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b-MINIMALITY

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We introduce a new notion of tame geometry for structures admitting an abstract notion of balls. The notion is named *b*-minimality and is based on definable families of points and balls. We develop a dimension theory and prove a cell decomposition theorem for *b*-minimal structures. We show that *b*-minimality applies to the theory of Henselian valued fields of characteristic zero, generalizing work by Denef–Pas [25, 26]. Structures which are *o*-minimal, *v*-minimal, or *p*-minimal and which satisfy some slight extra conditions are also *b*-minimal, but *b*-minimality leaves more room for nontrivial expansions. The *b*-minimal setting is intended to be a natural framework for the construction of Euler characteristics and motivic or *p*-adic integrals. The *b*-minimal cell decomposition is a generalization of concepts of Cohen [11], Denef [15], and the link between cell decomposition and integration was first made by Denef [13].

1. Introduction

Originally introduced by Cohen [11] for real and *p*-adic fields, cell decomposition techniques were developed by Denef and Pas as a useful device for the study of *p*-adic integrals [13–15, 25, 26]. Roughly speaking, the basic idea is to cut a definable set

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into a finite number of cells each of which is like a family of balls or points. For general Henselian valued fields of residue characteristic zero, Denef and Pas proved a cell decomposition theorem where the families of balls or points are parameterized by residue field and value group variables in a definable way.[a] An integral over the p-adic field is then replaced by a corresponding sum over these residue field and value group variables since the measure of a ball is clear and points have measure zero. Denef–Pas cell decomposition plays a fundamental role in our recent work [9] where we lay new general foundations for motivic integration. When we started in 2002 working on the project that led to [9], we originally intended to work in the framework of an axiomatic cell decomposition of which Denef–Pas cell decomposition would be an avatar, but we finally decided to keep on the safe side by staying within the Denef–Pas framework and we postponed the axiomatic approach to a later occasion. The present paper is an attempt to lay the fundamentals of a tame geometry based upon a cell decomposition into basic families of points and "abstract" balls. A key point in our approach, already present in [9, 25], is to work in a many sorted language with a unique main sort and possibly many auxiliary sorts that will parameterize the families of balls in a definable way. The theory is designed so that no field structure or topology is required. Instead, only a notion of balls is needed, whence the naming b-minimality. The collection of balls in a model is given by definition as the set of fibers of a predicate B of the basic language consisting of one symbol B. The notion of b-minimality is then based on three axioms, named (b1), (b2) and (b3), where (b1) rather directly imposes the theory to allow cell decomposition and where (b2) and (b3) imply a good dimension theory and exclude pathological behavior.

We show that every o-minimal structure is b-minimal, but more exotic expansions of o-minimal structures, like the field of real numbers with a predicate for the integer powers of 2 considered by van den Dries in [17], are also b-minimal, relative to the right auxiliary sorts. Also v-minimal theories of algebraically closed valued fields, defined by Hrushovski and Kazhdan in [21], and p-minimal theories defined by Haskell and Macpherson in [20] are b-minimal, under some slight extra conditions for the p-minimal case. Our framework is intended to be versatile enough to encompass promising candidate expansions, like entire analytic functions on valued and real fields, but still strong enough to provide cell decomposition and a nice dimension theory. For p-minimality, for example, cell decomposition is presently missing in the theory and there seem to be few candidate expansions in sight. For C-minimality and v-minimality, expansions by nontrivial entire analytic functions are not possible since these have infinitely many zeros in an algebraically closed valued field. As already indicated, another goal of the theory is the study of Grothendieck rings and more specifically the construction of additive Euler characteristics and motivic integrals. We intend to go further into that direction in some future work.

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[a]In [11, 13, 15, 25, 26], cells were very technically defined, and their more simple presentation as families of balls and points appears in [9].

Let us briefly review the content of the paper. In Sec. 2, the basic axioms are introduced and discussed and in Sec. 3, cell decomposition is proved. Section 4 is devoted to dimension theory. In the next two sections, more specific properties are considered: "preservation of balls" (which is a consequence of the monotonicity theorem in the *o*-minimal case) and *b*-minimality with centers (a kind of definable functions approximating the balls in the families). In Sec. 7, we show that the theory of Henselian valued fields of characteristic zero is *b*-minimal by adapting the Cohen–Denef proof scheme. In particular, we give (as far as we know) the first written instance of cell decomposition in mixed characteristic for unbounded ramification. Moreover, we prove that all definable functions are essentially given by terms. In Sec. 8, we compare *b*-minimality with *p*-minimality, *v*-minimality and C-minimality. We conclude the paper with some preliminary results on Grothendieck semirings associated to *b*-minimal theories.

2. *b*-Minimality

2.1. *Preliminary conventions*

A language may have many sorts, some of which are called main sorts, the others being called auxiliary sorts. An expansion of a language may introduce new sorts. In this paper, we shall only consider languages admitting a unique main sort and a distinguished collection of auxiliary sorts. If a model is named $\mathcal{M}$, then the main sort of $\mathcal{M}$ is denoted by M.

Usually, for a predicate B of a first order language $\mathcal{L}$, one has to specify that it is n-ary for some n, and one has to fix sorts $S_1, \ldots, S_n$ such that only for x_i running over S_i, one is allowed to write $B(x_1, \ldots, x_n)$. We will not consider this as information of $\mathcal{L}$ itself, but instead this information will be fixed by $\mathcal{L}$-theories by an axiom of the form

$$\forall x \ (x \in B \to (x \in S_1 \times \cdots \times S_n))$$

for some n and some sorts S_i. There is no harm in doing so.

By *definable*, we shall always mean definable with parameters, as opposed to $\mathcal{L}(A)$-definable or A-definable, which means definable with parameters in A. By a *point*, we mean a singleton. A definable set is called *auxiliary* if it is a subset of a finite Cartesian product of (the universes of) auxiliary sorts.

If S is a sort, then its Cartesian power S^0 is considered to be a point and to be $\varnothing$-definable. If S_1 and S_2 are sorts, then by convention S_1^0 is the same $\varnothing$-definable point as S_2^0, so in our models there is always at least one $\varnothing$-definable point.

Recall that *o*-minimality is about expansions of the language $\mathcal{L}_<$ with one predicate $<$, with the requirement that the predicate $<$ defines a dense linear order without endpoints. In the present setting, we shall study expansions of a language $\mathcal{L}_B$ consisting of one predicate B, which is nonempty and which has fibers in the M-sort (by definition called balls). In both cases of tame geometries (*o*-minimal and *b*-minimal), the expansion has to satisfy extra properties.

2.2.

Let $\mathcal{L}_B$ be the language with one predicate B. We require that B is interpreted in any $\mathcal{L}_B$-model $\mathcal{M}$ with main sort M as a nonempty set $B(\mathcal{M})$ with

$$B(\mathcal{M}) \subset A_B \times M$$

where A_B is a finite Cartesian product of (the universes of) some of the sorts of $\mathcal{M}$. When $a \in A_B$, we write $B(a)$ for

$$B(a) := \{m \in M | (a, m) \in B(\mathcal{M})\},$$

and if $B(a)$ is nonempty, we call it a *ball* (in the structure $\mathcal{M}$), or B-ball when useful.

Definition 2.1. Let $\mathcal{L}$ be any expansion of $\mathcal{L}_B$. We call an $\mathcal{L}$-model $\mathcal{M}$ *b-minimal* when the following three conditions are satisfied for every set of parameters A (the elements of A can belong to any of the sorts), for every A-definable subset X of M, and for every A-definable function $F : X \to M$.

(b1) There exists an A-definable function $f : X \to S$ with S auxiliary such that for each $s \in f(X)$ the fiber $f^{-1}(s)$ is a point or a ball.

(b2) If g is a definable function from an auxiliary set to a ball, then g is not surjective.

(b3) There exists an A-definable function $f : X \to S$ with S auxiliary such that for each $s \in f(X)$ the restriction $F_{|f^{-1}(s)}$ is either injective or constant.

We call an $\mathcal{L}$-theory *b-minimal* if all its models are b-minimal.

We call a map f as in (b1) an *A-definable b-map on X*, or b-map for short, and call a map f as in (b3) *compatible with F*.

Typically, in algebraic geometry, one studies families of varieties by means of a map from one variety X to another variety Y, and then the family consists of the fibers of such a map. Such a family (of fibers) is then just called a variety over Y. Likewise, in this paper, a definable set X *over* some other definable set Y is nothing else than a definable map from X to Y and its interest lies in the study of the family of fibers of this map. Often, X will be a definable subset of $M^n \times Y$ and the definable map to Y will be the coordinate projection. This terminology should not be confused with X being Y-definable meaning that X is definable using parameters from Y. By further analogy, a definable map between two definable sets X_1 and X_2 over Y is nothing else than a definable function from X_1 to X_2 making commutative diagrams with the maps from X_1 and X_2 to Y which are implicitly understood when we say that the X_i are over Y.

With this terminology of definable sets *over* definable sets, we can define a relative version of b-maps, that is, b-maps over some given definable set. Namely, when Y and $X \subset M \times Y$ are A-definable sets, an (*A-definable*) *b-map on X over Y* is by definition an (A-definable) function $f : X \to S \times Y$ which makes a commutative diagram with both projections to Y and with S auxiliary, such that for each $a \in f(X)$

the fiber $f^{-1}(a)$ is a point or a ball, where each $f^{-1}(a)$ is naturally considered as a subset of M (namely, by projecting on the M-coordinate). *Compatibility* of f with some definable function $F : X \subset M \times Y \to M \times Y$ over Y is defined similarly.

Remark 2.2. Axiom (b2) expresses in a strong way that the auxiliary sorts are different from the main sort. In particular, the main sort is not interpretable in the auxiliary sorts by a definable function, in the sense of model theory. It follows from (b2) that if $f : X \to S$ is a b-map and X is a ball, then at least one fiber of f is not a point, and hence, no ball is a point.

Remark 2.3. Most of the theory can be developed with the slightly weaker axiom (b3′) instead of axiom (b3):

(b3′) There exists a finite partition X_i of X into A-definable parts and A-definable functions $f_i : X_i \to S_i$ with S_i auxiliary such that for each i and $s \in f_i(X_i)$ the restriction $F_{|f_i^{-1}(s)}$ is either injective or constant.

As soon as there are at least two $\varnothing$-definable auxiliary points, (b3) and (b3′) become equivalent.

Another variant with a similar theory would be to allow for more than one predicate B to define the balls, for example, predicates $B_i \subset A_{B_i} \times M$ and a ball being a fiber $B_i(a) \subset M$ for $a \in A_{B_i}$. We do not pursue this variant here.

Remark 2.4. Unlike many other notions of minimality or of tame geometries, the notion of b-minimality does not require the theory to be complete.

2.3. *Refinements*

As explained after Definition 2.1, a definable function f can be seen as the family of its fibers. The following notion of refinement captures the idea that each of the fibers of f gets partitioned into fibers of another function f', where the images of f and of f' are supposed to be auxiliary sets. This last condition is assumed in order to exclude trivial maps like $X \to X$.

Let $f : X \to S$ be a definable function on $X \subset M$, with S auxiliary. By a *refinement of* f, we mean a pair (f', g) with $f' : X \to S'$ and $g : f'(X) \to S$ definable functions and S' auxiliary, such that $g \circ f' = f$. Since g is uniquely determined by f', we shall write f' instead of (f', g) for a refinement of f and $f \geq f'$ if f' is a refinement of f. This induces a structure of partially ordered set on the set of all definable functions on X of the form $X \to S$ with S auxiliary.

In the relative setting, let $f : X \to S \times Y$ be a definable function on $X \subset M \times Y$ over Y (the terminology *over* Y means that f makes a commutative diagram with the projections to Y, see the discussion below Definition 2.1), with S auxiliary. By a *refinement of* f (over Y), we mean a pair (f', g) with $f' : X \to S' \times Y$ a definable function on X over Y, $g : f'(X) \to S \times Y$ a definable function over Y and S' auxiliary, such that $g \circ f' = f$. Since g is uniquely determined by f', we call f' a refinement of f and write $f \geq f'$.

