A GENERIC APPROACH TO MEASURING THE STRENGTH OF COMPLETENESS/COMPACTNESS OF VARIOUS TYPES OF SPACES AND ORDERED STRUCTURES

HANNA ĆMIEL, FRANZ-VIKTOR KUHLMANN AND KATARZYNA KUHLMANN

Abstract. With a simple generic approach, we develop a classification that encodes and measures the strength of completeness (or compactness) properties in various types of spaces and ordered structures. The approach also allows us to encode notions of functions being contractive in these spaces and structures. As a sample of possible applications we discuss metric spaces, ultrametric spaces, ordered groups and fields, topological spaces, partially ordered sets, and lattices. We describe several notions of completeness in these spaces and structures and determine their respective strengths. In order to illustrate some consequences of the levels of strength, we give examples of generic fixed point theorems which then can be specialized to theorems in various applications which work with contracting functions and some completeness property of the underlying space.

Ball spaces are nonempty sets of nonempty subsets of a given set. They are called spherically complete if every chain of balls has a nonempty intersection. This is all that is needed for the encoding of completeness notions. We discuss operations on the sets of balls to determine when they lead to larger sets of balls; if so, then the properties of the so obtained new ball spaces are determined. The operations can lead to increased level of strength, or to ball spaces of newly constructed structures, such as products. Further, the general framework makes it possible to transfer concepts and approaches from one application to the other; as examples we discuss theorems analogous to the Knaster–Tarski Fixed Point Theorem for lattices and theorems analogous to the Tychonoff Theorem for topological spaces. Finally, we present some generic multivalued fixed point theorems as well as coincidence theorems for ball spaces.

Date: June 7, 2020.

2010 *Mathematics Subject Classification.* Primary 54A05, 54H25; Secondary 03E75, 06A05, 06A06, 06B23, 06B99, 06F20, 12J15, 12J20, 13A18, 47H09, 47H10, 54C10, 54C60, 54E50.

Key words and phrases. complete, compact, spherically complete, ball space, metric space, ultrametric space, topological space, partially ordered set, lattice, ordered abelian group, ordered field, fixed point theorem, Caristi–Kirk Fixed Point Theorem, Knaster–Tarski Theorem, Tychonoff Theorem.

The first two authors are supported by Opus grant 2017/25/B/ST1/01815 from the National Science Centre of Poland.

Contents

1. Introduction

In view of the notions of completeness of metric spaces, spherical completeness of ultrametric spaces and compactness of topological spaces, the question arose how these notions can be "reconciled", which indicates the search for some "umbrella" notion. The question was triggered in the early 1990s by the appearance of an ultrametric version of Banach's Fixed Point Theorem (see [23]), which turned out to be a useful tool in valuation theory. An attempt at finding a generic fixed point theorem for "metric and order fixed point theory" was made by M. Kostanek and P. Waszkiewicz in an unpublished paper in the early 2010s. However, the structure they introduced for this purpose is quite involved.

While fixed point theory was the driving force behind the above question, the notions we have mentioned in the beginning are fundamental and have a multitude of other applications. Therefore, the main purpose of this paper is to present a simple basic approach that enables us to formulate an umbrella notion which is suitable to encode various completeness notions, and to measure and compare the strength of these notions. Fixed point theorems will be used to illustrate the consequences of the level of strength and to show how the umbrella notion makes it possible to formulate generic fixed point theorems which then can be specialized to theorems in the various applications.

The inspiration for the minimal structure that allows encoding of notions of completeness is taken from ultrametric spaces and the notions of "ultrametric ball" and "spherically complete". A **ball space** $(X, \mathcal{B})$ consists of a nonempty set X together with a nonempty family $\mathcal{B}$ of distinguished nonempty subsets B of X. Note that $\mathcal{B}$, a subset of the power set $\mathcal{P}(X)$, is partially ordered by inclusion; we will write $(\mathcal{B}, \subseteq)$ when we refer to this partially ordered set (in short: poset). A **nest of balls** in $(X, \mathcal{B})$ is a nonempty totally ordered subset of $(\mathcal{B}, \subseteq)$. The basic completeness notion for ball spaces is inspired by the corresponding notion for ultrametric spaces: a ball space $(X, \mathcal{B})$ is called **spherically complete** if every nest of balls has a nonempty intersection. We note that if this is the case and if $\mathcal{B}' \subseteq \mathcal{B}$, then also $(X, \mathcal{B}')$ is spherically complete.

The concept of ball spaces enables us to distinguish various levels of spherical completeness, which then provide a tool for measuring the strength of completeness in the spaces and ordered structures under consideration. On the one hand, we can specify what the intersection of a nest really is, apart from being nonempty. On the other hand, we can consider intersections of more general collections of balls than just nests. A **directed system of balls** is a nonempty collection of balls such that the intersection of any two balls in the collection contains a ball included in the collection. A **centered system of balls** is a nonempty collection of balls such that the intersection of any finite number of balls in the collection is nonempty. Note that every nest is a directed system, and every directed system is a centered system (but in general, the converses are not true).

We introduce the following hierarchy of spherical completeness properties:

$\mathbf{S}_1$: The intersection of each nest in $(X, \mathcal{B})$ is nonempty.

$\mathbf{S}_2$: The intersection of each nest in $(X, \mathcal{B})$ contains a ball.

$\mathbf{S}_3$: The intersection of each nest in $(X, \mathcal{B})$ contains maximal balls.

$\mathbf{S}_4$: The intersection of each nest in $(X, \mathcal{B})$ contains a largest ball.

$\mathbf{S}_5$: The intersection of each nest in $(X, \mathcal{B})$ is a ball.

$\mathbf{S}_i^d$: The same as $\mathbf{S}_i$, but with "directed system" in place of "nest".

$\mathbf{S}_i^c$: The same as $\mathbf{S}_i$, but with "centered system" in place of "nest".

Note that $\mathbf{S}_1$ is just the property of being spherically complete.

The strongest of these properties is $\mathbf{S}_5^c$; we will abbreviate it as $\mathbf{S}^*$ as it will play a central role, enabling us to prove useful results about several important ball spaces that have this property (it is the "star" among the above properties). In Section 5.6 we will define an even stronger property, namely that arbitrary intersections of balls are again balls.

We have the following implications:

$$
\begin{array}{ccccccc}
\mathbf{S}_1 & \Leftarrow & \mathbf{S}_1^d & \Leftarrow & \mathbf{S}_1^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_2 & \Leftarrow & \mathbf{S}_2^d & \Leftarrow & \mathbf{S}_2^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_3 & \Leftarrow & \mathbf{S}_3^d & \Leftarrow & \mathbf{S}_3^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_4 & \Leftarrow & \mathbf{S}_4^d & \Leftarrow & \mathbf{S}_4^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_5 & \Leftarrow & \mathbf{S}_5^d & \Leftarrow & \mathbf{S}_5^c & = & \mathbf{S}^*
\end{array} \tag{1}
$$

In Section 2 we exemplify the (explicit or implicit) use of spherical completeness and its stronger versions by presenting generic fixed point theorems for ball spaces. We discuss various ways of encoding the property of a function of being contractive in the ball space language. We demonstrate the flexibility of ball spaces, which allows us to taylor them to the specific function under consideration. In connection with Theorem 2.6 we introduce the idea of associating with every element $x \in X$ a ball $B_x \in \mathcal{B}$, leading to the very useful notion of "$\mathbf{B}_x$–ball space".

The proofs for the generic fixed point theorems will be given in Section 3. We use Zorn's Lemma as the main tool in two different ways: it can be applied to the set of all balls as well as to the set of all nests, as both are partially ordered by inclusion.

The properties of hierarchy (1) will be studied in more detail in Section 4. We introduce a refinement of the hierarchy; however, it will not be used further in the present paper. We clarify the connection between properties in the hierarchy and properties of posets. Finally we reveal the strong properties of ball spaces that are closed under various types of nonempty intersections of balls.

In Section 5 we discuss the ways in which ball spaces can be associated with metric spaces, ultrametric spaces, ordered groups and fields, topological spaces, partially ordered sets, and lattices. In each case we determine which completeness property is expressed by the spherical completeness of the associated ball space; an overview is given in the table below. We also study the properties of the associated ball spaces, in particular which of the properties in the hierarchy (1) they satisfy.

spaces	balls	completeness property
ultrametric spaces	all closed ultrametric balls	spherically complete
metric spaces	metric balls with radii in suitable sets of positive real numbers	complete
totally ordered sets, ordered abelian groups and fields	all intervals $[a, b]$ with $a \leq b$	symmetrically complete
posets	intervals $[a, \infty)$	inductively ordered
topological spaces	all nonempty closed sets	compact
metric spaces	Caristi–Kirk balls or Oettli–Théra balls	complete

The last entry, the second one for metric spaces, is different from all the other ones. In all other cases the table has to be read as saying that the completeness property of the given space is equivalent to the spherical completeness of one single associated ball space containing the indicated balls. But if we work with Caristi–Kirk balls or Oettli–Théra balls, then the completeness of the metric space is equivalent to the spherical completeness of a whole variety of Caristi–Kirk ball spaces or Oettli–Théra ball spaces that can be defined on it (see Section 5.3). While this may appear impracticable at first glance, it turns out that these types of balls offer a much better ball spaces approach to metric spaces than the metric balls.

Not only the specialization of the general framework to particular applications is important. It is also fruitful to develop the abstract theory of ball spaces, in particular the behaviour of the various levels of spherical completeness in the hierarchy (1) under basic operations on ball spaces.

In Section 6 we study our strongest, the $\mathbf{S}^*$ ball spaces. Examples are the compact topological spaces, where we take the balls to be the nonempty closed sets. Their ball spaces are closed under arbitrary nonempty intersections of balls, and we make use of the results of Section 4. $\mathbf{S}^*$ ball spaces allow the definition of what we call spherical closures of subsets. They help us to deal with ball space structures induced on subsets of the set underlying the ball space.

In Section 8 we consider set theoretic operations on ball spaces, such as their closure under finite unions or nonempty intersections of balls, and we study the behaviour of spherical completeness properties under these operations. We use these preparations to associate a topology to each ball space and show that it is compact if and only if the ball space is $\mathbf{S}_1^{\mathrm{c}}$.

Products of ball spaces will be studied in Section 9. In the paper [1], we discuss a notion of continuity for functions between ball spaces, as well as quotient spaces and category theoretical aspects of ball spaces. The products we define here turn out to be the products in a suitable category of ball spaces.

Further, the fact that a general framework links various quite different applications can help to transfer ideas, approaches and results from one to the other. For instance, the Knaster–Tarski Theorem in the theory of complete lattices presents a useful property of the set of fixed points: they form again a complete lattice. In Section 7, using our general framework and in particular the results from Section 6, we transfer this result to other applications, such as ultrametric and topological spaces. Similarly, in Section 9 the Tychonoff Theorem from topology is proven for ball spaces and then transferred to ultrametric spaces. To derive the topological Tychonoff Theorem from its ball spaces analogue, essential use is made of the results of Section 8.

Finally, the last section of our paper is devoted to a quick discussion of two types of theorems that are related to fixed point theorems (and in fact are generalizations, as fixed point theorems can be deduced from them). First, we present generic multivalued fixed point theorems for ball spaces. Such theorems deal with functions F from a nonempty set X to its power set $\mathcal{P}(X)$ and ask for criteria that guarantee the existence of a fixed point $x \in X$ in the sense that

$$x \in F(x) .$$

Multivalued ultrametric fixed point theorems have been successfully applied in logic programming (see [27, 5]).

Second, we present generic coincidence theorems for ball spaces. Coincidence theorems consider two or more functions $f_1, \ldots, f_n$ from a nonempty set X to itself and ask for criteria that guarantee the existence of a coincidence point $x \in X$ in the sense that

$$f_1(x) = \ldots = f_n(x) .$$

A number of coincidence theorems for ball spaces and ultrametric spaces have been proven in [17] (see also [25] for theorems on ultrametric spaces).

For both types of theorems we will use two approaches. Inspired by the theory of strongly contractive ball spaces which we develop in connection with Caristi–Kirk and Oettli–Théra ball spaces in Section 5.3, we first employ criteria for the existence of singleton balls with suitable properties. Thereafter, we prove variants which work with minimal balls instead.

We hope that we have convinced the reader that the advantage of a general framework is (at least) threefold:

- compare the strength of completenes properties in various spaces and ordered structures, and transfer concepts and results from one to another,
- provide generic proofs of results (such as generic fixed point theorems) which then only have to be specialized to various applications,
- exhibit the underlying principles that are essential for theorems working with some completeness notion in various spaces and ordered structures.

2. Generic fixed point theorems and the notion of "contractive function"

Fixed Point Theorems (FPTs) can be divided into two classes: those dealing with functions that are in some sense "contracting", like Banach's FPT and its ultrametric variant (cf. [23], [26]), and those that do not use this property (explicitly or implicitly), like Brouwer's FPT. In this section, we will be concerned with the first class.

Under which conditions do "contracting" functions have a fixed point? First of all, we have to say in which space we work, and we have to specify what we mean by "contracting". These specifications will have to be complemented by a suitable condition on the space, in the sense that it is "rich" or "complete" enough to contain fixed points for all "contracting" functions. Ball spaces constitute a simple minimal setting in which the necessary conditions on the function and the space can be formulated.

We will now give examples of generic FPTs for ball spaces; they will be proved in Section 3.4. More such theorems and related results such as coincidence theorems and so-called attractor theorems are presented in [13, 14, 15, 17]. In the present paper we will not discuss the uniqueness of fixed points; see the cited papers for this aspect. However, an exception will be made in Theorem 2.2, as this will be used later for an interesting comparison with a topological fixed point theorem proven in [31].

For the remainder of this section, we fix a function $f : X \to X$. We abbreviate $f(x)$ by fx. Further, we call a subset S of X **f-closed** if $f(S) \subseteq S$. An f-closed set S will be called **f-contracting** if it satisfies $f(S) \subsetneq S$ unless it is a singleton. In the search for fixed points, it is a possible strategy to try to find f-closed singletons $\{a\}$ because then the condition $f(\{a\}) \subseteq \{a\}$ implies that $fa = a$. The significance of this idea is particularly visible in the case of Caristi–Kirk and Oettli–Théra ball spaces discussed in Section 5.3.

Theorem 2.1. *Assume that the ball space $(X, \mathcal{B})$ is spherically complete.*

1) If every f-closed subset of X contains an f-contracting ball, then f has a fixed point in each f-closed set.

2) If every f-closed subset of X is *an f-contracting ball, then f has a* unique *fixed point.*

We will now give examples showing how some of the stronger notions of spherical completeness can be employed in general FPTs. In the next theorem, observe how stronger assumptions on the ball space and on f allow us to only talk about f-closed balls instead of f-closed subsets.

Theorem 2.2. *Assume that $(X, \mathcal{B})$ is an $\mathbf{S}_5$ ball space and that $f(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$.*

1) If every f-closed ball contains an f-contracting ball, then f has a fixed point in each f-closed ball.

2) If every f-closed ball is f-contracting, then f has a unique fixed point in each f-closed ball. If in addition $X \in \mathcal{B}$, then f has a unique fixed point.

The next theorem is a variation on the first parts of the previous two theorems.

Theorem 2.3. *Assume that $(X, \mathcal{B})$ is an $\mathbf{S}_2$ ball space. If every ball in $\mathcal{B}$ contains a fixed point or a smaller ball, then f has a fixed point in each ball.*

We can get around asking that the ball space be $\mathbf{S}_2$ by giving a condition on the intersection of nests; note that it is implicit in this condition that the ball space is spherically complete.

Theorem 2.4. *Take a ball space $(X, \mathcal{B})$ such that the intersection of every nest of balls in $\mathcal{B}$ contains a fixed point or a smaller ball $B \in \mathcal{B}$. Then f admits a fixed point in every ball of $\mathcal{B}$.*

A condition like "contains a fixed point or a smaller (f-closed) ball" may appear a little unusual at first. However, a possible algorithm for finding fixed points should naturally be allowed to stop when it has found one, so from this point of view the condition is quite natural. We also sometimes use a condition like "each f-closed ball is a singleton or contains a smaller f-closed ball". This implies "contains a fixed point or a smaller f-closed ball" because in an f-closed singleton $\{a\}$ the element a must be a fixed point. But this condition is too strong: as we will see below, there are cases where finding a ball with a fixed point is easier and more natural than finding a singleton. One example are partially ordered sets where the balls are taken to be sets of the form $[a, \infty)$. On the other hand, Section 5.3 shows that there are settings in which in a natural way we are led to finding f-closed singletons (cf. Proposition 3.8).

The assumptions of these theorems can be slightly relaxed by adapting them to the given function f. Instead of talking about the intersections of all nests of balls, we need information only about the intersections of nests of f-closed balls. Trivially, if $\emptyset \neq \mathcal{B}' \subseteq \mathcal{B}$, then also $(X, \mathcal{B}')$ is a ball space, and if $(X, \mathcal{B})$ is spherically complete, then so is $(X, \mathcal{B}')$. This flexibility of ball spaces appeared already implicitly in Theorem 2.2 where only f-closed balls are used; if nonempty, the subset of all f-closed balls is also a ball space, and it inherits important properties from the (possibly) larger ball space. Tayloring the assumptions on the ball space to the given function also comes in handy in the following refinement of Theorem 2.2. In its formulation, the condition "spherically complete" does not appear explicitly anymore, but is implicitly present for the ball space that is chosen in dependence on the function f.

Theorem 2.5. *Assume that there is a ball space $(X, \mathcal{B}^f)$ such that*
(B1) *each ball in $\mathcal{B}^f$ is f-closed,*
(B2) *the intersection of every nest of balls in $\mathcal{B}^f$ is a singleton or contains a smaller ball $B \in \mathcal{B}^f$.*
Then f admits a fixed point in every ball in $\mathcal{B}^f$.

At first glance, certain conditions of these theorems may appear somewhat unusual. But the reader should note that their strength lies in the fact that we can freely choose the ball space. For example, it does not have to be a topology, and in fact, for essentially all of our applications *it should not be*. This makes it possible to even choose the balls relative to the given function, which leads to results like the theorem above.

When uniqueness of fixed points is not required, then in certain settings (such as ultrametric spaces, see Section 5.1) the condition that a function be "contracting" on all of the space can often be relaxed to the conditions that the function just be "non-expanding" everywhere and "contracting" on orbits. Again, there is some room for relaxation, and this is why we will now introduce the following notion. For each $i \in \mathbb{N}$, f^i will denote the i-th iteration of f, that is, $f^0 x = x$ and $f^{i+1} x = f(f^i x)$. The function f will be called **ultimately contracting on orbits** if there is a function

$$X \ni x \mapsto B_x \in \mathcal{B} \tag{2}$$

such that for all $x \in X$, the following conditions hold:

(NB) $x \in B_x$,

(CO) $B_{fx} \subseteq B_x$, and if $x \neq fx$, then $B_{f^i x} \subsetneq B_x$ for some $i \geq 1$.

If in addition (CO) always holds with $i = 1$, then we call f **contracting on orbits**. Note that (NB) and (CO) imply that $f^i x \in B_x$ for all $i \geq 0$.

Now we can state our sixth basic theorem; its second assertion shows that instead of asking for general spherical completeness, the scope can be restricted to a particular kind of nests. We will say that a nest $\mathcal{N}$ of balls is an f**-nest** if $\mathcal{N} = \{B_x \mid x \in S\}$ for some set $S \subseteq X$ that is closed under f.

Theorem 2.6. *Assume that the function f on the ball space $(X, \mathcal{B})$ is ultimately contracting on orbits and that for every f-nest $\mathcal{N}$ in this ball space there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$. Then for every $x \in X$, f has a fixed point in B_x .*

The following is the ball spaces analogue of the Ultrametric Banach Fixed Point Theorem first proved in [23] (see Theorem 7.3 below).

Theorem 2.7. *Assume that the function f on the ball space $(X, \mathcal{B})$ is ultimately contracting on orbits and that the following holds:*

(C1) $B_y \subseteq B_x$ *for every* $y \in B_x$.

If $(X, \mathcal{B})$ is spherically complete, then f has a fixed point in every B_x .

A particularly elegant version of our approach can be given in the case of Caristi–Kirk and Oettli–Théra ball spaces (see Theorem 3.9 in Section 5.3). These ball spaces are used in complete metric spaces. Usually, proofs of fixed point theorems in this setting work with Cauchy sequences, while the use of metric balls is inefficient and complicated. For this reason, a ball spaces approach to metric spaces may seem pointless at first glance. However, it has turned out that ball spaces made up of Caristi–Kirk or Oettli–Théra balls have a particularly strong property (cf. Proposition 3.8), which makes the ball space approach in this case exceptionally successful, as demonstrated in Section 5.3 and the papers [2, 15].

To describe the properties of Caristi–Kirk and Oettli–Théra balls, we introduce the following notions for ball spaces. A ball space $(X, \mathcal{B})$ is a B_x–ball space if there is a function (2) such that $\mathcal{B} = \{B_x \mid x \in X\}$. We call a B_x–ball space $(X, \mathcal{B})$ **normalized** if it satisfies condition (NB), and **contractive** if for all $x, y \in X$ condition (C1) and the following condition hold:

(C2) if B_x is not a singleton, then there exists $y \in B_x$ such that $B_y \subsetneq B_x$.

A B_x–ball space $(X, \mathcal{B})$ is **strongly contractive** if it satisfies (C1) and:

(C2s) if $y \in B_x \setminus \{x\}$, then $B_y \subsetneq B_x$.

Note that condition (C2s) implies (C2) as well as that the function (2) is a bijection. In particular, every strongly contractive ball space is contractive. Proposition 5.10 will show that all Caristi–Kirk and Oettli–Théra ball spaces are strongly contractive normalized B_x–ball spaces.

It will turn out that condition (NB), while present in many applications, is not always necessary for our purposes. The next theorem has some similarity with Theorem 2.6, but it does not require the B_x–ball space to be normalized.

Theorem 2.8. *If $(X,\mathcal{B})$ is a spherically complete contractive B_x–ball space and the function f satisfies*

$$fx \in B_x \quad \text{for all } x \in X\,, \tag{3}$$

then it has a fixed point in every ball $B \in \mathcal{B}$.

We note that if $(X,\mathcal{B})$ is a strongly contractive B_x–ball space and the function f satisfies (3), then it also satisfies (CO) (with $i = 1$ for all x).

Interestingly, the exceptional strength of the Caristi–Kirk and Oettli–Théra ball spaces is shared by the ball space made up of the final segments $[a,\infty)$ on partially ordered sets. It would be worthwhile to find more examples of such strong ball spaces.

The proofs of our generic fixed point theorems are based on Zorn's Lemma. They will be given in Section 3 after first investigating the relation between partially ordered sets and ball spaces. In the present paper we are not interested in avoiding the use of the axiom of choice, nor is it our task to study its equivalence with certain fixed point theorems. For a detailed discussion of the case of Caristi–Kirk and Oettli–Théra ball spaces, see Remark 5.13.

3. Zorn's Lemma in the context of ball spaces

Consider a poset $(T,<)$. By a **chain** in T we mean a *nonempty* totally ordered subset of T. An element $a \in T$ is said to be an **upper bound** of a subset $S \subseteq T$ if $b \leq a$ for all $b \in S$. A poset is said to be **inductively ordered** if every chain has an upper bound.

