VALUATION THEORY OF HIGHER LEVEL *-SIGNATURES

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Higher level *-signatures on skew fields with involution are a common generalization of higher level signatures on commutative fields from [BHR] and $*$-orderings on skew fields with involution from [CS1]. We give an example of a higher level $*$-signature such that the corresponding set of bounded elements is not closed for addition. However, if we assume that there exists a central element i such that $i^2 = -1$ then the set of bounded elements is a valuation ring. In this case we can develop extension theory, Artin-Schreier theory and the weak isotropy principle as in the classical cases.

1. Introduction

Let $(R, *)$ be a domain with involution ($*$-domain for short) and $S = \mathrm{Sym}(R, *)$ its set of symmetric elements. A mapping $\sigma : S \to \mathbb{C}$ is a *$*$-signature of level m* if

(S_1) $\sigma(-1) = -1$,
(S_2) $\sigma(st + ts) = \sigma(s)\sigma(t)$ for every $s, t \in S$,
(S_3) $\sigma(rsr^*) = \sigma(s)\sigma(rr^*)$ for every $s \in S$ and $r \in R$,
(S_4) $\sigma((rr^*)^m) = 1$ for every nonzero $r \in R$,
(S_5) $\sigma^{-1}(1)$ is closed for addition.

Remarks 1.1. The following properties follow easily from the axioms:

(R_1) $\sigma(ks) = \sigma(s)$ for every $k \in \mathbb{N}$.
(R_2) $\sigma(rr^*) = \sigma(r^*r)$ for every $r \in R$.
$(\sigma(rr^*)^2 = \sigma(2(rr^*)(rr^*)) = \sigma(r(r^*r)r^*) = \sigma(rr^*)\sigma(r^*r).)$
(R_3) $\sigma((rr^*)^k) = \sigma(rr^*)^k$ for every $r \in R$ and every $k \in \mathbb{N}$.
$(\sigma((rr^*)^k) = \sigma(r(r^*r)^{k-1}r^*) = \sigma((r^*r)^{k-1})\sigma(rr^*) =$
$\sigma((rr^*)^{k-2})\sigma(r^*r)\sigma(rr^*) = \sigma((rr^*)^{k-2})\sigma(rr^*)^2 = \ldots = \sigma(rr^*)^k.)$
(R_4) $\sigma(s) = 0$ if and only if $s = 0$.
(If $s \neq 0$, then $\sigma(s)^{2m} = \sigma(s^{2m}) = 1 \neq 0$.
If $s = 0$ and $t = -1$ in axiom S_2, then $\sigma(0) = -\sigma(0)$.)
(R_5) $\sigma(S) \subseteq \mu_{2m} := \{0\} \cup \{z \in \mathbb{C} : \ |z|^{2m} = 1\}$.

Note that every one-sided Ore $*$-domain is two-sided and that the involution extends uniquely to its (two-sided) skew-field of fractions by

$(ab^{-1})^* = (b^*)^{-1}a^*$. For every element $x \in \mathrm{Sym}(D,*)$ there exists an element $r \in R$ such that $r^*xr \in \mathrm{Sym}(R,*)$.

Theorem 1.2. *Let R be an Ore domain, D its skew field of fractions and $\sigma : \mathrm{Sym}(R,*) \to \mathbb{C}$ a $*$-signature of level m. The following assertions are equivalent:*

(1) *σ extends uniquely to a $*$-signature $\tilde{\sigma} : \mathrm{Sym}(D,*) \to \mathbb{C}$ of level m.*
(2) *$\sigma(a^*c + c^*a) = \sigma(ac^* + ca^*)$ for every $a, c \in R$ such that $ab, cb \in \mathrm{Sym}(R,*)$ for some nonzero $b \in R$.*

Proof. We will proof first that (2) implies (1). Pick any $x \in \mathrm{Sym}(D,*)$ and $r \in R$ such that $r^*xr \in \mathrm{Sym}(R,*)$. For every $*$-signature $\tau : \mathrm{Sym}(D,*) \to \mathbb{C}$ which extends σ, we have $\tau(x) = \sigma(r^*xr)\sigma(r^*r)^{-1}$, proving uniqueness. To prove existence write $\tilde{\sigma}(x) = \sigma(r^*xr)\sigma(r^*r)^{-1}$ for x and r as above. We claim that $\tilde{\sigma}$ is a $*$-signature on $(D,*)$ of level m extending σ.

Suppose that $x \in \mathrm{Sym}(D,*)$ and that $r_1^*xr_1, r_2^*xr_2 \in \mathrm{Sym}(R,*)$ for nonzero $r_1, r_2 \in R$. The Ore property gives nonzero $u_1, u_2 \in R$ such that $r_1u_1 = r_2u_2$. By axiom S_3, it follows that

$$\sigma(r_1^*xr_1)\sigma(u_1^*u_1) = \sigma(u_1^*r_1^*xr_1u_1) = \sigma(u_2^*r_2^*xr_2u_2) = \sigma(r_2^*xr_2)\sigma(u_2^*u_2),$$

in particular $\sigma(r_1^*r_1)\sigma(u_1^*u_1) = \sigma(r_2^*r_2)\sigma(u_2^*u_2)$. Dividing, we get

$$\sigma(r_1^*xr_1)\sigma(r_1^*r_1)^{-1} = \sigma(r_2^*xr_2)\sigma(r_2^*r_2)^{-1},$$

hence $\tilde{\sigma}$ is well-defined. Clearly $\tilde{\sigma}$ extends σ, in particular $\tilde{\sigma}(-1) = -1$.

For any $x, y \in \mathrm{Sym}(D,*)$ there exists a nonzero element $r \in R$ such that $r^*xr \in \mathrm{Sym}(R,*)$ and $r^*yr \in \mathrm{Sym}(R,*)$. If $\tilde{\sigma}(x) = \tilde{\sigma}(y) = 1$, then $\sigma(r^*xr) = \sigma(r^*yr) = \sigma(r^*r)$. By axiom S_5 $\sigma(r^*(x+y)r) = \sigma(r^*r)$, so that $\tilde{\sigma}(x+y) = 1$.

Suppose $x \in \mathrm{Sym}(D,*)$ and $d \in D$. Pick nonzero $r, u \in R$ such that $dr \in R$ and $u^*r^*d^*xdru \in R$. Since $dru \in R$, we have

$$\tilde{\sigma}(d^*xd) = \sigma(u^*r^*d^*xdru)\sigma(u^*r^*ru)^{-1} =$$
$$\sigma(u^*r^*d^*xdru)\sigma(u^*r^*d^*dru)^{-1}\sigma(u^*r^*d^*dru)\sigma(u^*r^*ru)^{-1} = \tilde{\sigma}(x)\tilde{\sigma}(dd^*).$$

As in Remark R_3, it follows that

$$\tilde{\sigma}((dd^*)^m) = \tilde{\sigma}(dd^*)^m = \sigma(u^*r^*d^*dru)^m\sigma(u^*r^*ru)^{-m} = 1.$$

For any $x, y \in D$ there exists a nonzero $r \in R$ such that $xr, yr \in R$. If $x, y \in \mathrm{Sym}(D,*)$, then $r^*x, r^*y \in R$, $r^*xr, r^*yr \in \mathrm{Sym}(R,*)$ and

$$\tilde{\sigma}(x)\tilde{\sigma}(y)\sigma(r^*r)^2 = \sigma(r^*xr)\sigma(r^*yr) =$$
$$\sigma(r^*xrr^*yr + r^*yrr^*xr) = \sigma(xrr^*y + yrr^*x)\sigma(r^*r).$$

Applying assertion (2) with $a = r^*x$, $c = r^*y$ and $b = r$, we get

$$\sigma(xrr^*y + yrr^*x) = \sigma(r^*yxr + r^*xyr) = \tilde{\sigma}(yx + xy)\sigma(r^*r).$$

It follows that $\tilde{\sigma}(yx + xy) = \tilde{\sigma}(x)\tilde{\sigma}(y)$. Therefore, $\tilde{\sigma}$ is a $*$-signature of level m.

The proof that (1) implies (2) depends on the following claim:
Claim: $\tau(usv + vsu) = \tau(u)\tau(s)\tau(v)$ *for any* $*$*-signature* τ *on any* $*$*-domain* A *and for any* $u, s, v \in \mathrm{Sym}(A, *)$
By axioms S_2 and S_3 we have

$$\begin{aligned}\tau((sus)v + v(sus)) = \tau(v)\tau(sus) = \tau(v)\tau(u)\tau(s)^2,\\ \tau(svs)\tau(u) = \tau(u(svs) + (svs)u) = \tau(v)\tau(u)\tau(s)^2.\end{aligned}$$

By axiom S_5, it follows that

$$\tau((sus)v + v(sus) + u(svs) + (svs)u) = \tau(v)\tau(u)\tau(s)^2.$$

On the other hand

$$\begin{aligned}\tau((sus)v + v(sus) + u(svs) + (svs)u) =\\ \tau(s(usv + vsu) + (usv + vsu)s) = \tau(usv + vsu)\tau(s).\end{aligned}$$

Applying the Claim with $(A, *) = (D, *)$, τ any extension of σ, $u = ab = b^*a^*$, $s = (b^*b)^{-1}$ and $v = cb = b^*c^*$, we get

$$\begin{aligned}\sigma(ac^* + ca^*) = \tau(usv + vsu) = \tau(u)\tau(s)\tau(v) = \tau(uv + vu)\tau(s) =\\ \sigma(b^*a^*cb + b^*c^*ab)\sigma(b^*b)^{-1} = \sigma(a^*c + c^*a).\end{aligned}$$

□

Remarks 1.3. Special cases of $*$-signatures have already been considered in the literature:

(1) Our $*$-signatures of level 1 correspond to support zero $*$-orderings in [M1] or [CS1]. By [CS1], every $*$-signature of level 1 on an Ore domain can be extended uniquely to its skew field of fractions. We conjecture that this is true for all $*$-signatures.

    Let $D = \mathbb{H}$ be the skew field of real quaternions and $*$ the standard involution. Then $S = \mathbb{R}$ and $\mathrm{sign}(a) = 1$ if $a > 0$, $\sigma(0) = 0$ and $\mathrm{sign}(a) = -1$ if $a < 0$ defines a $*$-signature of level 1.

(2) If $*$ is the identity, then D is a commutative field and σ is a signature of level m from [B, BHR].