Lemma 2.5. *Let $\mathcal{M}$ be a model of a b-minimal theory. Let $F : X \to X'$ be a definable function over Y for some subsets X and X' of $M \times Y$ (that F is over Y means that F commutes with the projections to Y). Then any two definable functions $f : X \to S \times Y$ and $f' : X \to S' \times Y$ over Y with S and S' auxiliary admit a common refinement $f'' : X \to S'' \times Y$ over Y, such that f'' is moreover a b-map over Y and compatible with F. (In particular, the opposite category of the category associated to the partially ordered set of such maps on X over Y is filtering.) If moreover F, f, and f' are A-definable for some A, then f'' can be taken A-definable.*

Proof. For each $y \in Y$ there exists a map as in (b3) on the fiber $\pi^{-1}(y)$, with $\pi : X \to Y$ the projection, and similarly in all models of the theory. By compactness, one finds a definable $f_0 : X \to S_0 \times Y$ over Y, with S_0 auxiliary, which is compatible with F. Now define

$$f_1 : X \to S \times S' \times S_0 \times Y$$

by

$$x \mapsto p(f(x), f'(x), f_0(x))$$

with

$$p : S \times S' \times S_0 \times Y^3 \to S \times S' \times S_0 \times Y$$

the projection. For each $s \in S \times S' \times S_0 \times Y$, there exists a b-map on $f_1^{-1}(s)$ by (b1), and this holds in all models of the theory. By compactness, one finds a b-map $f'' : X \to S \times S' \times S_0 \times S'' \times Y$ on X over Y, for some auxiliary S'', whose composition with the projection to $S \times S' \times S_0 \times Y$ equals f_1. We obtain this way a map f'' as required.   □

2.4. *Some criteria*

The following two criteria are consequences of the lemmas and their corollary below.

Proposition 2.6. *Let $\mathcal{T}$ be an $\mathcal{L}$-theory with all its models satisfying (b1) and (b2). Suppose that there are at least two $\varnothing$-definable auxiliary points. Then $\mathcal{T}$ is b-minimal if and only if for all models $\mathcal{M}$ the following statement (∗) holds in $\mathcal{M}$:*

(∗) *if $F : X \subset M \to Y$ is a definable surjection with Y a ball, then not all fibers of F contain balls.*

Proposition 2.7. *Let $\mathcal{T}$ be an $\mathcal{L}$-theory with all its models satisfying (b1). Suppose that there are at least two $\varnothing$-definable auxiliary points. Then $\mathcal{T}$ is b-minimal if and only if for all models $\mathcal{M}$ the following statement (†) holds in $\mathcal{M}$:*

(†) *If $F : X \subset S \times M \to Y$ is a definable surjection with S auxiliary and Y a ball, then there exists $y \in F(X)$ such that $p(F^{-1}(y))$ does not contain a ball, with p the projection $X \to M$.*

Lemma 2.8. *Let $\mathcal{T}$ be an $\mathcal{L}$-theory with all its models satisfying (b2) and (b3). Then conditions (∗) and (†) are satisfied for each model $\mathcal{M}$ of $\mathcal{T}$.*

Proof. We first prove $(*)$. Let Y be a ball and let $F : X \subset M \to Y$ be a definable surjection. Consider a map $g : X \to S$ such that, for every s in $g(X)$, the restriction $F_{|g^{-1}(s)}$ is either injective or constant, given by (b3). Consider the definable subset S_1 of S consisting of all points s such that $F_{|g^{-1}(s)}$ is injective and set $X_1 := g^{-1}(S_1)$, $S_2 := g(X) \backslash S_1$ and $X_2 := g^{-1}(S_2)$. Suppose that for some $y \in Y$ the set $F_{|X_1}^{-1}(y)$ contains a ball T. Then the restriction of g to T is injective, which contradicts (b2). Hence, no set of the form $F_{|X_1}^{-1}(y)$ contains a ball. Now, for every s in S_2, $g^{-1}(s)$ is contained in a (unique) fiber of $F_{|X_2}$. It follows that F induces a definable surjection $S_2 \to F(X_2)$. If $Y = F(X_2)$, then we get a contradiction to (b2). If $Y \neq F(X_2)$, then there exists a point $y \in Y$ with $F^{-1}(y) = F_{|X_1}^{-1}(y)$ which does not contain a ball as shown above.

We now prove $(\dagger)$, in a similar way. Let Y be a ball and let $F : X \subset S \times M \to Y$ be a definable surjection with S auxiliary. By (b3) and by compactness (as in the proof of Lemma 2.5), we can consider a map $g : X \to S \times S'$ over S such that, for every t in $g(X)$, the restriction $F_{|g^{-1}(t)}$ is either injective or constant. For s in S, set $X_s = X \cap (\{s\} \times M)$. Consider the definable subset S_1 of $g(X)$ consisting of all points t such that $F_{|g^{-1}(t)}$ is injective and set $X_1 := g^{-1}(S_1)$, $S_2 := g(X) \backslash S_1$ and $X_2 := g^{-1}(S_2)$. Suppose that for some $y \in Y$ the set $p(F_{|X_1}^{-1}(y))$ contains a ball T_0, with p the projection to M. Of course we can identify any subset of X_s with a subset of M by projecting on the M-coordinate. When we do so we have the following helpful claim.

Claim. *The set* $X_s \cap F_{|X_1}^{-1}(y)$ *contains a ball* T *for some* s *(after identification with a subset of* M *by projecting on the* M*-coordinate).*

We first prove the claim. Denote by C_s the set $X_s \cap F_{|X_1}^{-1}(y)$. Suppose by contradiction that, for all s, the set C_s contains no ball. Apply (b1) to the sets C_s for all s. Since C_s contains no balls, we find b-maps $f_s : C_s \to S_s$ for some auxiliary sets S_s such that the nonempty fibers of the f_s are points. But then by compactness we find a single b-map $f' : p(F_{|X_1}^{-1}(y)) \to S'$ for some auxiliary S' such that all nonempty fibers of f' are points. Since we have supposed that $p(F_{|X_1}^{-1}(y))$ contains the ball T_0, the restriction of f' to T_0 gives a definable bijection between T_0 and an auxiliary set. This is a contradiction to (b2) and the claim is proven. By the claim g is injective on T, which again gives a contradiction to (b2). Hence, no set of the form $p(F_{|X_1}^{-1}(y))$ contains a ball. Now, for every t in S_2, $g^{-1}(t)$ is contained in a (unique) fiber of $F_{|X_2}$. It follows that F induces a definable surjection $S_2 \to F(X_2)$. If $Y = F(X_2)$, then we have a contradiction to (b2). If $Y \neq F(X_2)$ then there exists a point $y \in Y$ with $F^{-1}(y) = F_{|X_1}^{-1}(y)$ which does not contain a ball as shown above. $\square$

Lemma 2.9. *Let* $\mathcal{T}$ *be an* $\mathcal{L}$*-theory with all its models satisfying* (b1), (b2) *and condition* $(*)$. *Suppose that there are at least two* $\varnothing$*-definable auxiliary points. Then* $\mathcal{T}$ *satisfies* (b3).

Proof. Let $\mathcal{M}$ be a model of $\mathcal{T}$ and let $F : X \to Y$ be A-definable with X, Y subsets of M. We may work piecewise on A-definable sets since there are at least two

$\varnothing$-definable auxiliary points. Let Y_1 be the definable subset of Y consisting of those $y \in Y$ such that $F^{-1}(y)$ contains a ball. Let $f_1 : Y_1 \to S_1$ be a b-map on Y_1. By (∗), all fibers of f_1 are points. Taking $X_1 := F^{-1}(Y_1)$ and $f_1' : X_1 \to S_1 : x \mapsto f_1(F(x))$, we see that f_1' is compatible with $F_{|X_1}$. Hence we may suppose that Y_1 is empty. Let Γ_F be the graph of F. Take a b-map $f_2 : \Gamma_F \to Y \times S$ of Γ_F over Y (thus not over X). Define

$$f : X \to S : x \mapsto p_S \circ f_2(x, F(x)),$$

with p_S the projection $Y \times S \to S$. Then clearly f is compatible with $F|_X$. Indeed, all fibers of f_2 are points, thus for $x_1 \neq x_2$ in X either $f(x_1) \neq f(x_2)$, or $F(x_1) \neq F(x_2)$.

The functions f_i and f can clearly be taken A-definable so (b3) follows.   □

Lemma 2.10. *Let $\mathcal{T}$ be any $\mathcal{L}$-theory with all its models satisfying condition (†). Then all models of $\mathcal{T}$ satisfy* (b2).

Proof. Suppose by contradiction to (b2) that Y is a ball and that

$$g : S \to Y$$

is definable and surjective, with S auxiliary. Let T be a ball. Then the map

$$F : X \to Y : (s,t) \mapsto g(s)$$

with $X = S \times T$ contradicts (†).   □

Corollary 2.11. *Let $\mathcal{T}$ be any $\mathcal{L}$-theory which satisfies* (b1) *and condition* (†) *for all its models $\mathcal{M}$. Suppose that there are at least two $\varnothing$-definable auxiliary points. Then $\mathcal{T}$ satisfies* (b3) *for all its models.*

Proof. Follows by Lemmas 2.9 and 2.10 by noticing that $\mathcal{M}$ satisfies (∗) whenever it satisfies (†).   □

3. Cell Decomposition

Let $\mathcal{L}$ be any expansion of $\mathcal{L}_B$, as before, and let $\mathcal{M}$ be an $\mathcal{L}$-model. Cells are defined by induction on the number of variables.

3.1. *Cells*

Let $X \subset M$ be definable and $f : X \to S$ a definable function with S auxiliary. If all fibers of f are balls, then we call (X, f) a (1)-cell with presentation f. If all fibers of f are points, then we call (X, f) a (0)-cell with presentation f. For short, we call such X a cell and f its presentation.

Let $X \subset M^n$ be definable and let $(j_1, \ldots, j_n)$ be in $\{0, 1\}^n$. Let $p : X \to M^{n-1}$ be a coordinate projection. We call X a $(j_1, \ldots, j_n)$-cell with presentation

$$f : X \to S$$

for some auxiliary S, if for each $\hat{x} := (x_1, \ldots, x_{n-1}) \in p(X)$, the set $p^{-1}(\hat{x}) \subset \{\hat{x}\} \times M$, identified with a subset of M via the projection $\{\hat{x}\} \times M \to M$, is a (j_n)-cell with presentation

$$p^{-1}(\hat{x}) \to S : x_n \mapsto f(\hat{x}, x_n)$$

and $p(X)$ is a $(j_1, \ldots, j_{n-1})$-cell with some presentation

$$f' : p(X) \to S'$$

satisfying $f' \circ p = p' \circ f$ for some definable $p' : S \to S'$.[b]

3.2. *Relative cells*

In the relative setting, when Y and $X \subset M^n \times Y$ are definable sets, we say that X together with a definable function $f : X \to S \times Y$ commuting with the projections $\pi : X \to Y$ and $S \times Y \to Y$ and with S auxiliary is a $(j_1, \ldots, j_n)$-cell over Y with presentation

$$f : X \to S \times Y$$

if the following holds with $p : M^n \times Y \to M^{n-1} \times Y$ a coordinate projection. For each $(\hat{x}, y) := (x_1, \ldots, x_{n-1}, y) \in p(X)$, the set $p^{-1}(\hat{x}, y) \subset \{\hat{x}\} \times M \times \{y\}$, identified with a subset of M via the projection $\{\hat{x}\} \times M \times \{y\} \to M$, is a (j_n)-cell with presentation

$$p^{-1}(\hat{x}, y) \to S : x_n \mapsto f(\hat{x}, x_n, y)$$

and $p(X)$ is a $(j_1, \ldots, j_{n-1})$-cell over Y with some presentation

$$f' : p(X) \to S' \times Y$$

satisfying $f' \circ p = p' \circ f$ for some $p' : S \times Y \to S' \times Y$.

3.3. *b-maps*

Let $X \subset M^n$ and $f : X \to S$ be definable with S auxiliary. By induction on the variables, with $p : X \to M^{n-1}$ the coordinate projection on the first $n-1$ variables, f is called a *b-map* on X when for each $\hat{x} := (x_1, \ldots, x_{n-1}) \in p(X)$, the function

$$p^{-1}(\hat{x}) \to S : x_n \mapsto f(\hat{x}, x_n)$$

is a *b*-map on $p^{-1}(\hat{x})$ as in Sec. 2.2, and there exists some *b*-map

$$f' : p(X) \to S'$$

satisfying $f' \circ p = p' \circ f$ for some $p' : S \to S'$.

