

COMPLETIONS OF VALUATION RINGS

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1. INTRODUCTION

Suppose that k is a field of characteristic zero, K is an algebraic function field over k , and V is a k -valuation ring of K (that is, $k \subset V$ and the quotient field of V is K). Zariski's theorem of local uniformization [23] shows that there exist algebraic regular local rings R_i with quotient field K which are dominated by V , and such that the direct limit

$$\cup R_i = V.$$

Now suppose that K^* is a finite algebraic extension of K and V^* is a k -valuation ring of K^* such that $V = V^* \cap K$. Let Γ^* be the value group of V^* , and Γ be the value group of V .

The first author has shown with Olivier Piltant in [9] that a relative local uniformization theorem holds for the extension K^* of K , which gives the strongest possible generalization of the classical ramification theory of Dedekind domains to general valuations. The following theorem is a summary of some of the conclusions of Theorem 6.3 [9].

Theorem 1.1. *Suppose that assumptions and notations are as above. Let k' be an algebraic closure of V^*/m_{V^*} . Then there exist a directed system of algebraic regular local rings S_i with quotient field K^* which are dominated by V^* , and a directed system of algebraic normal local rings R_i with quotient field K which are dominated by V such that*

- (1) $\cup S_i = V^*$ and $\cup R_i = V$.
- (2) S_i is a localization at a maximal ideal of the integral closure of R_i in K^* for all i .
- (3) There exist actions of Γ^*/Γ on $\hat{S}_i \hat{\otimes}_{S_i/m_{S_i}} k'$ which are compatible with the directed system so that

$$(\hat{S}_i \hat{\otimes}_{S_i/m_{S_i}} k')^{\Gamma^*/\Gamma} \cong \hat{R}_i \hat{\otimes}_{R_i/m_{R_i}} k'.$$

It was shown by an example of Abhyankar [2] that it is in general not possible to find an algebraic regular local ring S with quotient field K^* which is dominated by V^* such that there exists an algebraic regular local ring R with quotient field K such that S is a localization of the integral closure of R in K^* . The fact (proven in [9] and [7]) that normal local rings R exist satisfying this property proves the ‘‘local weak simultaneous resolution conjecture’’ of Abhyankar, posed by Abhyankar in [2] and [5]. The R_i found in the proof of Theorem 1.1 in fact have toric singularities. This is reflected in the fact stated above that their completions are abelian quotient singularities.

From this theorem we obtain the following.

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Theorem 1.2. *Let notations be as in Theorem 1.1. Let $U^* = \cup \hat{S}_i \hat{\otimes}_{S_i/m_{S_i}} k'$ and let $U = \cup \hat{R}_i \hat{\otimes}_{R_i/m_{R_i}} k'$. Then U^* and U are Henselian normal domains, and $Q(U^*)$ is a finite Galois extension of $Q(U)$ with Galois group Γ^*/Γ .*

We give a proof of Theorem 1.2 in Section 3.

In this paper we compare the “completion” of Theorem 1.2 with other notions of completion of a valuation ring ([16], [18], [19], [14], [21], [22]).

Let us briefly allow k to be an arbitrary field. We summarize some of the results of Section 4. Suppose that $\{R_i\}$ is a directed system of normal algebraic local rings which are dominated by V , and such that $\cup R_i = V$. The ring $T = \cup \hat{R}_i$ does not depend on our choice of $\{R_i\}$ whose union is V (Lemma 4.1), and is Henselian (Proposition 4.2). Thus T can be considered to be a “completion” of the valuation ring V . We give an example showing that T is in general not a valuation ring, and we show that T is itself a valuation ring if and only if for each i there exists a unique valuation ring V_i with quotient field K_i (where K_i is the quotient field of \hat{R}_i) which dominates V and \hat{R}_i . (Theorem 4.4). We make use of a theorem of Heinzer and Sally [14] on the uniqueness of extensions of valuations dominating a local ring to their completion in proving this result.

We give an example (Example 7.5) showing that even if V and T are rank 1 valuation rings, then T is in general not complete and in particular is not a maximal immediate extension, as defined in [18] and [16].

The essential obstruction to T being a valuation ring is the problem of the rank of the valuation increasing upon extending the valuation dominating a particular R to a valuation dominating its completion (Corollary to Theorem 4.4). In the case of rank 1 valuations, this problem can be handled in a very satisfactory way, and (in characteristic zero) we will obtain a good valuation theoretic explanation of Theorem 1.2.

In Section 5, we define the prime ideal $p(\hat{R})_\infty$ of elements of infinite value of the completion \hat{R} of an algebraic local ring R dominated by a rank 1 valuation V . This prime has previously been defined and considered in [6] and [22], as well as by Spivakovsky. The essential point here is that there is a unique extension of the valuation ring V to a valuation ring of the quotient field of $\hat{R}/p(\hat{R})_\infty$ which dominates $\hat{R}/p(\hat{R})_\infty$. We conclude that there is a unique valuation ring \bar{V} of the quotient field of the ring $\bar{T} = \cup \hat{R}_i/p(\hat{R}_i)_\infty$ which contains T .