    For example, for every $*$-signature on D of level m and every $s \in S$ is $\sigma|_{\mathbb{Q}(s)}$ a signature of level m on a commutative field $\mathbb{Q}(s)$.

Example 1.4. Let F be the field $\mathbb{Q}(\sqrt{2})$ with involution $(a+b\sqrt{2})^* = a - b\sqrt{2}$. For every $*$-signature τ on $(F,*)$ we have $\tau((1+\sqrt{2})(1+\sqrt{2})^*) = -1$. Therefore, $(F,*)$ has no $*$-signature of level 1. However $\sigma(a) = \mathrm{sign}(a)$ for every $a \in \mathrm{Sym}(F,*) = \mathbb{Q}$ is a signature of level 2.

The following two nontrivial examples are modifications of the examples from [M2].

Example 1.5. The complex Weyl algebra $A_1(\mathbb{C})$ is generated by two elements x and y which are subject to relation $yx = xy+1$. Recall that $A_1(\mathbb{C})$ is an Ore $*$-domain with involution $c^* = \overline{c}$, $x^* = x$, $y^* = -y$.

Every element $z \in A_1(\mathbb{C})$ can be expressed uniquely as $z = \sum_{k=0}^{r} f_k(x)y^k$. Let l_r be the leading coefficient of $f_r(x)$. If z is symmetric, then $\overline{l_r} = (-1)^r l_r$, hence $i^r l_r \in \mathbb{R}$. For every $m \in \mathbb{N}$, the mapping

$$\sigma : \mathrm{Sym}(A_1(\mathbb{R}),*) \to \mathbb{C}$$
$$\sigma(z) = \mathrm{sign}(i^r l_r)\exp(2\pi i r/2m)$$

is a $*$-signature of level m. It satisfies the condition (2) of Theorem 1.2, hence it can be extended to the Weyl field $W_1(\mathbb{R})$.

Example 1.6. Let L be a finite dimensional complex Lie algebra with involution. The involution can be extended uniquely to the enveloping algebra $U(L)$ and further to its skew field of fractions D. We can choose symmetric elements $x_1,\ldots,x_d \in L$ which generate L as a vector space over $\mathbb{C}$. The standard filtration of $U(L)$ corresponds to the total degree in $x_1,\ldots,x_d$, hence it is $*$-invariant. The corresponding graded ring is isomorphic to the polynomial ring $\mathbb{C}[t_1,\ldots,t_d]$ and the induced involution is given the conjugation of all coefficients. Therefore, $\mathrm{Sym}(U(L),*)$ is "projected" onto $\mathbb{R}[t_1,\ldots,t_n]$. Let τ be a signature of level m on $\mathbb{R}[t_1,\ldots,t_d]$ which is compatible with the total degree (e.g. $\tau(f(t_1,\ldots,t_d)) = \mathrm{sign}(\mathrm{lc}(f))\exp(2\pi i \deg(f)/m)$, where $\deg(f)$ is the total degree of f and $\mathrm{lc}(f)$ is the leading coefficient of f w.r.t. the total lexicographic ordering of monomials induced by $t_1 > \ldots > t_d$.) Then τ can be extended in a natural way to a signature $\sigma : \mathrm{Sym}(U(L),*) \to \mathbb{C}$ of level m. Clearly, σ satisfies condition (2) of Theorem 1.2, hence it can be extended to D.

2. Natural valuations of a $*$-signatures

For every *-signature $\sigma : S \to \mathbb{C}$ write

$$A(\sigma) = \{d \in D : \ \exists r \in \mathbb{Q}^+ : \sigma(r \pm dd^*) = 1\}.$$

The aim of this section is to prove the following theorem:

Theorem 2.1. *If σ is a *-signature on $(D,*)$ of level m and there exists $i \in Z(D)$ such that $i^2 = -1$ and $i^* = -i$, then $A(\sigma)$ is a *-valuation subring of D.*

We will split the proof into several lemmas and propositions. The assumption that $i \in Z(D)$ is used only in the proof that $A(\sigma)$ is closed for addition. The following example shows that it cannot be omitted even in the commutative case.

Example 2.2. Consider a field $D = \mathbb{Q}(X)(\sqrt{2})$ with involution $(p(X)+q(X)\sqrt{2})^* = p(X) - q(X)\sqrt{2}$. Then, $S = \mathrm{Sym}(D,*) = \mathbb{Q}(X)$. Every nonzero element $q(X) \in \mathbb{Q}(X)$ can be written as $q(X) = r(X)(X^2-2)^k$, where $r(\sqrt{2}) \neq 0$. Write $\sigma(q(X)) = \mathrm{sign}(r(\sqrt{2}))$ and $\sigma(0) = 0$. Note that $\sigma : S \to \mathbb{C}$ is a *-signature of level 2. Write $a = \frac{X+\sqrt{2}}{X-\sqrt{2}}$ and note that $aa^* = 1$ and $(1+a)(1+a)^* = \frac{4X^2}{X^2-2}$. It follows that $\sigma(2 \pm aa^*) = 1$ and that for every $r \in \mathbb{Q}^+$, $\sigma(r + (1+a)(1+a)^*) = 1$ and $\sigma(r - (1+a)(1+a)^*) = -1$. Hence $a \in A(\sigma)$ and $1 + a \notin A(\sigma)$.

Proposition 2.3. *Let σ be a *-signature of level m on $(D,*)$.*

1. *$\sigma(x) = \sigma(y^{-1}xy + y^*xy^{-*})$ for every $x \in S$ and $y \in D^\times := D \setminus \{0\}$.*
2. *If $a \in A(\sigma)$, then $a^* \in A(\sigma)$ for every $a \in D$.*
3. *If $a, b \in A(\sigma)$, then $ab \in A(\sigma)$ for every $a, b \in D$.*

Proof. (1) Axioms S_3 and S_2 imply that

$$\sigma(y^{-1}xy + y^*xy^{-*})\sigma(yy^*) = \sigma(y(y^{-1}xy + y^*xy^{-*})y^*) = \sigma(xyy^* + yy^*x) = \sigma(x)\sigma(yy^*).$$

Cancelling $\sigma(yy^*)$ we get $\sigma(y^{-1}xy + y^*xy^{-*}) = \sigma(x)$.

(2) Replacing $x = r \pm aa^*$ and $y = a$ in assertion (1), we get

$$\sigma(r \pm aa^*) = \sigma(a^{-1}(r \pm aa^*)a + a^*(r \pm aa^*)a^{-*}) = \sigma(2(r \pm a^*a))$$

Since $\sigma(r \pm aa^*) = 1$ if and only if $\sigma(r \pm a^*a) = 1$, it follows that $a \in A(\sigma)$ if and only if $a^* \in A(\sigma)$.

(3) Replacing $x = r^2 \pm abb^*a^*$ and $y = a$ in assertion (1), we get

$$\sigma(r^2 \pm abb^*a^*) = \sigma(a^{-1}(r^2 \pm abb^*a^*)a + a^*(r^2 \pm abb^*a^*)a^{-*}) = \sigma(2r^2 \pm (bb^*a^*a + a^*abb^*)) = \sigma(\tfrac{1}{2}(s_1 + s_2))$$

where

$$s_1 = (r \pm a^*a)(r + bb^*) + (r + bb^*)(r \pm a^*a)$$
$$s_2 = (r \mp a^*a)(r - bb^*) + (r - bb^*)(r \mp a^*a).$$

If $a, b \in A(\sigma)$, then $a^* \in A(\sigma)$ by assertion (2), hence there exists $r \in \mathbb{Q}^+$ such that $\sigma(r \pm a^*a) = 1$ and $\sigma(r \pm bb^*) = 1$. Axiom S_2 implies that $\sigma(s_1) = 1$ and $\sigma(s_2) = 1$. Now, $\sigma(\frac{1}{2}(s_1 + s_2)) = \sigma(s_1 + s_2) = 1$ by remark R_3 and axiom S_5. ☐

Proposition 2.4. *If σ is a *-signature on $(D, *)$, then*

(1) *For every $s \in S$ we have that $s \in A(\sigma)$ if and only if there exists $r \in \mathbb{Q}^+$ such that $\sigma(r \pm s) = 1$.*
(2) *The set $A(\sigma) \cap S$ is closed for addition and subtraction.*
(3) *For every $d \in D$ we have that $d \in A(\sigma)$ if and only if $dd^* \in A(\sigma)$ if and only if $(dd^*)^k \in A(\sigma)$ for some $k \in \mathbb{N}$.*
(4) *For every $d \in D$, either $d \in A(\sigma)$ or $d^{-1} \in A(\sigma)$.*

Proof. Write $B(\sigma) = \{s \in S : \exists r \in \mathbb{Q}^+ : \sigma(r \pm s) = 1\}$. For every $s \in S$, the restriction $\sigma|_{\mathbb{Q}(s)}$ is a signature in the sense of Becker. The set $B(\sigma) \cap \mathbb{Q}(s) = B(\sigma|_{\mathbb{Q}(s)})$ is a valuation subring of $\mathbb{Q}(s)$ by [B, Satz 2.2]. Since $B(\sigma) \cap \mathbb{Q}(s)$ is total, we have that either $s \in B(\sigma)$ or $s^{-1} \in B(\sigma)$. Since $B(\sigma) \cap \mathbb{Q}(s)$ is integrally closed in $\mathbb{Q}(s)$, we have that $s \in B(\sigma)$ if and only if $s^k \in B(\sigma)$ for some $k \in \mathbb{N}$.

(1) For every $s \in D$ we have $s \in A(\sigma)$ if and only if there exists $r \in \mathbb{Q}^+$ such that $\sigma(r \pm ss^*) = 1$ if and only if $s^2 \in B(\sigma)$ if and only if $s \in B(\sigma)$. Hence, $A(\sigma) \cap S = B(\sigma)$.

(2) Since $B(\sigma)$ is closed for addition and subtraction, so is $A(\sigma) \cap S$.

(3) If $d \in D$, then $d \in A(\sigma)$ if and only if $dd^* \in B(\sigma) = A(\sigma) \cap S$ if and only if $(dd^*)^k \in B(\sigma) = A(\sigma) \cap S$ for some k.