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[b]The condition $f' \circ p = p' \circ f$ for some p' could as well be left out from the definition of cells since f can always be refined to imply the existence of f' and p' with this property, see the proof of Lemma 3.3. We chose to include this condition so that a presentation f captures information about f' (and so on) as well. The same remark applies to the definition of relative cells and of *b*-maps in Sec. 3.3.

Working relatively, for $X \subset M^n \times Y$ a definable set, we say that a definable function $f : X \to S \times Y$ over Y is a *b-map* on X over Y if there is a projection $p : X \to M^{n-1} \times Y$ such that for every $(\hat{x}, y) \in p(X)$ the restriction of f to $p^{-1}(\hat{x}, y)$ (also here identified with a subset of M) is a *b*-map on $p^{-1}(\hat{x}, y)$ and there is a *b*-map $f' : p(X) \to S' \times Y$ on $p(X)$ over Y and a definable function $p' : S \times Y \to S' \times Y$ satisfying $f' \circ p = p' \circ f$.

Remark 3.1. The ordering of coordinates on M^n used for cells and *b*-maps, is usually implicitly chosen. Such a choice appears also in the definitions of *o*-minimal and *p*-adic cells.

Lemma–Definition 3.2 (Types of cells). *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y and $X \subset M^n \times Y$ be definable sets. If X is a $(i_1, \ldots, i_n)$-cell over Y, then X is not a $(i'_1, \ldots, i'_n)$-cell over Y (for the same ordering of the factors of M^n) for any tuple $(i'_1, \ldots, i'_n)$ different from $(i_1, \ldots, i_n)$. We call $(i_1, \ldots, i_n)$ the type of the cell X.*

Proof. By induction on n. For $n = 1$, this follows from (b2), cf. Remark 2.2. The image X' of X under the projection $p_n : M^n \times Y \to M^{n-1} \times Y$ is an $(i_1, \ldots, i_{n-1})$-cell over Y and by induction this type is unique. Assume now X is at the same time an $(i_1, \ldots, i_{n-1}, 0)$-cell over Y and an $(i_1, \ldots, i_{n-1}, 1)$-cell over Y. This means that X is at the same time a (1)-cell and a (0)-cell over $M^{n-1} \times Y$ which is impossible again by (b2). $\qquad \square$

Lemma–Definition 3.3 (Refinements). *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y and $X \subset M^n \times Y$ be definable. Then there exists a b-map on X over Y. Moreover, any two b-maps $f : X \to S \times Y$, $f' : X \to S' \times Y$ over Y have a common refinement, namely, a b-map $f'' : X \to S'' \times Y$ over Y with (automatically unique) definable maps $\lambda : f''(X) \to S \times Y$ and $\mu : f''(X) \to S' \times Y$ such that $\lambda \circ f'' = f$ and $\mu \circ f'' = f'$.*

Proof. By compactness (as in the proof of Lemma 2.5) and induction on n (as in the proof of Lemma 3.2). Indeed, for $n = 1$, the existence of a *b*-map on X over Y follows clearly by compactness. For $n > 1$, let $f_0 : X \to S_0 \times M^{n-1} \times Y$ be a *b*-map over $M^{n-1} \times Y$ which exists by the result for $n = 1$. Next, write $p(X)$ for the image of X under the coordinate projection $p : M^n \times Y \to M^{n-1} \times Y$. By induction, there exists a *b*-map $f' : p(X) \to S' \times Y$ over Y. Now let $f : X \to S' \times S_0 \times Y$ be the definable function $x \in X \mapsto (\pi' f'(p(x)), f_0(x))$ with $\pi' : S' \times Y \to S'$ the coordinate projection. Then f is a *b*-map over Y as desired since clearly $f' \circ p = p' \circ f$ with $p' : S' \times S_0 \times Y \to S' \times Y$ the coordinate projection. This proves the existence of *b*-maps on X over Y. The construction of the refinements is done as in the proof of Lemma 2.5. $\qquad \square$

Theorem 3.4 (Cell decomposition). *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y and $X \subset M^n \times Y$ be definable sets. Then there exists a finite partition of X into cells over Y.*

Proof. Same proof as for Lemma 3.3. □

Definition 3.5 (Refinements of cell decompositions). Let Y and $X \subset M^n \times Y$ be definable sets. Let $\mathcal{P}$ and $\mathcal{P}'$ be two finite partitions of X into cells (X_i, f_i), respectively, (X_j', f_j'), over Y. We call $\mathcal{P}'$ a *refinement of $\mathcal{P}$* when for each j there exists i such that

$$X_j' \subset X_i$$

and such that f_j' is a refinement of $f_{i|X_j'}$ in the sense of Lemma–Definition 3.3, or in other words, for each $b \in f_j'(X_j')$, there exists a (necessarily unique) $a \in f_{i|X_j'}(X_j')$ such that

$$f_j'^{-1}(b) \subset f_i^{-1}(a).$$

Lemma 3.6. *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y and $X \subset M^n \times Y$ be definable sets. Then any two cell decompositions of X over Y admit a common refinement.*

Proof. As for Lemma 2.5. □

Remark 3.7. In fact, the results of this section on cell decomposition already hold for a theory satisfying only (b1) and (b2) for all its models.

4. Dimension Theory

We now develop a dimension theory for b-minimal structures along similar lines as what is done for o-minimal theories, cf. [18]. In what follows, $\mathcal{L}$ is any expansion of $\mathcal{L}_B$ as before and $\mathcal{M}$ is an $\mathcal{L}$-model.

Definition 4.1. The dimension of a nonempty definable set $X \subset M^n$ is defined as the maximum of all sums

$$i_1 + \cdots + i_n,$$

where $(i_1, \ldots, i_n)$ runs over the types of all cells contained in X, for all orderings of the n factors of M^n. To the empty set we assign the dimension $-\infty$.

If $X \subset S \times M^n$ is definable with S auxiliary, the dimension of X is defined as the dimension of $p(X)$ with $p : S \times M^n \to M^n$ the projection.

When $F : X \to Y$ is an $\mathcal{L}$-definable function, the dimension of X over Y is defined as the maximum of the dimensions of the fibers $F^{-1}(y)$ over all $y \in Y$. (Of course, the dimension of X over Y depends on F.)

We write $\dim(X/Y)$ for the dimension of X over Y, and $\dim(X)$ for the dimension of X. The dimension of X over Y is also called the relative dimension of X

over Y (along F). Usually, F is implicit and X is just called a definable set over Y, see the discussion below Definition 2.1.

Proposition 4.2. *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y, W, Z be definable sets, let X be a $(i_1, \ldots, i_n)$-cell over Y, and let A, B, C be definable sets over Y with $A, B \subset C$. Then*

(0) $\dim(X/Y) = i_1 + \cdots + i_n$,
(1) $\dim(A \cup B/Y) = \max(\dim(A/Y), \dim(B/Y))$,
(2) $\dim(W \times Z) = \dim(W) + \dim(Z)$.

Proof. Let us prove (0). First we notice that for any definable subset $E \subset X$ which is an $(i_{E_1}, \ldots, i_{E_n})$-cell over Y, with the same ordering of coordinates, one has $i_{E_j} \leq i_j$ for $j = 1, \ldots, n$. For $n = 1$, this follows from (b2), and for $n > 1$, this is proven by induction on n similarly as in the proof of Lemma–Definition 3.2.

Next we show that for any definable subset $E \subset X$ which is an $(i_{E_1}, \ldots, i_{E_n})$-cell over Y with respect to a different order of the coordinates on M^n, one has $i_{E_1} + \cdots + i_{E_n} \leq i_1 + \cdots + i_n$. This is clear for $n = 1$, so let us consider the case $n = 2$. We may assume Y is a point. Assume first that $i_{E_1} = i_{E_2} = 1$. We want to prove that $i_1 = i_2 = 1$. By (b2), or rather by Remark 2.2, one finds $i_1 = 1$. Denote by $p_{2X} : M^2 \to M$ and $p_{1X} : M \to M^0$ the projections corresponding to the order of coordinates for the cell X, and by $p_{2E} : M^2 \to M$ and $p_{1E} : M \to M^0$ the projections corresponding to the order of coordinates for the cell E. We may assume that the image of E by p_{2E} is a ball T and that all fibers of the restriction of p_{2E} to E contain a ball, since E is a $(1,1)$-cell. Assume now that $i_2 = 0$. This means that X is a 0-cell over M with respect to the projection p_{2X}, thus, there exists an injective map $g : X \to S \times M$ over M, with S auxiliary. Hence, the map

$$F := p_{2E} \circ g_{|E}^{-1} : g(E) \to M$$

gives a surjection from $g(E)$ to the ball T. Moreover, $pF^{-1}(t)$ contains a ball for each $t \in T$ with $p : S \times M \to M$ the projection, which contradicts condition (†) of Lemma 2.7.

To conclude the case $n = 2$, it is enough to prove that if $i_1 = i_2 = 0$, then $i_{E_1} = i_{E_2} = 0$. But if $i_1 = i_2 = 0$, then X is definably isomorphic to a definable subset of some auxiliary sorts, which makes it impossible for E to contain a ball by (b2), hence forces $i_{E_1} = i_{E_2} = 0$. Now for general n, it is enough to consider a transposition $((x_1, \ldots, x_j, x_{j+1}), \ldots, x_n) \mapsto ((x_1, \ldots, x_{j+1}, x_j), \ldots, x_n)$ of two adjacent coordinates. By induction on n and by projecting onto the first $j + 1$ coordinates, $x_1, \ldots, x_{j+1}$, one may suppose that $j + 1 = n$ and one reduces to the cases already considered. Statement (0) follows.

Proving (1) amounts to showing that if X is an $(i_1, \ldots, i_n)$-cell over Y and X_j is a finite partition of X into $(i_{j1}, \ldots, i_{jn})$-cells with respect to the same ordering of the coordinates, then $\max_j(i_{j1} + \cdots + i_{jn}) = \dim X$, which is clear when $n = 1$ and follows by induction on n when $n > 1$. Property (2) is clear by the previous

properties since partitions of W and Z into cells induce a partition of $W \times Z$ into cells. □

Recall that, in our terminology, a definable set over another definable set has the meaning as explained after Definition 2.1.

Proposition 4.3. *Let $\mathcal{M}$ be a model of a b-minimal theory. Let Y be a definable set, let X and X' be definable sets over Y, and let $f : X \to X'$ be a definable function over Y, that is, compatible with the maps to Y. Then*

(3) *$\dim(X) \geq \dim(f(X))$, hence also $\dim(X/Y) \geq \dim(f(X)/Y)$.*
(4) *For each integer $d \geq 0$, the set $S_f(d) := \{x' \in X' | \dim(f^{-1}(x')/Y) = d\}$ is definable and*

$$\dim(f^{-1}(S_f(d))/Y) = \dim(S_f(d)/Y) + d,$$

with the convention $-\infty + d = -\infty$.
(5) *If Y is auxiliary, then $\dim(X/Y) = \dim(X)$.*

Proof. We first prove (5). We reduce to the case that X is a definable subset of $M^n \times Y$, as follows. By the definition of relative dimensions we may replace X be the graph of $X \to Y$ so that X becomes a definable subset of $M^n \times S \times Y$ for some auxiliary S and some $n \geq 0$. Again by the definition of relative dimensions, we may replace Y by $S \times Y$ to conclude our reduction to the case that X is a definable subset of $M^n \times Y$. By property (1) of Proposition 4.2 and Theorem 3.4, we may then suppose that X is an $(i_1, \ldots, i_n)$-cell over Y. Now (5) follows similarly to the way that (0) of Proposition 4.2 is proven. Namely, if $n = 1$, (5) follows from (b2), for $n = 2$, it follows from property (†), and for $n > 2$, one uses induction.

For (3) and (4), we may suppose that Y is a point, since relative dimension over Y is defined as the maximum of the dimensions of the fibers of $y \in Y$.