Zorn's Lemma states that every inductively ordered poset contains maximal elements. By restricting the assertion to the set of all elements in the chain and above it, we obtain the following more precise assertion:

Lemma 3.1. *In an inductively ordered poset, every chain has an upper bound which is a maximal element in the poset.*

Corollary 3.2. *In an inductively ordered poset, every element lies below a maximal element.*

Take a ball space $(X,\mathcal{B})$. If we order $\mathcal{B}$ by setting $B_1 < B_2$ if $B_1 \supsetneq B_2$, then we obtain a poset $(\mathcal{B},<)$. Under this transformation, nests of balls in $\mathcal{B}$ correspond to chains in the poset. A maximal element in the poset $(\mathcal{B},<)$ is a **minimal ball**, i.e., a ball that does not contain any smaller ball.

3.1. The case of $\mathbf{S}_2$ ball spaces.

The following observation is straightforward:

Lemma 3.3. *The ball space $(X,\mathcal{B})$ is $\mathbf{S}_2$ if and only if every chain in $(\mathcal{B},<)$ has an upper bound.*

From this fact, one easily deduces the following result.

Proposition 3.4. *In an $\mathbf{S}_2$ ball space, every ball and therefore also the intersection of every nest contains a minimal ball. If in addition every ball is either a singleton or contains a smaller ball, then every ball and therefore also the intersection of every nest contains a singleton ball.*

In view of Lemma 3.3 it is important to note that every $\mathbf{S}_1$ ball space $(X,\mathcal{B})$ can easily be extended to an $\mathbf{S}_2$ ball space by adding all singleton subsets of X: we define

$$\mathcal{B}_s := \mathcal{B} \cup \{\{a\} \mid a \in X\}\,.$$

The proof of the following result is straightforward.

Lemma 3.5. *The ball space $(X,\mathcal{B}_s)$ is $\mathbf{S}_2$ if and only if $(X,\mathcal{B})$ is $\mathbf{S}_1$.*

However, in many situations the point is exactly to prove that a given ball space admits singleton balls. This is in particular the case when we work with ball spaces that are adapted to a given function, as in Theorem 2.5. In such cases, instead of applying Zorn's Lemma to chains of balls, one can work with chains of nests instead, as we will discuss in Section 3.2.

3.2. **Posets of nests of balls.**

We call a poset **chain complete** if every chain of elements has a least upper bound (which we also call a **supremum**). Note that commonly the condition "nonempty" is dropped from the definition of chains, in which case a chain complete poset must have a least element. However, for our purposes it is more convenient to only consider chains as nonempty totally ordered sets.

Lemma 3.6. *For every ball space $(X,\mathcal{B})$, the set of all nests of balls, ordered by inclusion, is a chain complete poset.*

Proof: The union over a chain of nests of balls is again a nest of balls, and it is the smallest nest that contains all nests in the chain. $\square$

This shows that in particular every chain of nests that contains a given nest $\mathcal{N}_0$ has an upper bound. Hence Zorn's Lemma shows:

Corollary 3.7. *Every nest $\mathcal{N}_0$ of balls in a ball space is contained in a maximal nest.*

3.3. **The case of contractive $\mathbf{B}_x$–ball spaces.**

In general, a (strongly) contractive ball space $(X,\mathcal{B})$ may not contain balls of the form $\{a\}$ for every $a \in X$. Then we cannot apply Lemma 3.5 to acquire the equivalence between properties $\mathbf{S}_1$ and $\mathbf{S}_2$. However, the following lemma yields the existence of a "sufficient" amount of singleton balls to obtain this equivalence which moreover satisfy $B_a = \{a\}$ even if $(X,\mathcal{B})$ is not assumed to be normalized.

Proposition 3.8. *In a contractive B_x–ball space, the intersection of a maximal nest of balls, if nonempty, is a singleton ball of the form $B_a = \{a\}$.*

Proof: Let $\mathcal{M}$ be a maximal nest of balls and assume that $a \in \bigcap \mathcal{M}$ for some element $a \in X$. Since $a \in B$ for every ball $B \in \mathcal{M}$, we obtain from (C2) that $B_a \subseteq B$ for every $B \in \mathcal{M}$ and thus $B_a \subseteq \bigcap \mathcal{M}$. This means that $\mathcal{M} \cup \{B_a\}$ is a nest of balls, so by maximality of $\mathcal{M}$ we have that $B_a \in \mathcal{M}$. Consequently, $B_a = \bigcap \mathcal{M}$. Suppose that B_a is not a singleton. Then by condition (C3) there is some element b such that $B_b \subsetneq B_a$ whence $B_b \notin \mathcal{M}$. But then $\mathcal{M} \cup \{B_b\}$ is a nest which strictly contains $\mathcal{M}$. This contradiction to the maximality of $\mathcal{M}$ shows that B_a is a singleton. Since $a \in \bigcap \mathcal{M} = B_a$, we must have that $B_a = \{a\}$. $\square$

Since by Corollary 3.7 every nest is contained in a maximal nest, this proposition yields:

Theorem 3.9.
1) A contractive B_x–ball space is $\mathbf{S}_1$ if and only if it is $\mathbf{S}_2$.

2) In a spherically complete contractive B_x–ball space every ball B_x contains a singleton ball of the form $B_a = \{a\}$.

3.4. **Proofs of the fixed point theorems.**

Take a ball space $(X, \mathcal{B})$ and a function $f : X \to X$. By $\mathcal{B}^f$ we will denote the collection of all f-closed balls in $\mathcal{B}$, provided there exist any. From Corollary 3.7 we infer that every nest in $(X, \mathcal{B})$ and every nest in $(X, \mathcal{B}^f)$ is contained in a maximal nest.

Under various conditions on f and on $(X, \mathcal{B})$ or $(X, \mathcal{B}^f)$, we have to make sure that the intersections of such nests contain a fixed point for f. We observe:

Lemma 3.10. *1) If S is an f-closed set, then $f^2(S) \subseteq f(S)$ since $f(S) \subseteq S$, hence $f(S)$ is f-closed.*

2) The intersection over any collection of f-closed sets is again an f-closed set.

Proof of Theorem 2.1: Take any f-closed set S. By the assumption of the theorem we know that it contains an f-contracting ball B. By definition, B is f-closed. By Corollary 3.7 there exists a maximal nest $\mathcal{N}$ in the set $\mathcal{B}^f$ of all f-closed balls in $\mathcal{B}$ which contains the nest $\{B\}$. Then by part 2) of Lemma 3.10, $\bigcap \mathcal{N}$ is an f-closed set. By assumption, it contains an f-contracting ball B'. Suppose that B' is not a singleton. Then B' properly contains $f(B')$, which by part 1) of Lemma 3.10 is an f-closed set. Again by assumption, it contains an f-contracting and hence f-closed ball B''. Since $B'' \subseteq f(B') \subsetneq B' \subseteq \bigcap \mathcal{N}$, we find that $\mathcal{N} \cup \{B''\}$ is a larger nest than $\mathcal{N}$, which contradicts the maximality of $\mathcal{N}$. This proves that B' is an f-closed singleton contained in S and thus, S contains a fixed point. This proves part 1) of the theorem.

In order to prove part 2), assume that x and y are fixed points of f. Then the set $S = \{x, y\}$ is f-closed, hence by assumption it is f-contracting. Since $f(S) = S$, it must be a singleton, i.e., $x = y$. □

Proof of Theorem 2.2: Assume that $(X, \mathcal{B})$ is an $\mathbf{S}_5$ ball space and that $f(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$. Take some f-closed ball $B_0 \in \mathcal{B}$.

1): As in the previous proof, choose a maximal nest $\mathcal{N}$ in $\mathcal{B}^f$ which contains the nest $\{B_0\}$. Then $\bigcap \mathcal{N}$ is an f-closed set. As $(X, \mathcal{B})$ is assumed to be an $\mathbf{S}_5$ ball space, $\bigcap \mathcal{N}$ is also a ball, so $\bigcap \mathcal{N} \in \mathcal{B}^f$. Hence by assumption, $\bigcap \mathcal{N}$ contains an f-contracting ball B. This must be a singleton, because otherwise, it would contain the smaller f-closed ball $f(B)$, giving rise to the nest $\mathcal{N} \cup \{f(B)\}$ that properly contains $\mathcal{N}$, which contradicts the maximality of $\mathcal{N}$. Thus, $\bigcap \mathcal{N}$ is an f-closed singleton contained in B_0 and therefore, B_0 contains a fixed point.

2): Take any f-closed ball $B_0 \in \mathcal{B}$. Using transfinite induction, we build a nest $\mathcal{N}$ consisting of f-closed balls as follows. We set $\mathcal{N}_0 := \{B_0\}$. Having constructed $\mathcal{N}_\nu$ for some ordinal ν with smallest f-closed ball $B_\nu \in \mathcal{N}_\nu$, we set $B_{\nu+1} := f(B_\nu) \subseteq B_\nu$ and $\mathcal{N}_{\nu+1} := \mathcal{N}_\nu \cup \{B_{\nu+1}\}$. By part 1) of Lemma 3.10, also $B_{\nu+1}$ is f-closed, and by assumption, it is again a ball.

If λ is a limit ordinal and we have constructed $\mathcal{N}_\nu$ for all $\nu < \lambda$, we observe that the union over all $\mathcal{N}_\nu$ is a nest $\mathcal{N}'_\lambda$. We set $B_\lambda := \bigcap \mathcal{N}'_\lambda$ and $\mathcal{N}_\lambda := \mathcal{N}'_\lambda \cup \{B_\lambda\}$. Since $(X,\mathcal{B})$ is an $\mathbf{S}_5$ ball space, we know that $B_\lambda \in \mathcal{B}$, and by part 2) of Lemma 3.10, B_λ is f-closed.

The construction becomes stationary when we reach an f-closed ball B_μ that does not properly contain $f(B_\mu)$. By assumption, B_μ is f-contracting, so this means that $B_\mu \subseteq B_0$ is a singleton $\{x\}$. As it is f-closed, x is a fixed point contained in B_0.

If $x \neq y \in B_0$, then $y \notin B_\mu$ which means that there is some $\nu < \mu$ such that $y \in B_\nu$, but $y \notin B_{\nu+1} = f(B_\nu)$. This shows that y cannot be a fixed point of f. Therefore, x is the unique fixed point of f in B_0.

The second assertion of part 2) is an immediate consequence of the first. $\square$

Proof of Theorem 2.3: Assume that $(X,\mathcal{B})$ is an $\mathbf{S}_2$ ball space and that every ball in $\mathcal{B}$ contains a fixed point or a smaller ball. Take a ball $B \in \mathcal{B}$. By Proposition 3.4, B contains a minimal ball B_0. As B_0 cannot contain a smaller ball, it must contain a fixed point by assumption, which then is also an element of B. $\square$

Proof of Theorem 2.4: Take a ball $B \in \mathcal{B}$. As before, there exists a maximal nest $\mathcal{N}$ in $\mathcal{B}$ which contains the nest $\{B\}$. Now $\bigcap \mathcal{N}$ cannot contain a smaller ball since this would contradict the maximality of $\mathcal{N}$. Hence by assumption, $\bigcap \mathcal{N}$ and thus also B must contain a fixed point. $\square$

Proof of Theorem 2.5: Assume that $\mathcal{B}^f$ is a ball space of f-closed balls and that the intersection of every nest of balls in $\mathcal{B}^f$ is a singleton or contains a smaller ball $B \in \mathcal{B}^f$. Take a ball $B \in \mathcal{B}^f$. As in the previous proofs, there exists a maximal nest $\mathcal{N}$ in $\mathcal{B}^f$ which contains the nest $\{B\}$. The intersection $\bigcap \mathcal{N}$ cannot contain a smaller ball $B' \in \mathcal{B}^f$ since this would contradict the maximality of $\mathcal{N}$. Hence by assumption, $\bigcap \mathcal{N}$ must be a singleton. As it is also f-closed by part 2) of Lemma 3.10 and contained in B, we have proved that f has a fixed point in B. $\square$

Proof of Theorem 2.6: Take a function f on a ball space $(X,\mathcal{B})$ which is ultimately contracting on orbits and assume that for every f-nest $\mathcal{N}$ in this ball space there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$. For every $x \in X$, the set $\{B_{f^i x} \mid i \geq 0\}$ is an f-nest. The set of all f-nests is partially ordered in the following way. If $\mathcal{N}_1 = \{B_x \mid x \in S_1\}$ and $\mathcal{N}_2 = \{B_x \mid x \in S_2\}$ are f-nests with S_1 and S_2 closed under f, then we define $\mathcal{N}_1 \leq \mathcal{N}_2$ if $S_1 \subseteq S_2$. Then the union over an ascending chain of f-nests is again an f-nest since the union over sets that are closed under f is again closed under f. Hence by Corollary 3.2, for every $x_0 \in X$ there is a maximal f-nest $\mathcal{N}$ containing $\{B_{f^i x_0} \mid i \geq 0\}$. By the assumption of Theorem 2.6, there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$. We wish to show that z is a fixed point of f. If $z \neq fz$ would hold, then by (CO), $B_{f^i z} \subsetneq B_z \subseteq \bigcap \mathcal{N}$ for some $i \geq 1$, and the f-nest $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\}$ would properly contain $\mathcal{N}$. But this would contradict the maximality of $\mathcal{N}$. Hence, $z \in \bigcap \mathcal{N} \subseteq B_{x_0}$ is a fixed point of f. $\square$

Proof of Theorem 2.7: As in the previous proof, we obtain a maximal f-nest $\mathcal{N}$ containing B_{x_0}. Since $(X,\mathcal{B})$ is assumed to be spherically complete, there is some $z \in \bigcap \mathcal{N}$. Hence $z \in B_x$ for every B_x in $\mathcal{N}$. By condition (C1), it follows that $B_z \subseteq B_x$, whence $B_z \subseteq \bigcap \mathcal{N}$. Now we proceed as in the previous proof.

Proof of Theorem 2.8: Take a spherically complete contractive B_x–ball space $(X,\mathcal{B})$ and a function $f : X \to X$ such that $fx \in B_x$ for all $x \in X$. Then by part 2) of Theorem 3.9, every ball B_x contains a singleton ball of the form $B_a = \{a\}$. Since $fa \in B_a = \{a\}$, we find that a is a fixed point of f which is contained in B_x.

4. Some facts about the hierarchy of ball spaces

4.1. **A refinement of the hierarchy.**

By considering stronger properties of directed and centered systems of balls, we will now add further entries to the hierarchy (1).

We will say that a centered system of balls is

c' if the intersection of any finite number of balls in the system contains a ball,
c'' if the intersection of any finite number of balls in the system contains a largest ball,
c''' if the intersection of any finite number of balls in the collection is a ball.

We will say that a directed system of balls is

d' if the intersection of any finite number of balls in the system contains a ball which is again in the system,
d'' if the intersection of any finite number of balls in the system contains a largest ball which is again in the system,
d''' if the intersection of any finite number of balls in the system is a ball which is again in the system.

For $1 \le i \le 5$ we will say that a ball space is $\mathbf{S}_i^{d'}$ (or $\mathbf{S}_i^{d''}$, or $\mathbf{S}_i^{d'''}$) if it satisfies the definition of $\mathbf{S}_i^d$ with "directed system" replaced by "d' directed system" (or "d'' directed system", or "d''' directed system", respectively). Again for $1 \le i \le 5$, we will say that a ball space is $\mathbf{S}_i^{c'}$ (or $\mathbf{S}_i^{c''}$, or $\mathbf{S}_i^{c'''}$) if it satisfies the definition of $\mathbf{S}_i^c$ with "centered system" replaced by "c' centered system" (or "c'' centered system", or "c''' centered system", respectively).

By induction one shows that in the above definitions for d' and d''', "any finite number of" can be replaced by "any two" without changing the meaning. In particular, every directed system of balls is d'. We also note that every nest of balls is a d''' directed system of balls. This together with the obvious implications between the properties defined above gives us the following refinement of each row of the hierarchy (1):

$$
\begin{array}{ccccccccc}
\mathbf{S}_i & \Leftarrow & \mathbf{S}_i^{d'''} & \Leftarrow & \mathbf{S}_i^{d''} & \Leftarrow & \mathbf{S}_i^{d'} & = \mathbf{S}_i^{d} & \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & \mathbf{S}_i^{c'''} & \Leftarrow & \mathbf{S}_i^{c''} & \Leftarrow & \mathbf{S}_i^{c'} & \Leftarrow & \mathbf{S}_i^{c}
\end{array}
\tag{4}
$$

for $1 \le i \le 5$.

4.2. **Connection with posets.**

A **directed system** in a poset is a nonempty subset which contains an upper bound for any two of its elements. A poset is called **directed complete** if every directed system has a least upper bound. Note that commonly the condition "nonempty" is dropped; but for our purposes it is more convenient to only consider nonempty systems (cf. our remark in Section 3.2). As every chain is a directed system, every directed complete poset is chain complete.

The proof of the following observations is straightforward:

Proposition 4.1. *1) A ball space* $(X, \mathcal{B})$ *is* $\mathbf{S}_2$ *if and only if* $(\mathcal{B}, <)$ *is inductively ordered.*
2) A ball space $(X, \mathcal{B})$ *is* $\mathbf{S}_2^d$ *if and only if every directed system in* $(\mathcal{B}, <)$ *has an upper bound.*
3) A ball space $(X, \mathcal{B})$ *is* $\mathbf{S}_4$ *if and only if* $(\mathcal{B}, <)$ *is chain complete.*
4) A ball space $(X, \mathcal{B})$ *is* $\mathbf{S}_4^d$ *if and only if* $(\mathcal{B}, <)$ *is directed complete.*

Let us point out that the intersection of a system of balls may not be itself a ball, even if it is nonempty (but if it is a ball, then it is clearly the largest ball contained in all of the balls in the system). For this reason, in general, the properties $\mathbf{S}_1$, $\mathbf{S}_1^d$, $\mathbf{S}_5$ and $\mathbf{S}_5^d$ cannot be translated into a corresponding property of $(\mathcal{B}, <)$. This shows that ball spaces have more expressive strength than the associated poset structures.

A proof of the following fact can be found in [4, p. 33]. See also [19] for generalizations.

Proposition 4.2. *Every chain complete poset is directed complete.*

This proposition together with Proposition 4.1 yields:

Corollary 4.3. *Every* $\mathbf{S}_4$ *ball space is an* $\mathbf{S}_4^d$ *ball space.*

In the next sections, we will give further criteria for the equivalence of various properties in the hierarchy.

4.3. **Singleton balls.**

In many applications (e.g. metric spaces, ultrametric spaces, T_1 topological spaces) the associated ball spaces have the property that singleton sets are balls. The following observation is straightforward:

Proposition 4.4. *For a ball space in which all singleton sets are balls,* $\mathbf{S}_1$ *is equivalent to* $\mathbf{S}_2$, $\mathbf{S}_1^d$ *is equivalent to* $\mathbf{S}_2^d$ *, and* $\mathbf{S}_1^c$ *is equivalent to* $\mathbf{S}_2^c$ *.*

4.4. **Tree-like ball spaces.**

We will call a ball space $(X, \mathcal{B})$ **tree-like** if any two balls in $\mathcal{B}$ with nonempty intersection are comparable by inclusion. We will see in Section 5.1 (Proposition 5.1) that the ball spaces associated with classical ultrametric spaces are tree-like.

Proposition 4.5. *In a tree-like ball space, every centered system of balls is a nest. For such a ball space,* $\mathbf{S}_i$, $\mathbf{S}_i^d$ *and* $\mathbf{S}_i^c$ *are equivalent, for each* $i \in \{1, \ldots, 5\}$*. If in addition, in this ball space all singleton sets are balls, then* $\mathbf{S}_1$ *is equivalent to* $\mathbf{S}_2^c$ *.*

Proof: The first assertion follows from the fact that in a tree-like ball space, every two balls in a centered system have nonempty intersection and therefore are comparable by inclusion, so the system is a nest. From this, the second assertion follows immediately. The third assertion follows by way of Proposition 4.4. □

4.5. **Intersection closed ball spaces.**
A ball space $(X, \mathcal{B})$ will be called **finitely intersection closed** if $\mathcal{B}$ is closed under nonempty intersections of any finite collection of balls, **chain intersection closed** or **nest intersection closed** if $\mathcal{B}$ is closed under nonempty intersections of nests of balls, and **intersection closed** if $\mathcal{B}$ is closed under nonempty intersections of arbitrary collections of balls.

We deduce from Proposition 4.5:

Proposition 4.6. *Every chain intersection closed tree-like ball space is intersection closed.*

Proof: Every collection $\mathcal{C}$ of balls with nonempty intersection in an arbitrary ball space is a centered system. If the ball space is tree-like, then by Proposition 4.5, $\mathcal{C}$ is a nest. If in addition the ball space is chain intersection closed, then the intersection $\bigcap \mathcal{C}$ is a ball. Hence under the assumptions of the proposition, the ball space is intersection closed. □

The proofs of the following two propositions are straightforward:

Proposition 4.7. *Assume that the ball space $(X, \mathcal{B})$ is finitely intersection closed. Then by closing under finite intersections, every centered system of balls can be expanded to a directed system of balls which has the same intersection. Hence for a finitely intersection closed ball space, $\mathbf{S}_i^d$ is equivalent to $\mathbf{S}_i^c$, for $1 \leq i \leq 5$.*

Proposition 4.8. *For chain intersection closed ball spaces, the properties $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$ and $\mathbf{S}_5$ are equivalent.*

As can be expected, the intersection closed ball spaces are the strongest when it comes to equivalence of the properties in the hierarchy.

Theorem 4.9. *For an intersection closed ball space, $\mathbf{S}_1$ is equivalent to $\mathbf{S}^*$, so all properties in the hierarchy (1) are equivalent.*

Proof: Since $(X, \mathcal{B})$ is intersection closed, it is in particular chain intersection closed, hence by Proposition 4.8, $\mathbf{S}_1$ implies $\mathbf{S}_4$. By Corollary 4.3, $\mathbf{S}_4$ implies $\mathbf{S}_4^d$. Since $(X, \mathcal{B})$ is intersection closed, Proposition 4.7 shows that $\mathbf{S}_4^d$ implies $\mathbf{S}_4^c$. Again since $(X, \mathcal{B})$ is intersection closed, the intersection over every directed system of balls, if nonempty, is a ball; hence $\mathbf{S}_4^c$ implies $\mathbf{S}_5^c$. Altogether, we have shown that $\mathbf{S}_1$ implies $\mathbf{S}_5^c$, which shows that all properties in the hierarchy (1) are equivalent. □

Proposition 4.10. *Every $\mathbf{S}^*$ ball space is intersection closed.*

Proof: Take any collection of balls with nonempty intersection. Each element in the intersection lies in every ball, so the collection is a centered system. By assumption, the intersection is again a ball. □

In a poset, a set S of elements is **bounded** if and only if it has an upper bound. A poset is **bounded complete** if every nonempty bounded set has a least upper bound. A **bounded system of balls** is a nonempty collection of balls whose intersection contains a ball. Note that a bounded system of balls is a centered system, but the converse is in general not true (not even a nest of balls is necessarily a bounded system if the ball space is not $\mathbf{S}_2$).

