent sequence**) if

(PCS) $v(c_\tau - c_\sigma) \;>\; v(c_\sigma - c_\rho)$ whenever $\rho < \sigma < \tau < \lambda$.

We will say that an assertion **holds ultimately** for $(c_\nu)_{\nu<\lambda}$ if there is $\nu_0 < \lambda$ such that the assertion holds for all c_ν with $\nu \geq \nu_0$.

We set

$$\gamma_\nu \;:=\; v(c_{\nu+1} - c_\nu)\,.$$

If $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence, then $(\gamma_\nu)_{\nu<\lambda}$ is strictly increasing.

Lemma 2.10. *Let $(c_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in (K,v). Then*

$$v(c_\nu - c_\mu) \;=\; \gamma_\mu \quad \textit{whenever } \mu < \nu < \lambda\,. \tag{2.3}$$

If $x \in K$, then either

$$v(x - c_\mu) \;<\; v(x - c_\nu) \quad \textit{whenever } \mu < \nu < \lambda\,, \tag{2.4}$$

or there is $\nu_0 < \lambda$ *such that*

$$v(x - c_\nu) = v(x - c_{\nu_0}) \text{ whenever } \nu_0 \le \nu < \lambda\,.$$

Property (2.4) is equivalent to

$$v(x - c_\nu) = \gamma_\nu \text{ for all } \nu < \lambda\,. \tag{2.5}$$

In other words, if $(v(x - c_\nu))_{\nu<\lambda}$ *is not strictly increasing, then it is ultimately constant.*

Taking $x = 0$, we obtain:

Corollary 2.11. *For every pseudo Cauchy sequence* $(c_\nu)_{\nu<\lambda}$*, either* $(vc_\nu)_{\nu<\lambda}$ *is strictly increasing or ultimately constant.*

Note that if $(L|K, v)$ is an extension of valued fields and $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K, v), then it is also a pseudo Cauchy sequence in (L, v). An element $x \in L$ is called a **pseudo limit** (or just **limit**) of $(c_\nu)_{\nu<\lambda}$ if it satisfies (2.4), or equivalently, (2.5). Since $v(x - c_{\nu+1}) \ge \gamma_{\nu+1} > \gamma_\nu$ implies that $v(x - c_\nu) = \min\{\gamma_\nu, v(x - c_{\nu+1})\} = \gamma_\nu$, both conditions are equivalent to

$$v(x - c_\nu) \ge \gamma_\nu \text{ for all } \nu < \lambda\,. \tag{2.6}$$

We will only be interested in pseudo Cauchy sequences in (K, v) that have no limit in K and therefore do not have a last element. This justifies that from the start we have indexed pseudo Cauchy sequences by limit ordinals.

The following is Theorem 1 of [8].

Theorem 2.12. *If* $(L|K, v)$ *is an immediate extension, then every* $a \in L \setminus K$ *is limit of a pseudo Cauchy sequence in* (K, v) *without a limit in* K*.*

An analogue of this theorem for immediate approximation types will be proved in Lemma 3.7.

We consider a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in (K, v) and a polynomial $f \in K[X]$. We will say that $(c_\nu)_{\nu<\lambda}$ fixes the value of f if the sequence $vf(c_\nu)_{\nu<\lambda}$ is ultimately constant. If $(c_\nu)_{\nu<\lambda}$ fixes the value of every polynomial in $K[X]$, then it is said to be of **transcendental type**. If there is some $f \in K[X]$ whose value is not fixed by $(c_\nu)_{\nu<\lambda}$, then $(c_\nu)_{\nu<\lambda}$ is said to be of **algebraic type**. The following are Theorems 2 and 3 of [8].

Theorem 2.13. *For every pseudo Cauchy sequence* $(c_\nu)_{\nu<\lambda}$ *in* (K, v) *of transcendental type there exists a simple immediate transcendental extension* $(K(x), v)$ *such that* x *is a limit of* $(c_\nu)_{\nu<\lambda}$*. If* $(K(y), v)$ *is another extension field of* (K, v) *such that* y *is a limit of* $(c_\nu)_{\nu<\lambda}$*, then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *sending* x *to* y *is valuation preserving.*

Theorem 2.14. *Take a pseudo Cauchy sequence* $(c_\nu)_{\nu<\lambda}$ *in* (K, v) *of algebraic type and a polynomial* $f(X) \in K[X]$ *of minimal degree whose value is not fixed by* $(c_\nu)_{\nu<\lambda}$*. If* a *is a root of* f*, then there exists an extension of* v *from* K *to* $K(a)$ *such that* $(K(a)|K, v)$ *is an immediate extension and* a *is a limit of* $(c_\nu)_{\nu<\lambda}$*.*

If $(K(b), v)$ is another extension field of (K, v) such that b is a limit of $(c_\nu)_{\nu<\lambda}$, then any field isomorphism between $K(a)$ and $K(b)$ over K sending a to b will preserve the valuation.

In Section 3.3 we will prove analogues of the last two theorems for immediate approximation types. These will also provide proofs of the above theorems through the connection we set up between pseudo Cauchy sequences and approximation types in Section 3.4.

2.6. Ultrametric balls and nests.

We define the **closed ultrametric ball** in (K, v) of radius $\gamma \in vK\infty := vK \cup \{\infty\}$ centered at $c \in K$ to be

$$B_\gamma(c, K) \;=\; \{a \in K \mid v(a - c) \geq \gamma\} \,,$$

and the **open ultrametric ball** in (K, v) of radius $\gamma \in vK$ centered at $c \in K$ to be

$$B^\circ_\gamma(c, K) \;=\; \{a \in K \mid v(a - c) > \gamma\} \,.$$

Note that under the topology induced by the valuation, both types of balls are open and closed. Note further that all of these balls contain their center and are thus nonempty.

Lemma 2.15. *1) If $B = B_\gamma(c, K)$ or $B = B^\circ_\gamma(c, K)$, and if $b \in B$, then $B = B_\gamma(b, K)$ or $B = B^\circ_\gamma(b, K)$, respectively. In other words, every element in an (open or closed) ultrametric ball is its center.*

2) Any two closed or open ultrametric balls B, B' are either disjoint or comparable by inclusion.

Proof. 1): If $b \in B_\gamma(c, K)$, then $v(c - b) \geq \gamma$, hence for every $a \in B_\gamma(c, K)$, we have that $v(a - b) \geq \min\{v(a - c), v(b - c)\} \geq \gamma$. This proves that $B_\gamma(c, K) \subseteq B_\gamma(b, K)$. The reverse inclusion follows by symmetry.

The proof for the balls $B^\circ_\gamma(c, K)$ is analogous.

2): Assume that $B \cap B' \neq \emptyset$, and choose $c \in B \cap B'$. Then by part 1), c is a center of both B and B'. If one of the two has a smaller radius than the other, then by definition it contains the other. If both have the same radius, then the closed one contains the open one. The case of $B = B'$ is trivial. □

A **nest of balls** is a nonempty collection of closed and open balls linearly ordered by inclusion. A **full nest of balls** is a nest of balls $\mathcal{N}$ that contains every closed or open ball which contains some ball $B \in \mathcal{N}$, i.e.,

$$\mathcal{N} \;=\; \{B' \mid B' \text{ open or closed ultrametric ball containing some } B \in \mathcal{N}\} \,.$$

Part 2) of the above lemma shows that the set on the right hand side is indeed a nest. For any nest $\mathcal{N}$ of balls, we set

$$\bigcap \mathcal{N} \;:=\; \bigcap_{B \in \mathcal{N}} B \,. \tag{2.7}$$

If a full nest $\mathcal{N}$ contains a smallest ball B, then $\bigcap\mathcal{N} = B$ and $\mathcal{N}$ is **generated by** B in the sense that $\mathcal{N}$ consists of exactly all open and closed balls that contain B.

Lemma 2.16. *Take a nest of balls* $\mathcal{N}$.

1) If $B \in \mathcal{N}$ *is a ball with center* c *and radius* γ*, then every other ultrametric ball in* $\mathcal{N}$ *that contains* B *is a closed or open ultrametric ball of radius* $\leq \gamma$ *with center* c*; that is, every larger ball that appears in* $\mathcal{N}$ *is uniquely determined by* B.

2) For every $\gamma \in vK\infty$*, the nest* $\mathcal{N}$ *contains at most one closed ball and at most one open ball of radius* γ.

3) There is a uniquely determined full nest $\mathcal{N}'$ *containing* $\mathcal{N}$ *and such that each of its ultrametric balls contains an ultrametric ball from* $\mathcal{N}$*. The nest* $\mathcal{N}'$ *satisfies* $\bigcap\mathcal{N}' = \bigcap\mathcal{N}$.

Proof. 1): This follows from part 1) of Lemma 2.15.

2): Since every two balls in a nest are comparable, this follows from part 1) of our lemma.

3): The collection of all ultrametric balls that contain some ultrametric ball from $\mathcal{N}$ is a nest by part 2) of Lemma 2.15. It is clear that it is a full nest of balls. The last assertion follows from the fact that $\mathcal{N} \subseteq \mathcal{N}'$ and every ball in $\mathcal{N}'$ contains a ball from $\mathcal{N}$. □

Pseudo Cauchy sequences give rise to nests of balls:

Lemma 2.17. *Take a pseudo Cauchy sequence* $(c_\nu)_{\nu<\lambda}$ *in* (K, v)*. Then*

$$(B_{\gamma_\nu}(c_\nu, K))_{\nu<\lambda} \tag{2.8}$$

is a nest of balls in K*. The intersection over all balls in this nest is the set of all limits of* $(c_\nu)_{\nu<\lambda}$ *in* K.

Proof. Assume that $\mu < \nu < \lambda$. Then by (2.3), $v(c_\nu - c_\mu) = \gamma_\mu$, that is, $c_\nu \in B_{\gamma_\mu}(c_\mu, K)$. By part 2) of Lemma 2.15, this implies that (2.8) is a nest of balls.

An element $c \in K$ is a limit of $(c_\nu)_{\nu<\lambda}$ if and only if for all $\nu < \lambda$ we have that $v(c - c_\nu) = \gamma_\nu$, or in other words, $c \in B_{\gamma_\nu}(c_\nu, K)$. This in turn holds if and only if c lies in the intersection over all balls in the nest. This proves our second assertion. □

3. Approximation types

3.1. Definition of approximation types.

An **approximation type A over** (K, v) is either a full nest of open and closed balls in (K, v), or the empty set. It follows that

$$\mathrm{supp}\mathbf{A} := \{\gamma \mid \mathbf{A} \text{ contains a closed ball of radius } \gamma\}$$

is a (possibly empty) initial segment of $vK\infty$, called the **support** of **A**. If $\gamma \in \mathrm{supp}\mathbf{A}$, then by part 2) of Lemma 2.16, **A** contains a unique closed ball of radius γ, which we will denote by $\mathbf{A}_\gamma$. If $\mathbf{A}$ also contains an open ball of radius γ, then that too is unique, and we will denote it by $\mathbf{A}_\gamma^\circ$.

We may write $\mathbf{A}_\gamma = B_\gamma(c_\gamma, K)$ if $\gamma \in \mathrm{supp}\mathbf{A}$, and $\mathbf{A}_\gamma = \emptyset$ otherwise. Likewise, we may write $\mathbf{A}_\gamma^\circ = B_\gamma^\circ(c_\gamma, K)$ if $\mathbf{A}$ contains an open ball of radius γ, and $\mathbf{A}_\gamma^\circ = \emptyset$ otherwise; note that if $\mathbf{A}$ contains an open ball of radius γ, then by Lemma 2.15 we can choose c_γ to be any of its elements, and since the open ball of radius γ is contained in the closed ball $\mathbf{A}_\gamma$, again by Lemma 2.15 we can take the same c_γ also as a center of $\mathbf{A}_\gamma$. If γ is not the maximal element of $\mathrm{supp}\mathbf{A}$, then $\mathbf{A}_\gamma^\circ \neq \emptyset$, but if γ is the maximal element of $\mathrm{supp}\mathbf{A}$, then $\mathbf{A}_\gamma^\circ$ may or may not be empty. Further, note that for every $\gamma \in vK$ and $c \in K$, the balls $B_\gamma(c, K)$ and $B_\gamma^\circ(c, K)$ are nonempty, but they may not be contained in $\mathbf{A}$; in particular, $B_\gamma(c, K) \in \mathbf{A}$ *does not* imply $B_\gamma^\circ(c, K) \in \mathbf{A}$. See Example 3.20 below.

We note the following straightforward observation:

Lemma 3.1. *Two approximation types* $\mathbf{A}$ *and* $\mathbf{A}'$ *over* (K, v) *are equal if and only if* $\mathbf{A}_\gamma = \mathbf{A}'_\gamma$ *and* $\mathbf{A}_\gamma^\circ = \mathbf{A}'^\circ_\gamma$ *for all* $\gamma \in vK$.

Recall that $\bigcap \mathbf{A}$ denotes the intersection over all balls in $\mathbf{A}$. If $\mathbf{A} = \emptyset$, we set $\mathrm{supp}\mathbf{A} = \emptyset$ and $\bigcap \mathbf{A} = K$. In this case, conditions like $\alpha > \mathrm{supp}\mathbf{A}$ and $\alpha \geq \mathrm{supp}\mathbf{A}$ shall be understood to always be satisfied.

When we say that $\mathrm{supp}\mathbf{A}$ has no maximal element, then we will tacitly mean that it is nonempty. If this is the case, then for every $\gamma \in \mathrm{supp}\mathbf{A}$ there is a larger $\beta \in \mathrm{supp}\mathbf{A}$, so $\mathbf{A}_\beta \neq \emptyset$. Now part 1) of Lemma 2.16 proves:

Lemma 3.2. *If* $\mathbf{A} \neq \emptyset$ *and* $\mathrm{supp}\mathbf{A}$ *has no maximal element, then* $\mathbf{A}$ *is uniquely determined by its closed balls, and it is also uniquely determined by its open balls. In this case,*

$$\bigcap \mathbf{A} = \bigcap_{\gamma \in \mathrm{supp}\mathbf{A}} \mathbf{A}_\gamma = \bigcap_{\gamma \in \mathrm{supp}\mathbf{A}} \mathbf{A}_\gamma^\circ .$$

Take any extension $(L|K, v)$ and $x \in L$. For all $\gamma \in vK\infty$, we set

$$\mathrm{appr}_v(x, K)_\gamma := \{c \in K \mid v(x - c) \geq \gamma\} = B_\gamma(x, L) \cap K , \tag{3.1}$$

and for $\gamma \in vK$, we set

$$\mathrm{appr}_v(x, K)_\gamma^\circ := \{c \in K \mid v(x - c) > \gamma\} = B_\gamma^\circ(x, L) \cap K . \tag{3.2}$$

Further, we set

$$\begin{aligned} \mathrm{appr}_v(x, K) \;:=\; & \{\mathrm{appr}_v(x, K)_\gamma \mid \gamma \in vK\infty \text{ and } \mathrm{appr}_v(x, K)_\gamma \neq \emptyset\} \\ & \cup \{\mathrm{appr}_v(x, K)_\gamma^\circ \mid \gamma \in vK \text{ and } \mathrm{appr}_v(x, K)_\gamma^\circ \neq \emptyset\} . \end{aligned}$$

Note that $\mathrm{appr}_v(x, K) = \emptyset$ if and only if $vx < vK$.

Remark 3.3. As the right hand sides of (3.1) and (3.2) show, subtraction is not needed to define the approximation type of an element. Therefore, these approximation types can already be defined in ultrametric spaces without any further algebraic structure and can be used to study extensions of ultrametric spaces and other structures with underlying ultrametric spaces. #

Lemma 3.4. *For each $\gamma \in vK\infty$, $B_\gamma(x,L)\cap K$ is a closed ultrametric ball and $B^\circ_\gamma(x,L)\cap K$ is an open ultrametric ball in (K,v), if nonempty. The collection $\mathrm{appr}_v(x,K)$ is an approximation type over (K,v).*

Proof. Assume that $c\in B_\gamma(x,L)\cap K$. Then $v(x-c)\geq\gamma$ and by the ultrametric triangle law, for every $d\in K$ we have

$$d\in B_\gamma(x,L)\;\Leftrightarrow\; v(x-d)\geq\gamma\;\Leftrightarrow\; v(c-d)\geq\gamma\;\Leftrightarrow\; d\in B_\gamma(c,K)\,,$$

whence $B_\gamma(x,L)\cap K=B_\gamma(c,K)$. A similar argument yields that if $c\in B^\circ_\gamma(x,L)\cap K$, then $B^\circ_\gamma(x,L)\cap K=B^\circ_\gamma(c,K)$.