For (4), let $\Gamma_f \subset X' \times X$ be the graph of f (more precisely, the transpose of the graph). We have $X \subset M^n \times S$ and $X' \subset M^m \times S'$ for some auxiliary sets S and S' and some $m, n \geq 0$. We first prove the property when S and S' are singletons, that is, when $X \subset M^n$ and $X' \subset M^m$. By Proposition 4.2 and Theorem 3.4, we may suppose that Γ_f is an $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n})$-cell. Suppose first that $i_{m+1} + \cdots + i_{m+n} = d$. Then $X' = S_f(d)$, $X = f^{-1}(S_f(d))$, $i_1 + \cdots + i_m = \dim(X')$, and $i_1 + \cdots + i_{m+n} = \dim(\Gamma_f) = \dim(X)$ by Proposition 4.2. Hence (4) follows. When $i_{m+1} + \cdots + i_{m+n} \neq d$ the set $S_f(d)$ is empty and there is nothing to prove. The case of general S now follows from (5) and compactness. Finally, the case of general S' follows from compactness.

Statement (3) is a corollary of (4) and of (1) of Proposition 4.2. □

5. Preservation of Balls

The following property seems especially useful for Henselian valued fields in the context of motivic integration. Definition 5.1 is used for the change of variables in one

variable in motivic integration in [9]; in an *o*-minimal structure it is a consequence of the Monotonicity theorem.

Definition 5.1 (Preservation of balls). Let $\mathcal{M}$ be a *b*-minimal $\mathcal{L}$-structure. We say that $\mathcal{M}$ *preserves balls* if for every set of parameters A and every A-definable function

$$F : X \subset M \to Y \subset M$$

there is an A-definable *b*-map

$$f : X \to S$$

such that for each $s \in S$ the set

$$F(f^{-1}(s))$$

is either a ball or a point.

If moreover f can be chosen in a way that for every refining *b*-map $f_1 : X \to S_1$ the set

$$F(f_1^{-1}(s_1))$$

is also either a ball or a point for each $s_1 \in S_1$, then we say that $\mathcal{M}$ *preserves all balls*.

We say that a *b*-minimal theory *preserves balls* (respectively, *preserves all balls*) when all its models do.

Remark 5.2. In an $\mathcal{L}$-model $\mathcal{M}$ of a *b*-minimal theory which preserves all balls, one has the following property which is the *b*-minimal analogue of the Monotonicity theorem for *o*-minimal structures, and which can be proven by taking refinements of *b*-maps. For any A-definable function $F : X \to Y$, with X, Y subsets of M, there is an A-definable *b*-map $f : X \to S$ such that for each $s \in S$ the restriction $F_{|f^{-1}(s)}$ is either injective or constant and $F(f^{-1}(s))$ is either a ball or a point and such that similar properties hold for each refinement of f.

6. *b*-Minimality with Centers

This section is about approximating balls (occurring in definable families) by definable functions. Indeed, sometimes it is useful to "center" balls around some "center" points, not necessarily lying "in" the ball, but just lying "close" to the ball. In our treatment we are guided by the situation for characteristic zero Henselian valued fields. In an *o*-minimal context this notion seems irrelevant. In Henselian valued fields, centers are heavily used for the classification of semi-algebraic *p*-adic sets in [4], for the definition of motivic integrals in [9] and of exponential motivic integrals in [10], for the Fubini theorem for motivic integrals in [9], and for finding estimates for *p*-adic exponential sums in [6].

Define $\mathcal{L}'_B$ as

$$\mathcal{L}'_B := \mathcal{L}_B \cup \{B_n\}_n,$$

the language $\mathcal{L}_B$ together with predicates B_n for n running in some index set, to be interpreted in any model in such a way that

$$B_n \subset A_B \times M$$

for each index n, with A_B as defined in Sec. 2.

Let $\mathcal{L}$ be any extension of $\mathcal{L}'_B$ and let $\mathcal{M}$ be an $\mathcal{L}$-model. For $a \in A_B$, write

$$B_n(a) := \{m \in M \mid (a, m) \in B_n\}.$$

Definition 6.1. A point x in M is called a B_n-*center of a ball* Y when there is $a \in A_B$ such that $Y = B(a)$ and $x \in B_n(a)$. A point x in M is called a B_n-*center of a point* y when $x = y$.

Let $f : X \subset M \to S$ be a b-map. A map $c : f(X) \to M$ is called a B_n-*center of f* if $c(s)$ is a B_n-center of $f^{-1}(s)$ for each $s \in f(X)$. When there exists such n, we call c a *center* for f.

Centers for b-maps $f : X \subset Y \times M \to Y \times S$ over Y are defined similarly.

In the context of Henselian valued fields, the index n for a B_n-center of a ball describes the distance between the ball and the center, see Sec. 7.2, where $n > 0$ is an integer.

Definition 6.2. Let $\mathcal{M}$ be a b-minimal $\mathcal{L}$-model. Say that $\mathcal{M}$ is *b-minimal with $\{B_n\}_n$-centers*, if every set of parameters A and every A-definable b-map $f : X \subset M \to S$ has an A-definable refinement f' with an A-definable B_n-center for some n.

We say an $\mathcal{L}$-theory is *b-minimal with $\{B_n\}_n$-centers* (or b-minimal with centers for short), if all its models are.

The definition of cells can be adapted naturally to a definition for cells with $\{B_n\}_n$-centers, as follows.

Definition 6.3 (Cells with $\{B_n\}_n$-centers). Let $X \subset M \times Y$ be a cell over Y with presentation $f : X \to S \times Y$. A definable function

$$c : f(X) \to M$$

is called a B_n-center of the cell (X, f) when it is a B_n-center of the b-map f over Y. The triple (X, f, c) is called a cell over Y with B_n-center c (and presentation f).

Let $X \subset M^n \times Y$ be a cell over Y with presentation $f : X \to S \times Y$. Let $p : X \to M^{n-1} \times Y$ be the coordinate projection. By induction, call

$$(X, f, c_1, \ldots, c_n)$$

a cell over Y with $(B_{m_1}, \ldots, B_{m_n})$-center $(c_1, \ldots, c_n)$ if (X, f, c_n) is a cell over $M^{n-1} \times Y$ with B_{m_n}-center c_n and $(p(X), f', c_1, \ldots, c_{n-1})$ is a $(j_1, \ldots, j_{n-1})$-cell with $(B_{m_1}, \ldots, B_{m_{n-1}})$-center $(c_1, \ldots, c_{n-1})$ and some presentation

$$f' : p(X) \to S'$$

satisfying $f' \circ p = p' \circ f$ for some $p' : S \to S'$.

For short, call X a *cell with $(B_{m_j})_j$-center* (over Y) when there exist such f and c_i, or call X a *cell with $\{B_n\}_n$-center* (over Y) when there exists such tuple $(B_{m_j})_j$.

Theorem 6.4 (Cell decomposition with centers). *Let T be a b-minimal theory with $\{B_n\}_n$-centers and let $\mathcal{M}$ be a model. Let Y and $X \subset M^n \times Y$ be definable sets. Then there exists a finite partition of X into cells with $\{B_n\}_n$-centers (over Y).*

Proof. As for Theorem 3.4.   □

Remark 6.5. The analogue of Lemma 3.6 for cells with centers holds, with the same proof.

7. Some Examples of b-Minimal Structures

7.1. *o-minimal structures and non-o-minimal expansions*

Any o-minimal structure R admits a natural b-minimal expansion by taking as main sort R with the induced structure, the two point set $\{0, 1\}$ as auxiliary sort and two constant symbols to denote these auxiliary points. A possible interpretation for B is easy to find, for example,

$$
\begin{aligned}
B = \{(x, y, m) \in R^2 \times R | \; & x < m < y \text{ when } x < y, \\
& x < m \text{ when } x = y, \\
& \text{and } m < y \text{ when } x > y\},
\end{aligned}
$$

so that in the m variable one gets all open intervals as fibers of B above R^2. Property (b3) and preservation of all balls is in this case a corollary of the Monotonicity theorem for o-minimal structures.

The notion of b-minimality leaves much more room for expansions than the notion of o-minimality: some structures on the real numbers are not o-minimal but are naturally b-minimal, for example, the field of real numbers with a predicate for the integer powers of 2 is b-minimal by [17] when adding to the above language the set of integer powers of 2 as auxiliary sort and the natural inclusion of it into $\mathbf{R}$ as function symbol.

Let us make this more precise. Following van den Dries [17], let us consider $(\mathbf{R}, 2^{\mathbf{Z}})$ as an ordered field with the multiplicative group $2^{\mathbf{Z}}$ as a distinguished subset, so $(\mathbf{R}, 2^{\mathbf{Z}})$ is a structure in the language L of ordered rings augmented with a 1-variable predicate A. In the paper [17], van den Dries considers the set of axioms Σ for L-structures (R, A) expressing that R is a real closed field, A a multiplicative subgroup of positive elements, that 2 is in A, no x in $(1, 2)$ belongs to A and that every $x > 0$ lies in $[y, 2y)$ for some y in A. He shows that Σ axiomatizes the complete theory of $(\mathbf{R}, 2^{\mathbf{Z}})$. This is done by adding to the language L new 1-variable predicates P_i, for $i = 1, \ldots, n, \ldots$, and a 1-variable function symbol λ and adding to Σ axioms expressing that $P_n(x)$ holds if x is the nth power of an element in A, that $\lambda(x) = 0$ for $x \leq 0$ and that, for $x > 0$, $\lambda(x)$ lies in A and $\lambda(x) \leq x < 2\lambda(x)$,

and proving that the L^*-theory Σ^* obtained by adding the new axioms to Σ admits elimination of quantifiers.

Let us consider the two sorted language $\mathcal{L}$ having as main sort $R = M$ with the language of ordered rings, auxiliary sort A' with the language of Presburger groups $(0, 1, +, -, \leq, \{\equiv_n\}_n)$, two function symbols $\lambda' : M \to A'$ and $a : A' \to M$, and the ball predicate B as above.

Let Σ' be the $\mathcal{L}$-theory saying that R is a real closed field, A' is a **Z**-group, $a : A' \to R^\times$ is a homomorphism of ordered groups, that $2 = a(1)$, that every $x \in R$ with $x > 0$ lies in the interval $[a(\lambda'(x)), 2a(\lambda'(x)))$, and that $\lambda'(x)$ is 0 for $x \leq 0$. (It follows from Σ' that $a(0) = 1$ and, since a preserves the order, there lies no element of the form $a(y)$ between 1 and 2.)

The following proposition is essentially a corollary of the quantifier elimination result of [17].

Proposition 7.1. *Use notation from Sec. 7.1. The theory Σ' is b-minimal, preserves all balls, and eliminates quantifiers of the sorts R and A'.*

Proof. Axiom (b2) is clear. Axioms (b1) and (b3), quantifier elimination and preservation of all balls follow from the quantifier elimination result of [17]. One can also use criteria 2.7 or 2.6 to prove (b3). □

It is tempting to hope that the field **R** admits nontrivial b-minimal expansions with entire analytic functions other than the entire exponential function. Results in this direction have been obtained by Miller in [23] and independently by Wilkie in [28], relatively to an auxiliary sort similar to $2^{\mathbf{Z}}$; Miller and Wilkie add functions like $\sin(\log x)$ on the positive real line to the real field. A more careful study is needed.

7.2. *Henselian valued fields of characteristic zero*

In this section, we prove that the theory of Henselian valued fields of characteristic zero is b-minimal, in a natural definitional expansion of the valued field language, by adapting the Cohen–Denef proof scheme of cell decomposition [11–15]. We present a shorter and somewhat different version of that proof, which, we hope, will enable the reader to see better the main points in the proof, and to recover results by Cohen, Denef and Pas [11, 15, 25, 26]. For a more detailed proof which is much closer to the original decision procedure by Cohen, see Pas [26]. (The more detailed proof in [26] is also word for word adaptable to the unbounded ramified case.) Note that the shortcut to prove p-adic cell decomposition as presented by Denef in [13] does not yet generalize to any other than the p-adic setting.

As far as we know, this is the first written instance of cell decomposition in mixed characteristic for unbounded ramification. (By bounded ramification we mean that there exists an integer $n > 0$ such that $\mathrm{ord}(x^n) > \mathrm{ord}(p)$ for any element x of the maximal ideal, unbounded ramification being the negation of this property.) We

moreover prove that all definable functions are essentially given by terms. Preservation of all balls follows.