(4) If $d \in D$ and $d \notin A(\sigma)$, then $dd^* \notin B(\sigma)$. Hence, $(d^*)^{-1}d^{-1} = (dd^*)^{-1} \in B(\sigma)$. It follows that $(d^*)^{-1} \in A(\sigma)$. Since $A(\sigma)$ is closed for involution, it follows that $d^{-1} \in A(\sigma)$. ☐

Lemma 2.5. *Let σ be a *-signature on $(D, *)$ and $i \in Z(D)$ such that $i^2 = -1$ and $i^* = -i$.*

(1) *If $s, t \in A(\sigma) \cap S$, then $st + ts, i(st - ts) \in A(\sigma) \cap S$.*
(2) *If $u, v \in D$ and $u + u^*, i(u - u^*), v + v^*, i(v - v^*) \in A(\sigma)$, then $uv + (uv)^*, i(uv - (uv)^*) \in A(\sigma)$.*

Proof. (1) If $s, t \in A(\sigma) \cap S$, then by the assertion (1) of Proposition 2.4, there exist $r_1, r_2 \in \mathbb{Q}^+$ such that $\sigma(r_1 \pm s) = 1$ and $\sigma(r_2 \pm t) = 1$. Note that

$$2r_1r_2 \pm (st + ts) = \tfrac{1}{2}(x_1 + x_2)$$
$$x_1 = (r_1 + s)(r_2 \pm t) + (r_2 \pm t)(r_1 + s)$$
$$x_2 = (r_1 - s)(r_2 \mp t) + (r_2 \mp t)(r_1 + s)$$

By axiom S_2, $\sigma(x_1) = 1$ and $\sigma(x_2) = 1$. By axiom S_5, $\sigma(x_1 + x_2) = 1$. Hence, $st + ts \in A(\sigma)$.

Write $z = i(st - ts)$. Since $A(\sigma)$ is multiplicative and $s, t \in A(\sigma)$, we have that $tst, stts, tsst \in A(\sigma)$. By the previous paragraph, $s, tst \in$

$A(\sigma) \cap S$ implies that $s(tst) + (tst)s \in A(\sigma)$. Now, the assertion (2) of Proposition 2.4 and the fact that $-stst - tsts, stts, tsst \in A(\sigma) \cap S$ imply that $zz^* = -(stst + tsts) + stts + tsst \in A(\sigma)$. It follows from the assertion (3) of Proposition 2.4, that $z \in A(\sigma)$.

(2) We can write $uv + (uv)^* = \frac{1}{4}(s_1 - s_2 - s_3 - s_4)$ and $i(uv - (uv)^*) = \frac{1}{4}(s_5 + s_6 + s_7 - s_8)$, where

$$\begin{aligned}
s_1 &= (u + u^*)(v + v^*) + (v + v^*)(u + u^*), \\
s_2 &= i(u - u^*)i(v - v^*) + i(v - v^*)i(u - u^*), \\
s_3 &= i((u + u^*)i(v - v^*) - i(v - v^*)(u + u^*)), \\
s_4 &= i(i(u - u^*)(v + v^*) - (v + v^*)i(u - u^*)), \\
s_5 &= (u + u^*)i(v - v^*) + i(v - v^*)(u + u^*), \\
s_6 &= i(u - u^*)(v + v^*) + (v + v^*)i(u - u^*), \\
s_7 &= i((u + u^*)(v + v^*) - (v + v^*)(u + u^*)), \\
s_8 &= i(i(u - u^*)i(v - v^*) - i(v - v^*)i(u - u^*)).
\end{aligned}$$

By (1) and the assumptions, $s_1, \ldots, s_8 \in A(\sigma) \cap S$. Now (2) follows from assertion (2) of Proposition 2.4. □

Let $k \in \mathbb{N}$ and $x, y \in D$. We say that y is a *permuted product of* $(xx^*)^k$ if y is a product of k copies of x and k copies of x^*, not necessarily in this order.

Lemma 2.6. *Let σ be a *-signature on $(D, *)$ and $i \in Z(D)$ such that $i^2 = -1$ and $i^* = -i$. If $x \in A(\sigma)$, $k \in \mathbb{N}$ and y is a permuted product of $(xx^*)^k$, then $y + y^*, i(y - y^*) \in A(\sigma)$.*

Proof. By induction on k. If $k = 1$, then either $y = xx^*$ or $y = x^*x$. In both cases $y + y^* = 2y \in A(\sigma)$ and $i(y - y^*) = 0 \in A(\sigma)$. Suppose that the assertion is true for $1, \ldots, k-1$ and take any permuted product y of $(xx^*)^k$. We distinguish four cases: If $y = xzx^*$ (similarly, if $y = x^*zx$), then z is a permuted product of $(xx^*)^{k-1}$. By the induction hypothesis, $z + z^*, i(z - z^*) \in A(\sigma)$. Since $A(\sigma)$ is multiplicative, it follows that $y + y^* = x(z + z^*)x^* \in A(\sigma)$ and $i(y - y^*) = x(i(z - z^*))x^* \in A(\sigma)$.

If $y = xzx$ (similarly, if $y = x^*zx^*$), then $y = uv$ where u and v are permuted products of $(xx^*)^l$ and $(xx^*)^{k-l}$ respectively and $0 < l < k$. (If this is not true, then $y = y_1y_2 \cdots y_{2k}$, where $y_1 = x$ and $y_2, \ldots, y_{2k} \in \{x, x^*\}$ and each of the words $y_1y_2, y_1y_2y_3y_4, \ldots, y_1 \cdots y_{2k-2}$ has more than one half of the letters equal to x. It follows that $y_{2k-1} = y_{2k} = x^*$, contrary to the assumption that $y_{2k} = x$.) By the induction hypothesis, we have that $u + u^*, i(u - u^*), v + v^*, i(v - v^*) \in A(\sigma)$. It follows from Lemma 2.5, that $y + y^* = uv + (uv)^* \in A(\sigma)$ and that $i(y - y^*) = i(uv - (uv)^*) \in A(\sigma)$. □

Write μ for the set of all rational complex numbers of modulus 1.

Lemma 2.7. *Let σ be a *-signature on $(D,*)$ of level m and $i \in Z(D)$ such that $i^2=-1$ and $i^*=-i$. There exist numbers $k \in \mathbb{N}$, $r_1,\ldots,r_k \in \mathbb{Q}^+$ and $\xi_1,\ldots,\xi_k \in \mu$ such that $\xi_1 = 1$ and*

$$\sum_{i=1}^{k} r_i \left[(1+\xi_i x)(1+\xi_i x)^*\right]^m \in A(\sigma).$$

Proof. Write $h_k(\xi) = \xi^k + \overline{\xi}^k$ and note that $h_k(\alpha\xi)+h_k(\overline{\alpha}\xi) = h_k(\alpha)h_k(\xi)$ for any $k \in \mathbb{N}$ and $\alpha, \xi \in \mathbb{C}$. For every $j = 1,\ldots,m$ pick any $\alpha_j \in \mu$ such that $h_j(\alpha_j) < 0$. Let $g : \mu \to D$ be a function defined by

$$g(\xi) = [(1+\xi x)(1+\xi x)^*]^m = [(1+\xi x)(1+\xi^{-1}x^*]^m = \textstyle\sum_{j=-m}^{m} f_j \xi^j.$$

The sequence of functions $g_1,\ldots,g_{m+1} : \mu \to D$ defined by

$$g_1(\xi) = g(\xi) + g(\overline{\xi}) \quad g_{j+1}(\xi) = g_j(\alpha_j\xi) + g_j(\overline{\alpha}_j\xi) - h_j(\alpha_j)g_j(\xi)$$

has the following properties:

(1) For every $j = 1,\ldots,m+1$, there exist elements $f_{j,0} \in A(\sigma)$ and $f_{j,j},\ldots,f_{j,m} \in D$ such that $g_j(\xi) = f_{j,0} + f_{j,j}h_j(\xi) + \ldots + f_{j,m}h_m(\xi)$.
(2) For every $j = 1,\ldots,m+1$, there exist $r_{j,i} \in \mathbb{Q}^+$ and $\beta_{j,i} \in \mu$ such that $\beta_{j,1} = 1$ and $g_j(\xi) = \sum_i r_{j,i}\left[(1+\beta_{j,i}\xi x)(1+\beta_{j,i}\xi x)^*\right]^m$.

The proof of (2) is a simple induction on j.

To prove (1) for $j = 1$, write

$$\begin{aligned} g_1(\xi) = g(\xi) + g(\overline{\xi}) = \textstyle\sum_{j=-m}^{m} f_j(\xi^j + \overline{\xi}^j) = \\ = f_0 + \textstyle\sum_{j=1}^{m}(f_j + f_{-j})(\xi^j + \overline{\xi}^j) = f_{1,0} + \textstyle\sum_{j=1}^{m} f_{1,j}h_j(\xi). \end{aligned}$$

Element $f_{1,0} = 2f_0 = f_0 + f_0^*$ is a sum of the elements of the form $y + y^*$, where y is a permuted product of $(xx^*)^k$ where $k = 0,\ldots,m$. Lemma 2.6 implies that $f_{1,0} \in A(\sigma)$.

The induction step in (1) follows from

$$\begin{aligned} g_{j+1}(\xi) = g_j(\alpha_j\xi) + g_j(\overline{\alpha}_j\xi) - h_j(\alpha_j)g_j(\xi) = \\ = f_{j,0} + f_{j,j}(h_j(\alpha_j\xi) + h_j(\overline{\alpha}_j\xi)) + \ldots + f_{j,m}(h_m(\alpha_j\xi) + h_m(\overline{\alpha}_j\xi)) \\ -h_j(\alpha_j)(f_{j,0} + f_{j,j}h_j(\xi) + \ldots + f_{j,m}h_m(\xi)) = \\ = f_{j,0} + f_{j,j}h_j(\alpha_j)h_j(\xi) + \ldots + f_{j,m}h_m(\alpha_j)h_m(\xi) \\ -(h_j(\alpha_j)f_{j,0} + f_{j,j}h_j(\alpha_j)h_j(\xi) + \ldots + f_{j,m}h_j(\alpha_j)h_m(\xi)) = \\ = (1 - h_j(\alpha_j))f_{j,0} + (h_{j+1}(\alpha_j) - h_j(\alpha_j))f_{j,j+1}h_{j+1}(\xi) + \\ \ldots + (h_m(\alpha_j) - h_j(\alpha_j))f_{j,m}h_m(\xi) = \\ = f_{j+1,0} + f_{j+1,j+1}h_{j+1}(\xi) + \ldots + f_{j+1,m}h_m(\xi). \end{aligned}$$

Note that $f_{j+1,0} \in A(\sigma)$.