Lemma 4.11. *The poset $(\mathcal{B},<)$ is bounded complete if and only if the intersection of every bounded system of balls in $(X,\mathcal{B})$ contains a largest ball. In an intersection closed ball space, the intersection of every bounded system of balls is a ball.*

5. Ball spaces and their properties in various applications

In what follows, we will give the interpretation of various levels of spherical completeness in our applications of ball spaces. At this point, let us define a notion that we will need repeatedly. In a (totally or partially) ordered set $(S,<)$ a subset S is a **final segment** if for all $s\in S$, $s<t$ implies $t\in S$; similarly, S is an **initial segment** if for all $s\in S$, $s>t$ implies $t\in S$.

5.1. Ultrametric spaces.

An **ultrametric** u on a set X is a function from $X\times X$ to a partially ordered set Γ with smallest element 0, such that for all $x,y,z\in X$ and all $\gamma\in\Gamma$,

(U1) $u(x,y)=0$ if and only if $x=y$,
(U2) if $u(x,y)\le\gamma$ and $u(y,z)\le\gamma$, then $u(x,z)\le\gamma$,
(U3) $u(x,y)=u(y,x)$ (symmetry).

The pair (X,u) is called an **ultrametric space**. Condition (U2) is the ultrametric triangle law.

We set $uX:=\{u(x,y)\mid x,y\in X\}$ and call it the **value set of** (X,u). If uX is totally ordered, we will call (X,u) a **classical ultrametric space**; in this case, (U2) is equivalent to:

(UT) $u(x,z)\le\max\{u(x,y),u(y,z)\}$.

We will now introduce three ways of deriving a ball space from an ultrametric space. A **closed ultrametric ball** is a set

$$B_\alpha(x)\;:=\;\{y\in X\mid u(x,y)\le\alpha\}\,,$$

where $x\in X$ and $\alpha\in\Gamma$. We obtain the **ultrametric ball space** $(X,\mathcal{B}_u)$ from (X,u) by taking $\mathcal{B}$ to be the set of all such balls $B_\alpha(x)$.

It follows from symmetry and the ultrametric triangle law that every element in a ball is a center, meaning that

$$B_\alpha(x)\;=\;B_\alpha(y)\;\text{ if } y\in B_\alpha(x)\,. \tag{5}$$

Further,

$$B_\beta(y)\;\subseteq\;B_\alpha(x)\quad\text{ if }\quad y\in B_\alpha(x)\text{ and }\beta\le\alpha\,. \tag{6}$$

A problem with the ball $B_\alpha(x)$ can be that it may not contain any element y such that $u(x,y)=\alpha$; if it does, it is called **precise**. It is therefore convenient to work with the precise balls of the form

$$B(x,y)\;:=\;\{z\in X\mid u(x,z)\le u(x,y)\}\,,$$

where $x,y\in X$. We obtain the **precise ultrametric ball space** $(X,\mathcal{B}_{[u]})$ from (X,u) by taking $\mathcal{B}$ to be the set of all such balls $B(x,y)$.

It follows from symmetry and the ultrametric triangle law that

$$B(x,y) \;=\; B(y,x)$$

and that

(7) $\quad B(t,z) \subseteq B(x,y)$ if and only if $t \in B(x,y)$ and $u(t,z) \leq u(x,y)$.

In particular,

$$B(t,z) \subseteq B(x,y) \;\text{ if }\; t,z \in B(x,y)\,. \tag{8}$$

More generally,

$$B(t,z) \;\subseteq\; B_\alpha(x) \;\text{ if }\; t,z \in B_\alpha(x)\,, \tag{9}$$

showing that ultramtric balls are convex with respect to the ultrametric.

Two elements γ and δ of Γ are **comparable** if $\gamma \leq \delta$ or $\gamma \geq \delta$. Hence if $u(x,y)$ and $u(y,z)$ are comparable, then $B(x,y) \subseteq B(y,z)$ or $B(y,z) \subseteq B(x,y)$. If $u(y,z) < u(x,y)$, then in addition, $x \notin B(y,z)$. We note:

$$u(y,z) \;<\; u(x,y) \;\Longrightarrow\; B(y,z) \;\subsetneq\; B(x,y)\,. \tag{10}$$

In classical ultrametric spaces every two values α, β are comparable. Hence in this case one can derive from (5) and (6) that every two ultrametric balls with nonempty intersection are comparable by inclusion.

From (6), we derive:

Proposition 5.1. *In a classical ultrametric space (X,u), any two balls with nonempty intersection are comparable by inclusion. Hence $(X,\mathcal{B}_{[u]})$ and $(X,\mathcal{B}_u)$ are tree-like ball spaces.*

We define (X,u) to be **spherically complete** if its ultrametric ball space $(X,\mathcal{B}_u)$ is spherically complete. For this definition, it actually makes no difference whether we work with $\mathcal{B}_u$ or $\mathcal{B}_{[u]}$:

Proposition 5.2. *The classical ultrametric ball space $(X,\mathcal{B}_u)$ is spherically complete if and only if the precise ultrametric ball space $(X,\mathcal{B}_{[u]})$ is.*

Proof: Since $\mathcal{B}_{[u]} \subseteq \mathcal{B}_u$, the implication "$\Rightarrow$" is clear. Now take a nest $\mathcal{N}$ of balls in $\mathcal{B}_u$. We may assume that it does not contain a smallest ball since otherwise this ball equals the intersection over the nest, which consequently is nonempty. Further, there is a coinitial subnest $(B_{\alpha_\nu}(x_\nu))_{\nu<\kappa}$ such that κ is an infinite limit ordinal and $\mu < \nu < \kappa$ implies that $B_{\alpha_\nu}(x_\nu) \subsetneq B_{\alpha_\mu}(x_\mu)$. It follows that this subnest has the same intersection as $\mathcal{N}$.

For every $\nu < \kappa$, also $\nu + 1 < \kappa$ and thus $B_{\alpha_{\nu+1}}(x_{\nu+1}) \subsetneq B_{\alpha_\nu}(x_\nu)$. Hence there is $y_{\nu+1} \in B_{\alpha_\nu}(x_\nu) \setminus B_{\alpha_{\nu+1}}(x_{\nu+1})$. It follows that

$$u(x_{\nu+1},y_{\nu+1}) \;>\; \alpha_{\nu+1}\,,$$

and from (6) we obtain that

$$B_{\alpha_{\nu+1}}(x_{\nu+1}) \;\subseteq\; B_{u(x_{\nu+1},y_{\nu+1})}(x_{\nu+1}) \;=\; B(x_{\nu+1},y_{\nu+1})\,.$$

Since $x_{\nu+1}, y_{\nu+1} \in B_{\alpha_\nu}(x_\nu)$, we know from (9) that

$$B(x_{\nu+1},y_{\nu+1}) \;\subseteq\; B_{\alpha_\nu}(x_\nu)\,.$$

It follows that

$$\bigcap \mathcal{N} \;=\; \bigcap_{\nu<\kappa} B_{\alpha_\nu}(x_\nu) \;=\; \bigcap_{\nu<\kappa} B(x_{\nu+1},y_{\nu+1})\,.$$

Consequently, if $\mathcal{B}_{[u]}$ is $\mathbf{S}_1$, then this intersection is nonempty and we have proved that also $\mathcal{B}_u$ is $\mathbf{S}_1$. □

Since uX contains the smallest element $0 := u(x,x)$, $\mathcal{B}_u$ contains all singletons $\{x\} = B_0(x)$. Therefore, each ultrametric ball space is already $\mathbf{S}_2$ once it is $\mathbf{S}_1$. The same is true for the precise ultrametric ball space $(X, \mathcal{B}_{[u]})$ in place of $(X, \mathcal{B}_u)$. However, these ball spaces will in general not be $\mathbf{S}_3$, $\mathbf{S}_4$ or $\mathbf{S}_5$ because even if an intersection of a nest is nonempty, it will not necessarily be a ball of the form $B_\alpha(x)$ or $B(x,y)$, respectively.

In a classical ultrametric space, every two balls are comparable by inclusion once they have nonempty intersection. Therefore, every centered system is already a nest of balls. This shows:

Proposition 5.3. *A classical ultrametric space (X, u) is spherically complete if and only if the ball space $(X, \mathcal{B}_u)$ (or equivalently, $(X, \mathcal{B}_{[u]})$) is an $\mathbf{S}_2^c$ ball space.*

If (X, u) is a classical ultrametric space, then we can obtain stronger completeness properties if we work with a larger set of ultrametric balls. Given $x \in X$ and an initial segment $S \neq \emptyset$ of uX, we define:

$$B_S(x) \;=\; \{y \in X \mid u(x,y) \in S\}\;.$$

Setting

$$\mathcal{B}_{u+} \;:=\; \{B_S(x) \mid x \in X \text{ and } S \text{ a nonempty initial segment of } uX\}\;,$$

we obtain what we will call the **full ultrametric ball space** $(X, \mathcal{B}_{u+})$. Note that $X = B_{uX}(x) \in \mathcal{B}_{u+}$. We leave it to the reader to prove:

$$B_S(x) \;=\; \bigcup_{\alpha \in S} B_\alpha(x) \;\subseteq\; \bigcap_{\beta \geq S} B_\beta(x) \tag{11}$$

where $\beta \geq S$ means that $\beta \geq \gamma$ for all $\gamma \in S$, and the intersection over an empty index set is taken to be X. We note that the inclusion on the right hand side is proper if and only if S has no largest element but admits a supremum α in uX and there is $y \in X$ such that $\alpha = u(x,y)$. Indeed, if $S = \{\beta \mid \beta < \alpha\}$, then $B_S(x)$ is the **open ultrametric ball**

$$B_\alpha^\circ(x) \;:=\; \{y \in X \mid u(x,y) < \alpha\}\;,$$

which is a proper subset of $B_\alpha(x) = \bigcap_{\beta \geq S} B_\beta(x)$ if and only if $B_\alpha(x)$ is precise.

We have that

$$\mathcal{B}_{[u]} \;\subseteq\; \mathcal{B}_u \;\subseteq\; \mathcal{B}_{u+}$$

where the second inclusion holds because $B_\alpha(x) = B_S(x)$ for the initial segment $S = [0, \alpha]$ of uX. We have an easy generalization of (9):

$$\text{if } B \in \mathcal{B}_{u+} \text{ and } t, z \in B\,, \text{ then } B(t,z) \;\subseteq\; B\;. \tag{12}$$

The following results are proven in [11]:

Theorem 5.4. *Let (X, u) be a classical ultrametric space. Then the following assertions hold.*

1) The intersection over every nest of balls in $(X, \mathcal{B}_{u+})$ is equal to the intersection over a nest of balls in $(X, \mathcal{B}_u)$ and therefore, $(X, \mathcal{B}_{u+})$ is chain intersection closed.

2) The ball space $(X, \mathcal{B}_{u+})$ is spherically complete if and only if $(X, \mathcal{B}_u)$ is.

3) The ball space $(X,\mathcal{B}_{u+})$ is tree-like and intersection closed. If $(X,\mathcal{B}_u)$ is spherically complete, then $(X,\mathcal{B}_{u+})$ is an $\mathbf{S}^*$ *ball space.*

By [10, Theorem 1.2], assertions 1) and 2) of Theorem 5.4 also hold for all ultrametric spaces (X,u) with countable narrow value sets uX; the condition **narrow** means that all sets of mutually incomparable elements in uX are finite. On the other hand, it is shown in [10] that the condition "narrow" cannot be dropped in this case. It is however an open question whether the condition "countable" can be dropped.

A large number of ultrametric fixed point and coincidence point theorems have been proven by S. Prieß-Crampe and P. Ribenboim (see e.g. [23, 24, 25, 26, 28]). Using ball spaces, some of them have been reproven and new ones have been proven in [13, 14, 17].

5.2. **Metric spaces with metric balls.**

In metric spaces (X,d) we can consider the closed metric balls

$$B_\alpha(x) \;:=\; \{y\in X \mid d(x,y)\le\alpha\}$$

for $x\in X$ and $\alpha\in\mathbb{R}^{\ge 0} := \{r\in\mathbb{R}\mid r\ge 0\}$. We set

$$\mathcal{B}_d \;:=\; \{B_\alpha(x)\mid x\in X\,,\ \alpha\in\mathbb{R}^{\ge 0}\}\,.$$

The following theorem will be deduced from Theorem 5.6 below:

Theorem 5.5. *If the ball space $(X,\mathcal{B}_d)$ is spherically complete, then (X,d) is complete.*

The converse is not true. Consider a rational function field $k(x)$ together with the x-adic valuation v_x. Choose an extension of v_x to a valuation v of the algebraic closure K_0 of $k(x)$. Then the value group is $\mathbb{Q}$. An ultrametric in the sense of Section 5.1 is obtained by setting, for instance,

$$u(a,b) \;:=\; e^{-v(a-b)}\,.$$

Take (K,u) to be the completion of (K_0,u). It can be shown that the balls

$$B_{\alpha_i}\left(\sum_{j=1}^{i-1} x^{-\frac{1}{j}}\right) \quad \text{with } \alpha_i \;=\; e^{\frac{1}{i}} \qquad (2\le i\in\mathbb{N})$$

have empty intersection in K. Hence (K,u) is not spherically complete, that is, the ultrametric ball space induced by u on K is not spherically complete. But this ultrametric is a complete metric.

Note that from Theorem 5.19 below it follows that the ball space $(X,\mathcal{B}_d)$ is spherically complete if every closed metric ball in (X,d) is compact under the topology induced by d, as the closed metric balls are closed in this topology.

In order to characterize complete metric spaces by spherical completeness, we have to choose smaller induced ball spaces. For any subset S of the set $\mathbb{R}^{>0}$ of positive real numbers, we define:

$$\mathcal{B}_S \;:=\; \{B_r(x)\mid x\in X\,,\ r\in S\}\,.$$

Theorem 5.6. *The following assertions are equivalent:*

a) (X,d) *is complete,*

b) the ball space $(X,\mathcal{B}_S)$ *is spherically complete for* some $S\subset\mathbb{R}^{>0}$ *which admits* 0 *as its only accumulation point,*

c) the ball space $(X,\mathcal{B}_S)$ *is spherically complete for* every $S\subset\mathbb{R}^{>0}$ *which admits* 0 *as its only accumulation point.*

Proof: a) $\Rightarrow$ c): Assume that (X,d) is complete and take a set $S\subset\mathbb{R}^{>0}$ which admits 0 as its only accumulation point. This implies that S is discretely ordered, hence every infinite descending chain in S with a maximal element can be indexed by the natural numbers.

Take any nest $\mathcal{N}$ of closed metric balls in $\mathcal{B}_S$. If the nest contains a smallest ball, then its intersection is nonempty; so we assume that it does not. If $B\in\mathcal{N}$, then $\mathcal{N}_B:=\{B'\in\mathcal{N}\mid B'\subseteq B\}$ is a nest of balls with $\bigcap\mathcal{N}=\bigcap\mathcal{N}_B$; therefore, we may assume from the start that $\mathcal{N}$ contains a largest ball. Then the radii of the balls in $\mathcal{N}$ form an infinite descending chain in S with a maximal element, and 0 is their unique accumulation point. Hence we can write $\mathcal{N}=\{B_{r_i}(x_i)\mid i\in\mathbb{N}\}$ with $r_j<r_i$ for $i<j$, and with $\lim_{i\to\infty}r_i=0$.

For every $i\in\mathbb{N}$ and all $j\geq i$, the element x_j lies in $B_{r_i}(x_i)$ and therefore satisfies $d(x_i,x_j)\leq r_i$. This shows that $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is complete, it has a limit x in X. We have that $d(x_i,x)\leq r_i$, so x lies in every ball $B_{r_i}(x_i)$. This proves that the nest has nonempty intersection.

c) $\Rightarrow$ b): Trivial.

b) $\Rightarrow$ a): Assume that $(X,\mathcal{B}_S)$ is spherically complete. Take any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X. By our assumptions on S, we can choose a sequence $(s_i)_{i\in\mathbb{N}}$ in $\{s\in S\mid s<s_0\}$ such that $0<2s_{i+1}\leq s_i$. Now we will use induction on $i\in\mathbb{N}$ to choose an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that the balls $B_i:=B_{s_i}(x_{n_i})$ form a nest.

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, we have that there is n_1 such that $d(x_n,x_m)<s_2$ for all $n,m>n_1$. Once we have chosen n_{i-1} , we choose $n_i>n_{i-1}$ such that $d(x_n,x_m)<s_{i+1}$ for all $n,m\geq n_i$. We show that the so obtained balls B_i form a nest. Take $i\in\mathbb{N}$ and $x\in B_{i+1}=B_{s_{i+1}}(x_{n_{i+1}})$. This means that $d(x_{n_{i+1}},x)\leq s_{i+1}$. Since $n_i,n_{i+1}\geq n_i$, we have that $d(x_{n_i},x_{n_{i+1}})<s_{i+1}$. We compute:

$$\begin{aligned}d(x_{n_i},x) &\leq d(x_{n_i},x_{n_{i+1}})+d(x_{n_{i+1}},x)\\ &\leq s_{i+1}+s_{i+1}=2s_{i+1}\leq s_i\ .\end{aligned}$$

Thus $x\in B_i$ and hence $B_{i+1}\subseteq B_i$ for all $i\in\mathbb{N}$. The intersection of this nest $(B_i)_{i\in\mathbb{N}}$ contains some y, by our assumption. We have that $y\in B_i$ for all $i\in\mathbb{N}$, which means that $d(x_{n_i},y)\leq s_i$. Since

$$\lim_{i\to\infty}s_i=0,$$

we obtain that

$$\lim_{i\to\infty}x_{n_i}=y,$$

which proves that (X,d) is a complete metric space. □

Proof of Theorem 5.5: Assume that $(X,\mathcal{B}_d)$ is spherically complete. Then so is $(X,\mathcal{B}')$ for every nonempty $\mathcal{B}'\subset\mathcal{B}_d$. Taking $\mathcal{B}'=\mathcal{B}_S$ with S as in Theorem 5.6, we obtain that (X,d) is complete. □

Remark 5.7. Theorems 5.5 and 5.6 remain true if instead of the closed metric balls the open metric balls

$$B_\alpha(x) \ := \ \{y \in X \mid d(x,y) < \alpha\}$$

are used for the metric ball space.

5.3. **Metric spaces with Caristi–Kirk balls or Oettli–Théra balls.** Consider a metric space (X,d). A function $\varphi : X \to \mathbb{R}$ is **lower semicontinuous** if for every $y \in X$,

$$\liminf_{x\to y} \varphi(x) \ \geq \ \varphi(y)\ .$$

If φ is lower semicontinuous and bounded from below, we call it a **Caristi–Kirk function on** X. For a fixed Caristi–Kirk function φ we consider **Caristi–Kirk balls** of the form

$$B_x^\varphi \ := \ \{y \in X \mid d(x,y) \leq \varphi(x) - \varphi(y)\}, \quad x \in X, \tag{13}$$

and the corresponding **Caristi–Kirk ball space** $(X, \mathcal{B}^\varphi)$ given by

$$\mathcal{B}^\varphi := \{B_x^\varphi \mid x \in X\}.$$

These ball spaces and their underlying theory can be employed to prove the Caristi–Kirk Theorem in a simple manner (see below). We found the sets that we call Caristi–Kirk balls in a proof of the Caristi–Kirk Theorem given by J.-P. Penot in [21].

We say that a function $\phi : X \times X \to (-\infty, +\infty]$ is an **Oettli–Théra function on** X if it satisfies the following conditions:

(a) $\phi(x,\cdot) : X \to (-\infty,+\infty]$ is lower semicontinous for all $x \in X$;

(b) $\phi(x,x) = 0$ for all $x \in X$;

(c) $\phi(x,y) \leq \phi(x,z) + \phi(z,y)$ for all $x,y,z \in X$;

(d) there exists $x_0 \in X$ such that $\inf\limits_{x\in X} \phi(x_0,x) > -\infty$.

This notion was, to our knowledge, first introduced by Oettli and Théra in [20]. An Oettli–Théra function ϕ yields balls of the form

$$B_x^\phi := \{y \in X \mid d(x,y) \leq -\phi(x,y)\}, \quad x \in X,$$

which will be called **Oettli–Théra balls**. If an element x_0 satisfies condition (d) above, then we will call it an **Oettli–Théra element for ϕ in X**. For a fixed Oettli–Théra element x_0 we define the associated **Oettli–Théra ball space** to be $(B_{x_0}^\phi, \mathcal{B}_{x_0}^\phi)$, where

$$\mathcal{B}_{x_0}^\phi := \{B_x^\phi \mid x \in B_{x_0}^\phi\}.$$

We observe that for a given Caristi–Kirk function $\varphi : X \to \mathbb{R}$, the mapping

$$\phi(x,y) := \varphi(y) - \varphi(x)$$

is an Oettli–Théra function. Furthermore, every Caristi–Kirk ball is also an Oettli–Théra ball.

In general the balls defined above are not metric balls. However, when working in complete metric spaces they prove to be a more useful tool than metric balls. As observed in the previous section, the completeness of a metric space need not imply spherical completeness of the space of metric balls $(X, \mathcal{B}_d)$. In the case of Caristi–Kirk and Oettli–Théra balls, completeness turns out to be equivalent to spherical completeness, as shown in the following two propositions.

Proposition 5.8. *Let (X,d) be a metric space. Then the following assertions are equivalent:*

a) The metric space (X,d) is complete.

b) Every Caristi–Kirk ball space $(X,\mathcal{B}^\varphi)$ is spherically complete.

c) For every continuous function $\varphi : X \to \mathbb{R}$ bounded from below, the Caristi–Kirk ball space $(X,\mathcal{B}^\varphi)$ is spherically complete.

Proposition 5.9. *A metric space (X,d) is complete if and only if the Oettli–Théra ball space $(B^\phi_{x_0},\mathcal{B}^\phi_{x_0})$ is spherically complete for every Oettli–Théra function ϕ on X and every Oettli–Théra element x_0 for ϕ in X.*

The proofs of Proposition 5.8 and Proposition 5.9 can be found in [15, Proposition 3] and in [2], respectively.

The easy proof of the next proposition is provided in [2].

Proposition 5.10. *Every Caristi–Kirk ball space $(X,\mathcal{B}^\varphi)$ and every Oettli–Théra ball space $(B^\phi_{x_0},\mathcal{B}^\phi_{x_0})$ is a strongly contractive normalized B_x–ball space.*

We will meet another strongly contractive ball space in the case of partially ordered sets; see Proposition 5.30.

The following is the **Caristi–Kirk Fixed Point Theorem**:

Theorem 5.11. *Take a complete metric space (X,d) and a lower semicontinuous function $\varphi : X \to \mathbb{R}$ which is bounded from below. If a function $f : X \to X$ satisfies the* **Caristi condition**

$$d(x,fx) \;\leq\; \varphi(x) - \varphi(fx)\,, \tag{14}$$

for all $x \in X$, then f has a fixed point on X.

Also in [2], the same tools (with Proposition 5.8 replaced by Proposition 5.9) are used to prove the following generalization:

Theorem 5.12. *Take a complete metric space (X,d) and ϕ an Oettli-Théra function on X. If a function $f : X \to X$ satisfies*

$$d(x,fx) \;\leq\; -\phi(x,fx), \tag{15}$$

for all $x \in X$, then f has a fixed point on X.

The conditions (14) and (15) guarantee that $fx \in B_x$ for every $B_x \in \mathcal{B}^\varphi$ or $B_x \in \mathcal{B}^\phi_{x_0}$, respectively. Hence Theorem 2.8 in conjunction with Propositions 5.8, 5.9 and 5.10 proves Theorems 5.11 and 5.12. Similar proofs were given in [2] (see also [15]). Note that conditions (14) and (15) do not necessarily imply that every ball B_x is f-closed.