Let Hen denote the collection of all Henselian valued fields of characteristic zero (hence, mixed characteristic, as well as equal characteristic zero are allowed).

For K in Hen, write K° for the valuation ring, Γ_K for the value group, ord: $K^\times \to \Gamma_K$ for the valuation, M_K for the maximal ideal of K°, and $\widetilde{K}$ for the residue field.

For $n > 0$ an integer, set $nM_K = \{nm \mid m \in M_K\}$ and consider the natural group morphism

$$rv_n : K^\times \to K^\times/1 + nM_K$$

which we extend to $rv_n : K \to (K^\times/1 + nM_K) \cup \{0\}$ by sending 0 to 0.

For every $n > 0$, we write $RV_n(K)$ for

$$RV_n(K) := (K^\times/1 + nM_K) \cup \{0\},$$

and we also write rv for rv_1 and RV for RV_1.

We use the norm notation $|\cdot|$ for the multiplicative norm associated to the additively written ord$(\cdot)$; any formula with $|\cdot|$ is an abbreviation of the analogous formula with ord instead of $|\cdot|$.

We define the family $B(K)$ of balls by

$$B(K) = \{(a, b, x) \in K^\times \times K^2 \mid |x - b| < |a|\}.$$

Hence, a ball is by definition any set of the form $B(a, b) = \{x \in K \mid |x - b| < |a|\}$ with a nonzero. For the centers we consider the family $\{B_n\}_n$ over integers $n > 0$, with

$$B_n(K) := \{(a, b, x) \in K^\times \times K^2 \mid |x - b| = |n^{-1}a|\}.$$

A center in $B_n(K)$ for a ball lies at distance $|n^{-1}|$ from that ball. Such a center is useful to describe the ball using definable parameters, as is explained in the following remark.

Remark 7.2 (Centers). For any $n > 0$, any nonzero $\xi \in RV_n$, and any $h \in K$, the set

$$X := \{x \in K \mid rv_n(x - h) = \xi\} \tag{7.1}$$

is an open ball of the form

$$\{x \in K \mid \text{ord}(x - b) > \text{ord}(n(b - h))\}$$

for any $b \in X$. Often, none of the points b is definable (over a certain set of parameters or in a certain definable family of balls) while h and ξ are definable (uniformly in the family). This is the advantage of the description (7.1) of the open ball X and justifies the use of centers. Indeed, with the notation of Secs. 2.2 and 6.1, X is the ball $B(n(b - h), b)$ and h is a center for X, that is, h lies in $B_n(n(b - h), b)$. If some theory has definable centers, then there is a definable choice for h and thus

one can still describe the ball X by Eq. (7.1), even if b is not definable. Given X and h, the element ξ is unique in Eq. (7.1). The converse is also true: if X is a ball $B(a, b)$ and h lies in $B_n(a, b)$, then X can be written as $\{x \in K | rv_n(x - h) = \xi\}$ for a unique ξ.

We consider the following language L_{Hen}: it consists of the language of rings $(+, -, \cdot, 0, 1)$ for the valued field sort which is the main sort, together with function symbols rv_n for integers $n > 0$ from the main sort into the RV_n which are the auxiliary sorts, and the inclusion language as defined below on the auxiliary sorts.

We denote by T_{Hen} the theory of all fields in Hen in the language L_{Hen}.

7.2.1. *The inclusion language on the RV_n*

Let K be in Hen. For $a_i \in RV_{n_i}(K)$, $b_j \in K$ and for f, g polynomials over $\mathbf{Z}$ in $n + m$ variables, we let the expression

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

correspond to the following subset of K

$$\{x \in K \mid (\exists y \in K^n)(f(y, b) = x \wedge_i rv_{n_i}(y_i) = a_i)\}.$$

Thus, the expression $f(a, b)$ stands for a kind of image of the restriction of f to a specific domain. By an inclusion

$$f(a, b) \subset g(a, b) \tag{7.2}$$

of such expressions, with a and b tuples as above, we shall mean the inclusions of the corresponding subsets of K.

The inclusion language $\mathcal{L}_{RV}$ on the sorts RV_n, $n > 0$, consists of the three symbols $+, \cdot, \subset$, interpreted as the restriction to (all Cartesian products of) the sorts RV_n of the relation explained in (7.2). (There are no terms in this language, only relations of the form $f(x) \subset g(x)$, for f and g polynomials formed with $+$ and $\cdot$ in variables x that run over some of the sorts RV_n.)

7.2.2. *An alternative language on the RV_n sorts*

Historically, in [1, 2, 27], other languages were considered for the auxiliary sorts RV_n for elimination of valued field quantifiers. We define a variant $\mathcal{L}_{RV}^{\text{alt}}$ of these languages on the RV_n sorts that is most closely related to the variant of Scanlon [27] (but without the structure of valued D-field in the terminology of [27], or, put otherwise, the language of [27] with trivial D-structure). We show that our language $\mathcal{L}_{RV}$ on the RV_n sorts is definitionally equivalent to that alternative language $\mathcal{L}_{RV}^{\text{alt}}$ in the sense that the same sets are definable in both languages. We also show that L_{Hen} eliminates valued field quantifiers. (The same reasoning holds if one would use the original Basarab language [1].)

The language $\mathcal{L}_{RV}^{\mathrm{alt}}$ puts on the RV_n the structure of partially ordered multiplicative semi-groups (it is a multiplicative group with an annihilating zero-element, hence a semi-group). The partial order is the one induced by ord on the valued field. That is, $\mathrm{ord}(a) < \mathrm{ord}(b)$ for $a, b \in RV_n$ if and only if for a', b' in K with $rv_n(a') = a$, $rv_n(b') = b$ one has $\mathrm{ord}(a') < \mathrm{ord}(b')$. Apart from this structure, there are for each m dividing n the natural projection map $RV_n \to RV_m$ (also denoted by rv_m which is harmless since rv_m on K factorizes through RV_n via rv_n) and a partial binary function $+_{n,m}$ from RV_n^2 into RV_m sending (a, b) to c if and only c is the unique element satisfying $rv_m(c') = c$ for any a', b', c' in K with $rv_n(a') = a$, $rv_n(b') = b$ and $a' + b' = c'$.

Clearly all symbols of $\mathcal{L}_{RV}^{\mathrm{alt}}$ are valued field quantifier free definable in our language $\mathcal{L}_{RV}$.

The alternative language $L_{\mathrm{Hen}}^{\mathrm{alt}}$ is then the language of rings $(+, -, \cdot, 0, 1)$ for the valued field sort together with the function symbols rv_n for integers $n > 0$ and together with the language $\mathcal{L}_{RV}^{\mathrm{alt}}$. We know by [27] (here just with trivial D-structure) that $L_{\mathrm{Hen}}^{\mathrm{alt}}$ eliminates valued field quantifiers. Hence, also our language L_{Hen} eliminates valued field quantifiers. Moreover, both $L_{\mathrm{Hen}}^{\mathrm{alt}}$ and L_{Hen} are definitional expansions of the language $\mathcal{L}_0$ which has the ring language for the valued field sort, the RV_n sorts, and the function symbols rv_n for $n > 0$. We claim that this implies that also the languages $\mathcal{L}_{RV}^{\mathrm{alt}}$ and $\mathcal{L}_{RV}$ are definitionally equivalent. We only have to prove one direction since the other is already shown. Take any $\mathcal{L}_{RV}$-definable subset X in the RV_n sorts. This is also definable in the language $\mathcal{L}_0$, possibly using valued field quantifiers, because L_{Hen} is a definitional expansion of $\mathcal{L}_0$. Hence, X is also definable in the language $L_{\mathrm{Hen}}^{\mathrm{alt}}$ because this is a definitional expansion of $\mathcal{L}_0$. By elimination of valued field quantifiers in $L_{\mathrm{Hen}}^{\mathrm{alt}}$ and since the elements $rv_n(k)$ with k integers are definable in $\mathcal{L}_{RV}^{\mathrm{alt}}$, the set X is $\mathcal{L}_{RV}^{\mathrm{alt}}$-definable.

We have proven the following variant of the results of [1, 2, 27].

Proposition 7.3 (Elimination of valued field quantifiers). *The theory T_{Hen} admits elimination of valued field quantifiers in the language L_{Hen}. To be precise, for any L_{Hen}-formula φ there exists an L_{Hen}-formula ψ without quantifiers running over the valued field so that φ and ψ are equivalent over T_{Hen}, that is, so that T_{Hen} proves $\varphi \leftrightarrow \psi$.*

Remark 7.4. Recently, Denef [16] gave a new, alternative, proof of quantifier elimination for Henselian valued fields based on monomialization (which is a strong kind of resolution of singularities, see [12]).

The main result of this section is the following.

Theorem 7.5. *The theory T_{Hen} is b-minimal with $\{B_n\}_n$-centers. Moreover, T_{Hen} preserves all balls.*

Before we come to the proofs of the main Theorems 7.5 and 7.8 of this section, we establish two technical lemmas which yield a variant of Cohen's proof [11] of cell decomposition. The first lemma is a corollary of Hensel's lemma.

Lemma 7.6. *Let K be in* Hen*. Let*

$$f(y) = \sum_{i=0}^{m} a_i y^i$$

be a polynomial in y with coefficients in K, let $n > 0$ an integer, and let $x_0 \neq 0$ be in $RV_n(K)$. Assume that there exist $i_0 > 0$ and $x \in K$ satisfying the following conditions (7.3)–(7.6),

$$rv_n(x) = x_0, \tag{7.3}$$

$$\mathrm{ord}(a_{i_0} x^{i_0}) \text{ is minimal among the } \mathrm{ord}(a_i x^i) \tag{7.4}$$

meaning that $\min_{0 \le i \le m} \mathrm{ord}(a_i x^i) = \mathrm{ord}(a_{i_0} x^{i_0})$,

$$\mathrm{ord}(f(x)) > \mathrm{ord}(n^2 a_{i_0} x^{i_0}), \tag{7.5}$$

and

$$\mathrm{ord}(f'(x)) \le \mathrm{ord}(n a_{i_0} x^{i_0 - 1}). \tag{7.6}$$

Then there exists a unique $y_0 \in K$ with

$$f(y_0) = 0 \quad and \quad rv_n(y_0) = x_0. \tag{7.7}$$

Furthermore, if one writes $f(y) = \sum_i b_i (y - y_0)^i$, then for any w satisfying

$$rv_n(w) = x_0, \tag{7.8}$$

one has moreover

$$\mathrm{ord} f(w) = \mathrm{ord}\, b_1 (w - y_0). \tag{7.9}$$

Proof. First consider the case where all coefficients a_i lie in the valuation ring K° and where a_{i_0} and x are units in K°. Then conditions (7.5) and (7.6) read $\mathrm{ord}(f(x)) > \mathrm{ord}(n^2)$ and $\mathrm{ord}(f'(x)) \le \mathrm{ord}(n)$. Thus, the existence of y_0 satisfying $y_0 \equiv x \bmod nM_K$ and $f(y_0) = 0$ follows from Hensel's lemma. But $y_0 \equiv x \bmod nM_K$ is equivalent to $rv_n(y_0) = x_0$ since x is a unit and $rv_n(x) = x_0$ and (7.7) follows. To prove (7.9), let w satisfy (7.8) and write $f(y) = \sum_i b_i (y - y_0)^i$. Write

$$f(w) = b_1 (w - y_0) + b_2 (w - y_0)^2 + \cdots,$$

and

$$f'(w) = b_1 + 2b_2 (w - y_0) + \cdots,$$

where the dots represent higher order terms. By (7.3), (7.7) and (7.8) and since $\mathrm{ord}(f'(x)) \le \mathrm{ord}(n)$, it follows that $\mathrm{ord}(f'(w)) \le \mathrm{ord}(n)$ and $\mathrm{ord}(y_0 - w) > n$. Hence, from the expression for $f'(w)$, one gets $\mathrm{ord}(b_1) \le \mathrm{ord}(n)$. Clearly $\mathrm{ord}(b_j) \ge 0$ for each j since the a_i and y_0 lie in K°. Now (7.9) is clear from the expression for $f(w)$.