Now, $g_{m+1}(1) = \sum_{i=1}^{k} r_i\left[(1+\xi_i x)(1+\xi_i x)^*\right]^m = f_{m+1,0} \in A(\sigma)$ where $r_i = r_{m+1,i}$ and $\xi_i = \beta_{m+1,i}$. Note that $\xi_1 = 1$. □

Proposition 2.8. *If σ is a signature on $(D,*)$ and there exists $i \in Z(D)$ such that $i^2 = -1$ and $i^* = -i$, then $A(\sigma)$ is closed for addition.*

Proof. It is easy to prove that if a sum of positive elements belongs to $A(\sigma)$, then each of these positive elements belongs to $A(\sigma)$. If $x \in A(\sigma)$, then it follows from Lemma 2.7 that $[(1+x)(1+x)^*]^m \in A(\sigma)$. Assertion (3) of Proposition 2.4 we see that $1+x \in A(\sigma)$.

If $a, b \in A(\sigma)$, then either $ab^{-1} \in A(\sigma)$ or $ba^{-1} \in A(\sigma)$. In the first case we have that $a+b = (1+ab^{-1})b \in A(\sigma)$ and in the second case we have that $a+b = (1+ba^{-1})a \in A(\sigma)$. □

Proposition 2.9. *Let σ be a signature on a *-field D. If $A(\sigma)$ is closed for addition, then it is invariant.*

Proof. Take any symmetric $s \in A(\sigma)$ and any $y \in D^\times$. There exists $r \in \mathbb{Q}^+$, such that $\sigma(r \pm s) = 1$. Since

$$\begin{gathered}\sigma(y^{-*}(r \pm s)y^*y(r \pm s)y^{-1}) = \sigma((r \pm s)y^*y(r \pm s))\sigma(y^{-*}y^{-1}) = \\ \sigma(y^*y)\sigma((r \pm s)^2)\sigma((yy^*)^{-1}) = \sigma(yy^*)\sigma((r \pm s)^2)\sigma(yy^*)^{-1} = \\ \sigma((r \pm s)^2) = \sigma(r \pm s)^2 = 1,\end{gathered}$$

it follows that

$$\begin{gathered}\sigma(2r^2 + 2y^{-*}sy^*ysy^{-1}) = \\ \sigma((y^{-*}(r+s)y^*y(r+s)y^{-1} + y^{-*}(r-s)y^*y(r-s)y^{-1}) = 1\end{gathered}$$

Since $t := (r-s)(r+s)^{-1} = (r+s)^{-1}(r-s) \in S$ and $\sigma(t) = 1$, also

$$\begin{gathered}\sigma(2r^2 - 2y^{-*}sy^*ysy^{-1}) = \\ \sigma(y^{-*}(r+s)y^*y(r-s)y^{-1} + y^{-*}(r-s)y^*y(r+s)y^{-1}) = \\ \sigma((r+s)y^*y(r-s) + (r-s)y^*y(r+s))\sigma(y^{-*}y^{-1}) = \\ \sigma((r+s)(y^*yt + ty^*y)(r+s))\sigma(y^{-*}y^{-1}) = \\ \sigma(y^*yt + ty^y)\sigma((r+s)^2)\sigma(y^{-*}y^{-1}) = \\ \sigma(y^*y)\sigma(t)\sigma((r+s)^2)\sigma(y^{-*}y^{-1}) = 1.\end{gathered}$$

Therefore, $ysy^{-1} \in A(\sigma)$.

Take any antisymmetric $k \in A(\sigma)$ and any $y \in D^\times$. Since $k^2 \in A(\sigma)$ is symmetric, it follows by the previous paragraph that $(yky^{-1})^2 = yk^2y^{-1} \in A(\sigma)$. If $yky^{-1} \notin A(\sigma)$, then $(yky^{-1})^{-1} \in A(\sigma)$ and we get $yky^{-1} = (yky^{-1})^2(yky^{-1})^{-1} \in A(\sigma)$.

Finally, take any $x \in A(\sigma)$ and $y \in D^\times$. Additivity of $A(\sigma)$ implies that $s := \frac{1}{2}(x+x^*) \in A(\sigma)$ and $k := \frac{1}{2}(x-x^*) \in A(\sigma)$. By the previous paragraphs, we have that $ysy^{-1} \in A(\sigma)$ and $yky^{-1} \in A(\sigma)$. It follows that $yxy^{-1} = ysy^{-1} + yky^{-1} \in A(\sigma)$. □

Corollary 2.10. *If $A(\sigma)$ is additive, then $a^*a^{-1} \in A(\sigma)^\times$ for every $a \in D^\times$.*

Proof. If $a^*a^{-1} \notin A(\sigma)$ for some $a \in D^\times$, then $(a^*a^{-1})^{-1} \in A(\sigma)$ and $a^{-1}a^* = (a^*a^{-1})^{-*} \in A(\sigma)$. By Proposition 2.9, we have that $a^*a^{-1} = a(a^{-1}a^*)a^{-1} \in A(\sigma)$. Hence, $a^*a^{-1} \in A(\sigma)$ for every $a \in A(\sigma)$. Replacing a by a^{-1}, we get $a^{-*}a \in A(\sigma)$. By Proposition 2.9, it follows that $(a^*a^{-1})^{-1} = aa^{-*} = a(a^{-*}a)a^{-1} \in A(\sigma)$ for every $a \in A(\sigma)$. □

3. Extension theorem for higher level $*$-orderings

Let D be a skew field with involution, S its set of symmetric elements and $\sigma : S \to \mathbb{C}$ a signature of level m. A subset $M \subseteq S$ is a *σ-module* if

(SM_1) $1 \in M$,
(SM_2) $M + M \subseteq M$,
(SM_3) $\sigma^{-1}(\sigma(M)) = M$,
(SM_4) $M \cap -M = \{0\}$.

The smallest σ-module is $P := \sigma^{-1}(\{0,1\})$. A σ-module M is a *σ-semiordering* if it satisfies

(SM_5) $M \cup -M = S$.

We need a version of the intersection theorem.

Lemma 3.1. *The intersection of all σ-semiorderings is equal to $P = \sigma^{-1}(\{0,1\})$.*

Proof. We claim that every σ-module which is not a σ-semiordering is an intersection of two strictly larger σ-modules. It follows that for every $z \notin P$, every maximal σ-module which avoids z is a σ-semiordering. Note that there are only finitely many σ-semiorderings, so that the Zorn's Lemma is not needed.

If M is a σ-module which is not a σ-semiordering, then there exists an element $a \in S \setminus (M \cup -M)$. We claim that $M' := M + \sigma^{-1}(\sigma(a))$ and $M'' := M - \sigma^{-1}(\sigma(a))$ are σ-modules strictly larger than M and satisfying $M' \cap M'' = M$. The nontrivial part is to verify axiom SM_3 of a σ-module. Pick any $x \in \sigma^{-1}(\sigma(M'))$. There exist $m \in M$ and $b \in \sigma^{-1}(\sigma(a))$ such that $\sigma(x) = \sigma(m+b)$. Write $s := mb^{-1} + b^{-1}m$. Since $\sigma|_{\mathbb{Q}(s)}$ is an ordering of higher level in the sense of [B] and $\sigma(s) \neq -1$, it follows that $\sigma(1+s) \in \{1, \sigma(s)\}$. Therefore, $\sigma(x) \in \{\sigma(m), \sigma(b)\}$. Both cases imply that $x \in M'$. The same argument works for M''. □

Write $B = \{(a,p) \in S \times P^\times : \ \exists r \in \mathbb{Q}^+ : rp \pm a \in P\}$. For every σ-semiordering M we have a mapping $\phi_M : B \to \mathbb{R}$ defined by

$$\phi_M(a,p) = \inf\{r \in \mathbb{Q} : \ rp - a \in M\}.$$

As usually, we establish the following properties:

(1) $\phi_M(p,p) = 1$,
(2) $\phi_M(-a,p) = -\phi_M(a,p)$,
(3) $\phi_M(a+b,p) = \phi_M(a,p) + \phi_M(b,p)$,
(4) $\phi_M(a,p) = \phi_M(a,q)\phi_M(q,p)$,
(5) $\phi_M(ab+ba, pq+qp) = \phi_M(a,p)\phi_M(b,q)$,

where $a,b \in S$, $p,q \in P^\times$ are such that arguments on the right side belong to B.

We have that $\phi_M(a,p) \geq 0$ for every $a \in M$ and $p \in P^\times$. If $a \notin M$, then $-a \in M$. Since $\phi_M(-a,p) \geq 0$ and hence $\phi_M(a,p) \leq 0$. In other words, if $\phi_M(a,p) > 0$, then $a \in M$. If $\phi_M(a,p) = 0$, then $\phi_M(rp \pm a, p) > 0$ for every $r \in \mathbb{Q}^+$. Therefore, $rp \pm a \in M$ for every $r \in \mathbb{Q}^+$.

Lemma 3.2. *Suppose $i \in Z(D)$. If $p \in D$ is positive symmetric, $q \in D$ is antisymmetric and $v(q) = v(p)$, then $\phi_M(pq - qp, p^2) = 0$ and $\phi_M((pq+qp)^2, p^4) = 4\phi_M(q^2,p^2)$ for every σ-semiordering M.*

Proof. Since $v(p) = v(q)$, it follows that $qp^{-1} \in A(\sigma)$. Consequently, $i(qp^{-1} + p^{-1}q) \in A(\sigma) \cap M$. By assertion (1) of Proposition 2.4, there exist $r \in \mathbb{Q}^+$ such that $2r \pm i(qp^{-1} + p^{-1}q) \in P$. It follows that $\sigma(rp \pm iq) = \sigma(rp \pm iq)\sigma(p^{-1}) = \sigma((rp \pm iq)p^{-1} + p^{-1}(rp \pm iq)) = \sigma(2r \pm i(qp^{-1} + p^{-1}q)) = 1$. Thus $(iq, p) \in B$.

For every σ-semiordering M on S $\phi_M(qpq, p^3) = \phi_M(q^2, p^2)$ since $2\phi_M(qpq,p^3) + 2\phi_M(q^2,p^2)\phi_M(p,p) = 2\phi_M(qpq,p^3) + \phi_M(pq^2+q^2p, p^3) = \phi_M(2qpq + pq^2 + q^2p, p^3) = -\phi_M((p(iq) + (iq)p)(iq) + (iq)(p(iq) + (iq)p), p^3) = -2\phi_M(p(iq)+(iq)p, p^2)\phi_M(iq,p) = -4\phi_M(p,p)\phi_M(iq,p)^2 = 4\phi_M(p,p)\phi_M(q^2,p^2)$. Similarly, we prove that $\phi_M(qp^2q, p^4) = \phi_M(q^2,p^2)$ and $\phi_M(pq^2p, p^4) = \phi_M(q^2,p^2)$.