In the case when V has rank greater than 1 there is no natural ideal in \hat{R} which contains the obstruction to the jumping of the rank of an extension of V to \hat{R} , although this obstruction is obtained in a series of prime ideals in quotient rings of \hat{R} .

For the remainder of this introduction we assume that k has characteristic zero, and V has rank 1. We prove that \bar{T} is in fact a valuation ring in Theorem 7.3, and that $(\bar{T}, Q(\bar{T}))$ is an immediate Henselian extension of (V, K) in Theorem 7.4.

We further show (in Theorem 7.4) that we can choose our system of regular local rings R_i so that each $\hat{R}_i/p(\hat{R}_i)_\infty$ is a regular local ring. The main new technical result used in this statement is Theorem 6.5, which shows that we can simultaneously resolve the primes of infinite value in a finite extension. In this case the “finite extension” is just the identity, but we will need this more general result later.

We now turn to an analysis of our finite extension K^* over K , in the case when V^* (and $V = V^* \cap K$) are rank 1 valuation rings and k has characteristic zero. We make essential use of Theorem 6.5 (on simultaneous resolution of the primes of infinite value). We obtain in Theorems 8.1 and 8.2 a generalization of Theorem 5.1 [6] and Theorem 1.1 (Theorem 6.3 [9]) in this context. Let k' be an algebraic closure of

V^*/m_{V^*} . We find a system of regular local rings S_i whose union is V^* , and a system of normal local rings R_i whose union is V , such that for all i , S_i is a localization at a maximal ideal of the integral closure of R_i in K^* . If q_i and p_i are the respective primes of infinite value, then \hat{S}_i/q_i is a regular local ring and \hat{R}_i/p_i is a normal local ring with toric singularities. There are compatible actions of Γ^*/Γ on $(\hat{S}_i/q_i) \hat{\otimes}_{S_i/m_{S_i}} k'$ such that

$$((\hat{S}_i/q_i) \hat{\otimes}_{S_i/m_{S_i}} k')^{\Gamma^*/\Gamma} \cong (\hat{R}_i/p_i) \hat{\otimes}_{R_i/m_{R_i}} k'.$$

From Theorems 8.1 and 8.2 we obtain the following.

Theorem 1.3. *Let notations be as in Theorem 1.1. Assume that V^* (and $V = V^* \cap K$) have rank 1, and that $k = V^*/m_{V^*}$ is algebraically closed of characteristic zero. Then there exist directed systems of algebraic local rings R_i and S_i satisfying the conclusions of Theorem 1.1 and such that \hat{R}_i/p_i and \hat{S}_i/q_i , where q_i and p_i are the primes of elements of infinite value, satisfy the conclusions of the above paragraph. Let $\bar{U}^* = \cup \hat{S}_i/q_i$ and let $\bar{U} = \cup \hat{R}_i/p_i$. Then \bar{U}^* and \bar{U} are Henselian valuation rings, such that $(\bar{U}^*, Q(\bar{U}^*))$ and $(\bar{U}, Q(\bar{U}))$ are immediate extensions of (V^*, K^*) and (V, K) respectively, and $Q(\bar{U}^*)$ is a finite Galois extension of $Q(\bar{U})$ with Galois group Γ^*/Γ .*

We give the proof of Theorem 1.3 in Section 8.

2. NOTATIONS

We will denote the maximal ideal of a local ring R by m_R or $m(R)$. We will denote the quotient field of a domain R by $Q(R)$. Suppose that $R \subset S$ is an inclusion of local rings. We will say that R dominates S if $m_S \cap R = m_R$. Suppose that K is an algebraic function field over a field k . We will say that a subring R of K is algebraic if R is essentially of finite type over k . Suppose that K^* is a finite extension of an algebraic function field K , R is a local ring with quotient field K and S is a local ring with quotient field K^* . We will say that S lies over R and R lies below S if S is a localization at a maximal ideal of the integral closure of R in K^* . If R is a local ring, \hat{R} will denote the completion of R at its maximal ideal.

Good introductions to the valuation theory which we require in this paper can be found in Chapter VI of [24] and in [3]. A valuation ν of K will be called a k -valuation if $\nu(k) = 0$. We will denote by V_ν the associated valuation ring, which necessarily contains k . A valuation ring V of K will be called a k -valuation ring if $k \subset V$. The residue field V/m_V of a valuation ring V will be denoted by $k(\nu)$. The value group of a valuation ν with valuation ring V will be denoted by Γ_ν or Γ_V . If R is a subring of V_ν then the center of ν (the center of V_ν) on R is the prime ideal $R \cap m_{V_\nu}$.