The general case follows after changing coordinates. See Pas [25, Lemma 3.5] or [26] for explicit change of variables. In short, one replaces the variable y by

$z := y/x_1$ where x_1 is an arbitrary but fixed element satisfying (7.3)–(7.6) for $x = x_1$. Now $g(z) := f(y)/a_{i_0}x_1^{i_0}$ is as in the first case of the proof with the candidate zero of g equal to 1. Hence there exists a unique root z_0 of $g(z)$ with $rv_n(z_0) = rv_n(1)$. Moreover, if one writes $g(z) = \sum_i d_i(z - z_0)^i$, then $\mathrm{ord}g(v) = \mathrm{ord}d_1(v - z_0)$ for any v with $rv_n(v) = rv_n(1)$. Thus $y_0 := z_0x_1$ is a root of $f(y)$ with $rv_n(y_0) = rv_n(x_1) = x_0$. If y_0 with $f(y_0) = 0$ and $rv_n(y_0) = x_0$ were not unique then this would contradict the uniqueness of z_0 and thus (7.7) follows. To prove (7.9), we may suppose that $\mathrm{ord}a_{i_0}x_1^{i_0} = 0$. For w satisfying (7.8) put $v = w/x_1$. Then $rv_n(v) = rv_n(1)$ and we compute

$$\mathrm{ord}f(w) = \mathrm{ord}g(v) = \mathrm{ord}d_1(v - z_0) = \mathrm{ord}\frac{d_1}{x_1}(w - y_0) = \mathrm{ord}b_1(w - y_0)$$

since $b_1 = d_1/x_1$. Indeed,

$$f(y) = \sum_i b_i(y - y_0)^i = \sum_i x_1^i b_i(y/x_1 - y_0/x_1)^i = \sum_i x_1^i b_i(z - z_0)^i = \sum_i d_i(z - z_0)^i.$$

This proves (7.9). □

Definition 7.7 (Henselian functions). Let K be in Hen. For all integers $m \geq 0$, $n > 0$, define the function

$$h_{m,n} : K^{m+1} \times RV_n(K) \to K$$

as the function sending the tuple $(a_0, \ldots, a_m, x_0)$ with nonzero x_0 to b if there exist i_0 and x that satisfy the conditions (7.3)–(7.6) of Lemma 7.6 and where b is the unique element satisfying (7.7), and sending $(a_0, \ldots, a_m, x_0)$ to 0 in all other cases.

Define L^*_{Hen} as the union of the language L_{Hen} together with all the functions $h_{m,n}$.

A second main result of this section is the following generalization of some results of [7–9].

Theorem 7.8 (Term structure). *Let K be in* Hen. *Let $f : X \to K$ be an $L_{\mathrm{Hen}}(A)$-definable function for some set of parameters A. Then there exists an $L_{\mathrm{Hen}}(A)$-definable function $g : X \to S$ with S auxiliary such that*

$$f(x) = t(x, g(x)) \tag{7.10}$$

for each $x \in X$ and where t is an $L^*_{\mathrm{Hen}}(A)$-term.

Remark 7.9. Note that neither L^*_{Hen} nor L_{Hen} have a symbol for the field inverse on $K^\times$. Indeed, the field inverse is not needed for Theorem 7.8 since the term $h_{1,1}(-1, x, \xi)$ with $rv(x)\xi = 1$ yields a field inverse on $K^\times$.

The following lemma is a one-parameter variant of [25, Theorem 3.1].

Lemma 7.10. *Let K be in* Hen *and let $f(y)$ be a polynomial over K in the K-variable y of degree d. Write A for the subset of K consisting of the coefficients*

of f. Then:

(i) *There exist an integer $k > 0$ and an $L_{\mathrm{Hen}}(A)$-definable b-map*

$$\lambda : K \to S$$

with B_k-center

$$c : \lambda(K) \to K$$

such that, if one writes, for $s = \lambda(y)$,

$$f(y) = \sum_i a_i(s)(y - c(s))^i,$$

then

$$\mathrm{ord} f(y) \leq \min_{i=0}^{d} \mathrm{ord} k a_i(s)(y - c(s))^i.$$

(ii) *One can ensure that c is given by an $L_{\mathrm{Hen}}^*(A)$-term.*

Proof. We work by induction on d. For $d = 0$, the statements are trivial, so suppose $d > 0$. Let $f'(y)$ be the derivative of f with respect to y. Apply the induction hypothesis to f'. This way, we find a b-map

$$\lambda_0 : K \to S_0$$

with B_{k_0}-center

$$c_0 : S_0 \to K$$

for some k_0 which satisfy (i) and (ii) for f'.

Since c_0 is an $L_{\mathrm{Hen}}^*(A)$-term, there are L_{Hen}^*-terms $a_i(s)$ such that for all $y \in K$ and $s = \lambda_0(y)$, one has

$$f(y) = \sum_{i=0}^{d} a_i(s)(y - c_0(s))^i.$$

Let y' be the function

$$y' : K \to K : y \mapsto y - c_0 \circ \lambda_0(y)$$

and write a_i' for the function

$$a_i' : K \to K : y \mapsto a_i(\lambda_0(y)).$$

Claim 1. *The set $\lambda_0^{-1}(s)$ for $s \in \lambda_0(K)$ is equal to*

$$\{y \in K \mid rv_{k_0}(y'(y)) = s_1\},$$

for s_1 in RV_{k_0} only depending on s. Hence, s_1 depends in an $L_{\mathrm{Hen}}(A)$-definable way on s.

Claim 2. *For every nonzero multiple ℓ of k_0, we may assume that c_0 is a B_ℓ-center (satisfying (i) and (ii) for f').*

Let us prove the claims. Claim 1 follows from the fact that c_0 is a k_0-center of λ_0 and the description of centers and balls in Remark 7.2. Indeed, for each s in $\lambda_0(K)$, either $\lambda_0^{-1}(s)$ is a ball $B(a,b)$ for some a, b (in the notation of Sec. 2.2) and then $c_0(s)$ lies in $B_{k_0}(a,b)$ and thus one can use the description of Remark 7.2, or, $\lambda_0^{-1}(s)$ is a singleton and then $rv_{k_0}(y'(y)) = 0$. That s_1 depends in an $L_{\text{Hen}}(A)$-definable way on s follows since s_1 is uniquely determined by $\lambda_0^{-1}(s)$ and c_0 as explained in Remark 7.2.

For Claim 2 and ℓ a nonzero multiple of k_0, replace

$$\begin{aligned}
S_0 &\quad \text{by } RV_\ell(K) \times S_0, \\
\lambda_0 &\quad \text{by } (rv_\ell \circ y', \lambda_0), \\
c_0 &\quad \text{by } c_0 \circ p, \text{ and} \\
k_0 &\quad \text{by } \ell,
\end{aligned}$$

with p the projection $RV_\ell(K) \times S_0 \to S_0$. Then (i) and (ii) still hold for f' and the new λ_0, c_0 and k_0. Indeed, since the new λ_0 has as new component function the function $rv_\ell \circ y'$, that c_0 is now a B_ℓ-center follows from a similar description of balls and centers as explained in Remark 7.2 and in the proof of Claim 1. This proves the claims.

We may work piecewise on $L_{\text{Hen}}(A)$-definable pieces since terms on each piece can be combined to one term on the union, cf. the proof of Theorem 7.8 below. We will work on pieces that we denote by X.

By Claim 1, the set $\lambda_0^{-1}(s)$ for $s \in \lambda_0(K)$ is a ball if and only if $rv_{k_0}(y'(y)) \neq 0$ for y with $\lambda_0(y) = s$. Hence, we may suppose that each piece X in the finite partition is a cell with presentation $\lambda_{0|X}$ and center $c_{0|\lambda_0(X)}$. Moreover, we may focus on a single piece X which is a (1)-cell because the case of a (0)-cell is trivial.

By partitioning further, we may suppose that

$$X \subset \Big\{y \in K \mid \bigwedge_{i,j=0}^{d} (\operatorname{ord} a_i'(y)y'(y)^i \square_{ij} \operatorname{ord} a_j'(y)y'(y)^j \tilde{\square}_j + \infty)\Big\},$$

for some $(\square_{ij}) \in \{<,>,=\}^{(d+1)^2}$, $(\tilde{\square}_j)_j \in \{<,=\}^{d+1}$, with the convention that $\operatorname{ord}(0) = +\infty$. The case where $(\square_{0j})_j$ is $(=,<,\ldots,<)$ is trivial, with k equal to k_0. Hence, we may suppose that there exists $i_0 > 0$ such that

$$+\infty \neq \operatorname{ord}\big(a_{i_0}'(y)y'(y)^{i_0}\big) \leq \operatorname{ord}\big(a_i'(y)y'(y)^i\big) \text{ for all } y \in X \text{ and all } i. \quad (7.11)$$

By the previous application of the induction hypothesis to f', we have for y in X

$$\operatorname{ord} f'(y) \leq \min_i \operatorname{ord}(k_0 i a_i'(y)y'(y)^i) \leq \operatorname{ord}(k_0 i_0 a_{i_0}'(y)y'(y)^{i_0}). \quad (7.12)$$

By Claim 2, we may

$$\text{replace } k_0 \text{ by } k_0 i_0. \quad (7.13)$$

Then, by (7.12) and (7.13), one has for all $y \in X$ that

$$\operatorname{ord}(f'(y)) \leq \operatorname{ord}(k_0 a_{i_0}'(y)y'(y)^{i_0}). \quad (7.14)$$

Define the set $B_1 \subset X$ by

$$B_1 := \left\{ y \in X \mid \forall z \in K \ \left(rv_{k_0}(z) = rv_{k_0}(y'(y)) \right) \rightarrow \mathrm{ord} f(z) \leq \mathrm{ord} k_0^2 a'_{i_0}(y) z^{i_0} \right) \right\},$$

and set $B_2 := X \setminus B_1$. Note that in particular for $y \in B_1$ one has

$$\mathrm{ord}(f(y)) \leq \mathrm{ord}(k_0^2 a'_{i_0}(y) y'(y)^{i_0}). \tag{7.15}$$

We may suppose that either $X = B_1$ or $X = B_2$, since B_1 and B_2 are clearly $L_{\mathrm{Hen}}(A)$-definable.

Case 1. $X = B_1$.
By Claim 2, we may replace k_0 by k_0^2. Now (i) for f follows from (7.15) with $\lambda = \lambda_0$, $c = c_0$, and $k = k_0$ and (ii) is clear by construction.

Case 2. $X = B_2$.
By the definition of B_2, one has

$$\mathrm{ord}(f(z)) > \mathrm{ord}(k_0^2 a'_{i_0}(y) z^{i_0}) \text{ for } y \in X \text{ and some } z \text{ with } rv_{k_0}(z) = rv_{k_0}(y'(y)).$$
$$\tag{7.16}$$

We will need a new center. First replace λ_0 as is done in the proof of Claim 2, but just with $\ell = k_0$ (and replace c_0 and S_0 accordingly). Let $p : S_0 \rightarrow RV_{k_0}$ be the projection on the $rv_{k_0} \circ y'$ component of S_0, which exists since we have replaced λ according to the proof of Claim 2 with $\ell = k_0$. Define the $L^*_{\mathrm{Hen}}(A)$-term $d : S_0 \rightarrow K$ by

$$d(s) := c_0(s) + h_{d,k_0}\big(a_0(s), \ldots, a_d(s), p(s)\big),$$

with notation from Definition 7.7. By Lemma 7.6 and by (7.14) and (7.16), for each $y \in X$ and $s = \lambda_0(y)$, the element $d(s)$ lies in the ball $\lambda_0^{-1}(s)$ and $f(d(s)) = 0$. Define

$$S := S_0 \times RV_1,$$
$$\lambda : X \rightarrow S : y \mapsto (\lambda_0(y), rv_1(y - d(\lambda_0(y)))),$$
$$c := d \circ \pi,$$

with $\pi : S \rightarrow S_0$ the projection, and define $k = 1$. Clearly c satisfies (ii). Also, for $y \in X$ and $s_0 = \lambda_0(y)$, one has

$$rv_n(y - c_0(s_0)) = rv_n(d(s_0) - c_0(s_0)), \tag{7.17}$$

since $d(s_0)$ lies in the ball $\lambda_0^{-1}(s_0)$ which equals

$$\{z \mid rv_n(z - c_0(s_0)) = rv_n(y - c_0(s_0))\},$$

see Claim 1. Let us check that condition (i) for f holds on X for the present choice of λ, c, and k. Writing

$$f(y) = \sum b_i(s)(y - c(s))^i \text{ for } y \in X \quad \text{and} \quad s = \lambda(y),$$

one has by (7.17) and by (7.9) of Lemma 7.7 that

$$\operatorname{ord} f(y) = \operatorname{ord} b_1(s)(y - c(s)), \tag{7.18}$$

which proves the last part of (i). Since $d(s_0)$ lies in $\lambda_0^{-1}(s_0)$, by the description of balls and centers in Remark 7.2, and by the definition of λ, it follows that d is a B_1-center for λ. Hence, (i) for f holds on X for this λ and c.   □

Proof of Theorem 7.5. The proof of b-minimality is based on Lemma 7.10. In particular, the proof of axiom (b1) for K is derived from Lemma 7.10. Let $X \subset K$ be $L_{\mathrm{Hen}}(A)$-definable for some set of parameters A. By Proposition 7.3, we may suppose that X is given by an $L_{\mathrm{Hen}}(A)$ formula φ without valued field quantifiers. Let f_j be the polynomials appearing in φ. We may suppose that in φ, the polynomials f_j only appear in the form

$$rv_m(f_j) \tag{7.19}$$

for some $m > 0$ since the expression $f_j(x) = 0$ is equivalent to $rv_m(f_j(x)) = 0$.