A variant of part 2) of Theorem 3.9 is used in [2] to give quick proofs of several theorems that are known to be equivalent to the Caristi–Kirk Fixed Point Theorem (see [20, 21, 22] for presentations of these equivalent results and generalizations).

Remark 5.13. Assume that $(X,\mathcal{B})$ is a contractive B_x–ball space. Then we can define a partial ordering on X by setting

$$x \,\prec\, y \;:\Leftrightarrow\; B_y \subsetneq B_x\,.$$

If $(X, \mathcal{B})$ is strongly contractive, then the function $x \mapsto B_x$ is injective, and X together with the reverse of the partial order we have defined is order isomorphic to $\mathcal{B}$ with inclusion, that is, the function $x \mapsto B_x$ is an order isomorphism from $(X, \prec)$ onto $(\mathcal{B}, <)$ where the latter is defined as in the beginning of Section 3.

If the B_x are the Caristi–Kirk balls defined in (13), then we have that

$$x \prec y \Leftrightarrow d(x,y) < \varphi(x) - \varphi(y) ,$$

which means that $\prec$ is the **Brønsted ordering** on X. The **Ekeland Variational Principle** (cf. [2]) states that if the metric space is complete, then $(X, \prec)$ admits maximal elements, or in other words, $\mathcal{B}$ admits minimal balls. The Brønsted ordering has been used in several different proofs of the Caristi–Kirk Fixed Point Theorem. However, at least in the proofs that also define and use the Caristi–Kirk balls (such as the one of Penot in [21]), it makes more sense to use directly their natural partial ordering (as done in [15]). But the main incentive to use the balls instead of the ordering is that it naturally subsumes the metric case in the framework of fixed point theorems in several other areas of mathematics which is provided by the general theory of ball spaces as laid out in the present paper (see also [13, 14, 17]).

It has been shown that the Ekeland Variational Principle can be proven in the Zermelo Fraenkel axiom system ZF plus the **axiom of dependent choice** DC which covers the usual mathematical induction (but not transfinite induction, which is equivalent to the full axiom of choice). Conversely, it has been shown in [3] that the Ekeland Variational Principle implies the axiom of dependent choice.

Several proofs have been provided for the Caristi–Kirk FPT that work in ZF+DC. Kozlowski has given a proof that is **purely metric** as defined in his paper [9], which implies that the proof works in ZF+DC. The proofs of Proposition 5.8 in [15] and of Proposition 5.9 in [2] are purely metric. The existence of singleton balls in Caristi–Kirk and Oettli-Théra ball spaces over complete metric spaces can also be shown directly by purely metric proofs and this result can be used to give quick proofs of many principles that are equivalent to the Caristi–Kirk FPT in ZF+DC (cf. [2]). However, in other settings it may not be possible to deduce the existence in ZF+DC, so then the axiom of choice is needed. Therefore, in view of the number of possible applications even beyond the scope as presented in this paper, we do not hesitate to use Zorn's Lemma for the proofs of our generic fixed point theorems.

We should point out that proofs have been given that apparently prove the Caristi–Kirk FPT in ZF (see [18, 7]). This means that the Caristi–Kirk FPT and the Ekeland Variational Principle are equivalent in ZF+DC, but *not* in ZF. For the topic of axiomatic strength, see the discussions in [6, 8, 9].

5.4. **Totally ordered sets, abelian groups and fields.**

Take any ordered set $(I, <)$. We define the **closed interval ball space** associated with $(I, <)$ to be $(I, \mathcal{B}_{\text{int}})$ where $\mathcal{B}_{\text{int}}$ consists of all closed intervals $[a, b]$ with $a, b \in I$. By a **cut** in $(I, <)$ we mean a partition (C, D) of I such that $c < d$ for all $c \in C$, $d \in D$ and C, D are nonempty. The **cofinality** of a totally ordered set is the least cardinality of all cofinal subsets, and the **coinitiality** of a totally ordered set is the cofinality of this set under the reverse ordering. A cut (C, D) is **asymmetric** if the cofinality of C is different from the coinitiality of D. For example, every cut in $\mathbb{R}$ is asymmetric. The following fact was first proved in [30] for ordered fields, and then in [16] for any totally ordered sets.

Lemma 5.14. *The ball space $(I, \mathcal{B}_{\mathrm{int}})$ associated with the totally ordered set $(I, <)$ is spherically complete if and only if every cut (C, D) in $(I, <)$ is asymmetric.*

Totally ordered sets, abelian groups or fields whose cuts are all asymmetric are called **symmetrically complete**. By our above remark, $\mathbb{R}$ is symmetrically complete. The following theorem was proved in [16]; its first assertion follows from the previous lemma. The second assertion addresses the **natural valuation** of an ordered abelian group or field, which is the finest valuation compatible with the ordering; it is nontrivial if and only if the ordering is nonarchimedean.

Theorem 5.15. *A totally ordered set, abelian group or field is symmetrically complete if and only if its associated closed interval ball space is spherically complete. The ultrametric ball space associated with the natural valuation of a symmetrically complete ordered abelian group or field is a spherically complete ball space.*

In [30] it was shown that arbitrarily large symmetrically complete ordered fields exist. With a different construction idea, this was reproved and generalized in [16] to the case of ordered abelian groups and totally ordered sets, and a characterization of symmetrically complete ordered abelian groups and fields has been given.

In order to give an example of a fixed point theorem that can be proven in this setting, it is enough to consider symmetrically complete ordered abelian groups, as the additive group of a symmetrically complete ordered field is a symmetrically complete ordered abelian group. The following is Theorem 21 of [13] (see also [16]).

Theorem 5.16. *Take an ordered abelian group $(G, <)$ and a function $f : G \to G$. Assume that every nonempty chain of closed intervals in G has nonempty intersection and that f has the following properties:*
1) f is nonexpanding:

$$|fx - fy| \;\leq\; |x - y| \;\; \text{for all } x, y \in G\,,$$

2) f is contracting on orbits: there is a positive rational number $\frac{m}{n} < 1$ with $m, n \in \mathbb{N}$ such that

$$n|fx - f^2x| \;\leq\; m|x - fx| \;\; \text{for all } x \in G\,.$$

Then f has a fixed point.

As in the case of ultrametric spaces, all singletons in $\mathcal{B}_{\mathrm{int}}$ are balls: $\{a\} = [a, a]$. So also here, $(I, \mathcal{B}_{\mathrm{int}})$ is $\mathbf{S}_2$ as soon as it is $\mathbf{S}_1$. But again as in the case of ultrametric spaces, $\mathbf{S}_2$ does not necessarily imply $\mathbf{S}_5$ or even $\mathbf{S}_3$. For example, consider a nonarchimedean ordered symmetrically complete field. The set of infinitesimals is the intersection of balls $[-a, a]$ where a runs through all positive elements that are not infinitesimals. This intersection is not a ball, nor is there a largest ball contained in it.

Further, we note:

Lemma 5.17. *Assume that* $(I,<)$ *is a totally ordered set and its associated ball space* $(I,\mathcal{B}_{\rm int})$ *is an* $\mathbf{S}_1^d$ *or* $\mathbf{S}_3$ *ball space. Then* $(I,<)$ *is cut complete, that is, for every cut* (C,D) *in* $(I,<)$*,* C *has a largest or* D *has a smallest element.*

Proof: First assume that $(I,\mathcal{B}_{\rm int})$ is an $\mathbf{S}_1^d$ ball space, and take a cut (C,D) in I. If $a,c\in C$ and $b,d\in D$, then $\max\{a,c\}\in C$ and $\min\{b,d\}\in D$ and $[a,b]\cap[c,d]=[\max\{a,c\},\min\{b,d\}]$. This shows that

$$\{[c,d]\mid c\in C\,,\; b\in D\}$$

is a directed system in $\mathcal{B}_{\rm int}$. Hence its intersection is nonempty; if a is contained in this intersetion, it must be the largest element of C or the least element of D. Hence $(I,<)$ is cut complete.

Now assume that $(I,<)$ is not cut complete; we wish to show that $(I,\mathcal{B}_{\rm int})$ is not an $\mathbf{S}_3$ ball space. Take a cut (C,D) in I such that C has no largest element and D has no least element. Pick some $c\in C$. Then

$$\{[c,d]\mid d\in D\}$$

is a nest of balls in $(I,\mathcal{B}_{\rm int})$. Its intersection is the set $\{a\in C\mid c\le a\}$. Since C has no largest element, this set does not contain a maximal ball. This shows that $(I,\mathcal{B}_{\rm int})$ is not an $\mathbf{S}_3$ ball space. □

It is a well known fact that the only cut complete densely ordered abelian group or ordered field is $\mathbb{R}$. So we have:

Proposition 5.18. *The associated ball space of the reals is* $\mathbf{S}^*$*. For all other densely ordered abelian groups and ordered fields the associated ball space can at best be* $\mathbf{S}_2$*.*

Proof: Take any centered system $\{[a_i,b_i]\mid i\in I\}$ of intervals in $\mathbb{R}$. We set $a:=\sup_{i\in I}a_i$ and $b:=\inf_{i\in I}b_i$. Then

$$\bigcap_{i\in I}[a_i,b_i]\;=\;[a,b]\,.$$

We have to show that $[a,b]\neq\emptyset$, i.e., $a\le b$. Suppose that $a>b$. Then there are $i,j\in I$ such that $a_i>b_j$. But by assumption, $[a_i,b_i]\cap[a_j,b_j]\neq\emptyset$, a contradiction. We have now proved that the associated ball space of the reals is $\mathbf{S}^*$.

The second assertion follows from Lemma 5.17. □

5.5. **Topological spaces.**

If $\mathcal{X}$ is a topological space on a set X, then we will take its associated ball space to be $(X,\mathcal{B})$ where $\mathcal{B}$ consists of all nonempty closed sets. Since the intersections of arbitrary collections of closed sets are again closed, this ball space is intersection closed.

The following theorem shows how compact topological spaces are characterized by the properties of their associated ball spaces; note that we use "compact" in the sense of "quasi-compact", that is, it does not imply the topology being Hausdorff.

Theorem 5.19. *The following are equivalent for a topological space $\mathcal{X}$:*

a) $\mathcal{X}$ is compact,

b) the nonempty closed sets in $\mathcal{X}$ form an $\mathbf{S}_1$ ball space,

c) the nonempty closed sets in $\mathcal{X}$ form an $\mathbf{S}^*$ ball space.

Proof: a) $\Rightarrow$ b): Assume that $\mathcal{X}$ is compact. Take a nest $(X_i)_{i\in I}$ of balls in $(X,\mathcal{B})$ and suppose that $\bigcap_{i\in I} X_i = \emptyset$. Then $\bigcup_{i\in I} X \setminus X_i = X$, so $\{X \setminus X_i \mid i \in I\}$ is an open cover of $\mathcal{X}$. It follows that there are $i_1,\ldots,i_n \in I$ such that $X \setminus X_{i_1} \cup \ldots \cup X \setminus X_{i_n} = X$, whence $X_{i_1} \cap \ldots \cap X_{i_n} = \emptyset$. But since the X_i form a nest, this intersection equals the smallest of the X_{i_j}, which is nonempty. This contradiction proves that the nonempty closed sets in $\mathcal{X}$ form an $\mathbf{S}_1$ ball space.

b) $\Rightarrow$ c): This follows from Theorem 4.9.

c) $\Rightarrow$ a): Assume that the nonempty closed sets in $\mathcal{X}$ form an $\mathbf{S}^*$ ball space. Take an open cover Y_i, $i \in I$, of $\mathcal{X}$. Since $\bigcup_{i\in I} Y_i = X$, we have that $\bigcap_{i\in I} X \setminus Y_i = \emptyset$. As the ball space is $\mathbf{S}^*$, this means that $\{X \setminus Y_i \mid i \in I\}$ cannot be a centered system. Consequently, there are $i_1,\ldots,i_n \in I$ such that $X \setminus Y_{i_1} \cap \ldots \cap X \setminus Y_{i_n} = \emptyset$, whence $Y_{i_1} \cup \ldots \cup Y_{i_n} = X$. □

Some of the assertions of the following topological fixed point theorems were already proven in [13, Theorem 11]. We will give their modified and improved proofs here as they illustrate applications of Theorems 2.6 and 2.2.

Theorem 5.20. *Take a compact space X and a closed function $f : X \to X$. Assume that for every $x \in X$ with $fx \neq x$ there is a closed subset B of X such that $x \in B$ and $x \notin f(B) \subseteq B$. Then f has a fixed point in B.*

Proof: For every $x \in X$ we consider the following family of balls:

$$\mathfrak{B}_x := \{B \mid B \text{ closed subset of } X,\ x \in B \text{ and } f(B) \subseteq B\}.$$

Note that $\mathfrak{B}_x$ is nonempty because it contains X. We define

$$B_x := \bigcap \mathfrak{B}_x\,. \tag{16}$$

We see that $x \in B_x$ and that $f(B_x) \subseteq B_x$ by part 2) of Lemma 3.10. Further, B_x is closed, being the intersection of closed sets. This shows that B_x is the smallest member of $\mathfrak{B}_x$.

For every $B \in \mathfrak{B}_x$ we have that $fx \in B$ and therefore, $B \in \mathfrak{B}_{fx}$. Hence we find that $B_{fx} \subseteq B_x$.

Assume that $fx \neq x$. Then by hypothesis, there is a closed set B in X such that $x \in B$ and $x \notin f(B) \subseteq B$. Since f is a closed function, $f(B)$ is closed. Moreover, $f(f(B)) \subseteq f(B)$ and $fx \in f(B)$, so $f(B) \in \mathfrak{B}_{fx}$. Since $x \notin f(B)$, we conclude that $x \notin B_{fx}$, whence $B_{fx} \subsetneq B_x$. We have proved that f is contracting on orbits. Our theorem now follows from Theorem 2.6 in conjunction with Theorem 5.19. □

Note that if B satisfies the assumptions of the theorem, then $B \in \mathfrak{B}_x$. Hence the set B_x defined in (16) satisfies $B_x \subseteq B$, $f(B_x) \subseteq f(B)$ and therefore $x \notin f(B_x)$. This shows that B_x is the smallest of all closed subsets B of X for which $x \in B$ and $x \notin f(B) \subseteq B$.

An interesting interpretation of the ball B_x defined in (16) will be given in Remark 6.6 below.

The next theorem follows immediately from part 1) of Theorem 2.2 in conjunction with Theorem 5.19.

Theorem 5.21. *Take a compact space X and a closed function $f : X \to X$.*

1) If every nonempty closed and f-closed subset B of X contains a closed f-contracting subset, then f has a fixed point in X.

2) If every nonempty closed and f-closed subset B of X is f-contracting, then f has a unique fixed point in X.

The condition that every f-closed ball is f-contracting may appear to be quite strong. Yet there is a natural example in the setting of topological spaces where this condition is satisfied in a suitable collection of closed sets. In [31], Steprans, Watson and Just define the notion of "J-contraction" for a continuous function $f : X \to X$ on a topological space X as follows. An open cover $\mathcal{U}$ of X is called **J-contractive for** f if for every $U \in \mathcal{U}$ there is $U' \in \mathcal{U}$ such that the image of the closure of U under f is a subset of U'. Then f is called a **J-contraction** if any open cover $\mathcal{U}$ has a J-contractive refinement for f. We will use two important facts about J-contractions f on a connected compact Hausdorff space X which the authors prove in the cited paper:

(J1) If B is a closed subset of X with $f(B) \subseteq B$, then the restriction of f to B is also a J-contraction ([31, Proposition 1, p. 552]);
(J2) If f is onto, then $|X| = 1$ ([31, Proposition 4, p. 554]).

The following is Theorem 4 of [31]:

Theorem 5.22. *Take a connected compact Hausdorff space X and a continuous J-contraction $f : X \to X$. Then f has a unique fixed point.*

We will deduce our theorem from Theorem 2.2. We take $\mathcal{B}$ to be the set of all nonempty closed connected subsets of X; in particular, $X \in \mathcal{B}$. Take any $B \in \mathcal{B}$. As f is a continuous function on the compact Hausdorff space X, it is a closed function, so $f(B)$ is closed. Since B is connected and f is continuous, $f(B)$ is also connected. Hence $f(B) \in \mathcal{B}$.

Further, the intersection of any chain of closed connected subsets of X is closed and connected. This shows that $\mathcal{B}$ is chain intersection closed. By Theorem 5.19 the ball space consisting of all nonempty closed subsets of the compact space X is $\mathbf{S}^*$. As it contains $\mathcal{B}$, $(X, \mathcal{B})$ is $\mathbf{S}_1$ and it follows from Proposition 4.8 that $(X, \mathcal{B})$ is an $\mathbf{S}_5$ ball space.

Finally, we have to show that every f-closed ball $B \in \mathcal{B}$ is f-contracting. As B is closed in X, it is also compact Hausdorff, and it is connected as it is a ball in $\mathcal{B}$. By (J1), the restriction of f to B is also a J-contraction. Therefore, we can replace X by B and apply (J2) to find that if f is onto, then B is a singleton; this shows that B is f-contracting. Now Theorem 5.22 follows from part 2) of Theorem 2.2 as desired.

It should be noted that J-contractions appear in a natural way in the metric setting. The following is the content of Theorems 2 and 3 of [31]:

Theorem 5.23. *Any contraction on a compact metric space is a J-contraction. Conversely, if f is a J-contraction on a connected compact metrizable space X, then X admits a metric under which f is a contraction.*

5.6. **Partially ordered sets.**

Take any nonempty partially ordered set $(T,<)$. We will associate with it two different ball spaces; first, the ball space of principal final segments, and then later the segment ball space.

A **principal final segment** is a set $[a,\infty) := \{c \in T \mid a \le c\}$ with $a \in T$. Then the **ball space of principal final segments** is $(T,\mathcal{B}_{\rm pfs})$ where $\mathcal{B}_{\rm pfs} := \{[a,\infty) \mid a \in T\}$. The following proposition gives the interpretation of spherical completeness for this ball space:

Proposition 5.24. *The following assertions are equivalent:*

a) the poset $(T,<)$ *is inductively ordered,*

b) the ball space $(T,\mathcal{B}_{\rm pfs})$ *is spherically complete,*

c) $(T,\mathcal{B}_{\rm pfs})$ *is an* $\mathbf{S}_2$ *ball space.*

Proof: We observe that $\{a_i \mid i \in I\}$ is a chain in T if and only if $\mathcal{N} = ([a_i,\infty))_{i\in I}$ is a nest of balls in $\mathcal{B}_{\rm pfs}$.

a) $\Rightarrow$ c): Take a nest $\mathcal{N} = ([a_i,\infty))_{i\in I}$. Since $(T,<)$ is inductively ordered, the chain $\{a_i \mid i \in I\}$ admits an upper bound $a \in T$. Then for all $i \in I$, $a_i \le a$, whence $[a,\infty) \subseteq [a_i,\infty)$. Thus, $[a,\infty) \subseteq \bigcap \mathcal{N}$, which proves that $(T,\mathcal{B}_{\rm pfs})$ is an $\mathbf{S}_2$ ball space.

c) $\Rightarrow$ b): This holds by the general properties of the hierarchy.

b) $\Rightarrow$ a): Take a chain $\{a_i \mid i \in I\}$ in T. Since $(T,\mathcal{B}_{\rm pfs})$ is spherically complete, the intersection of the nest $\mathcal{N} = ([a_i,\infty))_{i\in I}$ is nonempty. If $a \in \bigcap \mathcal{N}$, then for all $i \in I$, $a \in [a_i,\infty)$, whence $a_i \le a$. Thus, a is an upper bound of $\{a_i \mid i \in I\}$, which proves that $(T,<)$ is inductively ordered. □

We leave it to the reader to show that $(T,\mathcal{B}_{\rm pfs})$ is an $\mathbf{S}_3$ (or $\mathbf{S}_3^d$ or $\mathbf{S}_3^c$) ball space if and only if every chain (or directed system, or centered system, respectively) has minimal upper bounds.

If $\{a_i \mid i \in I\}$ is a subset of T, then $\sup_{i\in I} a_i$ will denote its supremum, if it exists. We will need the following observation:

Lemma 5.25. *The equality*

$$[a,\infty) = \bigcap_{i\in I} [a_i,\infty)$$

holds if and only if $a = \sup_{i\in I} a_i$. *Further,* $\bigcap_{i\in I}[a_i,\infty)$ *is the (possibly empty) set of all upper bounds for* $\{a_i \mid i \in I\}$.

Proof: We have $a \in \bigcap_{i\in I}[a_i,\infty)$ if and only if $a \in [a_i,\infty)$ and hence $a \ge a_i$ for all i, which means that a is an upper bound for the a_i. Hence, $\bigcap_{i\in I}[a_i,\infty)$ is the set of all upper bounds of the a_i, and this set is equal to $[a,\infty)$ if and only if a is the least upper bound. □

An element a in a poset is called **top element** if $b \le a$ for all elements b in the poset, and **bottom element** if $b \ge a$ for all elements b in the poset. A top element is commonly denoted by $\top$, and a bottom element by $\perp$. A poset $(T,<)$ is an **upper semilattice** if every two elements in T have a supremum, and a **complete upper semilattice** if every nonempty set of elements in T has a supremum.

Proposition 5.26. *1)* $(T,\mathcal{B}_{\rm pfs})$ *is finitely intersection closed if and only if every nonempty finite bounded subset of* T *has a supremum.*

2) $(T, \mathcal{B}_{\rm pfs})$ *is intersection closed if and only if every nonempty bounded subset of* T *has a supremum, i.e.,* $(T,<)$ *is bounded complete.*

3) If $(T,<)$ *has a top element, then* $(T,<)$ *is an upper semilattice if and only if* $(T, \mathcal{B}_{\rm pfs})$ *is finitely intersection closed,*

4) $(T,<)$ *is a complete upper semilattice if and only if* $(T,<)$ *has a top element and* $(T, \mathcal{B}_{\rm pfs})$ *is intersection closed.*

Proof: 1), 2): Assume that $(T, \mathcal{B}_{\rm pfs})$ is (finitely) intersection closed and take a nonempty (finite) subset $\{a_i \mid i \in I\}$ of T. If this set is bounded, then $\bigcap_{i\in I}[a_i,\infty)$ is nonempty, and thus by assumption it is equal to $[a,\infty)$ for some $a \in T$. By Lemma 5.25, this implies that $a = \sup_{i\in I} a_i$, showing that $\{a_i \mid i \in I\}$ has a supremum.

Now assume that every nonempty (finite) bounded subset of T has a supremum. Take a nonempty (finite) set $\{[a_i,\infty) \mid i \in I\}$ of balls in $\mathcal{B}_{\rm pfs}$ with nonempty intersection. Take $b \in \bigcap_{i\in I}[a_i,\infty)$. Then b is an upper bound of $\{a_i \mid i \in I\}$. By assumption, there exists $a = \sup_{i\in I} a_i$ in T. Again by Lemma 5.25, this implies that $\bigcap_{i\in I}[a_i,\infty) = [a,\infty)$. Hence, $(T, \mathcal{B}_{\rm pfs})$ is (finitely) intersection closed.