Note that $\phi_M(pqpq + qpqp, p^4) = 2\phi_M(qpq,p^3)\phi_M(p,p) = 2\phi_M(q^2,p^2)$ by property (5) and the previous paragraph. Hence $\phi_M((pq-qp)^2, p^4) = \phi_M(pqpq+qpqp, p^4) - \phi_M(pq^2p, p^4) - \phi_M(qp^2q, p^4) = 2\phi(q^2,p^2) - \phi_M(q^2,p^2) - \phi_M(q^2,p^2) = 0$ and $\phi_M((pq+qp)^2, p^4) = 4\phi_M(q^2,p^2)$. □

Proposition 3.3. *Let v be the valuation corresponding to $A(\sigma)$. If $x_1, \ldots, x_k \in D$ and $\sigma(x_1^* x_1) = \ldots = \sigma(x_k^* x_k) = \sigma(x_i^* x_j + x_j^* x_i)$ for every $i,j = 1,\ldots,k$, then $v(x_1 + \ldots + x_k) = \min(v(x_1), \ldots, v(x_k))$.*

Proof. The inequality $\geq$ is clear. To prove the opposite inequality, take any $j = 1,\ldots,k$ and note that

$$\begin{gathered}\sigma(1 \pm (x_1^* + \ldots + x_k^*)^{-1} x_j^* x_j (x_1 + \ldots + x_k)^{-1}) = \\ = \sigma((x_1^* + \ldots + x_k^*)^{-1}((x_1^* + \ldots + x_k^*)(x_1 + \ldots + x_k) - \\ - x_j^* x_j)(x_1 + \ldots + x_k)^{-1}) = \\ = \sigma((x_1^* + \ldots + x_k^*)(x_1 + \ldots + x_k) \pm x_j^* x_j) \cdot \\ \cdot \sigma((x_1^* + \ldots + x_k^*)^{-1}(x_1 + \ldots + x_k)^{-1}) = 1\end{gathered}$$

because $\sigma(x_1^*x_1) = \ldots = \sigma(x_k^*x_k) = \sigma(x_i^*x_j + x_j^*x_i)$ implies that

$$\sigma((x_1^* + \ldots + x_k^*)(x_1 + \ldots + x_k) \pm x_j^*x_j) = \sigma(x_1^*x_1)$$
$$\sigma((x_1^* + \ldots + x_k^*)(x_1 + \ldots + x_k)) = \sigma(x_1^*x_1)$$

It follows that $x_j(x_1+\ldots+x_k)^{-1} \in A(\sigma)$. Hence $v(x_j) \geq v(x_1+\ldots+x_k)$ for every $j = 1, \ldots, k$. □

Corollary 3.4. *Let v be the valuation corresponding to $A(\sigma)$. For every element $x \in D$ such that $\sigma(xx^*) = \sigma(x^2 + (x^*)^2)$ we have that $v(x + x^*) = v(x)$. In particular*

(1) *$v((st)^k + (ts)^k) = v((st)^k)$ for any symmetric $s, t \in D$ and every $k \in \mathbb{N}$,*
(2) *$v(d^{-1}sd + d^*sd^{-*}) = v(s)$ for every symmetric $s \in D$ and any nonzero $d \in D$.*

Proof. Write $x_1 = x$ and $x_2 = x^*$ and note that $\sigma(x_1^*x_1) = \sigma(x_2^*x_2) = \sigma(x_1^*x_2 + x_2^*x_1)$ since $\sigma(x^*x) = \sigma(xx^*) = \sigma(x^2 + (x^*)^2)$. By Proposition 3.3, it follows that $v(x + x^*) = v(x)$.

Writing $x = (st)^k$ we have that $\sigma(x^2 + (x^*)^2) = \sigma((st)^{2k} + (ts)^{2k}) = \sigma(s)\sigma(t(st)^{2k-1}) = \sigma(s)\sigma(t^2)\sigma(s)^2 \cdots \sigma(t)^2\sigma(s) = \sigma(s)^{2k}\sigma(t)^{2k} = \sigma(x^*x)$. Therefore $v((st)^k + (ts)^k) = v(x + x^*) = v(x) = v((st)^k)$ by the first paragraph.

If $x = d^{-1}sd$, then $\sigma(x^2 + (x^*)^2) = \sigma(d^{-1}s^2d + d^*s^2d^{-*}) = \sigma(s^2) = \sigma(s^2)\sigma(d^{-*}d^{-1})\sigma(d^*d) = \sigma(sd^{-*}d^{-1}s)\sigma(d^*d) = \sigma(d^*sd^{-*}d^{-1}sd) = \sigma(x^*x)$. Therefore, $v(d^{-1}sd + d^*sd^{-*}) = v(x + x^*) = v(x) = v(d^{-1}sd) = v(s)$ by the first paragraph. □

Corollary 3.5. *Let v be the valuation corresponding to $A(\sigma)$.*

(1) *If $s, t \in D$ are symmetric and $v(s) = v(t)$ and $\sigma(s) = \sigma(t)$, then $v(s + t) = v(s) = v(t)$.*
(2) *If $s, t \in D$ are symmetric, then $v(s^nt - ts^n) = v(s^{n-1}(st - ts))$.*

Proof. To prove (1) write $x_1 = s$ and $x_2 = t$ and note that $\sigma(x_1^*x_1) = \sigma(s)^2$, $\sigma(x_2^*x_2) = \sigma(t)^2 = \sigma(s)^2$ and $\sigma(x_1^*x_2 + x_2^*x_1) = \sigma(st + ts) = \sigma(s)\sigma(t) = \sigma(s)^2$. Hence $v(s + t) = v(s)$ by Proposition 3.3.

To prove (2) write

$$s^nt - ts^n = \textstyle\sum_{i=1}^n x_i, \quad x_i = s^{n-i-1}(st - ts)s^i, \quad i = 1, \ldots, n$$

For every $i,j=1,\dots,n$ such that $i\le j$ we have that

$$\begin{gathered}\sigma(x_i^*x_j+x_j^*x_i)=\\ =\sigma(s^i(st-ts)^*s^{2n-2-i-j}(st-ts)s^j+\\ +s^j(st-ts)^*s^{2n-2-i-j}(st-ts)s^i)=\\ =\sigma\big((st-ts)^*s^{2n-2-i-j}(st-ts)s^{j-i}+\\ +s^{j-i}(st-ts)^*s^{2n-2-i-j}(st-ts)\big)\sigma(s^i)^2=\\ =\sigma((st-ts)^*s^{2n-2-i-j}(st-ts))\sigma(s^{j-i})\sigma(s^i)^2=\\ =\sigma((st-ts)^*(st-ts))\sigma(s^{2n-2-i-j})\sigma(s^{j-i})\sigma(s^i)^2=\\ =\sigma((st-ts)^*(st-ts))\sigma(s)^{2n-2}\end{gathered}$$

Therefore, $v(x_1+\ldots+x_n)=v(x_1)$ by Proposition 3.3. □

Proposition 3.6. *Let v be the valuation corresponding to $A(\sigma)$. For every nonzero symmetric $a,b\in D$ we have that $v(ab-ba)>v(ab)$.*

Proof. Since $v(ab-ba)=v(a(ba^{-1}-a^{-1}b)a)=v(ba^{-1}-a^{-1}b)+2v(a)$ and $v(ab)=v(ba^{-1})+2v(a)$, we have that $v(ab-ba)>v(ab)$ if and only if $v(ba^{-1}-a^{-1}b)>v(ba^{-1})$. Hence, we may assume that $a,b\in A(\sigma)$. Since $v(a^nb-ba^n)=v(a^{n-1})+v(ab-ba)$ by the assertion (2). of Corollary 3.5, and since $v(a^nb)=v(a^{n-1})+v(ab)$, we have that $v(ab-ba)>v(ab)$ if and only if $v(a^nb-ba^n)>v(a^nb)$. Hence, we may assume that $a,b\in P$.

Take any nonzero $a,b\in A(\sigma)\cap P$ and write

$$p_k=\tfrac{1}{2}((ab)^{2^k}+(ba)^{2^k}),\quad q_k=\tfrac{1}{2}((ab)^{2^k}-(ba)^{2^k}),\quad k=0,1,\dots$$

Clearly, $\sigma(p_k)=\sigma(a)\sigma(b(ab)^{2^k-1})=\sigma(a)\sigma(b)^2\sigma(a)^2\cdots\sigma(b)^2\sigma(a)=1$ for every $k=0,1,\dots$. Since $p_{k+1}+q_{k+1}=(ab)^{2^{k+1}}=((ab)^{2^k})^2=(p_k+q_k)^2=(p_k^2+q_k^2)+(p_kq_k+q_kp_k)$, it follows that $p_{k+1}=p_k^2+q_k^2$ and $q_{k+1}=p_kq_k+q_kp_k$. By the assertion (1) of Corollary 3.4, we have that $v(p_k)=v((ab)^{2^k})\le v(q_k)$. We want to show that $v(q_0)>v(p_0)$. If this is not true, then $q_0p_0^{-1}\in A(\sigma)\setminus I(\sigma)$. Therefore, there exists a σ-semiordering M such that $\phi_M(q_0^2,p_0^2)\neq 0$.

We claim that there exist $r>0$ and $\theta\neq m\pi$ such that

$$\phi_M(p_k,p_0^{2^k})=r^{2^k}\cos(2^k\theta)\quad \phi_M(-q_k^2,p_0^{2^{k+1}})=r^{2^{k+1}}\sin^2(2^k\theta)$$

for every $k=0,1,\dots$. Pick $r,\theta\in\mathbb{R}$ such that $\phi_M(p_0,p_0)=1=r\cos(\theta)$ and $\phi_M(-q_0^2,p_0^2)=r^2\sin(\theta)^2$, Since $\phi_M(q_0^2,p_0^2)\neq 0$, it follows that $\theta\neq m\pi$. This proves the claim for $k=0$. If the claim is true for k, then $\phi_M(p_{k+1},p_0^{2^{k+1}})=\phi_M(p_k^2+q_k^2,p_0^{2^{k+1}})=\phi_M(p_k,p_0^{2^k})^2+\phi_M(-q_k^2,p_0^{2^{k+1}})=r^{2^{k+1}}\cos^2(2^k\theta)-r^{2^{k+1}}\sin^2(2^k\theta)=r^{2^{k+1}}\cos(2^{k+1}\theta)$, $\phi_M(-q_{k+1}^2,p_0^{2^{k+2}})=\phi_M(-(p_kq_k+q_kp_k)^2,p_0^{2^{k+2}})=\phi_M(-(p_kq_k+q_kp_k)^2,p_k^4)\phi_M(p_k^4,p_0^{2^{k+2}})=-4\phi_M(q_k^2,p_k^2)\phi_M(p_k^2,p_0^{2^{k+1}})\phi_M(p_k,p_0^{2^k})^2=-4\phi_M(q_k^2,p_0^{2^{k+1}})\phi_M(p_k,p_0^{2^k})^2=4r^{2^{k+1}}\sin^2(2^k\theta)(r^{2^k}\cos(2^k\theta))^2=r^{2^{k+2}}\sin^2(2^{k+1}\theta)$.