Our assertion that $\mathrm{appr}_v(x,K)$ is an approximation type follows from the facts that if $B^\circ_\gamma(x,L)\cap K\neq\emptyset$, then $B_\gamma(x,L)\cap K\neq\emptyset$, and if $\gamma>\beta\in vK$ and $B_\gamma(x,L)\cap K\neq\emptyset$, then $B_\beta(x,L)\cap K\neq\emptyset$. □

We call $\mathrm{appr}_v(x,K)$ the **approximation type of x over** (K,v). For the sake of completeness, we state the following criteria for the equality of approximation types.

Lemma 3.5. *Take an extension $(L|K,v)$ and $x,x'\in L$. Then*
$\mathrm{appr}_v(x,K)=\mathrm{appr}_v(x',K)\;\Rightarrow\; v(x-x')\geq\mathrm{supp}\,\mathrm{appr}_v(x,K)=\mathrm{supp}\,\mathrm{appr}_v(x',K)\,.$
Conversely, if $v(x-x')>\mathrm{supp}\,\mathrm{appr}_v(x,K)\cup\mathrm{supp}\,\mathrm{appr}_v(x',K)$ or if $v(x-x')\in vK$ and $v(x-x')>\mathrm{supp}\,\mathrm{appr}_v(x,K)$, then $\mathrm{appr}_v(x,K)=\mathrm{appr}_v(x',K)$.

Proof. Assume that $\mathrm{appr}_v(x,K)=\mathrm{appr}_v(x',K)$. Then

$$\mathrm{supp}\,\mathrm{appr}_v(x,K)\;=\;\mathrm{supp}\,\mathrm{appr}_v(x',K)$$

and for every $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K)$ we have that $\mathrm{appr}_v(x,K)_\gamma=\mathrm{appr}_v(x',K)_\gamma$. The latter implies that for $c\in\mathrm{appr}_v(x,K)_\gamma$, $v(x-c)\geq\gamma$ and $v(x'-c)\geq\gamma$, whence $v(x-x')\geq\gamma$.

Now assume that $v(x-x')>\mathrm{supp}\,\mathrm{appr}_v(x,K)\cup\mathrm{supp}\,\mathrm{appr}_v(x',K)$. Take $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K)$. For every $c\in\mathrm{appr}_v(x,K)_\gamma$ we have that $v(x-c)\geq\gamma$ and since $v(x-x')>\gamma$, we find that $v(x'-c)\geq\gamma$. This shows that $\mathrm{appr}_v(x,K)_\gamma\subseteq\mathrm{appr}_v(x',K)_\gamma$ and $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x',K)$. In particular,

$$\mathrm{supp}\,\mathrm{appr}_v(x,K)\;\subseteq\;\mathrm{supp}\,\mathrm{appr}_v(x',K)\,. \tag{3.3}$$

By symmetry, we can interchange x and x', so we find that $\mathrm{supp}\,\mathrm{appr}_v(x,K)=\mathrm{supp}\,\mathrm{appr}_v(x',K)$ and $\mathrm{appr}_v(x,K)_\gamma=\mathrm{appr}_v(x',K)_\gamma$ for every $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K)$. For each such γ and every $c\in\mathrm{appr}_v(x,K)^\circ_\gamma$ we have that $v(x-c)>\gamma$, and since $v(x-x')>\gamma$, it follows that $v(x'-c)>\gamma$. This shows that $\mathrm{appr}_v(x,K)^\circ_\gamma\subseteq\mathrm{appr}_v(x',K)^\circ_\gamma$, and by symmetry, we obtain equality. From Lemma 3.1 we now obtain that $\mathrm{appr}_v(x,K)=\mathrm{appr}_v(x',K)$.

Finally, assume that $v(x-x')\in vK$ and $v(x-x')>\mathrm{supp}\,\mathrm{appr}_v(x,K)$. As above, we show that (3.3) holds. Suppose that the reverse inclusion were not true. Then there is some $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x',K)$ such that $\gamma>\mathrm{supp}\,\mathrm{appr}_v(x,K)$. We pick some

$c \in K$ such that $v(x'-c) = \gamma$. Now $v(x-c) \geq \min\{v(x-x'), v(x'-c)\} =: \delta$. By our assumption, $\delta \in vK$ with $\delta > \operatorname{supp} \operatorname{appr}_v(x,K)$. However, $v(x-c) \geq \delta$ implies that $c \in \operatorname{appr}_v(x,K)_\delta$, a contradiction. Hence the supports of the two approximation types are equal.

For every $\gamma \in \operatorname{supp} \operatorname{appr}_v(x,K)$ we have that $v(x-x') > \gamma$ and therefore, $v(x-c) \geq \gamma \Leftrightarrow v(x'-c) \geq \gamma$ and $v(x-c) > \gamma \Leftrightarrow v(x'-c) > \gamma$, showing that $\operatorname{appr}_v(x,K)_\gamma = \operatorname{appr}_v(x',K)_\gamma$ and $\operatorname{appr}_v(x,K)^\circ_\gamma = \operatorname{appr}_v(x',K)^\circ_\gamma$. Again from Lemma 3.1 we obtain that $\operatorname{appr}_v(x,K) = \operatorname{appr}_v(x',K)$. □

The next lemma demonstrates the connection between the supports of approximation types and the sets $v(x-K)$.

Lemma 3.6. *Take an extension $(L|K,v)$ and $x \in L$. Then*

$$\operatorname{supp} \operatorname{appr}_v(x,K) \;=\; v(x-K) \cap vK\,. \tag{3.4}$$

Proof. Take $\gamma \in \operatorname{supp} \operatorname{appr}_v(x,K)$, hence $\operatorname{appr}_v(x,K)_\gamma \neq \emptyset$. Pick some element $c \in \operatorname{appr}_v(x,K)_\gamma$. Then $v(x-c) \geq \gamma$. Now part 1) of Lemma 2.8 shows that $\gamma \in v(x-K) \cap vK$.

For the converse, take $\gamma \in v(x-K)\cap vK$ and choose $c \in K$ such that $\gamma = v(x-c)$. Then $c \in \operatorname{appr}_v(x,K)_\gamma$ and therefore, $\gamma \in \operatorname{supp} \operatorname{appr}_v(x,K)$. □

If $\mathbf{A}$ is an approximation type over (K,v) and there exists an element x in some valued extension field (L,v) such that $\mathbf{A} = \operatorname{appr}_v(x,K)$, then we say that x **realizes** $\mathbf{A}$ (in (L,v)). If $\mathbf{A}$ is realized by some $c \in K$, then $\mathbf{A}$ will be called **trivial**. This holds if and only if $\mathbf{A}_\infty \neq \emptyset$ (i.e., $\infty \in \operatorname{supp}\mathbf{A}$), in which case $\mathbf{A}_\infty = \{c\}$. As $\mathbf{A}_\infty$ can contain at most one element, a trivial approximation type over (K,v) can be realized by only one element in K.

3.2. **Immediate approximation types.**

Take an approximation type $\mathbf{A}$ over (K,v). Then $\mathbf{A}$ will be called **immediate** if

$$\bigcap \mathbf{A} \;=\; \emptyset\,.$$

If $\mathbf{A}$ is immediate, then $\mathbf{A} \neq \emptyset$ and $\operatorname{supp}\mathbf{A}$ cannot have a maximal element, hence Lemma 3.2 shows that $\mathbf{A}$ is uniquely determined by its nonempty closed ultrametric balls. In particular, an immediate approximation type cannot be trivial.

To simplify notation, we can represent immediate approximation types as

$$\mathbf{A} \;=\; \{\mathbf{A}_\gamma \mid \gamma \in \operatorname{supp}\mathbf{A}\}\,,$$

and if $\mathbf{A} = \operatorname{appr}_v(x,K)$, then we can write

$$\operatorname{appr}_v(x,K) \;:=\; \{\operatorname{appr}_v(x,K)_\gamma \mid \gamma \in vK \text{ and } \operatorname{appr}_v(x,K)_\gamma \neq \emptyset\}\,. \tag{3.5}$$

Lemma 3.7. *Let $(L|K,v)$ be an extension of valued fields.*
1) If $x \in K$, then $\mathrm{appr}_v(x,K)$ is trivial, hence not immediate.
2) If $x \in L \setminus K$, then $\mathrm{appr}_v(x,K)$ is immediate if and only if the set $v(x-K)$ has no maximal element.
3) If $\mathrm{appr}_v(x,K)$ is immediate, then its support is equal to $v(x-K)$.
4) The extension $(L|K,v)$ is immediate if and only if $\mathrm{appr}_v(x,K)$ is immediate for every $x \in L \setminus K$.

Proof. 1): If $x \in K$, then

$$\bigcap_{\gamma \in \mathrm{supp}\,\mathrm{appr}_v(x,K)} \mathrm{appr}_v(x,K)_\gamma \;=\; \mathrm{appr}_v(x,K)_\infty \;=\; \{x\} \;\neq\; \emptyset\,.$$

2): Assume that $\mathrm{appr}_v(x,K)$ is immediate and that c is an arbitrary element of K. Then by definition there is some γ such that $c \notin \mathrm{appr}_v(x,K)_\gamma \neq \emptyset$, so $v(x-c) < \gamma$. Choosing some $c' \in \mathrm{appr}_v(x,K)_\gamma$, we obtain that $v(x-c) < \gamma \leq v(x-c')$. This shows that $v(x-K)$ has no maximal element.

Now assume that $v(x-K)$ has no maximal element. Take any $c \in K$; we have to show that there is some $\gamma \in \mathrm{supp}\,\mathrm{appr}_v(x,K)$ such that $c \notin \mathrm{appr}_v(x,K)_\gamma$. By assumption, there are $c', c'' \in K$ such that

$$v(x-c'') \;>\; v(x-c') \;>\; v(x-c)\,.$$

By the ultrametric triangle law we obtain that $v(x-c) < v(x-c') = v(c''-c')$. Hence $\mathrm{appr}_v(x,K)_{v(c''-c')}$ is nonempty, but does not contain c.

3): This follows from Lemma 3.6 together with part 2) of our lemma and part 1) of Lemma 2.9.

4): This follows from part 2) of our lemma together with parts 2) and 3) of Lemma 2.9. □

The following result is a direct consequence of part 2) of the previous lemma together with part 5) of Lemma 2.9 and Lemma 3.1.

Corollary 3.8. *Take an extension $(L|K,v)$ and $x,x' \in L$. If $\mathrm{appr}_v(x,K)$ is immediate, then*

$$\mathrm{appr}_v(x,K) = \mathrm{appr}_v(x',K) \iff v(x-x') \geq \mathrm{supp}\,\mathrm{appr}_v(x,K)\,. \tag{3.6}$$

□

For our work with immediate approximation types, we introduce the following useful notation. Take an immediate approximation type $\mathbf{A}$ over (K,v), and some formula φ in one free variable. Then the sentence

$$\varphi(c) \text{ for } c \nearrow \mathbf{A}$$

will denote the assertion

$$\text{there is } \gamma \in \mathrm{supp}\mathbf{A} \text{ such that } \varphi(c) \text{ holds for all } c \in \mathbf{A}_\gamma\,.$$

Note that if $\varphi_1(c)$ for $c \nearrow \mathbf{A}$ and $\varphi_2(c)$ for $c \nearrow \mathbf{A}$, then also $\varphi_1(c)\wedge\varphi_2(c)$ for $c \nearrow \mathbf{A}$. In the case of $\mathbf{A} = \mathrm{appr}_v(x, K)$, we will also write "$c \nearrow x$" in place of "$c \nearrow \mathbf{A}$".

If $\alpha = \alpha(c) \in vK$ is a value that depends on $c \in K$ (e.g., the value $vf(c)$ for a polynomial $f \in K[X]$), then we will say that α **increases for** $c \nearrow x$ if there exists some δ in the support of $\mathrm{appr}_v(x, K)$ such that for every choice of $c' \in \mathrm{appr}_v(x, K)_\delta$,

$$\alpha(c) > \alpha(c') \quad \text{for} \quad c \nearrow x\,.$$

Similarly, we will say that α **is fixed for** $c \nearrow x$ if there exists some δ in the support of $\mathrm{appr}_v(x, K)$ such that α is constant on $\mathrm{appr}_v(x, K)_\delta$.

Lemma 3.9. *Take an immediate approximation type* $\mathbf{A}$ *over* (K, v)*, an extension* $(L|K, v)$*, and an element* $x \in L$*. Then the following assertions are equivalent:*

a) x *realizes* $\mathbf{A}$*;*

b) there is a cofinal subset $S \subseteq \mathrm{supp}\mathbf{A}$ *such that for every* $\gamma \in S$*,* $v(x - c) \geq \gamma$ *for some* $c \in \mathbf{A}_\gamma$*;*

c) for every $\gamma \in \mathrm{supp}\mathbf{A}$*,* $v(x - c) \geq \gamma$ *for some* $c \in \mathbf{A}_\gamma$*;*

d) $v(x - c)$ *is not fixed for* $c \nearrow \mathbf{A}$*.*

Proof. a)⇒d): Assume that x realizes $\mathbf{A}$, that is, $\mathbf{A} = \mathrm{appr}_v(x, K)$. Take any $\gamma \in \mathrm{supp}\mathbf{A}$; we will show that for every $c \in \mathbf{A}_\gamma = \mathrm{appr}_v(x, K)_\gamma$ there is some $d \in \mathrm{appr}_v(x, K)_\gamma$ such that $v(x - c) < v(x - d)$. Since $\mathrm{appr}_v(x, K)$ is immediate, we know from part 2) of Lemma 3.7 that $v(x - K)$ has no maximal element. Hence there is $d \in K$ such that $v(x - c) < v(x - d)$. Since $v(x - c) \geq \gamma$, we find that $v(x - d) \geq \gamma$ and therefore, $d \in \mathrm{appr}_v(x, K)_\gamma$.

d)⇒c): Assertion d) means that for all $\gamma \in \mathrm{supp}\mathbf{A}$ there are $c, d \in \mathbf{A}_\gamma$ such that $v(x - d) > v(x - c)$. This implies that $v(x - d) > \min\{v(x - c), v(c - d)\}$, whence $v(x - c) = v(c - d) \geq \gamma$.

c)⇒b): Trivial.

b)⇒a): Take any $\gamma \in S$. Then $\mathbf{A}_\gamma$ is a closed ultrametric ball of radius γ in (K, v). By assertion b), there is $c \in \mathbf{A}_\gamma$ with $v(x - c) \geq \gamma$. We obtain that $\mathbf{A}_\gamma = B_\gamma(c, K)$. By the ultrametric triangle inequality,

$$\begin{aligned} d \in \mathbf{A}_\gamma = B_\gamma(c, K) \quad &\Longleftrightarrow \quad v(c - d) \geq \gamma \iff v(x - d) \geq \gamma \\ &\Longleftrightarrow \quad d \in \{c' \in K \mid v(x - c') \geq \gamma\} = \mathrm{appr}_v(x, K)_\gamma\,. \end{aligned}$$

This shows that $\mathbf{A}_\gamma = \mathrm{appr}_v(x, K)_\gamma$ for all $\gamma \in S$. Since S is cofinal in $\mathrm{supp}\mathbf{A}$, part 1) of Lemma 2.16 shows that this also holds for all $\gamma \in \mathrm{supp}\mathbf{A}$. It remains to show that it also holds for $\gamma \notin \mathrm{supp}\mathbf{A}$. Since $\mathbf{A}$ is an immediate approximation type, we know that the intersection of all $\mathbf{A}_\gamma$ for $\gamma \in \mathrm{supp}\mathbf{A}$ is empty. By what we have shown already, this is equal to the intersection of all $\mathrm{appr}_v(x, K)_\gamma$ for $\gamma \in \mathrm{supp}\mathbf{A}$. If $\gamma \notin \mathrm{supp}\mathbf{A}$, then $\gamma > \mathrm{supp}\mathbf{A}$ and $\mathrm{appr}_v(x, K)_\gamma$ must be a subset of the intersection, hence empty and therefore again equal to $\mathbf{A}_\gamma$. It follows that $\mathrm{supp}\,\mathrm{appr}_v(x, K) = \mathrm{supp}\mathbf{A}$, which has no maximal element. Hence by what we have shown together with Lemma 3.2, $\mathrm{appr}_v(x, K) = \mathbf{A}$. □

3.3. **Algebraic and transcendental immediate approximation types.**

In this section we present the approximation type version of Kaplansky's Theorems 2 and 3 ([8]), which show that each immediate approximation type can be realized in a simple immediate extension.