3) and 4) follow from 1) and 2), respectively, because if $(T,<)$ has a top element, then every nonempty subset is bounded. □

We add to our hierarchy (1) an even stronger property: we say that the ball space $(X, \mathcal{B})$ is an $\mathbf{S}^{**}$ ball space if $\mathcal{B}$ is closed under *a*rbitrary intersections; in particular, this implies that intersections of arbitrary collections of balls are nonempty. Every $\mathbf{S}^{**}$ ball space is an $\mathbf{S}^{*}$ ball space. Note that every complete upper semilattice has a top element.

Proposition 5.27. *1) Assume that* $(T,<)$ *has a top element* $\top$. *Then every intersection of balls in* $(T, \mathcal{B}_{\rm pfs})$ *contains the ball* $[\top,\infty)$, *and* $(T, \mathcal{B}_{\rm pfs})$ *is an* $\mathbf{S}_2^c$ *ball space. Moreover,* $(T, \mathcal{B}_{\rm pfs})$ *is an* $\mathbf{S}^{*}$ *ball space if and only if it is an* $\mathbf{S}^{**}$ *ball space.*

2) $(T, \mathcal{B}_{\rm pfs})$ *is an* $\mathbf{S}^{**}$ *ball space if and only if* $(T,<)$ *has a top element and* $(T, \mathcal{B}_{\rm pfs})$ *is intersection closed.*

3) $(T,<)$ *is a complete upper semilattice if and only if* $(T, \mathcal{B}_{\rm pfs})$ *is an* $\mathbf{S}^{**}$ *ball space.*

Proof: 1): The first two statements are obvious. If $(T,<)$ has a top element, then every collection of balls in $\mathcal{B}_{\rm pfs}$ is a centered system. Hence if $(T, \mathcal{B}_{\rm pfs})$ is an $\mathbf{S}^{*}$ ball space, then it is an $\mathbf{S}^{**}$ ball space.

2): Assume that $(T, \mathcal{B}_{\rm pfs})$ is an $\mathbf{S}^{**}$ ball space. Then it follows directly from the definition that it is intersection closed. Further, the intersection over $\{[a,\infty) \mid a \in T\}$ is a ball $[b,\infty)$. By Lemma 5.25, b is the supremum of T and thus a top element.

Now assume that $(T,<)$ has a top element $\top$ and $(T, \mathcal{B}_{\rm pfs})$ is intersection closed, and take an arbitrary collection of balls in $\mathcal{B}_{\rm pfs}$. As all of the balls contain $\top$, their intersection is nonempty, and hence by our assumption, it is a ball.

3): This follows from part 2) of our proposition together with part 4) of Proposition 5.26. □

Now we can characterize chain complete and directed complete posets by properties from our hierarchy:

Theorem 5.28. *Take a poset $(T,<)$. Then the following are equivalent:*

a) $(T,<)$ *is chain complete,*

b) $(T,<)$ *is directed complete,*

c) $(T,\mathcal{B}_{\mathrm{pfs}})$ *is an* $\mathbf{S}_5$ *ball space,*

d) $(T,\mathcal{B}_{\mathrm{pfs}})$ *is an* $\mathbf{S}_5^d$ *ball space.*

If every nonempty finite bounded subset of T has a supremum, then the above properties are also equivalent to

e) $(T,\mathcal{B}_{\mathrm{pfs}})$ *is an* $\mathbf{S}^*$ *ball space.*

Proof: The equivalence of assertions a) and b) follows from Proposition 4.2.

b) $\Rightarrow$ d): Assume that $(T,<)$ is directed complete and take a directed system $S=\{[a_i,\infty)\mid i\in I\}$ in $\mathcal{B}_{\mathrm{pfs}}$. Then also $\{a_i\mid i\in I\}$ is a directed system in $(T,<)$. By our assumption on $(T,<)$ it follows that $\{a_i\mid i\in I\}$ has a supremum a in T. By Lemma 5.25, $[a,\infty)=\bigcap_{i\in I}[a_i,\infty)$, which shows that the intersection of S is a ball.

d) $\Rightarrow$ c) holds by the general properties of the hierarchy.

c) $\Rightarrow$ a): Take a chain $\{a_i\mid i\in I\}$ in T. Since $(T,\mathcal{B}_{\mathrm{pfs}})$ is an $\mathbf{S}_5$ ball space, the intersection of the nest $\mathcal{N}=([a_i,\infty))_{i\in I}$ is a ball $[a,\infty)$. It follows by Lemma 5.25 that a is the supremum of the chain, which proves that $(T,<)$ is chain complete.

If every nonempty finite bounded subset of T has a supremum, then by part 1) of Proposition 5.26, $(T,\mathcal{B}_{\mathrm{pfs}})$ is finitely intersection closed, hence by Proposition 4.7, properties $\mathbf{S}_5^d$ and $\mathbf{S}^*$ are equivalent. $\square$

Remark 5.29. Note that we define chains to be *nonempty* totally ordered sets and similarly, consider directed systems to be nonempty. If we drop this convention, then the theorem remains true if we require in c) and d) that $(T,<)$ has a least element.

The ball space $(T,\mathcal{B}_{\mathrm{pfs}})$ shares an important property with Caristi–Kirk and Oettli–Théra ball spaces:

Proposition 5.30. *The ball space $(T,\mathcal{B}_{\mathrm{pfs}})$ is a normalized strongly contractive B_x-ball space.*

Proof: We define

$$B_x := [x,\infty) \in \mathcal{B}_{\mathrm{pfs}}\,.$$

Then $x\in B_x$ for every $x\in T$. If $y\in B_x$, then $x\le y$ and therefore $[y,\infty)\subseteq[x,\infty)$; if in addition $x\ne y$, then $x<y$ so that $x\notin[y,\infty)$ and $[y,\infty)\subsetneq[x,\infty)$. $\square$

A function f on a poset $(T,<)$ is **increasing** if $f(x)\ge x$ for all $x\in T$. The following result is an immediate consequence of Zorn's Lemma, but can also be seen as a corollary to Propositions 5.24 and 5.30 together with Theorem 2.8:

Theorem 5.31. *Every increasing function $f:X\to X$ on an inductively ordered poset $(T,<)$ has a fixed point.*

Note that this theorem implies the **Bourbaki-Witt Theorem**, which differs from it by assuming that every chain in $(T,<)$ even has a supremum.

A function f on a poset $(T,<)$ is called **order preserving** if $x \leq y$ implies $fx \leq fy$. The following result is an easy consequence of Theorem 5.31:

Theorem 5.32. *Take an order preserving function f on a nonempty poset $(T,<)$ which contains at least one x such that $fx \geq x$ (in particular, this holds when $(T,<)$ has a bottom element). Assume that $(T,<)$ is chain complete. Then f has a fixed point.*

Proof: Take $S := \{x \in T \mid fx \geq x\} \neq \emptyset$. Then also S is chain complete. Indeed, if $(x_i)_{i\in I}$ is a chain in S, hence also in T, then it has a supremum $z \in T$ by assumption. Since $z \geq x_i$ and f is order preserving, we have that $fz \geq fx_i \geq x_i$ for all $i \in I$, so fz is also an upper bound for $(x_i)_{i\in I}$. Therefore, $fz \geq z$ since z is the supremum of the chain, showing that $z \in S$.

Further, S is closed under f, because if $x \in S$, then $fx \geq x$, hence $f^2x \geq fx$ since f is assumed to be order preserving; this shows that $fx \in S$ Now the existence of a fixed point follows from Theorem 5.31. □

The second ball space we associate with posets will be particularly useful for the study of lattices. We define the **principal segment ball space** $(T,\mathcal{B}_{\rm ps})$ of the poset $(T,<)$ by taking $\mathcal{B}_{\rm ps}$ to contain all **principal segments** (which may also be called "closed convex subsets"), that is, the closed intervals $[a,b] := \{c \in T \mid a \leq c \leq b\}$ for $a,b \in T$ with $a \leq b$, the principal initial and final segments $\{c \in T \mid c \leq a\}$ and $\{c \in T \mid a \leq c\}$ for $a \in T$, and T itself. Note that all of these sets are of the form $[a,b]$ if and only if T has a top element $\top$ and a bottom element $\bot$. Even if T does not have these elements, we will still use the notation $[\bot,b]$ for $\{c \in T \mid c \leq b\}$ and $[a,\top]$ for $\{c \in T \mid a \leq c\}$. Hence,

$$\mathcal{B}_{\rm ps} = \{[a,b] \mid a \in T \cup \{\bot\},\, b \in T \cup \{\top\}\}\,.$$

If $\bot,\top \in T$ (as is the case for complete lattices), this is a generalization to posets of the closed interval ball space $\mathcal{B}_{\rm int}$ that we defined for totally ordered sets. We will thus also talk again of "closed intervals" $[a,b]$.

A greatest lower bound of a subset S of T will also be called its **infimum**. If $\{a_i \mid i \in I\}$ is a subset of T, then $\inf_{i\in I} a_i$ will denote its infimum, if it exists.

Lemma 5.33. *Take subsets $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ of T such that $a_i \leq b_j$ for all $i,j \in I$. If $a = \sup_{i\in I} a_i$ and $b = \inf_{i\in I} b_i$ exist, then $a \leq b$ and*

$$[a,b] \;=\; \bigcap_{i\in I}[a_i,b_i]\,.$$

Proof: We can write

$$\bigcap_{i\in I}[a_i,b_i] \;=\; \bigcap_{i\in I}([a_i,\top] \cap [\bot,b_i]) \;=\; \bigcap_{i\in I}[a_i,\top] \,\cap\, \bigcap_{i\in I}[\bot,b_i]$$

Applying Lemma 5.25, we obtain that $[a,\top] = \bigcap_{i\in I}[a_i,\top]$, and applying it to L with the reverse order, we obtain that $[\bot,b] = \bigcap_{i\in I}[\bot,b_i]$. Hence the above intersection is equal to $[a,b]$, which we will now show to be nonempty.

By the assumption of our lemma, every b_j is an upper bound of the set $\{a_i \mid i \in I\}$. Since a is the least upper bound of this set, we find that $a \leq b_i$ for all $i \in I$. As b is the greatest lower bound of the set $\{b_i \mid i \in I\}$, it follows that $a \leq b$. □

5.7. **Lattices.**

A **lattice** is a poset in which every two elements have a supremum and an infimum (greatest lower bound). It then follows that all finite sets in a lattice $(L,<)$ have a supremum and an infimum. A **complete lattice** is a poset in which all nonempty sets have a supremum and an infimum. Lemma 5.33 implies the following analogue to Proposition 5.26:

Proposition 5.34. *The ball space $(L,\mathcal{B}_{\mathrm{ps}})$ associated to a lattice $(L,<)$ is finitely intersection closed. The ball space $(L,\mathcal{B}_{\mathrm{ps}})$ associated to a complete lattice $(L,<)$ is intersection closed.*

For a lattice $(L,<)$, we denote by $(L,>)$ the lattice endowed with the reverse order. We will now characterize complete lattices by properties from our hierarchy.

Theorem 5.35. *For a poset $(L,<)$, the following assertions are equivalent.*

a) $(L,<)$ is a complete lattice,

b) $(L,<)$ and $(L,>)$ are complete upper semilattices,

c) the principal final segments of $(L,<)$ and of $(L,>)$ form $\mathbf{S}^{**}$ ball spaces,

d) $(L,\mathcal{B}_{\mathrm{ps}})$ is an $\mathbf{S}^*$ ball space and $(L,<)$ admits a top and a bottom element,

e) $(L,\mathcal{B}_{\mathrm{ps}})$ is an $\mathbf{S}^*$ ball space and every finite set in $(L,<)$ has an upper and a lower bound.

Proof: The equivalence of a) and b) follows directly from the definitions. The equivalence of b) and c) follows from part 3) of Proposition 5.27.

a) $\Rightarrow$ d): Assume that $(L,<)$ is a complete lattice. Then it admits a top element (supremum of all its elements) and a bottom element (infimum of all its elements). Take a centered system $\{[a_i,b_i] \mid i \in I\}$ in $(L,\mathcal{B}_{\mathrm{ps}})$. Then for all $i,j \in I$, $[a_i,b_i] \cap [a_j,b_j] \neq \emptyset$, so $a_i \leq b_j$. Since $(L,<)$ is a complete lattice, $a := \sup_{i\in I} a_i$ and $b := \inf_{i\in I} b_i$ exist. From Lemma 5.33 it follows that $\bigcap_{i\in I}[a_i,b_i] = [a,b] \neq \emptyset$, which consequently is a ball in $\mathcal{B}_{\mathrm{ps}}$. We have proved that $(L,\mathcal{B}_{\mathrm{ps}})$ is an $\mathbf{S}^*$ ball space.

d) $\Rightarrow$ e): A top element is an upper bound and a bottom element a lower bound for every set of elements.

e) $\Rightarrow$ a): Take a poset $(L,<)$ that satisfies the assumptions of e), and any subset $S \subseteq L$. If S_0 is a finite subset of S, then it has an upper bound b by assumption. Hence the balls $[a,\top]$, $a \in S_0$, have a nonempty intersection, as it contains b. This shows that $\{[a,\top] \mid a \in S\}$ is a centered system of balls. Since $(L,\mathcal{B}_{\mathrm{ps}})$ is an $\mathbf{S}^*$ ball space, its intersection is a ball $[c,d]$, where we must have $d = \top$. By Lemma 5.25, c is the supremum of S.

Working with the reverse order, one similarly shows that S has an infimum since $(L,\mathcal{B}_{\mathrm{ps}})$ is an $\mathbf{S}^*$ ball space. Hence, $(L,<)$ is a complete lattice. □

For our next theorem, we will need one further lemma:

Lemma 5.36. *For a lattice $(L,<)$, the following are equivalent:*
a) $(L,<)$ *is a complete lattice,*
b) $(L,<)$ *and* $(L,>)$ *are directed complete posets,*
c) $(L,<)$ *and* $(L,>)$ *are chain complete posets.*

Proof: The implication a) $\Rightarrow$ b) is trivial as every nonempty set in a complete lattice has a supremum and an infimum.
b) $\Rightarrow$ a): Take a nonempty subset S of L. Let S' be the closure of S under suprema and infima of arbitrary finite subsets of S. Then S' is a directed system in both $(L,<)$ and $(L,>)$. Hence by b), S' has an infimum a and a supremum b. These are lower and upper bounds, respectively, for S. Suppose there was an upper bound $c<b$ for S. Then there would be a supremum d of some finite subset of S such that $d>c$. But as c is also an upper bound of this finite subset, we must have that $d\leq c$. This contradiction shows that b is also the supremum of S. Similarly, one shows that a is also the infimum of S. This proves that $(L,<)$ is a complete lattice.
b) $\Leftrightarrow$ c) follows from Proposition 4.2. □

Now we can prove:

Theorem 5.37. *For a lattice $(L,<)$, the following are equivalent:*
a) $(L,<)$ *is a complete lattice,*
b) $(L,\mathcal{B}_{\rm ps})$ *is an* $\mathbf{S}_5$ *ball space,*
c) $(L,\mathcal{B}_{\rm ps})$ *is an* $\mathbf{S}^*$ *ball space.*

Proof: a) $\Rightarrow$ c): This follows from Theorem 5.35.
c) $\Rightarrow$ b) holds by the general properties of the hierarchy.
b) $\Rightarrow$ a): By Lemma 5.36 it suffices to prove that $(L,<)$ and $(L,>)$ are chain complete posets. Take a chain $\{a_i\mid i\in I\}$ in $(L,<)$. Then $\{[a_i,\top]\mid i\in I\}$ is a nest of balls in $(L,\mathcal{B}_{\rm ps})$. Since $(L,\mathcal{B}_{\rm ps})$ is an $\mathbf{S}_5$ ball space, the intersection of this nest is a ball $[a,b]$ for some $a,b\in L$; it must be of the form $[a,\top]$ since the intersection contains $\top$. From Lemma 5.25 we infer that $a=\sup_{i\in I}a_i$. This shows that $(L,<)$ is a chain complete poset. The proof for $(L,>)$ is similar. □

An example for a fixed point theorem that holds in complete lattices is the Knaster–Tarski Theorem, which we will discuss in Section 7.

6. Sub-ball spaces and spherical closures

In this section we will study the relation between ball spaces $(X,\mathcal{B})$ and ball spaces on subsets Y of X. This is important for the Knaster–Tarski type theorems that we will prove in Section 7. First, we will introduce two particular cases of a good relation between the two ball spaces. Thereafter, we will consider ball spaces induced on subsets of $\mathbf{S}^*$ ball spaces.

6.1. **Induced and liftable sub-ball spaces.**

Take a ball space $(X,\mathcal{B}_X)$ and a subset $Y\subseteq X$. Basically, there are two different ways to obtain a related ball space $\mathcal{B}_Y$ on Y. Following the concept of subspace topology, we can define a ball space on Y by intersecting the balls of $\mathcal{B}$ with Y. However, in the case where the ball space on X is associated with some given structure on X (like metric, ultrametric, ordering), we prefer to consider Y with the restriction of this structure and then associate a ball space $(Y,\mathcal{B}_Y)$ to it. That ball space may not be induced in the way we have discussed before (think of closed intervals that may lose their endpoints when intersected with a subset). Nevertheless, we will hope for a somewhat "tight" relation between the two ball spaces. One such relation will be captured in the definition of "liftable sub-ball space".

Assume first that there is at least one ball $B\in\mathcal{B}_X$ such that $Y\cap B\neq\emptyset$. Then with

$$\mathcal{B}_X\cap Y \;:=\; \{B\cap Y\mid B\in\mathcal{B}_X\}\setminus\{\emptyset\}\,,$$

$(Y,\mathcal{B}_X\cap Y)$ is a ball space. We call it the **induced sub-ball space** on Y, induced by the ball space of X. For example, if X is equipped with a topology, then the subspace topology on Y can be obtained by taking the intersections with Y of the closed subsets of X to be the closed subsets of Y. Hence the ball space on Y associated with the induced subspace topology of Y is the sub-ball space on Y induced by the ball space on X which is associated with the topology of X.

Now assume that a ball space $\mathcal{B}_Y$ on Y is already given. We then call $(Y,\mathcal{B}_Y)$ a **liftable sub-ball space** of $(X,\mathcal{B}_X)$ if there is an assignment $\mathcal{B}_Y\ni B\mapsto B^X\in\mathcal{B}_X$ such that

(LSB1) for each $B\in\mathcal{B}_Y$ we have that $B^X\cap Y=B$,

(LSB2) for all $B_1,B_2\in\mathcal{B}_Y$, $B_1\subseteq B_2\Rightarrow B_1^X\subseteq B_2^X$.

Note that for every $B\in\mathcal{B}_Y$, the equality $B^X\cap Y=B$ implies that

$$B\subseteq B^X\,.$$

As an example, take (X,u) to be an ultrametric space and $(X,\mathcal{B}_{[u]})$ its associated ball space consisting of all precise balls. Then the restriction of u to Y renders the ultrametric space (Y,u). Denote by $(Y,\mathcal{B}_{Y,[u]})$ its associated ball space of precise balls. Then in general, there will exist $a,b\in X$ such that $B(a,b)\cap Y$ is not a precise ball. Then $(Y,\mathcal{B}_{Y,[u]})$ will not be an induced sub-ball space. But the following holds:

Lemma 6.1. *For every ultrametric space (X,u) and every nonempty subset $Y\subseteq X$, the precise ball space $(Y,\mathcal{B}_{Y,[u]})$ is a liftable sub-ball space of $(X,\mathcal{B}_{[u]})$, where for each $B(a,b)\in\mathcal{B}_{Y,[u]}$ we take $B(a,b)^X$ to be the precise ball generated by a,b in (X,u).*

Proof: It is clear that $B(a,b)^X\cap Y=B(a,b)$. If $B(a,b)\subseteq B(c,d)$ for $c,d\in Y$, then $a,b\in B(c,d)^X$ and hence $B(a,b)^X\subseteq B(c,d)^X$ by (8). $\square$

We will now study how sub-ball spaces can inherit properties of spherical completeness from $(X,\mathcal{B}_X)$. For this it is important to know whether nests of balls (or directed or centered systems) in $(Y,\mathcal{B}\cap Y)$ can be lifted to nests of balls (or directed or centered systems, respectively) in $(X,\mathcal{B}_X)$.

Lemma 6.2. *Assume that conditions (LSB1) and (LSB2) are satisfied.*

1) The assignment $\mathcal{B}_Y\ni B\mapsto B^X\in\mathcal{B}_X$ preserves inclusion in the strong sense that

$$B_1\subseteq B_2\;\Leftrightarrow\;B_1^X\subseteq B_2^X\quad\text{and}\quad B_1\neq B_2\;\Leftrightarrow\;B_1^X\neq B_2^X\,. \tag{17}$$

2) Take any collection $(B_i)_{i\in I}$ of balls in $\mathcal{B}_Y$. Then

$$\bigcap_{i\in I}B_i\;=\;\left(\bigcap_{i\in I}B_i^X\right)\cap Y\,. \tag{18}$$

3) If $(B_i)_{i\in I}$ is a nest of balls (or directed or centered system) in $(Y,\mathcal{B}_Y)$, then also $(B_i^X)_{i\in I}$ is a nest of balls (or directed or centered system, respectively) in $(X,\mathcal{B}_X)$.

Proof: 1): The "$\Rightarrow$" directions in (17) follow from (LSB2) and (LSB1), respectively. The "$\Leftarrow$" direction in the first statement of (17) follows from (LSB1), and the "$\Leftarrow$" direction in the second statement holds because to each $B\in\mathcal{B}_Y$ exactly one B^X is assigned.
2): Equation (18) holds since by (LSB1),

$$\bigcap_{i\in I} B_i \;=\; \bigcap_{i\in I}(B_i^X\cap Y) \;=\; \left(\bigcap_{i\in I} B_i^X\right)\cap Y\,.$$

3): If $(B_i)_{i\in I}$ is a nest of balls, then also $(B_i^X)_{i\in I}$ is totally ordered by inclusion by (LSB1), and hence a nest of balls.

If $(B_i)_{i\in I}$ is a centered system, then for all choices of $i_1,\ldots,i_n\in I$ we have that $\emptyset\neq B_{i_1}\cap\ldots\cap B_{i_n}\subseteq B_{i_1}^X\cap\ldots\cap B_{i_n}^X$; hence also $(B_i^X)_{i\in I}$ is a centered system. A similar proof works for directed systems. □

Using this lemma, we prove:

Proposition 6.3. *Assume that $(Y,\mathcal{B}_Y)$ is a liftable sub-ball space of $(X,\mathcal{B}_X)$ such that each ball in $\mathcal{B}_X$ has nonempty intersection with Y. If $(X,\mathcal{B}_X)$ is an $\mathbf{S}_2$ (or $\mathbf{S}_2^d$ or $\mathbf{S}_2^c$) ball space, then $(Y,\mathcal{B}_Y)$ is an $\mathbf{S}_1$ (or $\mathbf{S}_1^d$ or $\mathbf{S}_1^c$, respectively) ball space.*

Proof: Take a nest of balls $(B_i)_{i\in I}$ in $\mathcal{B}_Y$. Then by the previous lemma, $(B_i^X)_{i\in I}$ is a nest of balls in $\mathcal{B}_X$. By assumption, its intersection contains a ball B, and the intersection of this ball with Y is nonempty. In view of (18), this proves that $(Y,\mathcal{B}_Y)$ is an $\mathbf{S}_1$ ball space. The same proof works for the assertion with $\mathbf{S}_2^d$ and $\mathbf{S}_2^c$ in place of $\mathbf{S}_2$. □

In general, induced sub-ball spaces will not be liftable, and vice versa. However, we will show in the next section that sub-ball spaces induced by $\mathbf{S}^*$ ball spaces are always liftable. We will use this fact in Section 7 for the proof of a generic Knaster–Tarski Theorem for ball spaces.