Apply Lemma 7.10 to each of the polynomials f_j to find numbers $k_j > 0$, b-maps λ_j and centers c_j. As for Claim 1 in the proof of Lemma 7.10, we may replace each of the k_j by $k := m \cdot \ell$, with ℓ the least common multiple of the k_j. It then follows by (i) of Lemma 7.10 that $rv_m(f_j(x))$ factorizes through λ_j for each j. In other words, $rv_m(f_j(x))$ does only depend on $\lambda_j(x)$.

Define λ as the product map of the λ_j, that is,

$$\lambda : K \to \prod_j S_j : x \mapsto (\lambda_j(x))_j$$

with S_j the image of λ_j.

Since a finite intersection of balls is a ball, the map λ is a b-map. Since clearly the characteristic function of X factorizes through λ, (b1) follows.

We prove (b2). Examining valued field quantifier free formulas in one valued field variable as above, one finds that if $S \to K$ is a definable function from an auxiliary set S into K, then its image is finite and contained in the zero set of a polynomial. Property (b2) thus follows.

For (b3), one uses Proposition 2.6 and $(*)$ is clear by looking at quantifier free formulas in two valued field variables which give a definable function. Clearly the graph of such a function must lie in an algebraic set of dimension ≤ 1. This proves the b-minimality.

The preservation of all balls is a special case of Theorem 7.11, which is derived in [7] from Weierstrass division in rings of analytic functions.   □

Proof of Theorem 7.8. Note that it is enough to work piecewise. Namely, with the functions $h_{1,1}$ one can make terms which are characteristic functions of $rv(1)$, $rv(2)$, and so on, hence one can always paste a finite number of terms on finitely many disjoint pieces together. An example of a characteristic function of $rv(1)$ is the function $h_{1,1}(1, -1, \cdot) : RV \to K : a \mapsto h_{1,1}(1, -1, a)$. If one has two terms t_j,

$j = 1, 2$ on subsets B_j of some set C, one can replace the B_j and C by $B_j \times \{rv(j)\}$ and $C \times \{rv(1), rv(2)\}$ and construct the single term $\sum_j \chi_{rv(j)} t_j$ with $\chi_{rv(j)}$ the characteristic function of $rv(j)$.

Now let f be an $L_{\mathrm{Hen}}(A)$-definable function. In the theory of b-minimality, one derives general cell decomposition from property (b1) by compactness, see the proof of Theorem 3.4. In the proof of Theorem 7.5, property (b1) is derived from (i) of Lemma 7.10. Similarly to the mentioned application of Lemma 7.10 and the proof of general b-minimal cell decomposition by compactness, it follows that a cell decomposition theorem holds where all the centers are given by $L_{\mathrm{Hen}}^*(A)$-terms. That is, an $L_{\mathrm{Hen}}(A)$-definable set can be partitioned into $L_{\mathrm{Hen}}(A)$-definable cells whose centers are given by $L_{\mathrm{Hen}}^*(A)$-terms. Partition the graph of f into such cells to find the desired piecewise terms. $\qquad\square$

7.3. *Analytic structure on Henselian valued fields of characteristic zero*

The search for an expansion of T_{Hen} with a nontrivial entire analytic function is open and challenging. For the real line, Wilkie and Miller answered this quest with exp and other entire functions, see Sec. 7.1.

Working with analytic functions with bounded domain, the following is proven in [7], where the analytic functions have as domains products of the valuation ring and the maximal ideal. The work in [7] generalizes and axiomatizes [5, 8].

Theorem 7.11 [7]. *Let $\mathcal{L}_{\mathrm{an}}$ be any of the analytic expansions of L_{Hen} as described in the part of [7] on Henselian valued fields and let $\mathcal{T}_{\mathrm{an}}$ be the corresponding $\mathcal{L}_{\mathrm{an}}$-expansion of T_{Hen}.*

Then the theory $\mathcal{T}_{\mathrm{an}}$ is b-minimal with $\{B_n\}_n$-centers, preserves all balls, and allows elimination of valued field quantifiers in the language $\mathcal{L}_{\mathrm{an}}$. The analogue of Theorem 7.8 holds for the language $\mathcal{L}_{\mathrm{an}}^*$, the union of the language $\mathcal{L}_{\mathrm{an}}$ with all the functions $h_{m,n}$.

8. Comparison with v-Minimality, p-Minimality, and C-Minimality

The comparison with o-minimality and a generalization are already worked out in Sec. 7.1.

Proposition 8.1. *Let T be a p-minimal theory, as defined in [20]. Let T' be the theory T with as extra auxiliary sorts the residue field and the value group and the natural maps into them from the valued field (that is, the residue modulo the maximal ideal on the valuation ring and zero outside it, respectively, the valuation). Suppose that T has definable Skolem functions. Let B and B_n be as in Sec. 7.2. Then T' is b-minimal with $\{B_n\}_n$-centers (with respect to the auxiliary sorts the residue field and the value group).*

Proof. The statement follows from the main result of Mourgues [24] and the theory of p-minimal fields by Haskell and Macpherson [20], as follows. If T is any p-minimal

theory (not necessarily having definable Skolem functions), then T' satisfies properties (b2) and (b3) by the results of [20] on dimensions in p-minimal fields. Indeed, (b2) follows from [20, Theorem 3.3] and (b3) follows from [20, Proposition 2.6, Theorem 6.3] and properties of algdim as defined in [20]. By the main result of [24], for any p-minimal theory T with definable Skolem functions, the theory T' has cell decomposition with centers. From this, property (b1) and the center property 6.2 for T' follow. Thus, if T has definable Skolem functions, then T' satisfies property (b1) of Definition 2.1 and has the center property 6.2, by the main result of [24].   □

Proposition 8.2. *Let T be a v-minimal theory of algebraically closed valued fields, as defined in [21] (hence the residue characteristic is zero and the auxiliary sort is RV). Let B and B_n be as in Sec. 7.2. Then T is b-minimal with B_1-centers and preserves all balls.*

Proof. Property (b1) and the center property 6.2 follows from [21, Lemma 3.31] by noting that any definable subset of the valued field and main sort K is a finite Boolean combination of (open or closed) balls and points. Namely, let X be an A-definable subset of K. Write X as a finite Boolean combination of (open or closed) balls and points. By [21, Lemma 3.9], there exists a definable bijection h between any given definable finite set and an auxiliary set. Let h be such a bijection between these finitely many points in the Boolean combination and an auxiliary set. Extend h on K by zero to some map $h : K \to S$ with S auxiliary, or extend it on K in some trivial definable way to an auxiliary set S. Define the collection X' of closed balls in K as the union of all the closed balls in this Boolean combination and for each occurring open ball in the Boolean combination the minimal closed ball containing this ball. Then X' is an A-definable finite set of closed balls. By [21, Lemma 3.31], one can take a finite definable set Y such that in each of the closed balls of X' there lies exactly one point of Y. Let Y' be the (finite) set of all averages of points in Y. Let $g : Y' \to R$ be a definable bijection with R auxiliary, which exists by [21, Lemma 3.9]. Now let f be the definable map $x \mapsto (rv(x-y), h(z), g(y))$ into the auxiliary set $RV \times S \times R$ where $y \in Y'$ is the average of the points in Y that lie closest to x and z the average of the finitely many distinguished points in the Boolean combination that lie closest to x. Then f is a b-map as required for (b1) as follows from the description of balls in Remark 7.2.

Property (b2) is clear from [21, Lemma 3.41]. Property (b3) follows by criterium 2.6 of Sec. 2, and by the additivity property (a property similar to (4) of Proposition 4.3) of the VF-dimension of [21] following from [21, Corollary 3.58].

Preservation of all balls follows from [21, Proposition 5.1]. To see this, note that by the first sentence of the proof of [21, Proposition 5.1], for a finite A-definable equivalence relation $\sim$ on K there exists an A-definable $f : K \to S$ for some auxiliary S such that $x \sim y$ if and only if $f(x) = f(y)$.   □

Remark 8.3 (C-minimality). Let $\mathcal{T}$ be a C-minimal theory (see [19, 22]) of algebraically closed valued fields of characteristic zero and let $\mathcal{T}'$ be the union of

$\mathcal{T}$ with $\mathcal{T}_{\text{Hen}}$, so, the auxiliary sorts are the RV_n. It is not clear to us whether $\mathcal{T}'$ automatically satisfies (b3). [19, Proposition 6.1] seems to give a sufficient condition for (b3) via one of the criteria of Sec. 2. [19, Theorem 3.11] goes in the direction of preservation of all balls for $\mathcal{T}'$, but is only local. Axiom (b1) encompasses somehow the lack of Skolem functions, see also [19, Lemma 6.6]. In positive characteristic already (b1) is problematic.

Examples 8.4. (1) In the p-adic situation, one can use a technical description of p-adic cells based on the definition given by Denef in [13], see [5] for such a description. Write P_3 for the nonzero cubes in $\mathbf{Q}_p$ and then $X = P_3 \cap \mathbf{Z}_p$ is an example of a p-adic cell as in [5]. Suppose for simplicity that $p > 3$. To see that X is also a cell in the b-minimal setting, let $f : X \to \mathbf{Z}^2$ be the definable function given by $x \mapsto (\text{ord}(x), \overline{\text{ac}}(x))$, where $\overline{\text{ac}}(x)$ is the first nonzero coefficient of x in the p-adic expansion $x = \sum_{i \in \mathbf{N}} a_i p^i$ with $a_i \in \{0, \dots, p-1\}$. (This first nonzero coefficient can be controlled piecewise by using predicates P_k of nonzero kth powers for well chosen $k > 0$, and their cosets, see [15].) Clearly f is a b-map whose fibers are all balls, hence, X is a b-minimal (1)-cell with presentation f. For a center for (X, f) one can take the zero function $c : f(X) \to \mathbf{Q}_p : z \mapsto 0$.

(2) In an algebraically closed valued field K with residue field of characteristic zero, it is well known that a definable subset of K in the language of valued fields is a (finite) Boolean combination of balls and points, namely by quantifier elimination. We give two examples that in this case such a Boolean combination is also a finite union of b-minimal cells. Let R be the valuation ring of K, and t an element of the maximal ideal of R. Then $X = R \setminus (t)$ is clearly a simple Boolean combination of balls. Let $f : X \to RV$ be the definable function $x \mapsto rv(x)$. Then again f is a b-function whose fibers are all balls, hence X is a b-minimal (1)-cell with presentation f. For the set $Y = R \setminus \{t\}$, one can use the b-map $g : Y \to RV : y \mapsto rv(y - t)$ whose fibers are again all balls, hence also Y is a (1)-cell. For a center for (X, f) one can take the zero function, and for a center for (Y, g) one can take the constant function with value t, see Remark 7.2.