We consider an immediate approximation type $\mathbf{A}$ over (K,v) and a polynomial $f\in K[X]$. We will say that **A fixes the value of** f if there is $\delta\in\mathrm{supp}\mathbf{A}$ such that the value $vf(c)$ is constant for $c\in\mathbf{A}_\delta$. If $\mathbf{A}=\mathrm{appr}_v(x,K)$, then this means that $vf(c)$ is fixed for $c\nearrow x$. If $\mathbf{A}$ fixes the value of every polynomial in $K[X]$, then it is said to be **transcendental**. If there is some $f\in K[X]$ whose value is not fixed by $\mathbf{A}$, then $\mathbf{A}$ is said to be **algebraic**.

If there exists any polynomial $f\in K[X]$ whose value is not fixed by $\mathbf{A}$, then there also exists a monic polynomial of the same degree having the same property (since this property is not lost by multiplication with non-zero constants from K). If $f(X)$ is a monic polynomial of minimal degree such that $\mathbf{A}$ does not fix the value of f, then it will be called an **associated minimal polynomial** for $\mathbf{A}$ and its degree $\mathbf{d}:=\deg f$ will be called the **degree of A**. We set $\mathbf{d}:=\infty$ if $\mathbf{A}$ is transcendental. Note that an associated minimal polynomial f for $\mathbf{A}$ is always irreducible over K. Indeed, if $g,h\in K[X]$ are of degree less than $\deg f$, then $\mathbf{A}$ fixes the value of g and h and thus also of $g\cdot h$.

We will now study the behaviour of polynomials with respect to immediate approximation types $\mathrm{appr}_v(x,K)$. We use the Taylor expansion

$$f(X)=\sum_{i=0}^{n}\partial_i f(c)(X-c)^i \tag{3.7}$$

where $\partial_i f$ denotes the i-th **Hasse-Schmidt derivative**.

We need the following lemma for ordered abelian groups, which is a reformulation of Kaplansky's Lemma 4 in [8]. For archimedean ordered groups, it was proved by Ostrowski in [15].

Lemma 3.10. *Take elements $\alpha_1,\ldots,\alpha_m$ of an ordered abelian group Γ and a subset $\Upsilon\subset\Gamma$ without maximal element. Let $t_1,\ldots,t_m$ be distinct integers. Then there exists an element $\beta\in\Upsilon$ and a permutation σ of the indices $1,\ldots,m$ such that for all $\gamma\in\Upsilon$, $\gamma\geq\beta$,*

$$\alpha_{\sigma(1)}+t_{\sigma(1)}\gamma>\alpha_{\sigma(2)}+t_{\sigma(2)}\gamma>\ldots>\alpha_{\sigma(m)}+t_{\sigma(m)}\gamma\,.$$

If the immediate approximation type $\mathbf{A}$ is of degree $\mathbf{d}$ and $f\in K[X]$ is of degree at most $\mathbf{d}$, then $\mathbf{A}$ fixes the value of every Hasse–Schmidt derivative $\partial_i f$ of f ($1\leq i\leq\deg f$), since every such derivative has degree less than $\mathbf{d}$. So we can define β_i to be the fixed value $v\partial_i f(c)$ for $c\nearrow x$. In certain cases, a derivative may be identically 0. In this case, we have $\beta_i=\infty$. However, the Taylor expansion of f shows that not all derivatives vanish identically, and the vanishing ones will not play a role in our computations.

By use of Lemma 3.10, we can now prove:

Lemma 3.11. *Take an immediate approximation type* $\mathbf{A} = \mathrm{appr}_v(x, K)$ *of degree* $\mathbf{d}$ *over* (K, v) *and* $f \in K[X]$ *a polynomial of degree at most* $\mathbf{d}$*. Further, let* β_i *denote the fixed value* $v\partial_i f(c)$ *for* $c \nearrow x$*. Then there is a positive integer* $\mathbf{h} \leq \deg f$ *such that*

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) < \beta_i + i \cdot v(x - c) \tag{3.8}$$

whenever $i \neq \mathbf{h}$, $1 \leq i \leq \deg f$ *and* $c \nearrow x$*. Hence,*

$$v(f(x) - f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) \quad \text{for } c \nearrow x \,. \tag{3.9}$$

Consequently, if $\mathbf{A}$ *fixes the value of* f*, then*

$$v(f(x) - f(c)) > vf(x) = vf(c) \quad \text{for } c \nearrow x \,,$$

and if $\mathbf{A}$ *does not fix the value of* f*, then*

$$vf(x) > vf(c) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) \quad \text{for } c \nearrow x \,.$$

Proof. Set $n = \deg f$. We consider the Taylor expansion

$$f(x) - f(c) = \partial_1 f(c)(x - c) + \ldots + \partial_n f(c)(x - c)^n \tag{3.10}$$

with $c \in K$. We have that $v\partial_i f(c)(x - c)^i = \beta_i + i \cdot v(x - c)$ for $c \nearrow x$. So we apply the foregoing lemma with $\alpha_i = \beta_i$ and $t_i = i$, and with $\Upsilon = \mathrm{supp}\mathbf{A}$ (which has no maximal element since $\mathbf{A}$ is an immediate approximation type). We find that there is an integer $\mathbf{h} \leq \deg f$ such that $\beta_{\mathbf{h}} + \mathbf{h}v(x - c) < \beta_i + iv(x - c)$ for $c \nearrow x$ and $i \neq \mathbf{h}$. This is Equation (3.8), which in turn implies Equation (3.9).

If $\mathbf{A}$ fixes the value of f, then $vf(x) \neq vf(c)$ is impossible for $c \nearrow x$ since otherwise, the left hand side of (3.9) would be equal to $\min\{vf(x), vf(c)\}$ and thus fixed while the right hand side of (3.9) increases for $c \nearrow x$. This proves that $vf(x) = vf(c)$ and thus also $v(f(x) - f(c)) \geq vf(x)$ for $c \nearrow x$. But since the left hand side increases, we find that $v(f(x) - f(c)) > vf(x)$ for $c \nearrow x$.

If $\mathbf{A}$ does not fix the value of f, then $vf(x) \neq vf(c)$ and thus $v(f(x) - f(c)) = \min\{vf(x), vf(c)\}$ for $c \nearrow x$. Since $v(f(x) - f(c))$ increases for $c \nearrow x$ and $vf(x)$ is fixed, the minimum must be $vf(c)$. □

If $g \in K[X]$ has a degree smaller than the degree of $\mathbf{A}$, then by Lemma 3.11, the value of $g(x)$ in $(K(x), v)$ is given by $vg(x) = vg(c)$ for $c \nearrow x$. Since $g(c) \in K$, that means that the value of $g(x)$ is uniquely determined by $\mathbf{A}$ and the restriction of v to K. If g is a nonzero polynomial, then $g(c) \neq 0$ for $c \nearrow x$ (since there is a nonempty $\mathbf{A}_\gamma$ which does not contain the finitely many zeros of g, as $\mathbf{A}$ is immediate). Consequently, $g(x) \neq 0$, which shows that the elements $1, x, \ldots, x^{\mathbf{d}-1}$ are K-linearly independent.

We even know that $v(g(x) - g(c)) > vg(x)$ for $c \nearrow x$. This means that $(K, v) \subset (K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ is an immediate extension of valued vector spaces. If $\mathbf{d} = [K(x) : K] < \infty$, then $K(x) = K[x] = K + Kx + \ldots + Kx^{\mathbf{d}-1}$, so by Lemma 2.2, the extension $(K(x)|K, v)$ is immediate. If $\mathbf{d} = \infty$, then $(K, v) \subset (K[x], v)$ is immediate. But then again it follows that the extension $(K(x)|K, v)$ is immediate.

Indeed, if $v(g(x)-g(c)) > vg(x)$ and $v(h(x)-h(c)) > vh(x)$, then $vg(x) = vg(c)$, $vh(x) = vh(c)$ and

$$\begin{aligned} v\left(\frac{g(x)}{h(x)} - \frac{g(c)}{h(c)}\right) &= v\left[g(x)h(c) - g(c)h(x)\right] - vh(x)h(c) \\ &= v\left[g(x)h(c) - g(c)h(c) + g(c)h(c) - g(c)h(x)\right] - vh(x)h(c) \\ &= v\left[(g(x)-g(c))h(c) + g(c)(h(c)-h(x))\right] - vh(x)h(c) \\ &> vg(x)h(x) - vh(x)h(x) \;=\; v\,\frac{g(x)}{h(x)}\,. \end{aligned}$$

We have proved:

Lemma 3.12. *Take an immediate approximation type* $\mathbf{A} = \mathrm{appr}_v(x,K)$ *of degree* $\mathbf{d}$ *over* (K,v)*. Then the valuation on the valued* (K,v)*-vector subspace*

$$(K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$$

of $(K(x),v)$ *is uniquely determined by* $\mathbf{A}$ *because*

$$vg(x) = vg(c) \quad \text{for } c \nearrow x$$

for every $g(x) \in K + Kx + \ldots + Kx^{\mathbf{d}-1}$*. The elements* $1, x, \ldots, x^{\mathbf{d}-1}$ *are* K*-linearly independent. In particular,* x *is transcendental over* K *if* $\mathbf{d} = \infty$*.*

Moreover, the extension $(K,v) \subset (K+Kx+\ldots+Kx^{\mathbf{d}-1}, v)$ *of valued vector spaces is immediate. In particular, if* $\mathbf{d} = \infty$ *or if* $\mathbf{d} = [K(x):K] < \infty$*, then* $(K[x]|K,v)$ *is immediate and the same is consequently true for the extension* $(K(x)|K,v)$*.*

Theorem 3.13. (Theorem 2 of [8], approximation type version)
For every transcendental immediate approximation type $\mathbf{A}$ *over* (K,v) *there exists a simple immediate transcendental extension* $(K(x),v)$ *such that* $\mathrm{appr}_v(x,K) = \mathbf{A}$*.*

If $(K(y),v)$ *is another extension field of* (K,v) *such that* $\mathrm{appr}_v(y,K) = \mathbf{A}$*, then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *that sends* x *to* y *is valuation preserving.*

Proof. We take $K(x)|K$ to be a transcendental extension and define the valuation on $K(x)$ as follows. In view of the rule $v(g/h) = vg - vh$, it suffices to define v on $K[x]$. Take a nonzero polynomial $g \in K[X]$. By assumption, $\mathbf{A}$ fixes the value of g, that is, there is $\beta \in vK$ such that $vg(c) = \beta$ for $c \nearrow \mathbf{A}$. We set $vg(x) = \beta$. Our definition implies that $vg \neq \infty$ for every nonzero $g \in K[x]$.

Take $g, h \in K(X)$. Again by our definition, $vg(x) = vg(c)$ and $vh(x) = vh(c)$ for $c \nearrow \mathbf{A}$. Thus,

$$vg(x)h(x) \;=\; v(gh(x)) \;=\; v(gh(c)) \;=\; vg(c)h(c) \;=\; vg(c) + vh(c) \;=\; vg(x) + vh(x)$$

and

$$\begin{aligned} v(g(x)+h(x)) &= v((g+h)(x)) \;=\; v((g+h)(c)) \;=\; v(g(c)+h(c)) \\ &\geq \min\{vg(c), vh(c)\} \;=\; \min\{vg(x), vh(x)\} \end{aligned}$$

for $c \nearrow \mathbf{A}$. So indeed, our definition yields a valuation v on $K(x)$ which extends the valuation v of K. Under this valuation, we have that $\mathbf{A} = \mathrm{appr}_v(x, K)$; this is seen as follows. In view of Lemma 3.9, it suffices to prove that for every $\gamma \in \mathrm{supp}\mathbf{A}$, we have that $v(x - c_\gamma) \geq \gamma$ for each $c_\gamma \in \mathbf{A}_\gamma$. But this follows directly from our definition of $v(x-c_\gamma)$ because $c \in \mathbf{A}_\gamma$ for $c \nearrow \mathbf{A}$ and thus $v(x-c_\gamma) = v(c-c_\gamma) \geq \gamma$.

From Lemma 3.12, we now infer that $(K(x)|K, v)$ is an immediate extension. Take an element y in some valued field extension of (K, v) such that $\mathbf{A} = \mathrm{appr}(y, K)$. By hypothesis, the degree of $\mathbf{A}$ is ∞. From Lemma 3.12 we can thus deduce that y is transcendental over K. Hence, sending x to y induces an isomorphism from $K(x)$ onto $K(y)$. We have to show that this isomorphism is valuation preserving. For this, we only have to show that $vg(x) = vg(y)$ for every $g \in K[X]$. From Lemma 3.12 we infer that $vg(x) = vg(c) = vg(y)$ holds for $c \nearrow \mathbf{A}$; this proves the desired equality. □

Corollary 3.14. *If v is extended from a valued field (K, v) to a simple field extension $K(y)$ such that $\mathrm{appr}_v(y, K)$ is a transcendental immediate approximation type, then y is transcendental over K, the extension is uniquely determined by $\mathrm{appr}_v(y, K)$, and $(K(y)|K, v)$ is immediate.*

Proof. By the foregoing theorem, there is an immediate extension $(K(x)|K, v)$ such that $\mathrm{appr}_v(x, K) = \mathrm{appr}_v(y, K)$, with x transcendental over K. By the same theorem, there is a valuation preserving isomorphism of $K(x)$ and $K(y)$ over K. This proves our assertions. □

The next lemma will show that every algebraic immediate approximation type is realized by some element in an algebraic valued field extension.

Lemma 3.15. *Take an algebraic immediate approximation type $\mathbf{A}$ over (K, v), a polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, and a root a of f. Then there is an extension of v from K to $K(a)$ such that $\mathbf{A} = \mathrm{appr}(a, K)$.*

Proof. We choose some extension w of v from K to $K(a)$. We write $f(X) = d\prod_{i=1}^{\deg f}(X - a_i)$ with $d \in K$ and $a_i \in \tilde{K}$. If for all i, the values $w(c - a_i)$ would be fixed for $c \nearrow \mathbf{A}$, then $\mathbf{A}$ would fix the value of f, contrary to our assumption. Hence there is a root a_i of f such that $w(a_i - c)$ is not fixed for $c \nearrow \mathbf{A}$. Take some automorphism σ of $\tilde{K}|K$ such that $\sigma a = a_i$ and set $v := w \circ \sigma$. Then v extends the valuation of K, and $v(a - c) = w \circ \sigma(a - c) = w(\sigma a - c) = w(a_i - c)$ is not fixed for $c \nearrow \mathbf{A}$. By Lemma 3.9, $\mathbf{A} = \mathrm{appr}(a, K)$. □

The following is the analogue of Theorem 3.13 for immediate algebraic approximation types.

Theorem 3.16. (Theorem 3 of [8], approximation type version)
For every algebraic immediate approximation type $\mathbf{A}$ over (K, v) of degree $\mathbf{d}$ with associated minimal polynomial $f(X) \in K[X]$ and a root a of f, there exists an extension of v from K to $K(a)$ such that $(K(a)|K, v)$ is an immediate extension and $\mathrm{appr}(a, K) = \mathbf{A}$.