6.2. Spherical closures in $\mathbf{S}^*$ ball spaces.

Throughout this section, we consider an $\mathbf{S}^*$ ball space $(X,\mathcal{B}_X)$. As before, if $f:X\to X$ is a function, then $\mathcal{B}_X^f$ will denote the collection of all f-closed balls in $\mathcal{B}_X$. The following is a simple but useful observation. It follows from the fact that the intersection over any collection of f-closed sets is again f-closed, see part 2) of Lemma 3.10.

Lemma 6.4. *Also $(X,\mathcal{B}_X^f)$ is an $\mathbf{S}^*$ ball space, provided that $\mathcal{B}_X^f\neq\emptyset$.*

For every nonempty subset S of some ball in $\mathcal{B}_X$, we define

$$\mathrm{scl}_{\mathcal{B}_X}(S) \;:=\; \bigcap\{B\in\mathcal{B}_X\mid S\subseteq B\}$$

and call it the **(spherical) closure** of S in $\mathcal{B}_X$.

Lemma 6.5. *1) For every nonempty subset S of some ball in $\mathcal{B}_X$, $\mathrm{scl}_{\mathcal{B}_X}(S)$ is the smallest ball in $\mathcal{B}_X$ containing S.*
2) If $f:X\to X$ is a function, then for every nonempty subset S of some f-closed ball in $\mathcal{B}_X$, $\mathrm{scl}_{\mathcal{B}_X^f}(S)$ is the smallest f-closed ball containing S.

Proof: 1) The collection of all balls containing S is nonempty by our condition that S is a subset of a ball in $\mathcal{B}_X$. The intersection of this collection contains $S \neq \emptyset$, so it is a centered system, and since $(X, \mathcal{B}_X)$ is $\mathbf{S}^*$, its intersection is a ball. As all balls containing S appear in the system, the intersection must be the smallest ball containing S.

2) This follows from part 1) together with Lemma 6.4. □

Note that if $X \in \mathcal{B}_X$, then we can drop the condition that S is the subset of some ball (or some f-closed ball, respectively) in $\mathcal{B}_X$.

Remark 6.6. The ball B_x defined in (16) in the proof of Theorem 5.20 is equal to $\mathrm{scl}_{\mathcal{B}^f}(\{x\})$, where $\mathcal{B}_X^f$ is the set of all closed f-closed sets of the topological space under consideration.

The proof of the following observation is straightforward:

Lemma 6.7. *If $S \subseteq T$ are nonempty subsets of a ball in $\mathcal{B}_X$, then* $\mathrm{scl}_{\mathcal{B}}(S) \subseteq \mathrm{scl}_{\mathcal{B}_X}(T)$.

Now we take a subset Y of X, assume that $\mathcal{B}_Y := \mathcal{B}_X \cap Y \neq \emptyset$ and consider the induced ball space $(Y, \mathcal{B}_Y)$. We show that it is liftable:

Proposition 6.8. *The assignment $\mathcal{B}_Y \ni B \mapsto B^X := \mathrm{scl}_{\mathcal{B}_X}(B) \in \mathcal{B}_X$ satisfies conditions (LSB1) and (LSB2) and therefore, the induced sub-ball space $(Y, \mathcal{B}_Y)$ is liftable. Further, $\mathrm{scl}_{\mathcal{B}_X}(B)$ is the smallest ball in $\mathcal{B}_X$ that induces the ball $B \in \mathcal{B}_Y$.*

Proof: Since $B \in \mathcal{B}_Y = \mathcal{B}_X \cap Y$, there is some $B' \in \mathcal{B}_X$ such that $B = B' \cap Y \subseteq B'$ and therefore, $B^X = \mathrm{scl}_{\mathcal{B}_X}(B)$ is defined. It follows from the definition of $\mathrm{scl}_{\mathcal{B}_X}(B)$ that $B \subseteq \mathrm{scl}_{\mathcal{B}_X}(B)$, so $B \subseteq B^X \cap Y$. Since $\mathrm{scl}_{\mathcal{B}_X}(B)$ is the smallest ball in X containing B, it must be contained in B' and therefore, $B^X \cap Y = \mathrm{scl}_{\mathcal{B}_X}(B) \cap Y \subseteq B' \cap Y = B$. We have proved that B^X satisfies (LSB1) and that B^X is the smallest ball in $\mathcal{B}_X$ that induces B. By virtue of Lemma 6.7, also condition (LSB2) is satisfied. □

With the help of this proposition, we obtain:

Proposition 6.9. *Assume that $B \cap Y \neq \emptyset$ for every $B \in \mathcal{B}_X$. Then also $(Y, \mathcal{B}_X \cap Y)$ is an $\mathbf{S}^*$ ball space.*

Proof: Take a centered system of balls $(B_i)_{i\in\mathbb{N}}$ in $(Y, \mathcal{B}_X \cap Y)$. Then by Proposition 6.8 and part 3) of Lemma 6.2, $(\mathrm{scl}_{\mathcal{B}_X}(B_i))_{i\in\mathbb{N}}$ is a centered system of balls in $(X, \mathcal{B}_X)$ with $\bigcap_{i\in I} B_i = \left(\bigcap_{i\in I} \mathrm{scl}_{\mathcal{B}_X}(B_i)\right) \cap Y$. Since $(X, \mathcal{B}_X)$ is assumed to be $\mathbf{S}^*$, $\bigcap_{i\in I} \mathrm{scl}_{\mathcal{B}_X}(B_i)$ is a ball in $\mathcal{B}_X$. Therefore, $\bigcap_{i\in I} B_i = \left(\bigcap_{i\in I} \mathrm{scl}_{\mathcal{B}_X}(B_i)\right) \cap Y \neq \emptyset$ is a ball in $\mathcal{B}_X \cap Y$. □

In the special case considered in Section 7, the set Y is taken to be the set $\mathrm{Fix}(f)$ of fixed points of a given function $f : X \to X$. If $(X, \mathcal{B}_X)$ is an $\mathbf{S}^*$ ball space with $\mathcal{B}_X^f \neq \emptyset$ and every f-closed ball contains a fixed point, then it follows from Lemma 6.4 together with Proposition 6.9 that also

$$(\mathrm{Fix}(f), \mathcal{B}_X^f \cap \mathrm{Fix}(f))$$

is an $\mathbf{S}^*$ ball space. However, we are more interested in the possibly finer ball space

$$(\mathrm{Fix}(f), \mathcal{B}_X \cap \mathrm{Fix}(f)) .$$

The following proposition gives a criterion for the two ball spaces to be equal:

Proposition 6.10. *Take a function $f : X \to X$. If $\mathcal{B}_X \cap \mathrm{Fix}(f) \neq \emptyset$ and $B \in \mathcal{B}_X \cap \mathrm{Fix}(f)$ is such that $\mathrm{scl}_{\mathcal{B}_X}(B)$ is f-closed, then*

$$\mathrm{scl}_{\mathcal{B}_X}(B) \;=\; \mathrm{scl}_{\mathcal{B}_X^f}(B)\,. \tag{19}$$

If this holds for every $B \in \mathcal{B}_X \cap \mathrm{Fix}(f)$, then

$$\mathcal{B}_X^f \cap \mathrm{Fix}(f) \;=\; \mathcal{B}_X \cap \mathrm{Fix}(f)\,. \tag{20}$$

Proof: Pick $B \in \mathcal{B}_X \cap \mathrm{Fix}(f)$. By part 1) of Lemma 6.5, $\mathrm{scl}_{\mathcal{B}_X}(B)$ is the smallest of all balls in $\mathcal{B}_X$ that contain B. Consequently, if $\mathrm{scl}_{\mathcal{B}_X}(B)$ is f-closed, then it is also the smallest of all balls in $\mathcal{B}_X^f$ that contain B. Then by part 2) of Lemma 6.5, it must be equal to $\mathrm{scl}_{\mathcal{B}_X^f}(B)$.

Since $B = \mathrm{scl}_{\mathcal{B}_X}(B) \cap \mathrm{Fix}(f)$ by Proposition 6.8, the equality (19) implies that $B = \mathrm{scl}_{\mathcal{B}_X^f}(B) \cap \mathrm{Fix}(f) \in \mathcal{B}_X^f \cap \mathrm{Fix}(f)$. If the equality (19) holds for all $B \in \mathcal{B}_X \cap \mathrm{Fix}(f)$, then this implies the inclusion "$\supseteq$" in (20). The converse inclusion follows from the fact that $\mathcal{B}_X^f \subseteq \mathcal{B}_X$. □

Corollary 6.11. *Take a function $f : X \to X$. Assume that $f^{-1}(B) \in \mathcal{B}_X$ for every $B \in \mathcal{B}_X$ that contains a fixed point. Then (20) holds.*

Proof: Pick $B_0 \in \mathcal{B}_X \cap \mathrm{Fix}(f)$. Since $B := \mathrm{scl}_{\mathcal{B}_X}(B_0) \in \mathcal{B}_X$, we have by assumption that $f^{-1}(B) \in \mathcal{B}_X$. All fixed points contained in B are also contained in $f^{-1}(B)$, hence $B_0 \subseteq f^{-1}(B)$. As B is the smallest ball in $\mathcal{B}_X$ containing B_0, it follows that $B \subseteq f^{-1}(B)$ and thus $f(B) \subseteq f(f^{-1}(B)) \subseteq B$, i.e., B is f-closed. Hence by Proposition 6.10, (19) holds for arbitrary balls $B_0 \in \mathcal{B}_X \cap \mathrm{Fix}(f)$, which implies that (20) holds. □

7. Knaster–Tarski type theorems

7.1. **An analogue of the Knaster–Tarski Theorem for ball spaces.**

In 1927 B. Knaster and A. Tarski proved a set-theoretical fixed point theorem by which every inclusion preserving function on the family of all subsets of a given set has a fixed point. It can be used to prove the Cantor–Bernstein–Schröder Theorem (see, e.g., [29]). In 1955 Tarski generalized the result to the lattice-theoretical fixed point theorem which is now known as the **Knaster–Tarski Theorem** (cf. [32, Theorem 1]). It states:

Theorem 7.1. *Let L be a complete lattice and $f : L \to L$ an order-preserving function. Then the set $\mathrm{Fix}(f)$ of fixed points of f in L is also a complete lattice.*

We prove an analogue for ball spaces $(X, \mathcal{B})$ with a function $f : X \to X$. We call f **ball continuous** if for every ball B, also $f^{-1}(B)$ is a ball. As before, $\mathcal{B}^f$ will denote the collection of all f-closed balls in $\mathcal{B}$.

Theorem 7.2. *Take an $\mathbf{S}^*$ ball space $(X, \mathcal{B})$ and a function $f : X \to X$.*

1) Assume that every ball in $\mathcal{B}$ contains a fixed point or a smaller ball. Then every ball in $\mathcal{B}$ contains a fixed point, and $(\mathrm{Fix}(f), \mathcal{B} \cap \mathrm{Fix}(f))$ is an $\mathbf{S}^*$ ball space.

2) Assume that $\mathcal{B}$ contains an f-closed ball and every f-closed ball in $\mathcal{B}$ contains a fixed point or a smaller f-closed ball. Then every f-closed ball in $\mathcal{B}$ contains a fixed point, and $(\mathrm{Fix}(f), \mathcal{B}^f \cap \mathrm{Fix}(f))$ is an $\mathbf{S}^*$ ball space.

If in addition f is ball continuous, then $(\mathrm{Fix}(f), \mathcal{B}\cap \mathrm{Fix}(f))$ *is an* $\mathbf{S}^*$ *ball space.*

Proof: 1): It follows from our assumptions together with Theorem 2.3 that every $B\in\mathcal{B}$ contains a fixed point, that is, $B\cap\mathrm{Fix}(f)\neq\emptyset$. From Proposition 6.9 it follows that $(\mathrm{Fix}(f), \mathcal{B}^f\cap\mathrm{Fix}(f))$ is an $\mathbf{S}^*$ ball space.
2): By Lemma 6.4, $(X,\mathcal{B}^f)$ is an $\mathbf{S}^*$ ball space. Hence it follows from our assumptions together with part 1) of our theorem, applied to $\mathcal{B}^f$ in place of $\mathcal{B}$, that every f-closed ball B in $\mathcal{B}$ contains a fixed point and that $(\mathrm{Fix}(f), \mathcal{B}^f\cap\mathrm{Fix}(f))$ is an $\mathbf{S}^*$ ball space.
If in addition f is ball continuous, then Proposition 6.10 yields the equality $\mathcal{B}^f\cap\mathrm{Fix}(f)=\mathcal{B}\cap\mathrm{Fix}(f)$, which proves our last statement. □

In what follows, we will discuss some applications.

7.2. **The case of lattices.**

We show how to derive Theorem 7.1 from Theorem 7.2. We take a complete lattice $(L,<)$. By Theorem 5.37, the associated ball space $(L,\mathcal{B}_{\mathrm{ps}})$ is $\mathbf{S}^*$. Take an order-preserving function $f: L\to L$ and consider the set $\mathcal{B}_{\mathrm{ps}}^f$ of all f-closed balls in $\mathcal{B}_{\mathrm{ps}}$. It is nonempty since it contains $L=[\bot,\top]$. By Lemma 6.4, also $(L,\mathcal{B}_{\mathrm{ps}}^f)$ is $\mathbf{S}^*$.

Take an f-closed interval $[a,b]$. Since f is order preserving, it follows that $a\leq fa\leq fb\leq b$, and then $fa\leq f^2a\leq f^2b\leq fb$. If $fa=a$ or $fb=b$, then the interval contains a fixed point. If $fa\neq a$ or $fb\neq b$, then $[fa,fb]$ is an interval that is properly contained in $[a,b]$. It is also f-closed: if $c\in[fa,fb]$, then $f^2a\leq fc\leq f^2b$, whence $fc\in[f^2a,f^2b]\subseteq[fa,fb]$. We have shown that the assumptions of Theorem 7.2 hold, so we obtain that $(\mathrm{Fix}(f),\mathcal{B}_{\mathrm{ps}}^f\cap\mathrm{Fix}(f))$ is an $\mathbf{S}^*$ ball space.

Next, we show that $\mathcal{B}_{\mathrm{ps}}^f\cap\mathrm{Fix}(f)$ is exactly the set of all closed intervals $[a,b]_{\mathrm{Fix}(f)}$ in the poset $\mathrm{Fix}(f)$. Indeed, if a,b are fixed points, then $[a,b]$ is an f-closed interval in L with $[a,b]_{\mathrm{Fix}(f)}=[a,b]\cap\mathrm{Fix}(f)\in\mathcal{B}_{\mathrm{ps}}^f\cap\mathrm{Fix}(f)$. For the converse, take any $B\in\mathcal{B}_{\mathrm{ps}}^f$. By Lemma 6.8, $B\cap\mathrm{Fix}(f)=\mathrm{scl}_{\mathcal{B}_{\mathrm{ps}}^f}(B\cap\mathrm{Fix}(f))\cap\mathrm{Fix}(f)$. Now $\mathrm{scl}_{\mathcal{B}_{\mathrm{ps}}^f}(B\cap\mathrm{Fix}(f))$ is an f-closed interval $[a,b]$ in L. If a or b is not a fixed point, then $[f(a),f(b)]$ is an f-closed interval properly contained in $[a,b]$. But as it also contains the set $B\cap\mathrm{Fix}(f)$ which consists of fixed points, this is a contradiction to the minimality of the spherical closure. Hence, a,b are fixed points, and it follows that $[a,b]\cap\mathrm{Fix}(f)=[a,b]_{\mathrm{Fix}(f)}$.

We have shown that the closed interval ball space on $\mathrm{Fix}(f)$ is an $\mathbf{S}^*$ ball space. Applying what we have shown above to $B=L$ shows the existence of fixed points a,b such that $\mathrm{Fix}(f)=L\cap\mathrm{Fix}(f)=[a,b]_{\mathrm{Fix}(f)}$, i.e., $\mathrm{Fix}(f)$ has bottom element a and top element b. It now follows from Theorem 5.35 that $\mathrm{Fix}(f)$ is a complete lattice.

7.3. **The ultrametric case.**

Take a classical ultrametric space (X,u). A function $f: X\to X$ is called **nonexpanding** if $u(fx,fy)\leq u(x,y)$ for all $x,y\in X$. Further, f is called **contracting on orbits** if $u(fx,f^2x)<u(x,fx)$ for all $x\in X$ such that $x\neq fx$.

The following is the analogue of the Knaster–Tarski Theorem for ultrametric spaces; it extends the Ultrametric Banach Fixed Point Theorem that was first proved in [23].

Theorem 7.3. *Take a nonempty spherically complete classical ultrametric space (X,u) and a nonexpanding function $f : X \to X$ which is contracting on orbits. Then every f-closed ultrametric ball contains a fixed point and $(\mathrm{Fix}(f),u)$ is again a nonempty spherically complete ultrametric space.*

For the proof, we need the following auxiliary result:

Lemma 7.4. *Assume that $f : X \to X$ is nonexpanding and that $fx \in B(x,y)$. Then $B(x,y)$ is f-closed. In particular, every ball $B(x,fx)$ is f-closed, and the same holds for every $B(x,y)$ when x is a fixed point of f.*

Proof: Take $z \in B(x,y)$. Then $u(x,z) \le u(x,y)$ and since f is nonexpanding, $u(fx,fz) \le u(x,z) \le u(x,y)$. Since $fx \in B(x,y)$, we also have that $u(x,fx) \le u(x,y)$. By the ultrametric triangle law, this yields that $u(x,fz) \le \max\{u(x,fx),u(fx,fz)\} \le u(x,y)$, whence $fz \in B(x,y)$. □

Proof of Theorem 7.3: We set $B_x := B(x,fx)$ and let $\mathcal{B}$ be the B_x–ball space $\{B_x \mid x \in X\}$. Lemma 7.4 shows that $\mathcal{B} \subseteq \mathcal{B}^f_{[u]}$. Since (X,u) is assumed to be spherically complete, Proposition 5.3 shows that the ball space $(X,\mathcal{B}_{[u]})$ is spherically complete. Hence so are $(X,\mathcal{B}^f_{[u]})$ and $(X,\mathcal{B})$ because $\mathcal{B} \subseteq \mathcal{B}^f_{[u]} \subseteq \mathcal{B}_{[u]}$. What is more, $(X,\mathcal{B}^f_{[u]})$ is even an $\mathbf{S}_2$ ball space, as we will show now. Take any $B \in \mathcal{B}^f_{[u]}$ and $x \in B$. Then also $fx \in B$, hence by (9), $B(x,fx) \subseteq B$. If $\mathcal{N}$ is a nest in $\mathcal{B}^f_{[u]}$, then $\bigcap\mathcal{N} \neq \emptyset$ and we may pick some $x \in \bigcap\mathcal{N}$. Since $x \in B$ for every $B \in \mathcal{N}$, it follows that $B(x,fx) \subseteq B$ and consequently, $B(x,fx) \subseteq \bigcap\mathcal{N}$. This proves our claim.

We wish to show that $(X,\mathcal{B})$ satisfies the conditions of Theorem 2.7. By definition, (NB) holds. Since also $fx \in B_x$ and $u(fx,f^2x) < u(x,fx)$ by assumption, we infer from (10) that $B_{fx} = B(fx,f^2x) \subsetneq B(x,fx) = B_x$. Hence (CO) holds. Now take any $y \in B_x$. Then Lemma 7.4 shows that $fy \in B_x$, which by (7) shows that $B_y = B(y,fy) \subseteq B(x,fx) = B_x$. Hence also condition (C1) is satisfied. It thus follows from Theorem 2.7 that f has a fixed point in every B_x. In particular, $\mathrm{Fix}(f) \neq \emptyset$.

Take any $B \in \mathcal{B}^f_{[u]}$ and $x \in B$. Then as shown above, $B_x = B(x,fx) \subseteq B$. By what we have already proved, B_x contains a fixed point; consequently, so does B.

Take any $x,y \in \mathrm{Fix}(f)$. Then by Lemma 7.4, $B(x,y) \in \mathcal{B}^f_{[u]}$. Since $B(x,y) \cap \mathrm{Fix}(f)$ equals the precise ball $B(x,y)_{\mathrm{Fix}(f)}$ generated by x and y in the precise ball space of $(\mathrm{Fix}(f),u)$, this shows that (LSB1) holds. If $B(z,t)_{\mathrm{Fix}(f)} \subseteq B(x,y)_{\mathrm{Fix}(f)}$, then $z,t \in B(x,y)$ and therefore, $B(z,t) \subseteq B(x,y)$ by (8). This shows that also (LSB2) holds. Therefore, the precise ball space of $(\mathrm{Fix}(f),u)$ is a liftable sub-ball space of $(X,\mathcal{B}^f_{[u]})$. It now follows from Proposition 6.3 that this ball space is spherically complete. By Proposition 5.3, this proves that $(\mathrm{Fix}(f),u)$ is spherically complete. □

Working with $\mathcal{B}_{u+}$ in place of $B_{[u]}$, we can employ Theorem 7.2 too. The proof runs along the same lines as the above, using Theorem 5.4 and Lemma 6.4 in addition to Proposition 5.3 to deduce that $(X,\mathcal{B}^f_{u+})$ is an $\mathbf{S}^*$ ball space. Once we have obtained that every $B \in \mathcal{B}^f_{u+}$ contains a fixed point, one applies Proposition 6.9 and it remains to show:

Lemma 7.5. *The ball space* $(\mathrm{Fix}(f), \mathcal{B}_{u+}^f \cap \mathrm{Fix}(f))$ *is equal to the full ultrametric ball space of* $(\mathrm{Fix}(f), u)$.

Proof: For $x, y \in \mathrm{Fix}(f)$, denote by $B_F(x, y)$ the smallest ball in $(\mathrm{Fix}(f), u)$ that contains x and y. Then $B(x, y) \cap \mathrm{Fix}(f) = B_F(x, y)$.

Take any ball $B \in \mathcal{B}_{u+}^f$ and fix any element $x \in B \cap \mathrm{Fix}(f)$. If also $y \in B \cap \mathrm{Fix}(f)$, then by (12), $B(x, y) \subseteq B$; therefore,

$$B \cap \mathrm{Fix}(f) \;=\; \bigcup \{B_F(x, y) \mid y \in B \cap \mathrm{Fix}(f)\}\,.$$

This shows in particular that all balls in $\mathcal{B}_{u+}^f \cap \mathrm{Fix}(f)$ are balls in the full ultrametric ball space of $(\mathrm{Fix}(f), u)$.