Since $\theta \neq m\pi$, there exists an integer $k \geq 0$ such that $\cos(2^k\theta) < 0$. Since p_k and p_0 are positive, this is a contradiction. $\square$

Theorem 3.7. *Let σ be a *-signature on D with level m, v the valuation corersponding to $A(\sigma)$ and*

$$W = \{s + k|\ s^* = s, k^* = -k, v(k) > v(s)\} \cup \{0\},$$
$$\tau : W \to \mathbb{C} \quad \tau(x) = \sigma(x + x^*)$$

The ordered pair (W, τ) satisfies the following properties:

(E$_1$) *$S \subseteq W$ and $\tau|_S = \sigma$,*
(E$_2$) *if $x \in W$ then $x^* \in W$ and $\tau(x) = \tau(x^*)$,*
(E$_3$) *if $x \in W$ and $d \in D$ then $dxd^* \in W$ and $\tau(dxd^*) = \tau(x)\tau(dd^*)$,*
(E$_4$) *if $x, y \in W$ then $xy \in W$ and $\tau(xy) = \tau(x)\tau(y)$,*
(E$_5$) *if $x, y \in W$ and $\tau(x) = \tau(y) = 1$ then $x+y \in W$ and $\tau(x+y) = 1$.*

Proof. (E$_1$) Since $v(0) = \infty > v(s)$ for every nonzero s and $0 \in W$, it follows that $S \subseteq W$. Clearly, $\tau(s) = \sigma(2s) = \sigma(s)$ for every $s \in S$.

(E$_2$) If $s + k \in W$, then $v(-k) = v(k) > v(s)$, so that $s - k \in W$. Clearly, $\tau(s + k) = \sigma(2s) = \tau(s - k)$.

(E$_3$) If s is symmetric, k antisymmetric and $v(k) > v(s)$, then for every nonzero $d \in D$, dsd^* is symmetric, dkd^* is antisymmetric and $v(dkd^*) > v(dsd^*)$. Therefore, $dWd^* \subseteq W$ for every $d \in D$. If $x \in W$ and $d \in D$, then $\tau(dxd^*) = \sigma(dxd^* + (dxd^*)^*) = \sigma(d(x + x^*)d^*) = \sigma(x + x^*)\sigma(dd^*) = \tau(x)\tau(dd^*)$.

(E$_4$) If $x_1 = s_1 + k_1 \in W$ and $x_2 = s_2 + k_2 \in W$, then

$$x_1x_2 = s + k$$
$$s = \tfrac{1}{2}(x_1x_2 + x_2^*x_1^*) = s' + s'', \quad k = \tfrac{1}{2}(x_1x_2 - x_2^*x_1^*) = k' + k'',$$
$$s' = \tfrac{1}{2}(s_1s_2 + s_2s_1), s'' = \tfrac{1}{2}(s_1k_2 - k_2s_1 + k_1s_2 - s_2k_1 + k_1k_2 + k_2k_1),$$
$$k' = \tfrac{1}{2}(s_1s_2 - s_2s_1), k'' = \tfrac{1}{2}(s_1k_2 + k_2s_1 + s_2k_1 + k_1s_2 + k_1k_2 - k_2k_1).$$

Clearly, $v(s'') > v(s_1s_2)$ and $v(k'') > v(s_1s_2)$. By Corollary 3.4, $v(s') = v(s_1s_2)$ and by Proposition 3.6, $v(k') > v(s_1s_2)$. It follows that $v(s) = v(s') = v(s_1s_2) < \min(v(k'), v(k'')) \leq v(k)$. Hence $x_1x_2 \in W$ and

$$\tau(x_1x_2) = \sigma(s) = \sigma(s') = \sigma(s_1)\sigma(s_2) = \tau(x_1)\tau(x_2).$$

(E$_5$) If $x_1 = s_1 + k_1 \in W$ and $x_2 = s_2 + k_2 \in W$, then

$$x_1 + x_2 = s + k, \quad s = s_1 + s_2, \quad k = k_1 + k_2$$

Since $\sigma(s_1) = \sigma(s_2) = 1$, it follows from Corollary 3.5 that

$$v(s) = \min\{v(s_1), v(s_2)\} < \min\{v(k_1), v(k_2)\} \leq v(k).$$

Therefore, $x_1 + x_2 \in W$. Clearly, $\tau(x_1 + x_2) = \sigma(s_1 + s_2) = 1$. $\square$

4. Extended preorderings

Let D be a *-field and $S = \mathrm{Sym}(D, *)$. A subset T of D is an *extended *-preordering* with level m if

(EP_1) $-1 \notin T$,
(EP_2) $T + T \subseteq T$,
(EP_3) $TT \subseteq T$,
(EP_4) $T^* \subseteq T$,
(EP_5) $dTd^{-1} \subseteq T$ for every $d \in D^\times$,
(EP_6) $dsd^{-1}s^{-1} \in T$ for every $d \in D^\times$ and $s \in S^\times$.
(EP_7) $(dd^*)^m \subseteq T$, for every $d \in D$.

Clearly, $\mathbb{Q}^+ \subseteq T$ and $t^{-1} \in T$ for every nonzero $t \in T$. A subset Q of D is an *extended *-ordering* with level m if it is an extended $*$-preordering with level m and there exists a semigroup homomorphism $\phi : \Pi S \to \mathbb{C}$ with kernel $Q \cap \Pi S$. Theorem 3.7 has the following Corollary:

Corollary 4.1. *For every *-ordering P with level m on a $*$-field D there exists an extended *-ordering Q with level m on D such that $P = Q \cap S$.*

Motivated by this result we say that a subset $V \subseteq S$ is a *$*$-preordering* if there exists an extended $*$-preordering T on D such that $V = T \cap S$.

The aim of this section is to prove the following version of the Artin-Schreier Theorem.

Theorem 4.2. *Every $*$-preordering is equal to the intersection of all $*$-orderings containing it.*

An extended $*$-preordering T on a $*$-field D is *complete* if for every $a \in \Pi S$ such that $a^2 \in T$ we have $a \in T \cup -T$.

Lemma 4.3. *Every extended $*$-preordering which is not complete is equal to the intersection of two stricly larger extended $*$-orderings.*

Proof. If T is an extended $*$-preordering which is not complete then there exists an element $a \in \Pi S$ such that $a^2 \in T$ and $a \notin T \cup -T$. Clearly, the sets $T + aT$ and $T - aT$ are strictly larger then the set T. A long but straightforward verification of axioms shows that they are extended $*$-preorderings. We claim that $T = (T + aT) \cap (T - aT)$. If $x \in (T + aT) \cap (T - aT)$, then there exist $t_1, t_2, t_3, t_4 \in T$ such that $x = t_1 + at_2 = t_3 - at_4$. If $t_2 = 0$, then $x = t_1 \in T$. If $t_2 \neq 0$, then $t := t_2^{-1}t_4t_2 + 1 \in T^\times$ and $xt = t_1(t_2^{-1}t_4t_2) + t_3t_2 \in T$, so that $x \in T$. The opposite inclusion is clear. □

Proposition 4.4. *Every extended $*$-preordering is equal to the intersection of all complete extended $*$-preorderings that contain it.*

Proof. Let T_0 be an extended $*$-preordering (of some level) and $x \notin T_0$. By Zorn's Lemma, there exists a maximal extended *-preordering T containing T_0 and avoiding x. If T is not complete, then it is equal to the intersection of two stricly larger extended *-preorderings T_1 and T_2. By the choice of T, we have $x \in T_1$ and $x \in T_2$, which imply a contradiction $x \in T_1 \cap T_2 = T$. □

Lemma 4.5. *Let D be a *-field, S the set of its symmetric elements and T a complete extended *-preordering on D. Write $A(T) = \{s \in S: \exists r \in \mathbb{Q}^+ : r \pm s \in T\}$ and $I(T) = \{s \in S: \forall r \in \mathbb{Q}^+ : r \pm s \in T\}$.*

(1) *For every $s \in S$ we have either $s \in A(T)$ or $s^{-1} \in A(T)$.*
(2) *$A(T) \subseteq T \cup -T \cup I(T)$.*

Proof. Pick any $s \in S$ and note that $\mathbb{Q}(s)$ is a commutative field, $T \cap \mathbb{Q}(s)$ is a complete preordering on $\mathbb{Q}(s)$ in the sense of [B] and $A(T) \cap \mathbb{Q}(s) = A(T \cap \mathbb{Q}(s))$. By the results from [B], we have that either $s \in A(T \cap \mathbb{Q})$ or $s^{-1} \in A(T \cap \mathbb{Q}(s))$. This gives the assertion (1) The assertion (2) follows from the fact that $A(T \cap \mathbb{Q}(s)) \subseteq (T \cap \mathbb{Q}(s) \cup -(T \cap \mathbb{Q}(s)) \cup I(T \cap \mathbb{Q}(s))$ for every $s \in S$. □

Note that the set $U := \Pi(S^\times)$ is a normal subgroup of $D^\times$ which contains $1 + sts^{-1}t^{-1}$ for every $s, t \in S^\times$.

Lemma 4.6. *If T is a complete extended *-preordering on a *-field D and O is a subgroup of U which contains $T \cap U$ and avoids -1, then the set $O \cap S$ is additive and closed for Jordan multiplication.*

Proof. If $s, t \in O \cap S$, then $(1 + tst^{-1}s^{-1}) \in T \cap U \subseteq O$, so that $(s, t) := st + ts = (1 + tst^{-1}s^{-1})st \in O$. Hence $O \cap S$ is closed for the Jordan product.