If $(K(b),v)$ is another extension field of (K,v) such that $\mathrm{appr}(b,K)=\mathbf{A}$*, then any field isomorphism between $K(a)$ and $K(b)$ over K sending a to b preserves the valuation. (Note that such an isomorphism exists if and only if b is also a root of f.)*

Proof. We consider the valuation v of $K(a)$ given by Lemma 3.15. Then $\mathrm{appr}(a,K)=\mathbf{A}$. The fact that $(K(a)|K,v)$ is immediate follows from Lemma 3.12.

The last assertion of our theorem is shown in the same way as the corresponding assertion of Theorem 3.13: if $\mathrm{appr}(a,K)=\mathrm{appr}(b,K)$ and $g\in K[X]$ with $\deg g<\mathbf{d}$ then, again by Lemma 3.12, $vg(a)=vg(c)=vg(b)$ for $c\nearrow a$. Hence an isomorphism over K sending a to b will preserve the valuation. □

Proposition 3.17. *Assume that $(K'(x)|K',v)$ and $(K'|K,v)$ are extensions where $x\notin K'$ and (K,v) lies dense in (K',v). Then* $\mathrm{appr}_v(x,K')$ *and* $\mathrm{appr}_v(x,K)$ *have the same support, and if* $\mathrm{appr}_v(x,K')$ *is immediate, then so is* $\mathrm{appr}_v(x,K)$*. If in addition* $\mathrm{appr}_v(x,K')$ *is transcendental, then so is* $\mathrm{appr}_v(x,K)$*.*

Proof. Take $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K')$ and pick $c'_\gamma\in\mathrm{appr}_v(x,K')_\gamma$ so that $v(x-c'_\gamma)\geq\gamma$. Since $\gamma\neq\infty$ and (K,v) lies dense in (K',v), there is some $c_\gamma\in K$ such that $v(c'_\gamma-c_\gamma)>\gamma$. It follows that $v(x-c_\gamma)\geq\min\{v(x-c'_\gamma),v(c'_\gamma-c_\gamma)\}\geq\gamma$, whence $c_\gamma\in\mathrm{appr}_v(x,K)_\gamma$ and $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K)$. Thus $\mathrm{supp}\,\mathrm{appr}_v(x,K')\subseteq\mathrm{supp}\,\mathrm{appr}_v(x,K)$. The reverse inclusion follows from the fact that $B_\gamma(x,K(x))\cap K\subseteq B_\gamma(x,K'(x))\cap K'$. If $\mathrm{appr}_v(x,K')$ is immediate, then from

$$\bigcap_{\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K)}\mathrm{appr}_v(x,K)_\gamma\subseteq\bigcap_{\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K')}\mathrm{appr}_v(x,K')_\gamma=\emptyset$$

we see that $\mathrm{appr}_v(x,K)$ is immediate.

Now assume that $\mathrm{appr}_v(x,K')$ is transcendental immediate, and take $f\in K[X]$. Then $f\in K'[X]$ and so there is $\gamma\in\mathrm{supp}\,\mathrm{appr}_v(x,K')=\mathrm{supp}\,\mathrm{appr}_v(x,K)$ such that $vf(c)$ is constant for $c\in\mathrm{appr}_v(x,K')_\gamma$. Since $\mathrm{appr}_v(x,K)_\gamma\subseteq\mathrm{appr}_v(x,K')_\gamma$, the same holds for $c\in\mathrm{appr}_v(x,K)_\gamma$. This proves that $\mathrm{appr}_v(x,K)$ is transcendental. □

3.4. Immediate approximation types versus pseudo Cauchy sequences.

The following proposition reveals the connection between immediate approximation types and pseudo Cauchy sequences. Take a valued field (K,v), an immediate approximation type $\mathbf{A}$ over (K,v), and a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in (K,v). We will say that $\mathbf{A}$ and $(c_\nu)_{\nu<\lambda}$ are **associated** if in every extension $(L|K,v)$, $x\in L$ realizes $\mathbf{A}$ if and only if x is a limit of $(c_\nu)_{\nu<\lambda}$. Since trivial approximation types are not immediate, a pseudo Cauchy sequence in (K,v) associated with an immediate approximation type over (K,v) cannot have a limit in K.

Proposition 3.18. *1) Take an immediate approximation type* $\mathbf{A}$ *over* (K,v)*, a pseudo Cauchy sequence* $(c_\nu)_{\nu<\lambda}$ *in* (K,v) *without a limit in* K*. Then* $\mathbf{A}$ *and* $(c_\nu)_{\nu<\lambda}$ *are associated if and only if for all* $\nu<\lambda$*,*

$$\mathbf{A}_{\gamma_\nu} = B_{\gamma_\nu}(c_\nu,K)\;. \qquad (3.11)$$

If this is the case, then $\mathrm{supp}\mathbf{A}$ *is equal to the least initial segment of* vK *that contains all* γ_ν *.*

2) Every immediate approximation type over (K,v) *is associated with some pseudo Cauchy sequence in* (K,v)*, which consequently has no limit in* K*.*

3) Every pseudo Cauchy sequence in (K,v) *without a limit in* K *is associated with a unique immediate approximation type over* (K,v)*.*

Proof. 1): Assume first that (3.11) holds. Since $(c_\nu)_{\nu<\lambda}$ in (K,v) has no limit in K, we know from Lemma 2.17 that the intersection of the ultrametric balls $B_{\gamma_\nu}(c_\nu,K)$, $\nu<\lambda$, is empty; hence the same holds for the intersection over the $\mathbf{A}_{\gamma_\nu}$. Consequently, $\mathbf{A}_\gamma=\emptyset$ when $\gamma>\gamma_\nu$ for all $\nu<\lambda$. This shows that the set $\{\gamma_\nu\mid \nu<\lambda\}$ is cofinal in $\mathrm{supp}\mathbf{A}$, which implies the last assertion of part 1). We also obtain that the assumption of Lemma 3.9 is satisfied. Hence an element x in some extension of (K,v) realizes $\mathbf{A}$ if and only if for every $\nu<\lambda$ we have that $v(x-c)\geq\gamma_\nu$ for some $c\in\mathbf{A}_{\gamma_\nu}=B_{\gamma_\nu}(c_\nu,K)$, whence also $v(x-c_\nu)\geq\gamma_\nu$. By condition (2.6), this holds if and only if x is a limit of $(c_\nu)_{\nu<\lambda}$. We have proved that $\mathbf{A}$ and $(c_\nu)_{\nu<\lambda}$ are associated.

Assume now that $\mathbf{A}$ and $(c_\nu)_{\nu<\lambda}$ are associated. From Theorems 2.13 and 2.14 we know that there is at least one limit x of $(c_\nu)_{\nu<\lambda}$ in some extension (L,v) of (K,v). By condition (2.6), we have that $v(x-c_\nu)\geq\gamma_\nu$ for all $\nu<\lambda$. As x is supposed to also realize $\mathbf{A}$, we must have that $\mathbf{A}=\mathrm{appr}_v(x,K)$. Hence $\mathbf{A}_\gamma=\mathrm{appr}_v(x,K)_\gamma=\{c\in K\mid v(x-c)\geq\gamma\}$ for each $\gamma\in\mathrm{supp}\mathbf{A}$. In particular, $c_\nu\in\mathrm{appr}_v(x,K)_{\gamma_\nu}$ for all $\nu<\lambda$. By the ultrametric triangle law,

$$c\in B_{\gamma_\nu}(c_\nu,K)\;\Leftrightarrow\; v(c-c_\nu)\geq\gamma_\nu\;\Leftrightarrow\; v(x-c)\geq\gamma_\nu\;\Leftrightarrow\; c\in\mathrm{appr}_v(x,K)_{\gamma_\nu}=\mathbf{A}_{\gamma_\nu}\;.$$

This shows that (3.11) holds.

2): Take an immediate approximation type $\mathbf{A}$ over (K,v). By (possibly transfinite) induction we construct a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$. Pick any $c_0\in K$. Assume that we have constructed the sequence up to some c_ν that lies in $\mathbf{A}_{\gamma_\nu}$ for some $\gamma_\nu\in\mathrm{supp}\mathbf{A}$. Since $\mathbf{A}$ is immediate, there is some $\gamma_{\nu+1}\in\mathrm{supp}\mathbf{A}$ such that $c_\nu\notin\mathbf{A}_{\gamma_{\nu+1}}$, and we pick any $c_{\nu+1}\in\mathbf{A}_{\gamma_{\nu+1}}$. Assume that κ is a limit ordinal and we have chosen $c_\nu\in\mathbf{A}_{\gamma_\nu}\setminus\mathbf{A}_{\gamma_{\nu+1}}$ for all $\nu<\kappa$. If the values γ_ν, $\nu<\kappa$, are cofinal in $\mathrm{supp}\mathbf{A}$, we are done and set $\lambda=\kappa$. Otherwise, there is some $\gamma_\kappa\in\mathrm{supp}\mathbf{A}$ larger than all γ_ν, and we pick some $c_\kappa\in\mathbf{A}_{\gamma_\kappa}$. This procedure must stop at a limit ordinal λ as the cardinality of the set of values γ_ν is bounded by the cardinality of $\mathrm{supp}\mathbf{A}$. When it stops it means that the γ_ν we have constructed are cofinal in $\mathrm{supp}\mathbf{A}$.

Assume that $\rho<\sigma<\tau<\lambda$. By construction, $c_\rho\notin\mathbf{A}_{\gamma_\sigma}$ (since $\mathbf{A}_{\gamma_\sigma}\subseteq\mathbf{A}_{\gamma_{\rho+1}}$), $c_\sigma\in\mathbf{A}_{\gamma_\sigma}$, and $c_\tau\in\mathbf{A}_{\gamma_\tau}\subseteq\mathbf{A}_{\gamma_\sigma}$. We obtain that $v(c_\sigma-c_\rho)<\gamma_\sigma\leq v(c_\tau-c_\sigma)$. This shows that $(c_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence.

Since $c_\nu \in \mathbf{A}_{\gamma_\nu}$ by construction, it follows that $\mathbf{A}_{\gamma_\nu} = B_{\gamma_\nu}(c_\nu, K)$. Hence condition (3.11) holds for all $\nu < \lambda$, so by part 1), $\mathbf{A}$ and $(c_\nu)_{\nu<\lambda}$ are associated.

3): Take a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in (K, v) without a limit in K. From Lemma 2.17 we know that $\mathcal{N} := (B_{\gamma_\nu}(c_\nu, K))_{\nu<\lambda}$ is a nest of balls in K and that the intersection over all balls in this nest is empty. By part 3) of Lemma 2.16 there is a uniquely determined full nest $\mathcal{A}$ containing $\mathcal{N}$, which satisfies $\bigcap \mathcal{A} = \bigcap \mathcal{N}$. Consequently, $\mathcal{A}$ is an immediate approximation type, uniquely determined by the pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$. Condition (3.11) holds by definition of $\mathcal{N}$. □

The following result deals with an approximation type analogue of Cauchy sequences. We will say that $\mathbf{A}$ is a **completion type** over (K, v) if it is an immediate approximation type over (K, v) with $\mathrm{supp}\mathbf{A} = vK$.

Proposition 3.19. *Assume that* $\mathbf{A}$ *is a completion type over* (K, v)*. Then there is a unique element* y *in the completion of* (K, v) *that realizes* $\mathbf{A}$*. If* $(L|K, v)$ *is an extension containing two distinct elements* y, z *which both realize* $\mathbf{A}$*, then* $v(y-z) > vK$*.*

Proof. By Proposition 3.18 there is a pseudo Cauchy sequence that is associated with $\mathbf{A}$ and for which the values γ_ν are cofinal in $\mathrm{supp}\mathbf{A} = vK$. This sequence is hence a Cauchy sequence, and by the definition of the completion, has a limit in the completion. This limit realizes $\mathbf{A}$.

The last assertion follows from Corollary 3.8, and since the value group of the completion is equal to vK, it in turn proves the uniqueness in the first assertion. □

3.5. **Properties of arbitrary approximation types.**

In Section 3.2 we have exhibited the relation between immediate approximation types and extensions of valuations to simple field extensions. In this section, we will have a closer look at arbitrary approximation types $\mathbf{A}$ over a fixed valued field (K, v), in particular those that are not immediate. By definition, the latter means that $\bigcap \mathbf{A}$ is not empty.

Throughout, we will assume that $\mathbf{A}$ is non-trivial. We define: $\mathbf{A}$ is called **residue-immediate** if it satisfies

$$\mathbf{A}_\gamma^\circ \neq \emptyset \quad \text{for every } \gamma \in \mathrm{supp}\mathbf{A}\,, \tag{3.12}$$

and it is called **value-extending** if in addition $\bigcap \mathbf{A} \neq \emptyset$. Note that if $\gamma \in \mathrm{supp}\mathbf{A}$ such that $\mathbf{A}_\gamma^\circ = \emptyset$, then γ is the maximal element of $\mathrm{supp}\mathbf{A}$ since if there were $\delta \in \mathrm{supp}\mathbf{A}$ such that $\gamma < \delta$, then the nonempty set $\mathbf{A}_\delta$ would be contained in $\mathbf{A}_\gamma^\circ$.

Observe that the empty approximation type is value-extending.

Further, $\mathbf{A}$ is called **value-immediate** if it satisfies

$$\text{for every } c \in K \text{ there is } \gamma \in \mathrm{supp}\mathbf{A} \text{ such that } c \in \mathbf{A}_\gamma \setminus \mathbf{A}_\gamma^\circ\,, \tag{3.13}$$

and it is called **residue-extending** if in addition $\bigcap \mathbf{A} \neq \emptyset$.

Example 3.20. Choose any initial segment S of vK. Then

$$\{B_\gamma(0,K)\,,\, B^\circ_\gamma(0,K) \mid \gamma\in S\}$$

is a value-extending approximation type. For any $\gamma_0\in vK$,

$$\{B_\gamma(0,K) \mid \gamma\leq\gamma_0\}\cup\{B^\circ_\gamma(0,K) \mid \gamma<\gamma_0\}$$

is a residue-extending approximation type. If we adjoin $B^\circ_{\gamma_0}(0,K)$, then we obtain a value-extending approximation type. We will see that the value-extending approximation type describes an extension where $vx\notin vK$ induces a cut in vK with lower cut set S, which is $\{\gamma\in vK\mid\gamma\leq\gamma_0\}$ in the last example. In contrast, the residue-extending approximation type describes an extension where $vx=\gamma_0\in vK$ and for any $d\in K$ such that $vdx=0$ we have that $dxv\notin Kv$. We see that the difference between the two cases is revealed by an open ball; this is the reason why we take approximation types to contain both closed and open balls. $\diamondsuit$

Lemma 3.21. *1) Condition (3.12) holds if and only if* $\mathbf{A}=\emptyset$ *or*

$$\bigcap\mathbf{A} \;=\; \bigcap_{\gamma\,\in\,\mathrm{supp}\mathbf{A}} \mathbf{A}^\circ_\gamma\,. \tag{3.14}$$

2) $\mathbf{A}$ *is value-extending if and only if* $\mathbf{A}=\emptyset$ *or*

$$\bigcap_{\gamma\,\in\,\mathrm{supp}\mathbf{A}} \mathbf{A}^\circ_\gamma \;\neq\;\emptyset\,. \tag{3.15}$$

3) If $c\notin\mathbf{A}^\circ_\delta$ *for some* $\delta\in\mathrm{supp}\mathbf{A}$*, then there is* $\gamma\in\mathrm{supp}\mathbf{A}$ *such that* $c\in\mathbf{A}_\gamma\setminus\mathbf{A}^\circ_\gamma$*.*

4) $\mathbf{A}$ *is residue-extending if and only if there is* $\delta\in\mathrm{supp}\mathbf{A}$ *such that* $\bigcap\mathbf{A}=\mathbf{A}_\delta$ *and* $\mathbf{A}^\circ_\delta=\emptyset$*. If this is the case, then* δ *is the maximal element of* $\mathrm{supp}\mathbf{A}$*.*

5) Every non-trivial approximation type is immediate, value-extending or residue-extending. These three properties are mutually exclusive.