A further comparison with tame geometries in model theory and some further open questions can be found in [3].

9. Grothendieck Semirings

Let T be a b-minimal $\mathcal{L}$-theory and $\mathcal{M}$ a model. In this section, inspired by the treatments in [21, 9], we set some lines for a general study of Euler characteristics on b-minimal structures.

Let $\mathcal{S}$ be a collection of sets, and $\mathcal{F}_{\mathcal{S}}$ a collection of functions between some of the sets of $\mathcal{S}$. Write $A \sim_0 B$ for A and B in $\mathcal{S}$ if there exists $f : A \to B$ in $\mathcal{F}$. Let $\sim_{\mathcal{F}}$ be

the equivalence relation on $\mathcal{S}$ generated by the relation $\sim_0$. When $A \sim_{\mathcal{F}} B$ we say A and B in $\mathcal{S}$ are isomorphic.

We define the Grothendieck semigroup $K_0^+(\mathcal{S}, \mathcal{F})$ as the quotient of the free semigroup on isomorphism classes of $\sim_{\mathcal{F}}$ divided out by the relation

$$[A] + [B] = [A \cup B]$$

if $A, B \subset C$ are disjoint and A, B, C, and $A \cup B$ belong to $\mathcal{S}$.

Recall that in this section, T is a b-minimal $\mathcal{L}$-theory and $\mathcal{M}$ a model. For Y a definable set, let $\mathcal{S}(M/Y)$ be the collection of all $\mathcal{L}$-definable subsets of $M^n \times Y$ for $n \geq 0$, and $\mathcal{F}(M/Y)$ the collection of $\mathcal{L}$-definable bijections between such sets which commute with the projections to Y. Likewise, let $\mathcal{S}(\mathcal{M}/Y)$, respectively, $\mathcal{S}(\mathcal{M}_{\mathrm{Aux}}/Y)$, be the collection of all $\mathcal{L}$-definable subsets of $X \times Y$ for all definable X, respectively, for all auxiliary X, and $\mathcal{F}(\mathcal{M}/Y)$, respectively, $\mathcal{F}(\mathcal{M}_{\mathrm{Aux}}/Y)$, the collection of $\mathcal{L}$-definable bijections between such sets which commute with the projections to Y.

Then write

$$K_0^+(M/Y) \text{ for } K_0^+(\mathcal{S}(M/Y), \mathcal{F}(M/Y)),$$
$$K_0^+(\mathcal{M}/Y) \text{ for } K_0^+(\mathcal{S}(\mathcal{M}/Y), \mathcal{F}(\mathcal{M}/Y)),$$

and

$$K_0^+(\mathrm{Aux}/Y) \text{ for } K_0^+(\mathcal{S}(\mathcal{M}_{\mathrm{Aux}}/Y), \mathcal{F}(\mathcal{M}_{\mathrm{Aux}}/Y)).$$

These semigroups carry a multiplication induced by Cartesian product, hence they are endowed with a semiring structure.

Define $K_0^+(\mathrm{Aux}/Y)[\mathbf{N}]$ as the graded semigroup

$$\bigoplus_{i \in \mathbf{N}} K_0^+(\mathrm{Aux}/Y),$$

that is, the direct sum of countably many copies of $K_0^+(\mathrm{Aux}/Y)$ indexed by the nonnegative integers $\mathbf{N}$. Write $[X][i]$ for $(0, \ldots, 0, [X], 0, \ldots)$ if the entry $[X]$ occurs at the ith position in the tuple. This semigroup $K_0^+(\mathrm{Aux}/Y)[\mathbf{N}]$ is a semiring where multiplication of $[A][i]$ and $[B][j]$ with $A, B \in \mathcal{S}(\mathcal{M}_{\mathrm{Aux}}/Y)$ and $i, j \in \mathbf{N}$ is given by $[A \times B][i+j]$. Note that this semiring is generated as a semigroup by the elements $[X][i]$ for X in $\mathcal{S}(\mathcal{M}_{\mathrm{Aux}}/Y)$ and $i \geq 0$.

Consider a finite partition of a definable set $X \subset M^n \times Y$ into $(i_{j1}, \ldots, i_{jn})$-cells X_j over Y, with presentations

$$f_j : X_j \to S_j \times Y$$

over Y, and set $R_j := f_j(X_j)$. To such an X partitioned into cells with presentations f_j we associate the element

$$\chi(X, f_j)_j := \sum_j [R_j] \left[\sum_\ell i_{j\ell} \right]$$

in $K_0^+(\mathrm{Aux}/Y)[\mathbf{N}]$.

Write J for the ideal of $K_0^+(\mathrm{Aux}/Y)[\mathbf{N}]$ generated by the relations

$$\chi(X, f_j)_j = \chi(X, g_k)_k$$

for all X in $\mathcal{S}(M)$ and collections $\{g_k\}$ and $\{f_j\}$ of b-maps of cells corresponding to any two cell decompositions of X over Y.

Then there is a natural map

$$\chi : \mathcal{S}(M/Y) \to K_0^+(\mathrm{Aux}/Y)[\mathbf{N}]/J \tag{9.1}$$

sending X to the class of $\chi(X, f_j)_j$ for some presentations $\{f_j\}$ of a cell decomposition of X.

In this formalism, the following kind of change of variables for χ becomes almost trivial.

Proposition 9.1. *Suppose that $\mathcal{T}$ is b-minimal. Then χ factorizes through the natural projection*

$$\mathcal{S}(M/Y) \to K_0^+(M/Y).$$

Proof. Clearly χ is additive with respect to disjoint union. Let X and X' be subsets of $M^n \times Y$ and $M^\ell \times Y$ respectively and suppose that they are isomorphic over Y. Let $f : X \to X'$ be a definable bijection over Y and let Γ_f be its graph. Since a cell decomposition of X over Y induces one of Γ_f, it is clear that $\chi(\Gamma_f) = \chi(X)$. Similarly, $\chi(\Gamma_f) = \chi(X')$ and thus the proposition is proved. $\square$

The study of the map χ, in particular its image and its kernel, and of the ideal J seems to be quite fundamental in various settings. Preservation of balls plays an important role in the study of J. In the valued field setting, variants with isomorphisms in $\mathcal{F}(M, |\mathrm{Jac}|)$ or in $\mathcal{F}(M, rv(\mathrm{Jac}))$, consisting of definable bijections which are C^1 and have constant norm or constant rv of the Jacobian seem even more important.

In a v-minimal setting, such a study has been successfully achieved in [21]. In an o-minimal setting, χ is easily understood by results in [18] on the Euler characteristic and dimension.

Let $\mathcal{T}$ be a theory containing T_{Hen} which is b-minimal with centers of level $\{B_n\}_n$, for which definable functions are piecewise C^1, which preserves all balls (moreover respecting the Jacobian), which has the same sorts as T_{Hen}, and which induces no new structure on the value group in a strong sense. Then we expect that the whole construction of motivic integration of [9] goes through. Implementation of notation and constructions in this generality will be given elsewhere. This should fill in the gap for mixed characteristic[c] motivic integration, and allow to implement the analytic framework by the results of Sec. 7. In particular, the terminology of definable subassignments and the corresponding constructions make sense for any such b-minimal theory.

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[c]However, one can still imagine that by using some ingenious device there is a way to avoid the auxiliary sorts RV_n for $n > 1$ in mixed characteristic.

Acknowledgments

References

[1] Ş. Basarab, Relative elimination of quantifiers for Henselian valued fields, *Ann. Pure Appl. Log.* **53**(1) (1991) 51–74.

[2] Ş. Basarab and F.-V. Kuhlmann, An isomorphism theorem for Henselian algebraic extensions of valued fields, *Man. Math.* **77**(2–3) (1992) 113–126.

[3] R. Cluckers, An introduction to b-minimality, to appear in *Conference Proceedings of Logic Colloquium 2006*, Nijmegen, arXiv:math.LO/0610928.

[4] R. Cluckers, Classification of semialgebraic p-adic sets up to semialgebraic bijection, *J. Reine Angew. Math.* **540** (2001) 105–114.

[5] R. Cluckers, Analytic p-adic cell decomposition and integrals, *Trans. Amer. Math. Soc.* **356**(4) (2004) 1489–1499, arXiv:math.NT/0206161.

[6] R. Cluckers, Multi-variate Igusa theory: Decay rates of p-adic exponential sums, *Int. Math. Res. Not. IMRN*, No. 76 (2004) 4093–4108.

[7] R. Cluckers and L. Lipshitz, Fields with analytic structure, preprint, http://www.dma.ens.fr/∼cluckers/.

[8] R. Cluckers, L. Lipshitz and Z. Robinson, Analytic cell decomposition and analytic motivic integration, *Ann. Sci. École Norm. Sup.* **39**(4) (2006) 535–568, arxiv:math.AG/0503722.

[9] R. Cluckers and F. Loeser, Constructible motivic functions and motivic integration, to appear in *Invent. Math.* arxiv:math.AG/0410203.

[10] R. Cluckers and F. Loeser, Constructible exponential functions, motivic Fourier transform and transfer principle, to appear in *Ann. Math.* arXiv:math.AG/0512022.

[11] P. J. Cohen, Decision procedures for real and p-adic fields, *Commun. Pure Appl. Math.* **22** (1969) 131–151.

[12] S. D. Cutkosky, *Resolution of Singularities*, Graduate Studies in Mathematics, Vol. 63 (American Mathematical Society, 2004).

[13] J. Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, *Invent. Math.* **77** (1984) 1–23.

[14] J. Denef, On the evaluation of certain p-adic integrals, Théorie des nombres, *Sémin. Delange-Pisot-Poitou 1983–1984* **59** (1985) 25–47.

[15] J. Denef, p-adic semialgebraic sets and cell decomposition, *J. Reine Angew. Math.* **369** (1986) 154–166.

[16] J. Denef, Manuscript on quantifier elimination for Henselian valued fields, private communication (2006).

[17] L. van den Dries, The field of reals with a predicate for the powers of two, *Manuscripta Math.* **54**(1–2) (1985) 187–195.

[18] L. van den Dries, *Tame Topology and o-Minimal Structures*, Lecture Note Series, Vol. 248 (Cambridge University Press, 1998).

[19] D. Haskell and D. Macpherson, Cell decompositions of C-minimal structures, *Ann. Pure Appl. Log.* **66**(2) (1994) 113–162.

[20] D. Haskell and D. Macpherson, A version of o-minimality for the *p*-adics, *J. Symb. Log.* **62**(4) (1997) 1075–1092.

[21] E. Hrushovski and D. Kazhdan, Integration in valued fields, in *Algebraic Geometry and Number Theory*, Progress in Mathematics, Vol. 253 (Birkhäuser Boston, Boston, MA, 2006), pp. 261–405.

[22] D. Macpherson and C. Steinhorn, On variants of o-minimality, *Ann. Pure Appl. Log.* **79**(2) (1996) 165–209.

[23] C. Miller, Tameness in expansions of the real field, in *Logic Colloquium '01* (Urbana, IL), Lecture Notes in Logic, Vol. 20 (Association of Symbolic Logic, 2005), pp. 281–316.

[24] M.-H. Mourgues, Corps p-minimaux avec fonctions de Skolem définissables, Séminaire de structures algébriques ordonnées, 1999–2000, prépublication de l'équipe de logique mathématique de Paris 7, pp. 1–8.

[25] J. Pas, Uniform *p*-adic cell decomposition and local zeta functions, *J. Reine Angew. Math.* **399** (1989) 137–172.

[26] J. Pas, Cell decomposition and local zeta functions in a tower of unramified extensions of a *p*-adic field, *Proc. London Math. Soc. (3)* **60**(1) (1990) 37–67.

[27] T. Scanlon, Valuation theory and its applications, Vol. II, Fields Institute Communications Series, ch. Quantifier Elimination for the Relative Frobenius Conference, *Proceedings of the International Conference on Valuation Theory*, Saskatoon (1999) (American Mathematical Society, Providence, 2003), pp. 323–352.

[28] A. J. Wilkie, Adding a multiplicative group to a polynomially bounded structure (October 13, 2006), Lecture at the ENS, Paris.