We want to prove that $2 + (O \cap S) \subseteq O \cap S$. Pick any $s \in O \cap S$. By Lemma 4.5, we have either $s \in A(T) \subseteq T \cup -T \cup I(T)$ or $s^{-1} \in I(T)$. The case $s \in -T$ is not possible, since it implies a contradiction $-1 \in O$. If $s \in T$ or $s \in I(T)$ then $2 + s \in T \cap S \subseteq O$. If $s^{-1} \in I(T)$, then $2 + s^{-1} \in T \cap S^\times \subseteq O$. It follows that, $2 + s = s(1 + 2s^{-1}) \in O$.

If $s, t \in O \cap S$, then $(s^{-1}, t) \in O$ by the first paragraph and the second paragraph implies that $2 + (s^{-1}, t) \in O$. It follows that $s^{-1}(s + t) + (s + t)s^{-1} = 2 + (s^{-1}, t) \in O$. Since $0 \notin O$, we have that $s + t \neq 0$, so that $x = 1 + s^{-1}(s + t)s(s + t)^{-1}$ is well defined. We have that $x \in T \cap U \subseteq O$. On the other hand, $x(s + t) = (2 + (s^{-1}, t))s \in O$, hence $s + t \in O$ as desired. □

Corollary 4.7. *If T is a complete extended *-preordering on a *-field D and O is a subgroup of U such that $-1 \notin O$ and $T \cap U \subseteq O$ then the set $P := \{0\} \cup \Sigma O$ is an extended $*$-preordering such that $U \cap P = O$.*

Proof. The inclusion $O \subseteq U \cap \Sigma O$ is clear. The proof of the opposite inclusion depends on the following claim: *for every $u \in U$ we have $u \in O$ if and only if $u + u^* \in O$.* Pick any $u \in U \cap \Sigma O$. There exist $u_1, \ldots, u_k \in O$ such that $u = u_1 + \ldots + u_k$. The claim implies $u_1 + u_1^* \in O \cap S, \ldots, u_k + u_k^* \in O \cap S$. Lemma 4.6 says that the set $O \cap S$ is additive. It follows that $u + u^* = u_1 + u_1^* + \ldots + u_k + u_k^* \in O \cap S$. Finally, we get $u \in O$ by the claim.

We have $-1 \notin P$, otherwise we get a contradiction $-1 \in U \cap P = O$. Since T is an extended $*$-ordering, it follows that P also satisfies other axioms of an extended $*$-ordering. □

Proposition 4.8. *For every extended $*$-preordering T, the set $T \cap U$ is equal to the intersection of all extended *-orderings which contain it.*

Proof. Take any element $x \in U \setminus T$. The factor group $A := U/T \cap U$ is abelian. Let B be a subgroup of A generated by $\overline{x}$ and $\overline{-1}$. There exists a homomorphism $\psi : B \to \mathbb{C}^\times$ such that $\psi(\overline{-1}) = -1$ and $\psi(\overline{x}) \neq 1$. Since $\mathbb{C}^\times$ is divisible we can extend ψ from B to A. Let $\phi = \psi \circ \pi$ where $\pi : U \to A$ is the canonical projection. Write $O := \phi^{-1}(1)$ and $P := \{0\} \cup \Sigma O$. Since O is a subgroup of U such that $-1 \notin O$ and $T \cap U \subseteq O$, it follows by Corollary 4.7 that P is an extended $*$-ordering. Finally, $x \in U \setminus O$ and $P \cap U = O$ imply that $x \notin P$. □

5. Extended *-semiorderings

Let T be a preordering on D. A subset M of D is a T-module if

(TM_1) $-1 \notin M$,
(TM_2) $M + M \subseteq M$,
(TM_3) $1 \in M$,
(TM_4) $TM \subseteq M$,
(TM_5) $dMd^{-1} \subseteq M$ for every $d \in D^\times$.

Note that T is always a T-module. Properties TM_3 and TM_4 imply that $T \subseteq M$. Properties TM_4 and TM_5 imply that $MT \subseteq M$. For every T-module M write

$$A(M) = \{d \in D \mid \exists r \in \mathbb{Q}^+ : r \pm dd^* \in M\},$$

$$I(M) = \{d \in D \mid \forall r \in \mathbb{Q}^+ : r \pm dd^* \in M\}.$$

Identity $r - d^*d = d^*(r - dd^*)d^{-*}$ and property TM_5 imply that $A(M)^* \subseteq A(M)$ and $I(M)^* \subseteq I(M)$. A T-module M is a T-*semiordering* if $M \cup -M \supset \mathrm{Sym}(D, *)$.

For every element $s \in \mathrm{Sym}(D, *)$ and every level $m = \frac{n}{2}$ of a preordering T, there exist elements $s_n^+, s_n^- \in T$ such that $s = s_n^+ - s_n^-$.

This follows from the standard identity

$$n!s = \sum_{h=0}^{n-1} (-1)^{n-1-h}[(x+h)^n - h^n]$$

Proposition 5.1. *If M is a (Jordan) T-semiordering, then the set*

$$B(M) = \{s \in \mathrm{Sym}(D,*) | \ \exists r \in \mathbb{Q}^+ : r \pm s \in M\}$$

is a Jordan subring of $\mathrm{Sym}(D,*)$ *and the set*

$$J(M) = \{s \in \mathrm{Sym}(D,*) | \ \forall r \in \mathbb{Q}^+ : r \pm s \in M\}$$

is its Jordan ideal. Moreover, for any symmetric s, we have $s \notin B(M)$ if and only if $s^{-1} \in J(M)$.

Proof. Clearly, $B(M)$ and $J(M)$ are closed for addition. We know from the commutative theory that for every $s \in \mathrm{Sym}(D,*)$ the set $B(M) \cap \mathbb{Q}(s) = B(M \cap \mathbb{Q}(s))$ is a valuation subring of $\mathbb{Q}(s)$ with maximal ideal $J(M) \cap \mathbb{Q}(s) = J(M \cap \mathbb{Q}(s))$. This implies the last assertion in the proposition. It also implies that every integer polynomial (in particular every power) of every element from $B(M)$ belongs to $B(M)$. If $s,t \in B(M)$, then $s^2, t^2, (s+t)^2 \in B(M)$, so $st + ts = \frac{1}{2}((s+t)^2 - s^2 - t^2) \in B(M)$.

If $s \in J(M)$ and $t \in B(M) \cap T$, then $r_1 \pm s \in M$ for every $r_1 \in \mathbb{Q}^+$ and $r_2 \pm t \in M$ for some $r_2 \in \mathbb{Q}^+$. It follows that $2r_1r_2 \pm (st+ts) = ((r_1 \pm s)t + t(r_1 \pm s)) + (r_1(r_2-t)+(r_2-t)r_1) \in M+M \subseteq M$. If $s \in J(M)$ and $t \in B(M)$, then $t_n^+, t_n^- \in B(M) \cap T$, Hence, $st_n^+ + t_n^+ s \in J(M)$ and $st_n^- + t_n^- s \in J(M)$ implying that $st + ts \in J(M)$. □

Note that $B(M) = A(M) \cap \mathrm{Sym}(D,*)$ and $J(M) = I(M) \cap \mathrm{Sym}(D,*)$.

Theorem 5.2. *If D is a *-field with $i \in Z(D)$, T a preordering on D and M a T-semiordering, then $A(M)$ is an invariant *-valuation subring of D with maximal ideal $I(M)$.*

Proof. If $s \in B(M)$ and $t \in B(M) \cap T$, then $r_1 \pm s \in M$ and $r_2 \pm t \in M$ for some $r_1, r_2 \in \mathbb{Q}^+$. It follows that $r_1 r_2 \pm st = (r_1 \pm s)t + (r_2 - t)r_1 \in MT + MT \subseteq M$. If $s,t \in B(M)$, then $t_n^+, t_n^- \in B(M) \cap T$, so $2r - st = r - st_n^+ + r - st_n^- \in M + M \subseteq M$ for a suitable r. If $a, b \in A(M)$, then $a^*a, bb^* \in B(M)$, so $r \pm a^*abb^* \in M$ for some r. By property TM_5 of T-modules we have $r \pm ab(ab)^* \in M$, so $ab \in A(M)$. Similarly, we prove that $I(M)$ is an ideal of $A(M)$.

If $a \notin A(M)$, then $aa^* \notin B(M)$, so $a^{-*}a^{-1} = (aa^*)^{-1} \in B(M)$. By definition, $a^{-*} \in A(M)$. Since $A(M)^* \subseteq A(M)$, we have $a^{-1} \in A(M)$.

To prove that $A(M)$ is closed for addition, replace the proof of Lemma 2.5(a) with Proposition 5.1 and copy the proofs of Lemma 2.5(b), Lemma 2.6, Lemma 2.7 and Proposition 2.8.

It remains us to prove that $A(M)$ is invariant. If $t \in T \cap B(M)$ and $d \in D^\times$, then there exists $r \in \mathbb{Q}^+$ such that $r - t \in M$. It follows that $r^2 - d^{-1}tdd^*td^{-*} = \frac{1}{2}\left(d^{-1}(r-t)dd^*(r+t)d^{-*} + d^{-1}(r+t)dd^*(r-t)d^{-*}\right) \in d^{-1}Mdd^*Td^{-*} + d^{-1}Tdd^*Md^{-*} \subseteq MT + TM \subseteq M$. Obviously, $r^2 + d^{-1}tdd^*td^{-*} \in T + T \subseteq T \subseteq M$. Hence, $d^{-1}td \in A(M)$. It follows that for any $s \in B(M)$ and any $d \in D^\times$ we have $d^{-1}sd = d^{-1}s_n^+d - d^{-1}s_n^-d \in A(M)$. It follows that $d^{-1}kd \in A(M)$ for every antisymmetric $k \in A(M)$ and finally that $d^{-1}ad = \frac{1}{2}(d^{-1}(a+a^*)d + d^{-1}(a-a^*)d) \in A(M)$ for any $a \in A(M)$. □

We say that a T-semiordering M on a *-field D is *archimedean* if for every $s \in S = \mathrm{Sym}(D,*)$, there exists $r \in \mathbb{Q}^+$ such that $r \pm s \in M$. Note that $A(M) \cap M$ is always an archimedean T-semiordering as well as its projection $\overline{M}$ to the residue field $k = A(M)/I(M)$.