Proof. Our assertions are trivial if $\mathbf{A}=\emptyset$, so we may assume that $\mathbf{A}\neq\emptyset$.

1): The inclusion $\supseteq$ in (3.14) always holds. If condition (3.12) holds, then $\mathbf{A}^\circ_\gamma\neq\emptyset$ and thus $\bigcap\mathbf{A}\subseteq\mathbf{A}^\circ_\gamma$ for all $\gamma\in\mathrm{supp}\mathbf{A}$, hence the inclusion $\subseteq$ also holds in (3.14). If condition (3.12) does not hold, then there is $\delta\in\mathrm{supp}\mathbf{A}$ such that $\mathbf{A}^\circ_\delta=\emptyset$. As $\mathbf{A}^\circ_\delta$ contains $\mathbf{A}_\epsilon$ for every $\epsilon>\delta$, we then have that $\mathbf{A}_\epsilon=\emptyset$, whence δ is the maximal element of $\mathrm{supp}\mathbf{A}$ and $\bigcap\mathbf{A}=\mathbf{A}_\delta\neq\emptyset=\bigcap_{\gamma\in\mathrm{supp}\mathbf{A}}\mathbf{A}^\circ_\gamma$.

2): If $\mathbf{A}$ is value-extending, then by part 1) of our lemma,

$$\bigcap_{\gamma\in\mathrm{supp}\mathbf{A}} \mathbf{A}^\circ_\gamma \;=\; \bigcap\mathbf{A}\;\neq\;\emptyset\,.$$

On the other hand, if (3.15) holds, then also (3.14) holds. Then by part 1) of our lemma, (3.12) holds, showing that $\mathbf{A}$ is value-extending.

3): Take $c\in K$ and assume that $c\notin\mathbf{A}^\circ_\delta$ for some $\delta\in\mathrm{supp}\mathbf{A}$. If $c\in\mathbf{A}_\delta$, then our assertion holds for $\gamma=\delta$. Now assume that $c\notin\mathbf{A}_\delta$, pick $d\in\mathbf{A}_\delta$ and set $\gamma=v(d-c)$.

As $c \notin \mathbf{A}_\delta$, we have that $\delta > \gamma$ and therefore, $\mathbf{A}_\gamma = B_\gamma(d,K)$ and $\mathbf{A}^\circ_\gamma = B^\circ_\gamma(d,K)$. Since $v(d-c) = \gamma$, $c \in \mathbf{A}_\gamma \setminus \mathbf{A}^\circ_\gamma$. This proves part 3).

4): Assume that $\mathbf{A}$ is residue-extending, pick any $b \in \bigcap \mathbf{A}$ and $\delta \in \mathrm{supp}\mathbf{A}$ such that $b \in \mathbf{A}_\delta \setminus \mathbf{A}^\circ_\delta$. Then $b \notin \mathbf{A}_\varepsilon$ for any $\varepsilon \in vK$ with $\varepsilon > \delta$ since $\mathbf{A}_\varepsilon \subseteq \mathbf{A}^\circ_\delta$. As $b \in \bigcap \mathbf{A}$, this shows that $\mathbf{A}_\varepsilon = \emptyset$, hence δ is the maximal element of $\mathrm{supp}\mathbf{A}$. It follows from this together with (3.13) that for every $c \in \mathbf{A}_\delta$, we must have that $c \notin \mathbf{A}^\circ_\delta$, which proves that $\mathbf{A}^\circ_\delta = \emptyset$.

For the converse, assume that there is $\delta \in \mathrm{supp}\mathbf{A}$ such that $\bigcap \mathbf{A} = \mathbf{A}_\delta$ and $\mathbf{A}^\circ_\delta = \emptyset$. Since $\delta \in \mathrm{supp}\mathbf{A}$, we have that $\bigcap \mathbf{A} \neq \emptyset$. Pick any $c \in K$. Since $c \notin \emptyset = \mathbf{A}^\circ_\delta$, by part 3) there is $\gamma \in \mathrm{supp}\mathbf{A}$ such that $c \in \mathbf{A}_\gamma \setminus \mathbf{A}^\circ_\gamma$, showing that $\mathbf{A}$ is residue-extending. This finishes the proof of part 4).

5): Take a non-trivial approximation type $\mathbf{A}$ which is not immediate, i.e., $\bigcap \mathbf{A} \neq \emptyset$. If $\mathbf{A}$ is also not value-extending, then there is $\delta \in \mathrm{supp}\mathbf{A}$ such that $\mathbf{A}^\circ_\delta = \emptyset$. By part 4), this implies that $\mathbf{A}$ is residue-extending.

Now we prove the second assertion of part 5). If $\mathbf{A}$ is immediate, then $\bigcap \mathbf{A} = \emptyset$, so $\mathbf{A}$ can neither be value-extending nor residue-extending. If $\mathbf{A}$ is residue-extending, then by part 4) there is $\delta \in \mathrm{supp}\mathbf{A}$ such that $\bigcap \mathbf{A} = \mathbf{A}_\delta$ and $\mathbf{A}^\circ_\delta = \emptyset$, hence for this $\gamma = \delta$, condition (3.12) is violated. This shows that $\mathbf{A}$ cannot be value-extending. □

Remark 3.22. If $\mathbf{A}$ is residue-extending, then by part 4) of the previous lemma, $\mathrm{supp}\mathbf{A}$ admits a maximal element δ, and $\mathbf{A}^\circ_\delta = \emptyset$. Thus $\mathbf{A}_\delta$ is the smallest ball in $\mathbf{A}$, and $\mathbf{A}$ is generated by $\mathbf{A}_\delta$.

If $\mathbf{A}$ is value-extending and if $\mathrm{supp}\mathbf{A}$ admits a maximal element δ, then $\mathbf{A}^\circ_\delta \neq \emptyset$ is the smallest ball in $\mathbf{A}$ and $\mathbf{A}$ is generated by $\mathbf{A}^\circ_\delta$. However, not every value-extending approximation type $\mathbf{A}$ is such that $\mathrm{supp}\mathbf{A}$ admits a maximal element, in which case it is not generated by a single ball. #

Lemma 3.23. *Take an approximation type* $\mathbf{A}$ *over* (K,v)*, an extension* $(L|K,v)$*, and an element* $x \in L$*.*

1) Assume that $\mathbf{A}$ *is value-extending and pick some* $b \in \bigcap \mathbf{A}$*. Then* x *realizes* $\mathbf{A}$ *if and only if* $v(x-b) > \gamma$ *for all* $\gamma \in \mathrm{supp}\mathbf{A}$ *and* $v(x-c) < \varepsilon$ *for all* $\varepsilon \in vK \setminus \mathrm{supp}\mathbf{A}$ *and all* $c \in K$*.*

2) Assume that $\mathbf{A}$ *is residue-extending and pick some* $b \in \bigcap \mathbf{A}$*. Then* x *realizes* $\mathbf{A}$ *if and only if* $v(x-b) = \max \mathrm{supp}\mathbf{A}$ *and* $v(x-c) \leq v(x-b)$ *for all* $c \in K$*.*

Proof. 1): The condition "$v(x-c) < \varepsilon$ for all $\varepsilon \in vK \setminus \mathrm{supp}\mathbf{A}$ and all $c \in K$" is equivalent to $\mathrm{supp}\,\mathrm{appr}_v(x,K) \subseteq \mathrm{supp}\mathbf{A}$. Hence it holds when $\mathrm{appr}_v(x,K) = \mathbf{A}$, and this equality also implies that for all $\gamma \in \mathrm{supp}\mathbf{A}$ we have that $v(x-b) > \gamma$.

For the converse, assume that $v(x-b) > \gamma$ for all $\gamma \in \mathrm{supp}\mathbf{A}$. Then for all $\gamma \in \mathrm{supp}\mathbf{A}$, $b \in \mathrm{appr}_v(x,K)^\circ_\gamma$ and thus also $b \in \mathrm{appr}_v(x,K)_\gamma$, which implies that $\mathrm{appr}_v(x,K)^\circ_\gamma = B^\circ_\gamma(b,K) = \mathbf{A}^\circ_\gamma$ and $\mathrm{appr}_v(x,K)_\gamma = B_\gamma(b,K) = \mathbf{A}_\gamma$, and moreover, $\mathrm{supp}\mathbf{A} \subseteq \mathrm{supp}\,\mathrm{appr}_v(x,K)$. By what we have shown in the beginning, assuming also the condition "$v(x-c) < \varepsilon$ for all $\varepsilon \in vK \setminus \mathrm{supp}\mathbf{A}$ and $c \in K$" yields the reverse inclusion and thus equality of the supports. From Lemma 3.1 we now infer that $\mathrm{appr}_v(x,K) = \mathbf{A}$.

2): Since $\mathbf{A}$ is residue-extending, we know from part 4) of Lemma 3.21 that for $\delta = \max\mathrm{supp}\mathbf{A}$, $b \in \mathbf{A}_\delta$ and $\mathbf{A}_\delta^\circ = \emptyset$. Hence if $\mathrm{appr}_v(x,K) = \mathbf{A}$, then $v(x-b) = \delta = \max\mathrm{supp}\mathbf{A}$ and $v(x-c) \le \delta = v(x-b)$ for all $c \in K$. For the converse, assume that the latter holds. Then $\mathrm{appr}_v(x,K)_\delta^\circ = \emptyset = \mathbf{A}_\delta^\circ$, but $b \in \mathrm{appr}_v(x,K)_\gamma$ for all $\gamma \le \delta$ and therefore, $b \in \mathrm{appr}_v(x,K)_\gamma^\circ$ for all $\gamma < \delta$. It follows that $\mathrm{supp}\,\mathrm{appr}_v(x,K) = \mathrm{supp}\mathbf{A}$, $\mathrm{appr}_v(x,K)_\gamma = B_\gamma(b,K) = \mathbf{A}_\gamma$ for all $\gamma \le \delta$, and $\mathrm{appr}_v(x,K)_\gamma^\circ = B_\gamma^\circ(b,K) = \mathbf{A}_\gamma^\circ$ for all $\gamma < \delta$. From Lemma 3.1 we obtain that $\mathrm{appr}_v(x,K) = \mathbf{A}$. □

The following results justify the names "value-extending" and "residue-extending".

Lemma 3.24. *Take any extension $(K(x)|K,v)$ and set $\mathbf{A} := \mathrm{appr}_v(x,K)$. Then the following assertions hold:*

1) The approximation type $\mathbf{A}$ is value-extending if and only if there is $b \in K$ such that $v(x-b) \notin vK$. If this is the case, then

$$\mathrm{supp}\mathbf{A} \;=\; v(x-K)\cap vK \;=\; \{\gamma \in vK \mid \gamma < v(x-b)\}\,, \tag{3.16}$$

and $\bigcap\mathbf{A}$ is the set of all elements $b \in K$ for which $v(x-b) \notin vK$, or equivalently, $v(x-b)$ realizes the cut $(\mathrm{supp}\mathbf{A}\,,\, vK \setminus \mathrm{supp}\mathbf{A})$.

2) The approximation type $\mathbf{A}$ is residue-extending if and only if there are $b,d \in K$ such that $vd(x-b) = 0$ and $d(x-b)v \notin Kv$. If this is the case, then

$$\mathrm{supp}\mathbf{A} \;=\; v(x-K) \;=\; \{\gamma \in vK \mid \gamma \le v(x-b)\}\,, \tag{3.17}$$

and $\bigcap\mathbf{A}$ is the set of all $b \in K$ for which $v(x-b) = \max\mathrm{supp}\mathbf{A}$, or equivalently, $d(x-b)v \notin Kv$ for any $d \in K$ with $vd = -\max\mathrm{supp}\mathbf{A}$.

Proof. 1): Assume that $\mathbf{A}$ is value-extending and pick $b \in \bigcap\mathbf{A}$. Suppose that $v(x-b) \in vK$. Then for $\gamma = v(x-b)$ we would have that $b \in \mathbf{A}_\gamma \setminus \mathbf{A}_\gamma^\circ$. However, by (3.12), $\mathbf{A}_\gamma^\circ \neq \emptyset$, but $b \notin \mathbf{A}_\gamma^\circ$, which contradicts our choice of $b \in \bigcap\mathbf{A}$.

To prove the converse, assume that there is $b \in K$ such that $v(x-b) \notin vK$. We have that $b \in \mathbf{A}_\gamma$ and $b \in \mathbf{A}_\gamma^\circ$ for every $\gamma \in vK$ with $\gamma < v(x-b)$. Suppose that $\delta > v(x-b)$ and $d \in \mathbf{A}_\delta$. Then $v(x-d) \ge \delta > v(x-b)$, whence $v(d-b) = v(x-b) \notin vK$, which is a contradiction, showing that $\mathbf{A}_\delta = \emptyset$ and $\delta \notin \mathrm{supp}\mathbf{A}$. Consequently, $b \in \bigcap\mathbf{A}$ and (3.16) holds. For $\gamma \in \mathrm{supp}\mathbf{A}$ we have that $\gamma < v(x-b)$, hence $b \in \mathbf{A}_\gamma^\circ$ (and also $b \in \mathbf{A}_\gamma$ since $b \in \mathbf{A}_\gamma^\circ \subseteq \mathbf{A}_\gamma$), so we have that $\mathbf{A}_\gamma^\circ \neq \emptyset$. We have now proved that $\mathbf{A}$ is value-extending.

It remains to prove the last assertion of part 1). We have already shown that $b \in \bigcap\mathbf{A}$ if and only if $v(x-b) \notin vK$, and that the former implies (3.16). This means that $\mathrm{supp}\mathbf{A}$ is the lower cut set of the cut induced by $v(x-b)$ in vK, i.e., $v(x-b)$ realizes the cut $(\mathrm{supp}\mathbf{A}\,,\, vK \setminus \mathrm{supp}\mathbf{A})$. Conversely, if this is true, then $v(x-b) \notin vK$. This finishes the proof of part 1).

2): Assume that $\mathbf{A}$ is residue-extending. By part 4) of Lemma 3.21 we may pick $\delta \in \mathrm{supp}\mathbf{A}$ such that $\bigcap \mathbf{A} = \mathbf{A}_\delta$ and $\mathbf{A}_\delta^\circ = \emptyset$. Further, we pick $b \in \bigcap \mathbf{A}$. Since $b \notin \emptyset = \mathbf{A}_\delta^\circ$, we have that $v(x-b) = \delta \in vK$, so that we can pick $d \in K$ such that $vd = -\delta$. Then $vd(x-b) = 0$ and from part 5) of Lemma 2.8 we infer that $d(x-b)v \notin Kv$.

To prove the converse, assume that there are $b, d \in K$ such that $vd(x-b) = 0$ and $d(x-b)v \notin Kv$. Then by part 5) of Lemma 2.8, $\delta := v(x-b) = \max v(x-K) \in vK$. It follows that $\mathbf{A}_\delta^\circ = \emptyset$, $\bigcap \mathbf{A} = \mathbf{A}_\delta \neq \emptyset$, and from part 4) of Lemma 3.21 we obtain that $\mathbf{A}$ is residue-extending. It also follows that (3.17) holds, which means that $v(x-b) = \max \mathrm{supp}\mathbf{A}$.

It remains to prove the last assertion of part 2). We have already shown that for any $b \in \bigcap \mathbf{A}$ we have that $v(x-b) = \max \mathrm{supp}\mathbf{A}$ and if $d \in K$ with $-vd = v(x-b) = \max \mathrm{supp}\mathbf{A}$, then $d(x-b)v \notin Kv$. We have also shown that the latter implies that $v(x-b) = \max \mathrm{supp}\mathbf{A}$. Hence if $\delta = \max \mathrm{supp}\mathbf{A}$, then $b \in \mathbf{A}_\delta$. As $\mathbf{A}$ is residue-extending, we know from part 4) of Lemma 3.21 that $\mathbf{A}_\delta^\circ = \emptyset$, which shows that $b \in \bigcap \mathbf{A}$. □

For the sake of completeness, we include the following results.