For the converse, consider any ball B_F in the full ultrametric ball space of $(\mathrm{Fix}(f), u)$ and pick some $x \in B_F$. Then B_F can be written as

$$\begin{aligned} B_F \;&=\; \bigcup \{B_F(x, y) \mid y \in B_F\} \;=\; \bigcup \{B(x, y) \cap \mathrm{Fix}(f) \mid y \in B_F\} \\ &=\; \mathrm{Fix}(f) \cap \bigcup \{B(x, y) \mid y \in B_F\}\,. \end{aligned}$$

Note that $\{B(x, y) \mid y \in B_F\}$ is a nest of balls since all balls contain x, so its union is a ball in the full ultrametric ball space of (X, u). The second assertion of Lemma 7.4 shows that each ultrametric ball $B(x, y)$ with $x, y \in \mathrm{Fix}(f)$ is f-closed. Therefore, $\bigcup \{B(x, y) \mid y \in B_F\}$ is also f-closed. Hence $B_F \in \mathcal{B}_{u+}^f \cap \mathrm{Fix}(f)$. □

In fact, we could also have used Proposition 6.10. Indeed, it can be seen from the second part of the above proof that the full ultrametric ball space of $(\mathrm{Fix}(f), u)$ is equal to $(\mathrm{Fix}(f), \mathcal{B}_{u+} \cap \mathrm{Fix}(f))$. Further, if $B_0 \in \mathcal{B}_{u+} \cap \mathrm{Fix}(f)$ and $x \in B_0$, then

$$\mathrm{scl}_{\mathcal{B}_{u+}}(B_0) \;=\; \bigcup \{B(x, y) \mid y \in \mathrm{scl}_{\mathcal{B}_{u+}}(B_0)\}$$

is a union of balls which by Lemma 7.4 are f-closed and is thus itself f-closed. This shows that the assumption of Proposition 6.10 is satisfied and consequently,

$$\mathcal{B}_{u+} \cap \mathrm{Fix}(f) \;=\; \mathcal{B}_{u+}^f \cap \mathrm{Fix}(f)\,.$$

7.4. **The topological case.**

Take a compact topological space X and $(X, \mathcal{B})$ the associated ball space formed by the collection $\mathcal{B}$ of all nonempty closed sets. If $f : X \to X$ is any function, then $\mathcal{B}^f$ can be taken as the set of all nonempty closed sets of a (possibly coarser) topology, as arbitrary unions and intersections of f-closed sets are again f-closed. From Theorem 5.19, Lemma 6.4 and Theorem 7.2, we obtain:

Theorem 7.6. *Take a compact topological space* X *and a function* $f : X \to X$. *Assume that every nonempty closed,* f*-closed set contains a fixed point or a smaller closed,* f*-closed set. Then the topology on the set* $\mathrm{Fix}(f)$ *of fixed points of* f *having* $\mathcal{B}^f \cap \mathrm{Fix}(f)$ *as its collection of nonempty closed sets is itself compact.*

As we are rather interested in the topology on $\mathrm{Fix}(f)$ induced by the original topology of X, we ask for criteria on f which guarantee that $\mathcal{B}^f \cap \mathrm{Fix}(f) = \mathcal{B} \cap \mathrm{Fix}(f)$. If the function f is continuous in the topology of X, then it is ball continuous in the ball space $(X, \mathcal{B})$ and the equality follows from Corollary 6.11. Hence we obtain:

Corollary 7.7. *Take a compact topological space X and a continuous function $f : X \to X$. Assume that every nonempty closed, f-closed set contains a fixed point or a smaller closed, f-closed set. Then the induced topology on the set $\mathrm{Fix}(f)$ of fixed points of f is itself compact.*

8. Set theoretic operations on ball spaces

8.1. Subsets of ball spaces.

Proposition 8.1. *Take two ball spaces $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ on the same set X such that $\mathcal{B}_1 \subseteq \mathcal{B}_2$. If $(X, \mathcal{B}_2)$ is $\mathbf{S}_1$ (or $\mathbf{S}_1^d$ or $\mathbf{S}_1^c$), then also $(X, \mathcal{B}_1)$ is $\mathbf{S}_1$ (or $\mathbf{S}_1^d$ or $\mathbf{S}_1^c$, respectively). This does in general not hold for any other property in the hierarchy.*

Proof: The first assertion holds since every nest (or directed system, or centered system) in $\mathcal{B}_1$ is also a nest (or directed system, or centered system) in $\mathcal{B}_2$. To prove the second assertion one constructs an $\mathbf{S}^*$ ball space $(X, \mathcal{B}_2)$ and a nest (or directed system, or centered system) $\mathcal{N}$ such that the intersection $\bigcap \mathcal{N} \in \mathcal{B}_2$ does not lie in $\mathcal{N}$. Then to obtain $\mathcal{B}_1$ one removes all balls from $\mathcal{B}_2$ that lie in $\bigcap \mathcal{N}$. □

8.2. Unions of two ball spaces on the same set.

The easy proof of the following proposition is left to the reader:

Proposition 8.2. *If $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are $\mathbf{S}_1$ ball spaces on the same set X, then so is $(X, \mathcal{B}_1 \cup \mathcal{B}_2)$. The same holds with $\mathbf{S}_2$ or $\mathbf{S}_5$ in place of $\mathbf{S}_1$, and for all properties in the hierarchy if $\mathcal{B}_2$ is finite.*

Note that the assertion may become false if $\mathcal{B}_2$ is infinite and we replace $\mathbf{S}_1$ by $\mathbf{S}_3$ or $\mathbf{S}_4$. Indeed, the intersection of a nest in $\mathcal{B}_1$ may properly contain maximal balls which do not remain maximal balls contained in the intersection in $\mathcal{B}_1 \cup \mathcal{B}_2$.

It is also clear that in general infinite unions of $\mathbf{S}_1$ ball spaces on the same set X will not again be $\mathbf{S}_1$. For instance, ball spaces with just one ball are always $\mathbf{S}_1$, but by a suitable infinite union of such spaces one can build nests with empty intersection.

For any ball space $(X, \mathcal{B})$, we define the ball space $(X, \widehat{\mathcal{B}})$ by setting:

$$\widehat{\mathcal{B}} := \mathcal{B} \cup \{X\}.$$

Taking $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_2 = \{X\}$ in Proposition 8.2, we obtain:

Corollary 8.3. *A ball space $(X, \mathcal{B})$ is $\mathbf{S}_1$ if and only if $(X, \widehat{\mathcal{B}})$ is $\mathbf{S}_1$. The same holds for all properties in the hierarchy in place of $\mathbf{S}_1$.*

8.3. Closure under finite unions of balls.

Take a ball space $(X, \mathcal{B})$. By $\text{f-un}(\mathcal{B})$ we denote the set of all unions of finitely many balls in $\mathcal{B}$. The following lemma is inspired by Alexander's Subbase Theorem:

Lemma 8.4. *If $\mathcal{S}$ is a maximal centered system of balls in $\text{f-un}(\mathcal{B})$ (that is, no subset of $\text{f-un}(\mathcal{B})$ properly containg $\mathcal{S}$ is a centered system), then there is a subset $\mathcal{S}_0$ of $\mathcal{S}$ which is a centered system in $\mathcal{B}$ and has the same intersection as $\mathcal{S}$.*

Proof: It suffices to prove the following: if $B_1,\ldots,B_n\in\mathcal{B}$ such that $B_1\cup\ldots\cup B_n\in\mathcal{S}$, then there is some $i\in\{1,\ldots,n\}$ such that $B_i\in\mathcal{S}$.

Suppose that $B_1,\ldots,B_n\in\mathcal{B}\setminus\mathcal{S}$. By the maximality of $\mathcal{S}$ this implies that for each $i\in\{1,\ldots,n\}$, $\mathcal{S}\cup\{B_i\}$ is not centered. This in turn means that there is a finite subset $\mathcal{S}_i$ of $\mathcal{S}$ such that $\bigcap\mathcal{S}_i\cap B_i=\emptyset$. But then $\mathcal{S}_1\cup\ldots\cup\mathcal{S}_n$ is a finite subset of $\mathcal{S}$ such that

$$\bigcap(\mathcal{S}_1\cup\ldots\cup\mathcal{S}_n)\cap(B_1\cup\ldots\cup B_n)\;=\;\emptyset\,.$$

This yields that $B_1\cup\ldots\cup B_n\notin\mathcal{S}$, which proves our assertion. □

The centered systems of balls in a ball space form a poset under inclusion. Since the union of every chain of centered systems is again a centered system, this poset is chain complete. Hence by Corollary 3.2 every centered system is contained in a maximal centered system. We use this to prove:

Theorem 8.5. *If $(X,\mathcal{B})$ is an $\mathbf{S}_1^c$ ball space, then so is $(X,\text{f-un}(\mathcal{B}))$.*

Proof: Take a centered system $\mathcal{S}'$ of balls in $\text{f-un}(\mathcal{B})$. Take a maximal centered system $\mathcal{S}$ in $\text{f-un}(\mathcal{B})$ which contains $\mathcal{S}'$. By Lemma 8.4 there is a centered system $\mathcal{S}_0$ of balls in $\mathcal{B}$ such that $\bigcap\mathcal{S}_0=\bigcap\mathcal{S}\subseteq\bigcap\mathcal{S}'$. Since $(X,\mathcal{B})$ is an $\mathbf{S}_1^c$ ball space, we have that $\bigcap\mathcal{S}_0\neq\emptyset$, which yields that $\bigcap\mathcal{S}'\neq\emptyset$. This proves that $(X,\text{f-un}(\mathcal{B}))$ is an $\mathbf{S}_1^c$ ball space. □

In [1] it is shown that the theorem becomes false if "$\mathbf{S}_1^c$" is replaced by "$\mathbf{S}_1$".

In [1], the notion of "hybrid ball space" is introduced. The idea is to start with the union of two ball spaces as in Section 8.2 and then close under finite unions. The question is whether the resulting ball space is spherically complete if the original ball spaces are. On symmetrically complete ordered fields K we have two spherically complete ball spaces: $(K,\mathcal{B}_{\text{int}})$ and $(K,\mathcal{B}_u)$ where u is the ultrametric induced by the natural valuation of $(K,<)$ (cf. Theorem 5.15). But by Proposition 5.18, $(K,\mathcal{B}_{\text{int}})$ is not $\mathbf{S}_1^c$, hence Theorem 8.5 cannot be applied. Nevertheless, the following result is proven in [1] by a direct proof. The principles that make it work still remain to be investigated more closely.

Theorem 8.6. *Take a symmetrically complete ordered field K and $\mathcal{B}$ to be the set of all convex sets in K that are finite unions of closed intervals and ultrametric balls. Then $(K,\mathcal{B})$ is spherically complete.*

8.4. **Closure under nonempty intersections of balls.**

Take a ball space $(X,\mathcal{B})$. We define:

(a) $\text{ic}(\mathcal{B})$ to be the set of all nonempty intersections of arbitrarily many balls in $\mathcal{B}$,
(b) $\text{fic}(\mathcal{B})$ to be the set of all nonempty intersections of finitely many balls in $\mathcal{B}$,
(c) $\text{ci}(\mathcal{B})$ to be the set of all nonempty intersections of nests in $\mathcal{B}$.

Note that $(X,\mathcal{B})$ is intersection closed if and only if $\text{ic}(\mathcal{B})=\mathcal{B}$, finitely intersection closed if and only if $\text{fic}(\mathcal{B})=\mathcal{B}$, and chain intersection closed if and only if $\text{ci}(\mathcal{B})=\mathcal{B}$. If $(X,\mathcal{B})$ is $\mathbf{S}_5$, then $\text{ci}(\mathcal{B})=\mathcal{B}$. If $(X,\mathcal{B})$ is $\mathbf{S}^*$, then $\text{ic}(\mathcal{B})=\mathcal{B}$ by Proposition 4.10. We note:

Proposition 8.7. *Take an arbitrary ball space $(X,\mathcal{B})$. Then the ball space $(X,\text{ic}(\mathcal{B}))$ is intersection closed, and $(X,\text{fic}(\mathcal{B}))$ is finitely intersection closed.*

Proof: Take balls $B_i \in \mathrm{ic}(\mathcal{B})$, $i \in I$, and for every $i \in I$, balls $B_{i,j} \in \mathcal{B}$, $j \in J_i$, such that $B_i = \bigcap_{j\in J_i} B_{i,j}$. Then

$$\bigcap_{i\in I} B_i = \bigcap_{i\in I,\, j\in J_i} B_{i,j} \in \mathrm{ic}(\mathcal{B})\,.$$

If I is finite and $B_i \in \mathrm{fic}(\mathcal{B})$ for every $i \in I$, then every J_i can be taken to be finite and thus the right hand side is a ball in $\mathrm{fic}(\mathcal{B})$. □

In view of these facts, we call $(X, \mathrm{ic}(\mathcal{B}))$ the **intersection closure** of $(X, \mathcal{B})$, and $(X, \mathrm{fic}(\mathcal{B}))$ the **finite intersection closure** of $(X, \mathcal{B})$. If a chain intersection closed ball space $(X, \mathcal{B}')$ is obtained from $(X, \mathcal{B})$ by a (possibly transfinite) iteration of the process of replacing $\mathcal{B}$ by $\mathrm{ci}(\mathcal{B})$, then we call $(X, \mathcal{B}')$ a **chain intersection closure** of $(X, \mathcal{B})$. Chain intersection closures are studied in [10] and conditions are given for $(X, \mathrm{ci}(\mathcal{B}))$ to be the chain intersection closure of $(X, \mathcal{B})$. As stated already in part 1) of Theorem 5.4, this holds for classical ultrametric spaces. This result follows from a more general theorem (cf. [10, Theorem 2.2]):

Theorem 8.8. *If $(X, \mathcal{B})$ is a tree-like ball space, then $(X, \mathrm{ci}(\mathcal{B}))$ is its chain intersection closure, and if in addition $(X, \mathcal{B})$ is spherically complete, then so is $(X, \mathrm{ci}(\mathcal{B}))$.*

Since chain intersection closed spherically complete ball spaces are $\mathbf{S}_5$ we obtain:

Corollary 8.9. *If $(X, \mathcal{B})$ is a spherically complete tree-like ball space, then $(X, \mathrm{ci}(\mathcal{B}))$ is an $\mathbf{S}_5$ ball space.*

Also intersection closure can increase the strength of ball spaces:

Theorem 8.10. *If $(X, \mathcal{B})$ is an $\mathbf{S}_1^c$ ball space, then its intersection closure $(X, \mathrm{ic}(\mathcal{B}))$ is an $\mathbf{S}^*$ ball space.*

Proof: Take a centered system $\{B_i \mid i \in I\}$ in $(X, \mathrm{ic}(\mathcal{B}))$. Write $B_i = \bigcap_{j\in J_i} B_{i,j}$ with $B_{i,j} \in \mathcal{B}$. Then $\{B_{i,j} \mid i \in I, j \in J_i\}$ is a centered system in $(X, \mathcal{B})$: the intersection of finitely many balls $B_{i_1,j_1}, \ldots, B_{i_n,j_n}$ contains the intersection $B_{i_1} \cap \ldots \cap B_{i_n}$, which by assumption is nonempty. Since $(X, \mathcal{B})$ is $\mathbf{S}_1^c$, $\bigcap_i B_i = \bigcap_{i,j} B_{i,j} \neq \emptyset$. This proves that $(X, \mathrm{ic}(\mathcal{B}))$ is an $\mathbf{S}_1^c$ ball space. Since $(X, \mathrm{ic}(\mathcal{B}))$ is intersection closed, Theorem 4.9 now shows that $(X, \mathrm{ic}(\mathcal{B}))$ is an $\mathbf{S}^*$ ball space. □

8.5. Closure under finite unions and under intersections.

From Theorems 8.5 and 8.10 we obtain:

Theorem 8.11. *Take any ball space $(X, \mathcal{B})$. If $\mathcal{B}'$ is obtained from $\mathcal{B}$ by first closing under finite unions and then under arbitrary nonempty intersections, then:*

1) $\mathcal{B}'$ is closed under finite unions,

2) $\mathcal{B}'$ is intersection closed,

3) if in addition $(X, \mathcal{B})$ is an $\mathbf{S}_1^c$ ball space, then $(X, \mathcal{B}')$ is an $\mathbf{S}^*$ ball space.

Proof: 1): Take $S_1, \ldots, S_n \subseteq \text{f-un}(\mathcal{B})$ such that $\bigcap S_i \neq \emptyset$ for $1 \leq i \leq n$. Then

$$\left(\bigcap S_1\right) \cup \ldots \cup \left(\bigcap S_n\right) = \bigcap \{B_1 \cup \ldots \cup B_n \mid B_i \in S_i \text{ for } 1 \leq i \leq n\}\,.$$

Since $B_i \in \text{f-un}(\mathcal{B})$ for $1 \le i \le n$, we have that also $B_1 \cup \ldots \cup B_n \in \text{f-un}(\mathcal{B})$. This implies that $(\bigcap S_1) \cup \ldots \cup (\bigcap S_n) \in \mathcal{B}'$.

2): Since $\mathcal{B}'$ is an intersection closure, it is intersection closed.

3): By Theorems 8.5 and 8.10, $(X, \mathcal{B}')$ is an $\mathbf{S}^*$ ball space. □

8.6. **The topology associated with a ball space.**

Take any ball space $(X, \mathcal{B})$. Theorem 8.11 tells us that in a canonical way we can associate with it a ball space $(X, \mathcal{B}')$ which is closed under nonempty intersections and under finite unions. If we also add X and $\emptyset$ to $\mathcal{B}'$, then we obtain the collection of closed sets for a topology whose associated ball space is $(X, \mathcal{B}' \cup \{X\})$.

Theorem 8.12. *The topology associated with a ball space $(X, \mathcal{B})$ is compact if and only if $(X, \mathcal{B})$ is an $\mathbf{S}_1^c$ ball space.*

Proof: The "if" direction of the equivalence follows from Theorems 8.11 and 5.19. The other direction follows from Theorem 5.19 and Proposition 8.1. □

Example: the p-adics.

The field $\mathbb{Q}_p$ of p-adic numbers together with the p-adic valuation v_p is spherically complete. (This fact can be used to prove the original Hensel's Lemma via the ultrametric fixed point theorem, see [23], or even better, via the ultrametric attractor theorem, see [12].) The associated ball space is a classical ultrametric ball space and hence tree-like. It follows from Proposition 4.5 that it is an $\mathbf{S}_1^c$ ball space. Hence by Theorem 8.12 the topology derived from this ball space is compact.

However, $\mathbb{Q}_p$ is known to be locally compact, but not compact under the topology induced by the p-adic metric. But in this topology the ultrametric balls $B_\alpha(x)$ are basic open sets, whereas in the topology derived from the ultrametric ball space they are closed and their complements are the basic open sets. It follows that the balls $B_\alpha(x)$ are not open. It thus turns out that the usual p-adic topology on $\mathbb{Q}_p$ is strictly finer than the one we derived from the ultrametric ball space.

9. Tychonoff type theorems

9.1. **Products in ball spaces.**

In [1] it is shown that the category consisting of all ball spaces together with the ball continuous functions as morphisms allows products and coproducts. The products can be defined as follows.

Assume that $(X_j, \mathcal{B}_j)_{j\in J}$ is a family of ball spaces. Recall that $\hat{\mathcal{B}}_j = \mathcal{B}_j \cup \{X_j\}$. We set $X = \prod_{j\in J} X_j$ and define the **product** $(X_j, \mathcal{B}_j)_{j\in J}^{\text{pr}}$ to be $(X, (\mathcal{B}_j)_{j\in J}^{\text{pr}})$, where

$$(\mathcal{B}_j)_{j\in J}^{\text{pr}} := \left\{ \prod_{j\in J} B_j \mid \text{ for some } k \in J,\ B_k \in \mathcal{B}_k \text{ and } \forall j \neq k : B_j = X_j \right\}.$$

Further, we define the **topological product** $(X_j,\mathcal{B}_j)^{\text{tpr}}_{j\in J}$ to be $(X,(\mathcal{B}_j)^{\text{tpr}}_{j\in J})$, where

$$(\mathcal{B}_j)^{\text{tpr}}_{j\in J} := \left\{\prod_{j\in J} B_j \mid \forall j\in J:\ B_j\in\hat{\mathcal{B}}_j \text{ and } B_j = X_j \text{ for almost all } j\right\},$$

and the **box product** $(X_j,\mathcal{B}_j)^{\text{bpr}}_{j\in J}$ of the family to be $(X,(\mathcal{B}_j)^{\text{bpr}}_{j\in J})$, where

$$(\mathcal{B}_j)^{\text{bpr}}_{j\in J} := \left\{\prod_{j\in J} B_j \mid \forall j\in J:\ B_j\in\mathcal{B}_j\right\}.$$

Since the sets $\mathcal{B}_i$ are nonempty, it follows that $\mathcal{B}\neq\emptyset$, and as no ball in any $\mathcal{B}_i$ is empty, it follows that no ball in $(\mathcal{B}_j)^{\text{pr}}_{j\in J}$, $(\mathcal{B}_j)^{\text{tpr}}_{j\in J}$ and $(\mathcal{B}_j)^{\text{bpr}}_{j\in J}$ is empty.

We leave the proof of the following observations to the reader:

Proposition 9.1. *a) We have that*

$$(\mathcal{B}_j)^{\text{pr}}_{j\in J} \subseteq (\mathcal{B}_j)^{\text{tpr}}_{j\in J} = (\widehat{\mathcal{B}}_j)^{\text{tpr}}_{j\in J} \subseteq (\widehat{\mathcal{B}}_j)^{\text{bpr}}_{j\in J}.$$

b) The following equations hold:

$$\begin{aligned}\text{fic}\left((\widehat{\mathcal{B}}_j)^{\text{pr}}_{j\in J}\right) &= \text{fic}\left((\mathcal{B}_j)^{\text{tpr}}_{j\in J}\right) = (\text{fic}(\mathcal{B}_j))^{\text{tpr}}_{j\in J},\\ \text{ic}\left((\widehat{\mathcal{B}}_j)^{\text{pr}}_{j\in J}\right) &= \text{ic}\left((\mathcal{B}_j)^{\text{tpr}}_{j\in J}\right) = (\text{ic}(\widehat{\mathcal{B}}_j))^{\text{bpr}}_{j\in J}.\end{aligned}$$

The following theorem presents our main results on the various products.

Theorem 9.2. *The following assertions are equivalent:*

a) the ball spaces $(X_j,\mathcal{B}_j)$, $j\in J$, are spherically complete,

b) their box product is spherically complete,

c) their topological product is spherically complete.

d) their product is spherically complete.

The same holds with "$\mathbf{S}_1^d$" and "$\mathbf{S}_1^c$" in place of "spherically complete".

The equivalence of a) and b) also holds for all other properties in the hierarchy, and the equivalence of a) and d) also holds for $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$ and $\mathbf{S}_5$.