Theorem 5.3. *Let D be a *-field with $i \in Z(D)$ and M an archimedean T-semiordering. Then the set $M \cap (\Pi S) \cdot T$ is closed for multiplication.*

Proof. We claim that for any $a, b \in S$ such that $0 < a < b$ we have that $a^2 < b^2$. As in the commutative case, we can find $r \in \mathbb{Q}$ such that $a < r < b$ and show that $0 < a < r$ implies $0 < a^2 < r^2$ and $0 < r < b$ implies $0 < r^2 < b^2$. Therefore $a^2 < r^2 < b^2$. If $x, y \in Q \cap S$, then $0 < |x - y| < x + y$. The claim implies that $|x-y|^2 < (x+y)^2$, so $xy + yx \in M$.

If $d, e \in M \cap (\Pi S) \cdot T$, then $d + d^*, e + e^* \in M \cap S$. It follows that $(d+d^*)(e+e^*) + (e+e^*)(d+d^*) \in M$. It follows that $(d+d^*)(e+e^*) \in M$. This is obvious if either $d + d^* = 0$ or $e + e^* = 0$ and it follows from the fact that $1 + (d+d^*)^{-1}(e+e^*)^{-1}(d+d^*)(e+e^*) \in T \setminus 0$. Since $(1+d^*d^{-1})de(1+e^{-1}e^*) = (d+d^*)(e+e^*) \in M$ and $1+d^*d^{-1}, 1+e^{-1}e^* \in T \setminus 0$, we have $de \in M$. □

Theorem 5.4. *If Q is a T-semiordering and $Q' = \prod(A(Q)^* \cap Q) \cap (\Pi S) \cdot T$, then the set*

$$T \wedge \overline{Q} = \{\sum t_i u_i |\ t_i \in T, u_i \in Q'\},$$

is a preordering containing T and $Q \cap S$.

Proof. It is easy to verify properties EP_2-EP_7 of a preordering. Let $k = A(Q)/I(Q)$ and $\bar{\ } : A(Q) \to k$ be the natural projection. By Theorem 5.3, $\overline{Q}$ is an archimedean $\overline{T}$-semiordering on k, hence $\overline{Q'} \subseteq \overline{Q}$ is a multiplicative set. It follows that $\overline{Q'}^{-1} \subseteq \overline{Q}$. If $0 = t_1u_1 + \ldots + t_ku_k$, where $v(t_1) \leq \ldots \leq v(t_k)$, then $0 = 1 + t_1^{-1}t_2u_2u_1^{-1} + \ldots + t_1^{-1}t_ku_ku_1^{-1}$, so $-\overline{1} = \overline{t_1^{-1}t_2}\,\overline{u_2u_1}^{-1} + \ldots + \overline{t_1^{-1}t_k}\,\overline{u_ku_1}^{-1} \in \overline{S}$. So $-1 \in S + I(S)$, a contradiction with $-1 \notin S$. □

It follows immediately from the Theorem 5.3, that for any $a \in Q \cap S$ such that $v(a) \in v(T)$, we have that $a \in T \wedge \overline{Q}$.

We say that a preordering T is *compatible* with valuation v if $\overline{T} = (T \cap A_v)/I_v$ is a preordering on the residue field of v. In this case

$$T^v = \{\sum t_i(1+m_i) |\ t_i \in T_i, m_i \in I_v, 1+m_i \in (\Pi S)\cdot T\}$$

is again a preordering containing T.

6. Weak isotropy

Let D be a *-field, $S = \mathrm{Sym}(D,*)$ and ΠS the set of all products of elements from S. Let T be an extended *-preordering on D and Q a T-semiordering. Let v be a valuation corresponding to the valuation ring $A(Q)$.

Lemma 6.1. *If $x \in T \cdot \Pi S$, then $v(x) = v(x + x^*)$. In particular $v(st + ts) = v(s) + v(t)$ and $v(s^{-1}ts + sts^{-1}) = v(t)$ for any nonzero $s, t \in S$.*

Proof. Write $z = x^{-1}x^*$. Clearly, $v(z) = v(1) = 0$. If $v(1+z) > 0$, then $1 + z \in I(Q)$, so $\frac{1}{2} - (1+z)(1+z^*) \in Q$. Since $z, z^*, zz^* \in T \subseteq Q$, it follows that $-\frac{1}{2} = (\frac{1}{2}-(1+z)(1+z^*))+z+z^*+zz^* \in Q$, a contradiction. Now $v(x+x^*) = v(x(1+z)) = v(x) + v(1+z) = v(x)$. Take $x = st$ in the first special case and $x = sts^{-1}$ in the second special case □

Lemma 6.2. *If $a, b \in S$, $a \in Q$, $v(a) < v(b)$ and $v(a) \leq v(t) \leq v(b)$ for some $t \in T \cap S$, then $a - b \in Q$.*

Proof. Since $a \in Q$, we have $at^{-1} + t^{-1}a \in Q$. If the conclusion is false, then $b - a \in Q$, so $(b-a)t^{-1} + t^{-1}(b-a) \in Q$. Hence $0 < at^{-1} + t^{-1}a < bt^{-1} + t^{-1}b$. If $bt^{-1} + t^{-1}b \in A(Q)$ is bounded by some (every) positive rational, then $at^{-1} + t^{-1}a \in A(Q)$ is bounded by some (every) positive rational, too. On the other hand we have that either $v(a) < v(t) \leq v(b)$ or $v(a) \leq v(t) < v(b)$. The first case implies a contradiction $v(at^{-1}+t^{-1}a) < 0$ and $v(bt^{-1}+t^{-1}b) > 0$ and the second case implies a contradiction. $v(at^{-1} + t^{-1}a) \leq 0$ and $v(bt^{-1} + t^{-1}b) \geq 0$ □

Lemma 6.3. *If $q, x, u \in S$ satisfy $q \in Q$, $x \in I(Q)$, $u \in T$ and $0 \leq v(q^{-1}u) \leq v(x)$, then $(1-x)q \in Q$.*

Proof. Write $a = 2q$, $t = quq^{-1} + q^{-1}uq$ and $b = xq + qx$. Since $v(q) \leq v(u) \leq v(qx)$ and $v(a) = v(q)$, $v(t) = v(u)$, $v(b) = v(qx)$ by Lemma 6.1 and $v(a) < v(b)$, it follows by Lemma 6.2 that $(1-x)q + q(1-x) \in Q$. Element $1-x$ is invertible since $x \in T$ and $1+(1-x)^{-1}q^{-1}(1-x)q \in T$, so $(1-x)q \in Q$. □

Lemma 6.4. *Let Γ be an ordered abelian group and $n \geq 2$ a natural number. Let Λ be a subset of Γ such that $0 \notin \Lambda$ and $\Lambda + n\Gamma \subset \Lambda$. Then the set $\Sigma = \{\sigma \in \Gamma : |\sigma| < |\Lambda|\}$ is a convex subgroup of Γ.*

Proof. Clearly, Σ is a convex subset of Γ and $-\Sigma = \Sigma$. If we prove that $0 < \sigma \in \Sigma$ implies that $2\sigma \in \Sigma$, then the additivity of Σ follows from the convexity of Σ. We will actually prove that $0 < \sigma \in \Sigma$ implies that $\frac{n}{n-1}\sigma \in \Sigma$.

If $0 < \sigma \in \Sigma$, then $\sigma < \lambda$ for all $0 < \lambda \in \Lambda$. It follows that $n\sigma - (n-1)\lambda < \sigma$ for all $0 < \lambda \in \Lambda$. We also have $n\sigma - (n-1)\lambda = \lambda + n(\sigma - \lambda) \in \Lambda + n\Gamma \subseteq \Lambda$ for all $0 < \lambda \in \Lambda$. If $n\sigma - (n-1)\lambda > 0$ for some $0 < \lambda < \Lambda$, then σ is bounded from below by a positive element of Λ, a contradition with $0 < \sigma \in \Sigma$. □

Proposition 6.5. *For any $q_1, \dots, q_r \in Q$, there exists a valuation w (with valuation ring A_w, maximal ideal I_w and value group Γ_w) such that:*

1. *$A_w \supset A(Q)$,*
2. *$(1-x)q_i \in Q$ for any $x \in m_w$ and any $i = 1, \dots, r$.*
3. *If $v(q_j) \notin v(T)$ for some j, then $w(q_k) \notin w(T)$ for some k.*

Proof. For any $i = 1, \dots, r$ write $\Lambda_i = v(q_i^{-1}(T \cap S))$ and $\Sigma_i = \{\sigma \in \Gamma : |\sigma| < |\Lambda_i|\}$. Renumerating indices if necessary, we can assume that $\Sigma_1 \subseteq \ldots \subseteq \Sigma_r$. Let k be the largest index such that $0 \notin \Sigma_k$. By Lemma 6.4, Λ_k is a convex subgroup of Γ. Write $\Gamma_w = \Gamma/\Sigma_k$ and let $w : D \to \Gamma_w$ be the corresponding valuation. Since $\Sigma_k \cap \Lambda_k = \emptyset$, we have that $0 \notin w(q_k^{-1}T)$, so $w(q_k) \notin w(T)$. For any nonzero $x \in I_w$, we have $w(x) > 0$, so $v(x) \notin \Sigma_k$. Hence $v(x) \notin \Sigma_i$ for any $i = 1, \dots, k$. It follows that $v(x)$ is bounded from below by some element of $v(q_i^{-1}T)$ for every $i = 1, \dots, k$. By Lemma 6.3, it follows that $(1-x)q_i \in Q$ for every $x \in I_w \cap S$ and every $i = 1, \dots, k$. □

Now the same argument as in [C1, Theorem 5.2] implies the weak isotropy principle:

Theorem 6.6. *Let T be an extended $*$-semiordering on a $*$-field D and $a_1, \dots, a_n$ nonzero symmetric elements of D. The following are equivalent:*

1. *There exist elements $t_1, \dots, t_n \in T$ not all equal to zero such that $a_1 t_1 + \ldots + a_n t_n = 0$.*
2. *The following are satisfied:*
   - (a) *For every signature $\sigma : S \to \mathbb{C}$ such that $\sigma(T \cap S) = 1$, the set $\{\sigma(a_1), \dots, \sigma(a_n)\}$ has more than one element.*

(b) *For every valuation v which is compatible by T and satisfies $v(a_i) \not\equiv v(a_j) \mod v(\dot{T})$, there exist elements $t_1^v, \ldots, t_n^v \in T^v$ which are not all zero and satisfy $a_1 t_1^v + \ldots + a_n t_n^v = 0$.*

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