Proposition 3.25. *Take a non-trivial approximation type* $\mathbf{A}$ *over a valued field* (K, v). *Then the following assertions hold.*

1) $\mathbf{A}$ *is immediate if and only if it is value-immediate and residue-immediate.*

2) $\mathbf{A}$ *is value-immediate or residue-immediate.*

Proof. 1): Assume first that $\mathbf{A}$ is immediate. Then $\mathrm{supp}\mathbf{A}$ has no maximal element. Hence for every $\gamma \in \mathrm{supp}\mathbf{A}$ there is $\delta \in \mathrm{supp}\mathbf{A}$ such that $\delta > \gamma$ and therefore $\emptyset \neq \mathbf{A}_\delta \subseteq \mathbf{A}_\gamma^\circ$. This proves that $\mathbf{A}$ is residue-immediate.

Take any $c \in K$. Since $\bigcap \mathbf{A} = \emptyset$ there is some $\gamma \in \mathrm{supp}\mathbf{A}$ such that $c \notin \mathbf{A}_\gamma^\circ$. Now it follows from part 3) of Lemma 3.21 that condition (3.13) holds, so $\mathbf{A}$ is value-immediate.

For the proof of the reverse implication, assume that $\mathbf{A}$ is not immediate, so $\bigcap \mathbf{A} \neq \emptyset$. Assume that $\mathbf{A}$ is residue-immediate and thus value-extending. From part 5) of Lemma 3.21 we conclude that it cannot be residue-extending. Since $\bigcap \mathbf{A} \neq \emptyset$ holds, condition (3.13) must fail, showing that $\mathbf{A}$ is not value-immediate.

2): This follows from part 1) of our lemma together with part 5) of Lemma 3.21. □

4. Realization of approximation types

4.1. **Proof of Theorem 1.2.**

Let (K, v_0) be an arbitrary valued field. Recall that we denote by $\mathcal{V}$ the set of all extensions of v_0 to $K(x)$, and by $\mathcal{A}$ the set of all non-trivial approximation types over (K, v_0). The following is a more precise version of Theorem 1.2.

Theorem 4.1. *For every non-trivial approximation type* $\mathbf{A}$ *there is an extension* v *of* v_0 *to* $K(x)$ *such that* $\mathrm{appr}_v(x,K)=\mathbf{A}$*. That is, the function*

$$\mathcal{V}\longrightarrow\mathcal{A}\,,\qquad v\mapsto\mathrm{appr}_v(x,K) \tag{4.1}$$

is surjective. Moreover, the following assertions hold:

1) If $\mathbf{A}$ *is transcendental immediate, then the extension* v *is uniquely determined and immediate.*

2) If $\mathbf{A}$ *is algebraic immediate with* $\mathrm{supp}\mathbf{A}\neq v_0K$*, then the extension* v *can be chosen to be value-transcendental or residue-transcendental. If* $\mathbf{A}$ *is algebraic immediate with* $\mathrm{supp}\mathbf{A}=v_0K$*, then the extension* v *is uniquely determined and value-transcendental.*

3) If $\mathbf{A}$ *is value-extending, then the extension can be constructed in the following way: Take* $b\in\bigcap\mathbf{A}$*, and take* α *in some ordered abelian group containing* v_0K *such that* α *is not a torsion element modulo* v_0K *and realizes the cut* $\mathcal{C}$ *in* v_0K *that has lower cut set* $\mathrm{supp}\mathbf{A}$*. If* v *is the extension of* v_0 *to* $K(x)$ *obtained from Corollary 2.4 by assigning the value* α *to* $z=x-b$*, then* $\mathrm{appr}_v(x,K)=\mathbf{A}$*.*

Every extension $(K(x)|K,v)$ *constructed in this way is value-transcendental.*

4) If $\mathbf{A}$ *is residue-extending, then the extension can be constructed in the following way: Take* $b\in\bigcap\mathbf{A}=\mathbf{A}_\delta$*. If* v *is the extension of* v_0 *to* $K(x)$ *obtained from Corollary 2.4 by assigning the value* δ *to* $z=x-b$*, then* $\mathrm{appr}_v(x,K)=\mathbf{A}$*.*

Every extension $(K(x)|K,v)$ *constructed in this way is residue-transcendental.*

Proof. Let us assume first that $\mathbf{A}$ is immediate. If it is transcendental, then by Theorem 3.13 and Corollary 3.14 there is a uniquely determined extension v such that $\mathrm{appr}_v(x,K)=\mathbf{A}$. This proves part 1).

Now assume that $\mathbf{A}$ is algebraic. We first use Theorem 3.16 to obtain an extension v of v_0 to some algebraic extension $K(a)$ of K such that $\mathrm{appr}_v(a,K)=\mathbf{A}$. Then we choose some $\alpha>\mathrm{supp}\mathbf{A}$ that is an element either of v_0K, or of some ordered abelian group containing v_0K, in which case α should not be a torsion element modulo v_0K. With $z:=x-a$ we use Corollary 2.4 to extend v from $K(a)$ to $K(a,z)=K(a,x)$ by assigning the value α to the element z. Finally, we restrict this extension to $K(x)$ and infer from Corollary 3.8 that $\mathrm{appr}_v(x,K)=\mathrm{appr}_v(a,K)=\mathbf{A}$.

We can choose the extension $(K(a,x)|K(a),v)$ to be value-transcendental if we take $\alpha\notin v_0K$, and to be residue-transcendental if we take $\alpha\in v_0K$; however, if $\mathrm{supp}\mathbf{A}=v_0K$, then the condition that $\alpha>\mathrm{supp}\mathbf{A}$ forces $\alpha\notin v_0K$ (in which case it is automatically non-torsion modulo v_0K). If $(K(a,x)|K(a),v)$ is value-transcendental, then so is $(K(a,x)|K,v)$, that is, $vK(a,x)/v_0K$ is not a torsion group. Since $K(a,x)|K(x)$ is algebraic, $vK(a,x)/v_0K(x)$ is a torsion group, so we conclude that $vK(x)/v_0K$ cannot be a torsion group, i.e., $(K(x)|K,v)$ must be value-transcendental. A similar argument shows that if $(K(a,x)|K(a),v)$ is residue-transcendental, then so is $(K(x)|K,v)$, using the fact that $K(a,x)v/K(x)v$ is algebraic. We have proved the first assertion of part 2).

Assume that $\mathbf{A}$ is algebraic immediate with $\mathrm{supp}\mathbf{A} = v_0K$. Then the extension we have already constructed is value-transcendental. It remains to show that it is uniquely determined. To this end, let us assume that $(K(x)|K,v)$ is an extension such that $\mathrm{appr}_v(x,K) = \mathbf{A}$. We extend v from $K(x)$ to $\tilde{K}(x)$ and call this extension again v. Its restriction to $\tilde{K}$ provides us with an extension of v_0 from K to $\tilde{K}$.

By Theorem 3.16, there are $b \in \tilde{K}$ and an extension w of v_0 from K to $K(b)$ such that $\mathrm{appr}_v(x,K)$ is realized by b in $(K(b),w)$, that is, $\mathrm{appr}_v(x,K) = \mathrm{appr}_w(b,K)$. We extend w to $\tilde{K}$. As two extensions of v_0 from K to $\tilde{K}$ are conjugate, there is $\sigma \in \mathrm{Aut}\,\tilde{K}|K$ such that $w = v \circ \sigma$. We set $a := \sigma b$. Then for every $c \in K$ we have that

$$v(a-c) \;=\; v(\sigma b - c) \;=\; v\sigma(b-c) \;=\; w(b-c)\;,$$

which shows that $\mathrm{appr}_v(a,K) = \mathrm{appr}_w(b,K) = \mathrm{appr}_v(x,K)$. From Corollary 3.8 we infer that

$$v(x-a) \;\geq\; \mathrm{supp}\,\mathrm{appr}_v(x,K) \;=\; v_0K$$

holds in $(K(a,x),v)$, i.e., $v(x-a) > v_0K$. By Lemma 2.6, this uniquely determines the extension of v from $K(a)$ to $K(a,x)$.

If also $a' \in \tilde{K}$ realizes $\mathrm{appr}_v(x,K) = \mathrm{appr}_v(a,K)$, then again by Corollary 3.8, $v(x-a') \geq v_0K$. It follows that $v(a-a') \geq \min\{v(x-a), v(x-a')\} \geq v_0K$. As $a - a' \in \tilde{K}$, this means that $v(a-a') = \infty$, i.e., $a = a'$. We have now shown that once we fix an extension v of v_0 to $\tilde{K}$, then the element a and the extension of v from $K(a)$ to $K(a,x)$ are uniquely determined; then also by restriction, the extension of v_0 to $K(x)$ is uniquely determined.

If we choose another extension v_1 of v_0 to $\tilde{K}$, then with w and b as above, we have that $w = v_1 \circ \tau$ for some $\tau \in \mathrm{Aut}\,\tilde{K}|K$. Replacing a by $a_1 := \tau b$, the same construction as before yields a unique extension of v_1 from $K(a_1)$ to $K(a_1,x)$. The automorphism $\rho := \tau\sigma^{-1} \in \mathrm{Aut}\,\tilde{K}|K$ is an isomorphism from $K(a)$ to $K(a_1)$. As before for v, we have that $\mathrm{appr}_{v_1}(a_1,K) = \mathrm{appr}_w(b,K) = \mathrm{appr}_v(x,K)$. The uniqueness statement in Theorem 3.16 thus implies that ρ is an isomorphism from $(K(a),v)$ to $(K(a_1),v_1)$. We extend ρ to an isomorphism from $K(a,x)$ to $K(a_1,x)$ by setting $\rho x = x$. As the extensions of v from $K(a)$ to $K(a,x)$ and of v_1 from $K(a_1)$ to $K(a_1,x)$ are uniquely determined by the facts that $v(x-a) > v_0K$ and $v(x-a_1) > v_0K$ and as $\rho(x-a) = x - a_1$, we find that ρ is an isomorphism from $(K(a,x),v)$ to $(K(a_1,x),v_1)$. However, as the restriction of ρ to $K(x)$ is the identity, the restrictions of v and v_1 to $K(x)$ must coincide. This finishes the proof of the uniqueness assertion in the second part of statement 2).

Now assume that $\mathbf{A}$ is not immediate and pick some $b \in \bigcap \mathbf{A}$. By part 5) of Lemma 3.21, $\mathbf{A}$ is value-extending or residue-extending. Let us assume first that it is value-extending. Consider the cut in v_0K whose lower cut set is $\mathrm{supp}\mathbf{A}$ and choose an element α in some ordered abelian group containing v_0K that realizes this cut, i.e., $\alpha > \gamma$ for all $\gamma \in \mathrm{supp}\mathbf{A}$ and $\alpha < \varepsilon$ for all $\varepsilon \in v_0K \setminus \mathrm{supp}\mathbf{A}$. We can always choose α so that it is not a torsion element modulo v_0K; indeed, if the cut is induced by an element β in the divisible hull of v_0K, then we can replace β by

$\beta + \iota$ where ι is an infinitesimal, that is, $0 < \iota < \gamma$ for all γ in the divisible hull of v_0K. With $z := x - b$ we use Corollary 2.4 to extend v_0 from K to $K(z) = K(x)$ by assigning the value α to the element z. By our choice of α, the so constructed extension $(K(x)|K, v)$ is value-transcendental.

Moreover, for arbitrary $c \in K$ we have that $v(x - c) = v(x - b + b - c) = \min\{v(x - b), v(b - c)\} \le v(x - b) < \varepsilon$ for all $\varepsilon \in v_0K \setminus \mathrm{supp}\mathbf{A}$. Hence by part 1) of Lemma 3.23, $\mathrm{appr}_v(x, K) = \mathbf{A}$. This finishes the proof of part 3).

Finally, assume that $\mathbf{A}$ is residue-extending and set $\delta = \max \mathrm{supp}\mathbf{A} \in v_0K$. With $z := x - b$ we use Corollary 2.4 to extend v_0 from K to $K(z) = K(x)$ by assigning the value δ to the element z. From Corollary 2.4 it follows that the so constructed extension $(K(x)|K, v)$ is residue-transcendental.

By construction, for arbitrary $c \in K$ we have that $v(x - c) = v(x - b + b - c) = \min\{v(x - b), v(b - c)\} \le v(x - b) = \max \mathrm{supp}\mathbf{A}$. Hence by part 2) of Lemma 3.23, $\mathrm{appr}_v(x, K) = \mathbf{A}$. This finishes the proof of part 4). □

Remark 4.2. Assume that $\mathbf{A}$ is algebraic immediate. Let us describe in more detail the restriction to $K(x)$ of the value-transcendental extensions v on $K(a, x)$ that we have constructed in the above proof.

We fix an extension v of v_0 to $\tilde{K}$. The minimal polynomial f of b over K is an associated minimal polynomial for $\mathrm{appr}_v(x, K)$. As a and b are conjugate over K, f is also the minimal polynomial of a over K. We write

$$f(X) \;=\; \prod_{i=1}^{\deg f} (X - a_i)$$

in such a way that $v(x - a_i) \notin v\tilde{K}$ for $1 \le i \le n \le \deg f$ and $v(x - a_i) \in v\tilde{K}$ for $n < i \le \deg f$. Take any $i \le n$. If $v(x - a) \ne v(x - a_i)$, then $v(a - a_i) = \min\{v(x - a), v(x - a_i)\} \notin v\tilde{K}$, a contradiction. Therefore,

$$vf(x) \;=\; \sum_{i=1}^{\deg f} v(x - a_i) \;=\; nv(x - a) + \sum_{i=n+1}^{\deg f} v(x - a_i) \;=\; nv(x - a) + \alpha$$

with $\alpha \in v\tilde{K}$. Hence $vf(x)$ is not a torsion element modulo v_0K. Take any $h \in K[x]$ and present it in its f-adic expansion

$$h \;=\; \sum_{i=1}^{n} g_i(x) f^i(x)\,,$$

where every $g_i \in K[x]$ is of degree less than $\deg f = \deg \mathrm{appr}_v(x, K)$. By Lemma 3.12, the value of $g_i(x)$ is uniquely determined by $\mathrm{appr}_v(x, K)$ and lies in v_0K. Since $vf(x)$ is not a torsion element modulo v_0K, it follows that

$$vh \;=\; \min_{1 \le i \le n} vg_i(x) + ivf(x)$$

and that $vK(x) = v_0K \oplus \mathbb{Z}vf(x)$. Further, since the extension $(K(a)|K, v)$ is immediate, we have that $K(a)v = Kv_0$. From Corollary 2.4 we know that $K(a, x)v = K(a)v$, so we obtain that $K(a, x)v = Kv_0$ and thus also $K(x)v = Kv_0$.

Let us also mention that if the extension field $(K(a), v)$ we have constructed in the proof admits a transcendental immediate extension, then an immediate extension $(K(a, x)|K(a), v)$ can be constructed with $v(x - a) > \mathrm{supp}\mathbf{A}$. Then also $(K(a, x)|K, v)$ is immediate, and restricting v to $K(x)$ yields an immediate extension $(K(x)|K, v)$ with $\mathrm{appr}_v(x, K) = \mathbf{A}$. #

Remark 4.3. Value-extending approximation types determine the cut in v_0K that has to be filled by the value of an element like $x - b$ if x realizes the type, but if this cut can also be filled by an element that lies in the divisible hull of v_0K, then the approximation type does not determine whether it has to be filled by such an element or an element that is non-torsion modulo v_0K. We used this fact in the proof of Theorem 4.1. However, in the setting of this theorem this fact also implies that uniqueness of the extension v will in general fail.

The situation is similar for residue-extending approximation types. In general, they cannot determine whether the residue of an element like $d(x - c)$ has to be algebraic or transcendental over Kv_0.

These problems do not appear in the setting of Theorems 1.3 and 5.8, where v_0K is divisible and Kv_0 is algebraically closed. #

4.2. **Approximation types and model theoretic 1-types.**

In this section we exhibit the relation between approximation types and 1-types and the information that can be inferred from model theoretic algebra. For background on model theory and the notions we use we refer the reader to [5, 16]. In Theorem 4.5 we will show that under certain additional assumptions, a given approximation type over (K, v) can be realized by a transcendental element in some elementary extension of (K, v). (For simplicity, we will write "v" even for the valuation on K.)