Proof: Take ball spaces $(X_j,\mathcal{B}_j)$, $j\in J$, and in every $\mathcal{B}_j$ take a set of balls $\{B_{i,j}\mid i\in I\}$. Then we have:

$$\bigcap_{i\in I}\prod_{j\in J} B_{i,j} = \prod_{j\in J}\bigcap_{i\in I} B_{i,j}. \tag{21}$$

If $\mathcal{N} = (\prod_{j\in J} B_{i,j})_{i\in I}$ is a nest of balls in $(\prod_{j\in J} X_j,(\mathcal{B}_j)^{\text{bpr}}_{j\in J})$, then for every $j\in J$, also $(B_{i,j})_{i\in I}$ must be a nest in $(X_j,\mathcal{B}_j)$.

a) $\Rightarrow$ b): Assume that all ball spaces $(X_j,\mathcal{B}_j)$, $j\in J$, are spherically complete. Then for every $j\in J$, $(B_{i,j})_{i\in I}$ has nonempty intersection. By (21) it follows that $\bigcap\mathcal{N}\neq\emptyset$. This proves the implication a) $\Rightarrow$ b).

b) $\Rightarrow$ a): Assume that $(\prod_{j\in J} X_j,(\mathcal{B}_j)^{\text{bpr}}_{j\in J})$ is spherically complete. Take $j_0\in J$ and a nest of balls $\mathcal{N} = (B_i)_{i\in I}$ in $(X_{j_0},\mathcal{B}_{j_0})$. For each $i\in I$, set

$B_{i,j_0} = B_i$ and $B_{i,j} = B_{0,j}$ for $j \neq j_0$ where $B_{0,j}$ is an arbitrary fixed ball in $\mathcal{B}_j$. Then $(\prod_{j\in J} B_{i,j})_{i\in I}$ is a nest in $(\prod_{j\in J} X_j, (\mathcal{B}_j)^{\mathrm{bpr}}_{j\in J})$. By assumption,

$$\emptyset \neq \bigcap_{i\in I}\prod_{j\in J} B_{i,j} = \left(\bigcap_{i\in I} B_i\right) \times \left(\prod_{j_0\neq j\in J} B_{0,j}\right) ,$$

whence $\bigcap_{i\in I} B_i \neq \emptyset$. We have shown that for every $j \in J$, $(X_j, \mathcal{B}_j)$ is spherically complete. This proves the implication b) $\Rightarrow$ a).

a) $\Rightarrow$ c): Assume that all ball spaces $(X_j, \mathcal{B}_j)$, $j \in J$, are spherically complete. Then by Corollary 8.3, all ball spaces $(X_j, \widehat{\mathcal{B}}_j)$, $j \in J$, are spherically complete. By the already proven implication a) $\Rightarrow$ b), their box product $(X_j, \widehat{\mathcal{B}}_j)^{\mathrm{bpr}}_{j\in J}$ is spherically complete. By part a) of Proposition 9.1 together with Proposition 8.1, $(X, \mathcal{B})^{\mathrm{tpr}}_{j\in J}$ is spherically complete, too.

c) $\Rightarrow$ d): Again, by part a) of Proposition 9.1 together with Proposition 8.1, the product of the ball spaces $(X_j, \widehat{\mathcal{B}}_j)$, $j \in J$, is spherically complete, and as the product of the ball spaces $(X_j, \mathcal{B}_j)$, $j \in J$, is a subspace of this, it is also spherically complete.

d) $\Rightarrow$ a): Same as the proof of b) $\Rightarrow$ a), where we now take $B_{0,j} = X_j$.

These proofs also work when "spherically complete" is replaced by "$\mathbf{S}_1^d$" or "$\mathbf{S}_1^c$", as can be deduced from the following observations:

1) $\{\prod_{j\in J} B_{i,j} \mid i \in I\}$ is a centered system if and only if all sets $\{B_{i,j} \mid i \in I\}$, $j \in J$, are.

2) If $\{\prod_{j\in J} B_{i,j} \mid i \in I\}$ is a directed system, then so are $\{B_{i,j} \mid i \in I\}$ for all $j \in J$.

3) Fix $j_0 \in J$. If $\{B_{i,j_0} \mid i \in I\}$ is a directed system, then so is $\{\prod_{j\in J} B_{i,j} \mid i \in I\}$ when the balls are chosen as in the proof of b) $\Rightarrow$ a) or d) $\Rightarrow$ a).

A proof of the equivalence of a) and b) similar to the above also holds for all other properties in the hierarchy. For the properties $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$ and $\mathbf{S}_5$, one uses the fact that by definition, $\prod_{j\in J} B_j$ is a ball in $(\mathcal{B}_j)^{\mathrm{bpr}}_{j\in J}$ if and only if every B_j is a ball in $\mathcal{B}_j$ and that

4) $\prod_{j\in J} B'_j$ is a ball contained in $\prod_{j\in J} B_j$ if and only if every B'_j is a ball contained in B_j ,

5) $\prod_{j\in J} B'_j$ is a maximal (or largest) ball contained in $\prod_{j\in J} B_j$ if and only if every B'_j is a maximal (or largest, respectively) ball contained in B_j . $\square$

Example 9.3. *There are* $\mathbf{S}^*$ *ball spaces* $(X_j, \mathcal{B}_j)$, $j \in \mathbb{N}$, *such that the ball space* $(X, (\mathcal{B}_j)^{\mathrm{tpr}}_{j\in\mathbb{N}})$ *is not even* $\mathbf{S}_2$. Indeed, we choose a set Y with at least two elements, and for every $j \in \mathbb{N}$ we take $X_j = Y$ and $\mathcal{B}_j = \{B\}$ with $\emptyset \neq B \neq Y$. Then trivially, all ball spaces $(X_j, \mathcal{B}_j)$ are $\mathbf{S}^*$. For all $i, j \in \mathbb{N}$, define

$$B_i := \underbrace{B \times B \times \ldots \times B}_{i \text{ times}} \times Y \times Y \times \ldots \in (\mathcal{B}_j)^{\mathrm{tpr}}_{j\in\mathbb{N}}.$$

Then $\mathcal{N} = \{B_i \mid i \in I\}$ is a nest of balls in $(\mathcal{B}_j)^{\mathrm{tpr}}_{j\in\mathbb{N}}$, but the intersection $\bigcap \mathcal{N} = \prod_{j\in\mathbb{N}} B$ does not contain any ball in this ball space.

Example 9.4. *There are* $\mathbf{S}^*$ *ball spaces* $(X, \mathcal{B}_j)$, $j = 1, 2$, *such that the ball space* $(X, (\mathcal{B}_j)^{\mathrm{pr}}_{j\in\{1,2\}})$ *is not* $\mathbf{S}_2^c$. Indeed, we choose again a set Y with at least two elements and take $\mathcal{B}_1 = \mathcal{B}_2 = \{B\}$ with $\emptyset \neq B \neq Y$. Then as in the previous example, $(X_j, \mathcal{B}_j)$, $j = 1, 2$ are $\mathbf{S}^*$ ball spaces. Further, $(\mathcal{B}_j)^{\rm pr}_{j\in\{1,2\}} = \{Y \times Y, B \times Y, Y \times B\}$, which is a centered system whose intersection does not contain any ball.

9.2. **The ultrametric case.**

If (X_j, u_j), $j \in J$ are ultrametric spaces with value sets $u_jX_j = \{u_j(a, b) \mid a, b \in X_j\}$, and if $B_j = B_{\gamma_j}(a_j)$ is an ultrametric ball in (X_j, u_j) for each j, then

$$\prod_{j\in J} B_j \;=\; \{(b_j)_{j\in J} \mid \forall j \in J :\; u_j(a_j, b_j) \le \gamma_j\}\;.$$

This shows that the box product is the ultrametric ball space for the product ultrametric on $\prod_{j\in J} X_j$ which is defined as

$$u_{\rm prod}((a_j)_{j\in J}\,,\,(b_j)_{j\in J}) \;=\; (u_j(a_j, b_j))_{j\in J} \;\in \prod_{j\in J} u_jX_j\;.$$

The latter is a poset, but in general not totally ordered, even if all u_jX_j are totally ordered and even if J is finite. So the product ultrametric is a natural example for an ultrametric with partially ordered value set.

If the index set J is finite and all u_jX_j are contained in some totally ordered set Γ such that all of them have a common least element $0 \in \Gamma$, then we can define an ultrametric on the product $\prod_{j\in J} X_j$ which takes values in $\bigcup_{j\in J} u_jX_j \subseteq \Gamma$ as follows:

$$u_{\max}((a_j)_{j\in J}\,,(b_j)_{j\in J}) \;=\; \max_j u_j(a_j, b_j)$$

for all $(a_j)_{j\in J}, (b_j)_{j\in J} \in \prod_{j\in J} X_j$. We leave it to the reader to prove that this is indeed an ultrametric. The corresponding ultrametric balls are the sets of the form

$$\{(b_j)_{j\in J} \mid \forall j \in J :\; u_j(a_j, b_j) \le \gamma\}$$

for some $(a_j)_{j\in J} \in \prod_{j\in J} X_j$ and $\gamma \in \bigcup_{j\in J} u_jX_j$. Now the value set is totally ordered. It turns out that the collection of balls so obtained is a (usually proper) subset of the full ultrametric ball space of the product ultrametric. Therefore, if all (X_j, u_j) are spherically complete, then so is $(\prod_{j\in J} X_j, u_{\max})$ by Theorem 9.2 and Proposition 8.1.

Theorem 9.5. *Take ultrametric spaces (X_j, u_j), $j \in J$. Then the ultrametric space $(\prod_{j\in J} X_j, u_{\rm prod})$ is spherically complete if and only if all (X_j, u_j), $j \in J$, are spherically complete.*

If the index set J is finite and all u_jX_j are contained in some totally ordered set Γ such that all of them have a common least element, then the same also holds for $u_{\max}$ in place of $u_{\rm prod}$.

Proof: As was remarked earlier, the ultrametric ball space of the product ultrametric is the box product of the ultrametric ball spaces of the ultrametric spaces (X_j, u_j). Thus the first part of the theorem is a corollary to Theorem 9.2.

To prove the second part of the theorem, it suffices to prove the converse of the implication we have stated just before the theorem. Assume that the space $(\prod_{j\in J} X_j, u_{\max})$ is spherically complete and choose any $j_0 \in J$. Let $\mathcal{N}_{j_0} = \{B_{\gamma_i}(a_{i,j_0}) \mid i \in I\}$ be a nest of balls in (X_{j_0}, u_{j_0}). Further, for every

$j \in J \setminus \{j_0\}$ choose some element $a_j \in X_j$ and for every $i \in I$ set $a_{i,j} := a_j$ and

$$\begin{aligned} B_i &:= \{(b_j)_{j\in J} \in \prod_{j\in J} X_j \mid u_{\max}((a_{i,j})_{j\in J}, (b_j)_{j\in J}) \le \gamma_i\} \\ &= \{(b_j)_{j\in J} \in \prod_{j\in J} X_j \mid \forall j \in J : \ u_j(a_{i,j}, b_j) \le \gamma_i\} \,. \end{aligned}$$

In order to show that $\mathcal{N} := \{B_i \mid i \in I\}$ is a nest of balls in $(\prod_{j\in J} X_j, u_{\max})$, we have to show that any two balls B_i, B_k, $i,k \in I$, have nonempty intersection. Assume without loss of generality that $\gamma_i \le \gamma_k$. As $\{B_{\gamma_i}(a_{i,j_0}) \mid i \in I\}$ is a nest of balls, we have that $a_{i,j_0} \in B_{\gamma_k}(a_{k,j_0})$. It follows that $u_{j_0}(a_{k,j_0}, a_{i,j_0}) \le \gamma_k$, and since $a_{i,j} = a_j = a_{k,j}$ for every $j \in J \setminus \{j_0\}$,

$$(a_{i,j})_{j\in J} \ \in \ B_i \cap \{(b_j)_{j\in J} \in \prod_{j\in J} X_j \mid \forall j \in J : \ u_j(a_{k,j}, b_j) \le \gamma_k\} \ = \ B_i \cap B_k \,.$$

As $(\prod_{j\in J} X_j, u_{\max})$ is assumed to be spherically complete, there is some $(z_j)_{j\in J} \in \bigcap \mathcal{N}$; it satisfies $u_j(a_{i,j}, z_j) \le \gamma_i$ for all $i \in I$ and all $j \in J$. In particular, taking $j = j_0$, we find that $z_{j_0} \in B_{\gamma_i}(a_{i,j_0})$ for all $i \in I$ and thus, $z_{j_0} \in \bigcap \mathcal{N}_{j_0}$. □

9.3. **The topological case.**

In which way does Tychonoff's theorem follow from its analogue for ball spaces? The problem in the case of topological spaces is that the topological product ball space we have defined, while containing only closed sets of the product, does not contain all of them, as it is not necessarily closed under finite unions and arbitrary intersections. We have to close it under these operations.

If the topological spaces X_i, $i \in I$, are compact, then their associated ball spaces $(X_i, \mathcal{B}_i)$ are $\mathbf{S}_1^c$ (cf. Theorem 5.19). By Theorem 9.2 their topological product is also $\mathbf{S}_1^c$. Theorem 8.11 shows that the product topology of the topological spaces X_i is the closure of $(\mathcal{B}_j)_{j\in J}^{\mathrm{tpr}}$ under finite unions and under arbitrary nonempty intersections, when $\emptyset$ and the whole space are adjoined. By Theorem 8.12, this topology is compact.

We have shown that Tychonoff's Theorem follows from its ball spaces analogue.

10. Other results related to fixed point theorems

In this section, we will discuss two types of theorems that are related to fixed point theorems.

10.1. **Multivalued fixed point theorems.**

We take a function F from a nonempty set X to its power set $\mathcal{P}(X)$ and ask for criteria that guarantee the existence of a fixed point $x \in X$ in the sense that

$$x \ \in \ F(x) \,.$$

A very elegant approach to proving a generic multivalued fixed point theorem can be given by use of contractive ball spaces:

Theorem 10.1. *Take a spherically complete contractive B_x–ball space $(X,\mathcal{B})$ and a function $F : X \to \mathcal{P}(X)$. Assume that*

$$B_x \cap F(x) \neq \emptyset \text{ for all } x \in X .$$

Then F admits a fixed point in X.

Proof: By part 2) of Theorem 3.9, $\mathcal{B}$ contains a singleton ball $B_a = \{a\}$. Since by hypothesis $B_a \cap F(a) \neq \emptyset$, it follows that $a \in F(a)$. □

This theorem together with Proposition 5.9 and 5.10 can be used to prove the following result:

Theorem 10.2. *Take a complete metric space (X,d) and an Oettli-Théra function ϕ on X. If a function $F : X \to \mathcal{P}(X)$ satisfies*

$$\forall x \in X\, \exists y \in F(x) : \ d(x,y) \ \leq \ -\phi(x,y),$$

then F has a fixed point on X.

In [2] this theorem and its variants are proved using a version of part 2) of Theorem 3.9 together with Proposition 5.9.

The following is a slight generalization of Theorem 10.1, replacing the existence of singletons by that of minimal balls. Here again, as in Theorems 2.5 and 2.6, the general condition on the ball space is adapted to the given function: condition (C3) is replaced by a condition that depends on the function F.

Theorem 10.3. *Take a nonempty set X and a function $F : X \to \mathcal{P}(X)$. Assume that $(X,\mathcal{B})$ is a spherically complete normalized B_x–ball space such that for all $x,y \in X$,*

1) $B_x \cap F(x) \neq \emptyset$,

2) if $y \in B_x$, then $B_y \subseteq B_x$,

3) if $x \notin F(x)$, then there is some $z \in B_x$ such that $B_z \subsetneq B_x$.

Then F admits a fixed point in X.

Proof: A straightforward adaptation of the proof of Proposition 3.8 shows that the intersection of a maximal nest of balls, if nonempty, must be a minimal ball B_a which consequently must satisfy $a \in F(a)$. The assumption that the ball space is spherically complete guarantees that the intersection is nonempty. □

10.2. **Coincidence theorems.**

We take a nonempty set X and two or more functions $f_1,\ldots,f_n : X \to X$ and ask for criteria that guarantee the existence of a coincidence point $x \in X$ in the sense that

$$f_1(x) \ = \ \ldots \ = \ f_n(x)\,. \tag{22}$$

In order to obtain a generic coincidence theorem for ball spaces, one can again use the idea of showing the existence of singleton balls with suitable properties.

Theorem 10.4. *Take a spherically complete contractive B_x–ball space $(X,\mathcal{B})$ and functions $f_1,\ldots,f_n : X \to X$. Assume that*

$$f_1(x),\ldots,f_n(x) \in B_x \text{ for all } x \in X .$$

Then $f_1,\ldots,f_n$ admit a coincidence point in X.

Proof: By part 2) of Theorem 3.9, $\mathcal{B}$ contains a singleton ball B_a. Since by hypothesis $f_1(a), \dots, f_n(a) \in B_a$, it follows that $f_1(a) = \dots = f_n(a)$. $\Box$

As in the previous section, we prove a generalization that replaces the existence of singletons by that of minimal balls.

Theorem 10.5. *Take a nonempty set X and functions $f_1, \dots, f_n : X \to X$. Assume that there is a B_x–ball space $\mathcal{B}$ on X such that $(X, \mathcal{B})$ is an $\mathbf{S}_2$ ball space and for all $x \in X$, if (22) does not hold, then there is some $y \in X$ such that $B_y \subsetneq B_x$.*

Then $f_1, \dots, f_n$ admit a coincidence point in X.

Proof: Let $\mathcal{M}$ be a maximal nest of balls in $\mathcal{B}$ (it exists by Corollary 3.7). Since $(X, \mathcal{B})$ is an $\mathbf{S}_2$ ball space, there is a ball $B_x \subseteq \bigcap \mathcal{M}$. This means that $\mathcal{M} \cup \{B_x\}$ is a nest of balls, so by maximality of $\mathcal{M}$ we have that $B_x \in \mathcal{M}$. Consequently, $B_x = \bigcap \mathcal{M}$. Suppose that (22) does not hold. Then by hypothesis there is some element $y \in X$ such that $B_y \subsetneq B_x$ whence $B_y \notin \mathcal{M}$. But then $\mathcal{M} \cup \{B_y\}$ is a nest which strictly contains $\mathcal{M}$. This contradiction to the maximality of $\mathcal{M}$ shows that (22) must hold. $\Box$

Let us note that condition (22) can be replaced by any other condition on x. In this way, a generic theorem can be obtained that is neither a fixed point theorem nor a coincidence theorem but can be specialized to such theorems. This idea has been exploited in [17].

References

[1] Bartsch, R. – Kuhlmann, F.-V. – Kuhlmann, K.: *Construction of ball spaces and the notion of continuity*, submitted, `arXiv:1810.09275`

[2] Błaszkiewicz, P. – Ćmiel, H. – Linzi. A. – Szewczyk, P.: *Caristi–Kirk and Oettli–Théra ball spaces, and applications*, submitted, `arXiv:1901.03853`

[3] Brunner, N.: *Topologische Maximalprinzipien* (German) [Topological maximal principles], Z. Math. Logik Grundlag. Math. **33** (1987), 135–139

[4] Cohn, P. M.: *Universal algebra*, Harper and Row, New York, 1965

[5] Hitzler, P. – Seda, A. K.: *The fixed-point theorems of Priess-Crampe and Ribenboim in logic programming*, Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999), 219–235, Fields Inst. Commun. **32**, Amer. Math. Soc., Providence, RI, 2002

[6] Jachymski, J. R.: *Caristi's fixed point theorem and selections of set-valued contractions*, J. Math. Anal. Appl. **227** (1998), 55–67

[7] Jachymski, J. R.: *Order-theoretic aspects of metric fixed point theory*, Handbook of metric fixed point theory, 613641, Kluwer Acad. Publ., Dordrecht, 2001

[8] Kirk, W. A.: *Metric fixed point theory: a brief retrospective*, Fixed Point Theory Appl. 2015, article 215

[9] Kozlowski, W. M.: *A purely metric proof of the Caristi fixed point theorem*, Bull. Aust. Math. Soc. **95** (2017), 333–337

[10] Kubis, W. – Kuhlmann, F.-V.: *Chain intersection closures*, Topology and its Applications **262** (2019), 11–19

[11] Kubis, W. – Kuhlmann, F.-V.: *Chain union closures*, in preparation

[12] Kuhlmann, F.-V.: *Maps on ultrametric spaces, Hensel's Lemma, and differential equations over valued fields*, Comm. in Alg. **39** (2011), 1730–1776

[13] Kuhlmann, F.-V. – Kuhlmann, K.: *A common generalization of metric and ultrametric fixed point theorems*, Forum Math. **27** (2015), 303–327; and: Correction to "A common generalization of metric, ultrametric and topological fixed point theorems", Forum Math. **27** (2015), 329–330; alternative corrected version available at: `http://math.usask.ca/fvk/GENFPTAL.pdf`

[14] Kuhlmann, F.-V. – Kuhlmann, K.: *Fixed point theorems for spaces with a transitive relation*, Fixed Point Theory **18** (2017), 663–672

[15] Kuhlmann, F.-V. – Kuhlmann, K. – Paulsen, M.: *The Caristi–Kirk Fixed Point Theorem from the point of view of ball spaces*, Journal of Fixed Point Theory and Applications **20**, Art. 107 (2018)

[16] Kuhlmann, F.-V. – Kuhlmann, K. – Shelah, S.: *Symmetrically Complete Ordered Sets, Abelian Groups and Fields*, Israel J. Math. **208** (2015), 261–290

[17] Kuhlmann, F.-V. – Kuhlmann, K. – Sonallah, F.: *Coincidence Point Theorems for Ball Spaces and Their Applications*, to appear in: Ordered Algebraic Structures and Related Topics, CIRM, Luminy, France, October 12-16 2015, Contemporary Mathematics, AMS

[18] Mańka, R.: *Some forms of the axiom of choice*, Jbuch. Kurt-Gödel-Ges. 1988, 24–?34

[19] Markowsky, G.: *Chain-complete posets and directed sets with applications*, Algebra Universalis **6** (1976), 53–68

[20] Oettli, W. – Théra, M.: *Equivalents of Ekeland's principle*, Bull. Austral. Math. Soc. **48** (1993), 385–392

[21] Penot, J.-P.: *Fixed point theorems without convexity*, Analyse non convexe (Proc. Colloq., Pau, 1977). Bull. Soc. Math. France Mém. **60** (1979), 129–152

[22] Penot, J.-P.: *The drop theorem, the petal theorem and Ekeland's variational principle*, Nonlinear Anal. **10** (1986), 813–822

[23] Prieß-Crampe, S.: *Der Banachsche Fixpunktsatz für ultrametrische Räume*, Results in Mathematics **18** (1990), 178–186

[24] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Points, Combs and Generalized Power Series*, Abh. Math. Sem. Hamburg **63** (1993), 227–244

[25] Prieß-Crampe, S. – Ribenboim, P.: *The Common Point Theorem for Ultrametric Spaces*, Geom.Ded. **72** (1998), 105–110

[26] Prieß-Crampe, S. – Ribenboim, P.: *Fixed Point and Attractor Theorems for Ultrametric Spaces*, Forum Math. **12** (2000), 53–64

[27] Prieß-Crampe, S. – Ribenboim, P.: *Ultrametric spaces and logic programming*, J. Logic Programming **42** (2000), 59–70

[28] Prieß-Crampe, S. – Ribenboim, P.: *Ultrametric dynamics*, Illinois J. Math. **55** (2011), 287–303

[29] Shapiro, J. H.: *A fixed-point farrago*, Universitext. Springer, [Cham], 2016

[30] Shelah, S.: *Quite Complete Real Closed Fields*, Israel J. Math. **142** (2004), 261–272

[31] Steprāns, J. – Watson, S. – Just, W.: *A topological Banach fixed point theorem for compact Hausdorff spaces*, Canad. Bull. Math. **37**(4) (1994), 552–555

[32] Tarski, A.: *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. **5** (1955), 285-309

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: Hanna.Cmiel@stud.usz.edu.pl

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: fvk@usz.edu.pl

Institute of Mathematics, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: Katarzyna.Kuhlmann@usz.edu.pl