Take a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{S}$. Further, take a set $\mathcal{T}$ of $\mathcal{L}$-formulas in one variable X with parameters from the universe of $\mathcal{S}$. Let $\mathcal{S}'$ be an $\mathcal{L}$-structure with substructure $\mathcal{S}$ and x an element of the universe of $\mathcal{S}'$. We say that x **realizes** $\mathcal{T}$ **in** $\mathcal{S}'$ if $\varphi(x)$ holds in $\mathcal{S}'$ for every $\varphi(X) \in \mathcal{T}$.

A set $\mathcal{T}$ as above is called a **type** (or **1-type**) **over** $\mathcal{S}$ if it is consistent. A criterion for this is that every finite subset of $\mathcal{T}$ is realized in $\mathcal{S}$ by some a in the universe of $\mathcal{S}$. The **type of** x **over** $\mathcal{S}$ is the set of all $\mathcal{L}$-formulas $\varphi(X)$ in one variable X with parameters from the universe of $\mathcal{S}$ for which the sentence $\varphi(x)$ holds in $\mathcal{S}'$.

When studying valued fields, we work with a language of valued fields $\mathcal{L}$ that consists of the language $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$ of rings or alternatively, the language $\mathcal{L}_F = \{+, -, \cdot, 0, 1, ^{-1}\}$ of fields, together with either a unary relation symbol $\mathcal{O}(X)$ for membership in the valuation ring or a binary relation symbol $X \mid_v Y$ for valuation divisibility ($x \mid_v y \Leftrightarrow vx \leq vy$). Then "$v(X - c) \geq vd$", "$v(X - c) > vd$" and "$v(X - c) = vd$" are $\mathcal{L}$-formulas with parameters c, d. If $(L|K, v)$ is any valued field extension and $x \in L$, then the assertions "$c \in \mathrm{appr}_v(x,K)_\gamma$" and "$c \in \mathrm{appr}_v(x,K)^\circ_\gamma$" are expressed by the sentences "$v(x-c) \geq vd_\gamma$" and "$v(x-c) > vd_\gamma$", where $d_\gamma \in K$ with $vd_\gamma = \gamma$.

An ordered abelian group Γ is **dense** if for all $\gamma, \delta \in \Gamma$ with $\gamma < \delta$ there is $\beta \in \Gamma$ such that $\gamma < \beta < \delta$.

Proposition 4.4. *Take a valued field (K,v), and for every $\gamma \in vK$ choose some $d_\gamma \in K$ with $vd_\gamma = \gamma$. Take a non-trivial approximation type $\mathbf{A}$ over (K,v), and for every $\gamma \in \mathrm{supp}\mathbf{A}$, pick some $c_\gamma \in \mathbf{A}_\gamma$. If $\mathbf{A}$ is value-extending, then assume that vK is dense, and if $\mathbf{A}$ is residue-extending, then assume that Kv is infinite.*

If $\mathbf{A}$ is immediate, then set

$$\mathcal{T}_{\mathbf{A}} := \{v(X - c_\gamma) \geq vd_\gamma \mid \gamma \in \mathrm{supp}\mathbf{A}\}\,.$$

If $\mathbf{A}$ is value-extending, then pick some $b \in \bigcap \mathbf{A}$ and set

$$\begin{aligned}\mathcal{T}_{\mathbf{A}} \;:=\; & \{v(X-b) > vd_\gamma \mid \gamma \in \mathrm{supp}\mathbf{A}\} \\ & \cup \{\neg\, v(X-c) \geq vd_\varepsilon \mid \varepsilon \in vK \setminus \mathrm{supp}\mathbf{A} \text{ and } c \in K\}\,.\end{aligned}$$

If $\mathbf{A}$ is residue-extending, then pick some $b \in \bigcap \mathbf{A}$, set $\gamma_{\max} := \max \mathrm{supp}\mathbf{A}$ and

$$\mathcal{T}_{\mathbf{A}} := \{v(X-b) = vd_{\gamma_{\max}}\} \cup \{\neg\, v(X-c) > vd_{\gamma_{\max}} \mid c \in K\}\,.$$

In all three cases, the following assertions hold:

1) The set $\mathcal{T}_{\mathbf{A}}$ is finitely realizable in (K,v).

2) If x is an element in any valued field extension of (K,v) that realizes $\mathcal{T}_{\mathbf{A}}$, then x realizes $\mathbf{A}$, that is, $\mathrm{appr}_v(x,K) = \mathbf{A}$.

Proof. 1): Take a finite subset $\mathcal{T}_0 \subseteq \mathcal{T}_{\mathbf{A}}$. Assume first that $\mathbf{A}$ is immediate. Then we can write $\mathcal{T}_0 = \{v(X - c_{\gamma_i}) \geq vd_{\gamma_i} \mid 1 \leq i \leq n\}$. We set $\gamma := \max_i \gamma_i \in \mathrm{supp}\mathbf{A}$. For arbitrary $a \in \mathbf{A}_\gamma$, this implies that $a \in \mathbf{A}_{\gamma_i}$ and hence $v(a - c_{\gamma_i}) \geq vd_{\gamma_i}$ for $1 \leq i \leq n$. We have shown that a realizes $\mathcal{T}_0$ in (K,v).

Now assume that $\mathbf{A}$ is value-extending. We assume that $\mathrm{supp}\mathbf{A} \neq \emptyset$; the easy proof in the case of $\mathrm{supp}\mathbf{A} \neq \emptyset$ is left to the reader. We write $\mathcal{T}_0 = \{v(X-b) > vd_{\gamma_i} \mid 1 \leq i \leq n\} \cup \{\neg\, v(X - c_j) \geq vd_{\varepsilon_j} \mid 1 \leq j \leq m\}$. Set $\gamma := \max_i \gamma_i$ and $\varepsilon := \min\{\varepsilon_j\,,\, v(b-c_j) \mid 1 \leq j \leq m \text{ and } \gamma < v(b-c_j)\}$. Since vK is assumed to be dense, there is $\beta \in vK$ such that $\gamma < \beta < \varepsilon$. Choose $b' \in K$ with $vb' = \beta$ and set $a := b + b'$. Then for $1 \leq i \leq n$ we have that $v(a-b) = vb' = \beta > \gamma_i$. For $1 \leq j \leq m$ we have that $vb' = \beta \neq v(b-c_j)$, whence $v(a-c_j) = v(b'+b-c_j) = \min\{\beta, v(b-c_j)\} \leq \beta < \varepsilon_j$. We have shown that a realizes $\mathcal{T}_0$ in (K,v).

Finally, assume that $\mathbf{A}$ is residue-extending. Set $d := d_{\gamma_{\max}}$. As we may pass to a larger subset of $\mathcal{T}_{\mathbf{A}}$ as long as it remains finite, we can write $\mathcal{T}_0 = \{v(X-b) = vd\} \cup \{\neg\, v(X-c_j) > vd \mid 1 \leq j \leq m\}$. Since Kv is assumed to be infinite, we can choose some $c \in K$ such that $vc = 0$ and $cv \neq -d^{-1}(b-c_j)v$ for all j such that $vd = v(b-c_j)$. This implies that for those j we have that $v(c + d^{-1}(b-c_j)) = 0$. Consequently, $v(c + d^{-1}(b-c_j)) = \min\{vc, vd^{-1}(b-c_j)\} \leq 0$ for all $j \in \{1,\dots,m\}$.

We set $a := b + cd$. Then $v(a-b) = vcd = vd$ and $v(a-c_j) = v(cd+b-c_j) = vd + v(c + d^{-1}(b-c_j)) \leq vd$. Again, we have shown that a realizes $\mathcal{T}_0$ in (K,v).

2): Take x in any valued field extension of (K,v) that realizes $\mathcal{T}_{\mathbf{A}}$.

Assume first that $\mathbf{A}$ is immediate. Then $v(x-c_\gamma) \geq vd_\gamma = \gamma$ for all $\gamma \in \mathrm{supp}\mathbf{A}$. By Lemma 3.9 we conclude that x realizes $\mathbf{A}$.

Now assume that $\mathbf{A}$ is value-extending. Then $v(x-b) > vd_\gamma = \gamma$ for all $\gamma \in \mathrm{supp}\mathbf{A}$, and $v(x-c) < vd_\delta = \delta$ for all $\delta \notin \mathrm{supp}\mathbf{A}$ and $c \in K$. By part 1) of Lemma 3.23 we conclude that x realizes $\mathbf{A}$.

Finally, assume that $\mathbf{A}$ is residue-extending. Then $v(x-b) = \gamma_{\max} = \max \mathrm{supp}\mathbf{A}$ and $v(x-c) \leq v(x-b)$ for all $c \in K$. By part 2) of Lemma 3.23 we conclude that x realizes $\mathbf{A}$. □

From general model theory we infer that all types over an $\mathcal{L}$-structure $\mathcal{S}$ are realized in every card$(\mathcal{S})^+$-saturated elementary extension of $\mathcal{S}$, and that such extensions always exist. We take $\mathcal{L}$ to be the language of valued rings or fields, the $\mathcal{L}$-structure $\mathcal{S}$ to be a valued field (K,v), and $\mathbf{A}$ to be a non-trivial approximation type over (K,v). Then $\mathbf{A}$ cannot be realized by an element in K. Hence if an element x in some elementary extension realizes $\mathbf{A}$, then it will not lie in K and will thus be, by a well-known basic fact of model theory, be transcendental over K. This proves:

Theorem 4.5. *Take a valued field (K,v) and a non-trivial approximation type $\mathbf{A}$ over (K,v). If $\mathbf{A}$ is value-extending, then assume that vK is dense, and if $\mathbf{A}$ is residue-extending, then assume that Kv is infinite. Then $\mathbf{A}$ is realized in some elementary extension of (K,v) by an element x that is transcendental over K.*

5. Pure and almost pure extensions

In this section we work with a fixed valued field (K,v_0) and an element x that is transcendental over K.

5.1. Pure extensions.

Take an arbitrary extension $(K(x)|K,v)$ and $t \in K(x)$. If vt is not a torsion element modulo v_0K, then t will be called a **value-transcendental element**. If $vt = 0$ and tv is transcendental over Kv_0, then t will be called a **residue-transcendental element**. Further, t will be called a **valuation-transcendental element** if it is value-transcendental or residue-transcendental. In [10] we defined an extension $(K(x)|K,v)$ to be **pure (in** x), if one of the following cases holds:

- for some $b,d \in K$, $d(x-b)$ is valuation-transcendental,
- $\mathrm{appr}_v(x,K)$ is a transcendental immediate approximation type.

Note that if $d(x-c)$ is value-transcendental, then we may in fact choose $d=1$.

If the extension $(K(x)|K,v)$ is pure, then we will also say that v is a **pure extension** of v_0 from K to $K(x)$.

Lemma 5.1. *Take any extension $(K(x)|K,v)$ and $b,d\in K$.*

1) Assume that $d(x-b)$ is value-transcendental. Then $\mathrm{appr}_v(x,K)$ *is value-extending, and the valuation v on $K(x)$ is uniquely determined by (K,v_0) and the value $v(x-b)$. Further, we have that $vK(x)=v_0K\oplus\mathbb{Z}v(x-b)$ and that $K(x)v=Kv_0$.*

2) Assume that $d(x-b)$ is residue-transcendental. Then $\mathrm{appr}_v(x,K)$ *is residue-extending, and the valuation v on $K(x)$ is uniquely determined by (K,v_0) and the fact that $d(x-b)v$ is transcendental over Kv_0. Further, we have that $vK(x)=v_0K$ and that $K(x)v=Kv_0(d(x-b)v)$ is a rational function field over Kv_0.*

In both cases, v_0K is pure in $vK(x)$ (i.e., $vK(x)/v_0K$ is torsion free), and Kv_0 is relatively algebraically closed in $K(x)v$.

Proof. Lemma 3.24 shows that $\mathrm{appr}_v(x,K)$ is value-extending if $d(x-b)$ is value-transcendental, and that $\mathrm{appr}_v(x,K)$ is residue-extending if $d(x-b)$ is residue-transcendental. All remaining assertions follow from Corollary 2.4. □

Here is the "prototype" of pure extensions:

Proposition 5.2. *If K is algebraically closed and x is transcendental over K, then every extension $(K(x)|K,v)$ is pure.*

Proof. Assume first that the set $v(x-K)$ has no maximum. Then by part 2) of Lemma 3.7, $\mathrm{appr}_v(x,K)$ is immediate. Since K is algebraically closed, Theorem 3.16 shows that $\mathrm{appr}_v(x,K)$ must be transcendental.

Now assume that the set $v(x-K)$ has a maximum, say, $v(x-b)$ with $b\in K$. Then by part 5) of Lemma 2.8, $v(x-b)\notin v_0K$ or there is $d\in K$ such that $(d(x-b))v\notin Kv_0$. Since K is algebraically closed, v_0K is divisible. Hence if $v(x-b)\notin v_0K$, then it cannot be a torsion element modulo v_0K, which implies that $(K(x)|K,v)$ is value-transcendental.

Since K is algebraically closed, Kv_0 is also algebraically closed. Hence if the residue $(d(x-b))v$ is not in Kv_0, then it must be transcendental over Kv_0, which implies that $(K(x)|K,v)$ is residue-transcendental.

In all three cases, the extension is pure by definition. □

We will call **A** a **pure approximation type** if it is a transcendental immediate, value-extending or residue-extending approximation type. We denote by $\mathcal{V}_p$ the set of all pure extensions of v_0 to $K(x)$ and by $\mathcal{A}_p$ the set of all pure approximation types. For pure extensions, a stronger form of part 5) of Lemma 3.21 holds:

Theorem 5.3. *For a pure extension $(K(x)|K,v)$, there are the following three mutually exlusive cases: immediate, value-transcendental, residue-transcendental, and the following assertions hold:*

1) The extension is immediate if and only if $\mathrm{appr}_v(x,K)$ *is immediate. In this case, the extension v to $K(x)$ is uniquely determined by* $\mathrm{appr}_v(x,K)$.

2) The extension is value-transcendental if and only if $x-b$ is value-transcendental for some $b\in K$, and this holds if and only if $\mathrm{appr}_v(x,K)$ *is value-extending. If in* *addition* v_0K *is divisible, then the pure extension* v *to* $K(x)$ *is uniquely determined by* $\mathrm{appr}_v(x,K)$*.*

3) The extension is residue-transcendental if and only if $d(x-b)$ *is residue-transcendental for some* $b,d\in K$*, and this holds if and only if* $\mathrm{appr}_v(x,K)$ *is residue-extending. In this case, the pure extension* v *to* $K(x)$ *is uniquely determined by* $\mathrm{appr}_v(x,K)$*.*

4) If v_0K *is divisible, then the function*

$$\mathcal{V}_p \longrightarrow \mathcal{A}_p\,, \qquad v \mapsto \mathrm{appr}_v(x,K) \tag{5.1}$$

is a bijection.

Proof. 1): If $(K(x)|K,v)$ is immediate, then $\mathrm{appr}_v(x,K)$ is immediate by part 4) of Lemma 3.7. For the converse, assume that $\mathrm{appr}_v(x,K)$ is immediate. If there is a valuation-transcendental element $d(x-b)$, then Lemma 3.24 shows that $\mathrm{appr}_v(x,K)$ is value-extending or residue-extending, hence not immediate. Thus by the definition of "pure extension", $\mathrm{appr}_v(x,K)$ must be transcendental immediate. By Corollary 3.14, this implies that $(K(x)|K,v)$ is immediate, and that the valuation v on $K(x)$ is uniquely determined by $\mathrm{appr}_v(x,K)$.

2)&3): Assume that $(K(x)|K,v)$ is valuation-transcendental. Then it is not immediate, so by Corollary 3.14, $\mathrm{appr}_v(x,K)$ is not immediate. Hence by the definition of "pure extension", there are $b,d\in K$ such that $d(x-b)$ is valuation-transcendental. Assume that $(K(x)|K,v)$ is value-transcendental. Then this element cannot be residue-transcendental because otherwise from part 2) of Lemma 5.1 it would follow that $(K(x)|K,v)$ is not value-transcendental. Therefore, $d(x-b)$ is value-transcendental. From this it follows by part 1) of Lemma 5.1 that $\mathrm{appr}_v(x,K)$ is value-extending.

Assume that $(K(x)|K,v)$ is residue-transcendental. Similarly as before, one now concludes that $d(x-b)$ is residue-transcendental. From this it follows by part 2) of Lemma 5.1 that $\mathrm{appr}_v(x,K)$ is residue-extending.

Assume now that $\mathrm{appr}_v(x,K)$ is value-extending. Then by part 5) of Lemma 3.21, it cannot be immediate. Then by the definition of "pure extension", there are $b,d\in K$ such that $d(x-b)$ is valuation-transcendental. If it were residue-transcendental, then by what we have already shown, $\mathrm{appr}_v(x,K)$ were residue-extending, a contradiction. Hence $d(x-b)$ is value-transcendental, which implies that $(K(x)|K,v)$ is value-transcendental.

In order to show the uniqueness statement, assume that v_0K is divisible. Since $\mathbf{A}:=\mathrm{appr}_v(x,K)$ is value-extending, we know from part 1) of Lemma 3.24 that for an arbitrarily chosen $b\in\bigcap\mathbf{A}$, $v(x-b)$ realizes the cut $(\mathrm{supp}\mathbf{A}\,,\,v_0K\setminus\mathrm{supp}\mathbf{A})$ in v_0K. If also v' is an extension to $K(x)$ with $\mathrm{appr}_{v'}(x,K)=\mathrm{appr}_v(x,K)$, then again by part 1) of Lemma 3.24, $v'(x-b)$ realizes the cut $(\mathrm{supp}\mathbf{A}\,,\,v_0K\setminus\mathrm{supp}\mathbf{A})$. It follows from part 2) of Lemma 2.6 that v and v' are equivalent over v_0, and as we identify equivalent valuations, $v=v'$. This finishes the proof of part 2).

Finally, assume that $\mathrm{appr}_v(x,K)$ is residue-extending. Interchanging "residue-transcendental" and "value-transcendental" in the proof we just gave, we obtain that $(K(x)|K,v)$ is residue-transcendental. In order to show the uniqueness statement, take another pure extension v' such that $\mathbf{A} := \mathrm{appr}_v(x,K) = \mathrm{appr}_{v'}(x,K)$. From part 1) of Lemma 3.24 we infer that for an arbitrarily chosen $b \in \bigcap \mathbf{A}$ we have that $v(x-b) = \max \mathrm{supp}\mathbf{A} = v'(x-b)$ and that for $d \in K$ with $vd = -\max \mathrm{supp}\mathbf{A}$, $d(x-b)v$ and $d(x-b)v'$ do not lie in Kv_0. Since both extensions are pure, we know from Lemma 5.1 that Kv_0 is relatively algebraically closed in $K(x)v$ and in $K(x)v'$, which shows that both $d(x-b)v$ and $d(x-b)v'$ are transcendental over Kv_0. We set $y = d(x-b)$ and obtain from Proposition 2.3 that $v = v'$ on $K(y) = K(x)$. This finishes the proof of part 3).

Part 5) of Lemma 3.21 shows that the properties "immediate approximation type", "value-extending approximation type" and "residue-extending approximation type" are mutually exclusive. Hence by what we have proved so far, for pure extensions the properties "immediate", "value-transcendental" and "residue-transcendental" are mutually exclusive.

4): If $\mathbf{A}$ is a transcendental immediate approximation type, then by Theorem 3.13 and Corollary 3.14 there is a unique extension v of v_0 to $K(x)$ such that $\mathrm{appr}_v(x,K) = \mathbf{A}$. It follows that $v \in \mathcal{V}_p$.

Now assume that $\mathbf{A}$ is a value- or residue-extending approximation type. By Theorem 4.1 there is an extension $(K(x)|K,v)$ such that $\mathrm{appr}_v(x,K) = \mathbf{A}$, and we can choose it to be value-transcendental if $\mathbf{A}$ is a value-extending, and to be residue-transcendental if $\mathbf{A}$ is a residue-extending. In both cases, $v \in \mathcal{V}_p$. This proves the surjectivity of the function (5.1). The injectivity follows from what we have proven in parts 1), 2) and 3). □

Remark 5.4. Without the assumption that v_0K is divisible, the uniqueness statement in the value-transcendental case can in general not be achieved. Assume that $\gamma \in v_0K$ is not divisible by some $n \in \mathbb{N}$. Then the cut induced by $\frac{\gamma}{n}$ in v_0K may be equal to the cut $(\mathrm{supp}\mathbf{A},\, v_0K \setminus \mathrm{supp}\mathbf{A})$. We can fill this cut with an element α that is not torsion modulo v_0K by choosing a positive infinitesimal ι and setting $\alpha = \frac{\gamma}{n} - \iota$ or $\alpha = \frac{\gamma}{n} + \iota$. Assigning α to z as in Corollary 2.4 will lead to two distinct extensions: if $c \in K$ with $vc = -\gamma$, then in the first case, $vcz^n = -n\iota < 0$, and in the second case, $vcz^n = n\iota > 0$. #

5.2. **Almost pure extensions.**

In this section we generalize the notion "pure extension" in order to also capture the case where the base field (K,v_0) lies dense in its algebraic closure, i.e., the algebraic closure $\tilde{K}$ lies in the completion of (K,v_0) (and consequently, the completion is itself algebraically closed). First, we prove that this property does not depend on the chosen extension of v_0 to $\tilde{K}$:

Lemma 5.5. *Take a valued field (K,v_0) and extensions v_1 and v_2 of v_0 to $\tilde{K}$. Then (K,v_0) lies dense in $(\tilde{K},v_1)$ if and only if it lies dense in $(\tilde{K},v_2)$.*

Proof. Assume that (K,v_0) lies dense in $(\tilde{K},v_1)$. Take $a\in\tilde{K}$ and $\alpha\in v_2\tilde{K}$. As both $v_1\tilde{K}$ and $v_2\tilde{K}$ equal the divisible hull of v_0K, we know that $\alpha\in v_1\tilde{K}$. Since all extensions of v_0 to $\tilde{K}$ are conjugate, there is an automorphism σ of $\tilde{K}|K$ such that $v_2=v_1\circ\sigma$. By assumption, there is $c\in K$ such that $\alpha<v_1(\sigma a-c)=v_1\circ\sigma(a-c)=v_2(a-c)$. This proves that (K,v_0) lies dense in $(\tilde{K},v_2)$. By symmetry, also the converse holds, which proves our assertion. □

We define the extension $(K(x)|K,v)$ to be **almost pure (in** x**)** if it is pure or $\mathrm{appr}_v(x,K)$ is an algebraic completion type.

Proposition 5.6. *If (K,v_0) lies dense in its algebraic closure, then v_0K is divisible, Kv_0 is algebraically closed, and every extension $(K(x)|K,v)$ is almost pure.*

Proof. We extend v from $K(x)$ to $\widetilde{K(x)}$. Then the restriction of v to $\tilde{K}$ is an extension of v from K to $\tilde{K}$. Lemma 5.5 shows that K lies dense in $(\tilde{K},v)$.

The completion of (K,v_0) is an immediate extension, and it contains $(\tilde{K},v)$. We know that the value group of an algebraically closed valued field is divisible, and its residue field is algebraically closed. Hence the same holds for $v_0K=v\tilde{K}$ and $Kv_0=\tilde{K}v$.

From Proposition 5.2 we know that the extension $(\widetilde{K(x)}|\tilde{K},v)$ is pure. Assume first that $\mathrm{appr}_v(x,\tilde{K})$ is a transcendental immediate approximation type. Then by Lemma 3.17, also $\mathrm{appr}_v(x,K)$ is a transcendental immediate approximation type and $(K(x)|K,v)$ is pure.

Now we consider the case where $d(x-b)$ is valuation-transcendental for some $b,d\in\tilde{K}$. Assume that $vd(x-b)=0$ and $d(x-b)v$ is transcendental over $\tilde{K}v=Kv_0$. Since $d\in\tilde{K}$, we know that $v(x-b)=-vd\in v\tilde{K}=v_0K$. We choose $d'\in K$ such that $v(d-d')>vd$ and $b'\in K$ such that $v(b-b')>v(x-b)$. It follows that $vd=vd'$ and

$$\begin{aligned} v(d(x-b)-d'(x-b')) &\geq \min\{v(d(x-b)-d'(x-b))\,,\,v(d'(x-b)-d'(x-b'))\}\\ &= \min\{v(d-d')+v(x-b)\,,\,vd'+v(b-b')\}\\ &> vd+v(x-b)\;=\;0\,. \end{aligned}$$

Therefore, $d'(x-b')v=d(x-b)v$ is transcendental over Kv_0. This shows that if $(\widetilde{K(x)}|\tilde{K},v)$ is residue-transcendental, then $(K(x)|K,v)$ is residue-transcendental and pure.

Finally, assume that $vd(x-b)\notin v\tilde{K}=v_0K$, in which case we can assume that $d=1$. We distinguish two cases:

Case 1: there is $\alpha\in v_0K$ such that $\alpha>v(x-b)$. Then we choose $b'\in K$ such that $v(b-b')>\alpha$ and obtain that $v(x-b')=\min\{v(x-b),v(b-b')\}=v(x-b)\notin v_0K$. In this case, $(K(x)|K,v)$ is value-transcendental and pure.

Case 2: $v(x-b) > v_0K$. Then by Corollary 3.8, $\mathrm{appr}_v(x,K) = \mathrm{appr}_v(b,K)$. This implies that $\mathrm{supp}\,\mathrm{appr}_v(x,K) = \mathrm{supp}\,\mathrm{appr}_v(b,K) = v_0K$, so $(K(x)|K,v)$ is value-transcendental and almost pure. □

We will call **A** an **almost pure approximation type** if it is a pure approximation type or an algebraic completion type (note that every transcendental completion type is already a pure approximation type). We denote by $\mathcal{V}_{ap}$ the set of all almost pure extensions of v_0 to $K(x)$ and by $\mathcal{A}_{ap}$ the set of all almost pure approximation types.

Theorem 5.7. *For an almost pure extension $(K(x)|K,v)$, there are the following three mutually exlusive cases: immediate, value-transcendental, residue-transcendental, and parts 2), and 3) of Theorem 5.3 hold, as well as:*

1') The extension $(K(x)|K,v)$ is immediate if and only if $\mathrm{appr}_v(x,K)$ is transcendental immediate.

If $\mathrm{appr}_v(x,K)$ is algebraic immediate, then the extension $(K(x)|K,v)$ is value-transcendental.

4') If v_0K is divisible, then the function

$$\mathcal{V}_{ap} \longrightarrow \mathcal{A}_{ap}\,, \qquad v \mapsto \mathrm{appr}_v(x,K) \tag{5.2}$$

is a bijection.

Proof. From Theorem 5.3 we know that for pure extensions there are the three mutually exlusive cases immediate, value-transcendental, residue-transcendental. The only almost pure extensions that are not pure occur when the approximation type is an algebraic completion type. In this case we know from part 2) of Theorem 4.1 that if $(K(x)|K,v)$ is an extension in which x realizes the approximation type, then it must be value-transcendental. The latter also proves the second assertion of part 1'). Parts 2), and 3) of Theorem 5.3 hold because in these cases the extensions are value-transcendental and residue-transcendental, respectively, hence pure. We see that by the definition of almost pure extensions, the only remaining case where $(K(x)|K,v)$ can be immediate occurs when $\mathrm{appr}_v(x,K)$ is transcendental immediate. Conversely, when the latter is the case, then by Corollary 3.14, $(K(x)|K,v)$ is immediate. This finishes the proof of part 1').

Now we prove part 4'). In view of the bijection stated in part 4) of Theorem 5.3, we only have to deal with the valuations in $\mathcal{V}_{ap}\setminus\mathcal{V}_p$ and the approximation types in $\mathcal{A}_{ap}\setminus\mathcal{A}_p$. If $\mathbf{A}\in\mathcal{A}_{ap}\setminus\mathcal{A}_p$, then it is an algebraic completion type, and by part 2) of Theorem 4.1, it is realized by a uniquely determined extension v; by definition, $v\in\mathcal{V}_{ap}\setminus\mathcal{V}_p$. Hence $\mathcal{V}_{ap}\setminus\mathcal{V}_p \ni v\mapsto \mathrm{appr}_v(x,K)\in\mathcal{A}_{ap}\setminus\mathcal{A}_p$ is a bijection, which finishes the proof of part 4'). □

5.3. **Proof of Theorem 1.3.**

The following is a more precise version of Theorem 1.3.

Theorem 5.8. *Assume that K is algebraically closed, or that (K, v_0) lies dense in its algebraic closure. Then the function (4.1) is a bijection. If K is algebraically closed, then all assertions of Theorem 5.3 hold. If (K, v_0) lies dense in its algebraic closure, then all assertions of Theorem 5.7 hold.*

Proof. If K is algebraically closed, then v_0K is divisible and Proposition 5.2 shows that each extension $(K(x)|K, v)$ is pure, hence $\mathcal{V} = \mathcal{V}_p$ and all assertions of Theorem 5.3 hold. Combining the surjectivity of the function (4.1) proven in Theorem 4.1 with the bijectivity of the function in part 4) of Theorem 5.3 shows that $\mathcal{A} = \mathcal{A}_p$ and that the function (4.1) is a bijection.

If (K, v) lies dense in its algebraic closure, then by Proposition 5.6, v_0K is divisible and each extension $(K(x)|K, v)$ is almost pure, hence $\mathcal{V} = \mathcal{V}_{ap}$ and all assertions of Theorem 5.7 hold. Combining the surjectivity of the function (4.1) with the bijectivity of the function in part 4') of Theorem 5.7 shows that $\mathcal{A} = \mathcal{A}_{ap}$ and that also in this case, the function (4.1) is a bijection. □

References

[1] Alexandru, V. – Popescu, N. – Zaharescu, A.: *All valuations on $K(X)$*, J. of Math. Kyoto Univ. **30** (1990), 281–296

[2] Blaszczok, A.: *Distances of elements in valued field extensions*, manuscripta math. **159** (2019), 397–429

[3] Bourbaki, N.: Commutative algebra. Hermann, Paris (1972)

[4] Chabert, J.-L.: *On the polynomial closure in a valued field*, J. Number Theory **130** (2010), 458–468

[5] Chang, C. C. – Keisler, H. J.: *Model Theory*, Amsterdam – London (1973)

[6] Endler, O.: *Valuation theory*, Berlin (1972)

[7] Engler, A. – Prestel, A.: *Valued fields*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005.

[8] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

[9] Knaf, H. – Kuhlmann, F.-V.: *Abhyankar places admit local uniformization in any characteristic*, Ann. Scient. Ec. Norm. Sup. **38** (2005), 833–846

[10] Kuhlmann, F.-V.: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. **356** (2004), 4559–4600

[11] Kuhlmann, F.-V.: *A classification of Artin–Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[12] Kuhlmann, F.-V.: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43

[13] Kuhlmann, F.-V.: *Elimination of Ramification I: The Generalized Stability Theorem*, Trans. Amer. Math. Soc. **362** (2010), 5697–5727

[14] Kuhlmann, F.-V. – Vlahu, I.: *The relative approximation degree*, Mathematische Zeitschrift **276** (2014), 203-235

[15] Ostrowski, A.: *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z. **39**, 269–404 (1935)

[16] Prestel, A. – Delzell, C. N.: Mathematical logic and model theory. A brief introduction. Universitext. Springer, London, 2011
[17] Peruginelli, G. – Spirito, D.: *Extending valuations to the field of rational functions using pseudo-monotone sequences*, J. Algebra **586** (2021), 756–786
[18] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York – Heidelberg – Berlin (1960)
[19] Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam (1989)

Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15 70-451 Szczecin, Poland
Email address: fvk@usz.edu.pl