Suppose that R is a local domain. A monoidal transform $R \rightarrow R_1$ is a birational extension of local domains such that $R_1 = R[\frac{P}{x}]_m$ where P is a regular prime ideal of R , $0 \neq x \in P$ and m is a prime ideal of $R[\frac{P}{x}]$ such that $m \cap R = m_R$. $R \rightarrow R_1$ is called a quadratic transform if $P = m_R$.

If R is regular, and $R \rightarrow R_1$ is a monoidal transform, then there exists a regular system of parameters (x_1, \dots, x_n) in R and $r \leq n$ such that

$$R_1 = R \left[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1} \right]_m.$$

Suppose that ν is a valuation of the quotient field R with valuation ring V_ν which dominates R . Then $R \rightarrow R_1$ is a monoidal transform along ν (along V_ν) if ν dominates R_1 .

3. COMPLETION OF RELATIVE LOCAL UNIFORMIZATION

We now give the proof of Theorem 1.2.

Let $\bar{S}_i = \hat{S}_i \hat{\otimes}_{S_i/m_{S_i}} k'$, $\bar{R}_i = \hat{R}_i \hat{\otimes}_{R_i/m_{R_i}} k'$ for $i \in I$ and let $G = \Gamma^*/\Gamma$. U^* is the directed union of the \bar{S}_i and U is the directed union of the \bar{R}_i . The fact that R_i is a normal domain implies \hat{R}_i is a normal domain (Chapter VIII, Section 13, Theorem 32 [24]) and thus \bar{R}_i is a normal local domain (Proposition IV.6.7.4 [11]). Thus U (and U^*) are normal domains. The fact that U and U^* are Henselian follows from the proof of Proposition 4.2. The action of G on U^* extends to an action on $Q(U^*)$. Suppose that $h \in U^*$ and $\sigma(h) = h$ for all $\sigma \in G$. There exists i such that $h \in Q(\bar{S}_i)$. Since \bar{S}_i is finite over \bar{R}_i and $\bar{S}_i^G = \bar{R}_i$, it follows that $Q(\bar{S}_i)^G = Q(\bar{R}_i)$. Thus $h \in Q(U)$. We conclude that $Q(U^*)^G = Q(U)$, so that $Q(U^*)$ is a finite Galois extension of $Q(U)$ (c.f. Theorem V.2.15 [15]).

Remark 3.1. *The statement that $Q(U^*)$ is finite over $Q(U)$ can be seen directly from the fact that the minimal polynomial of each $Q(\hat{S}_i)$ over $Q(\hat{R}_i)$ is a factor of the minimal polynomial of an appropriate primitive element of K^* over K (by Proposition 1 [1]).*

The fact that the extension considered in Theorem 1.2 is Galois, even when the original field extension K^*/K is not, is a condition that can be easily seen in the case when U and U^* are valuation rings, as the first author realized with Franz-Viktor Kuhlmann in a discussion.

Theorem 3.2. *Suppose that V is a Henselian valuation ring of a field K , such that V contains an algebraically closed field k of characteristic zero, with $k \cong V/m_V$. If L is a finite extension of K , then there is a unique valuation ring W of L such that W dominates V , and L is Galois over K with Galois group Γ_W/Γ_V , where Γ_W and Γ_V are the respective value groups.*

Proof. Let J be a finite Galois extension of K which contains L . Let G be the Galois group of J over K . Since V is Henselian, there exists a unique valuation ring U of J such that U dominates V ((16.4), (16.6) [10]). Thus the splitting group $G^s(U/V) = G$ by Proposition 1.46 [3]. We have $U/m_U = V/m_V = k$ since k is algebraically closed. Thus the inertia group $G^i(U/V) = G^s(U/V) = G$ by Theorem 1.48 [3]. Finally,

$$G = G^i(U/V) \cong \Gamma_U/\Gamma_V$$

by Theorem 3 [17] or Chapter VI, Section 12, Corollary [24].

Since G is abelian, all intermediate subfields of J are Galois over K . Thus L is Galois over K , and the Galois group of L over K is Γ_W/Γ_V . \square

4. COMPLETIONS OF VALUATION RINGS

Suppose that K is an algebraic function field over a field k , and V is a valuation ring of K with maximal ideal m_V and value group Γ . Suppose that $\{R_i \mid i \in I\}$ is a directed system of normal local rings such that

- (a) $V = \cup_{i \in I} R_i$.
- (b) I has a minimum 0.
- (c) Each R_i is essentially of finite type over k and has quotient field K .
- (d) If $i < j$ then R_j dominates R_i .

Let K_i be the quotient field of \hat{R}_i . By Zariski's subspace theorem ((10.13) [4]) we have natural inclusions $\hat{R}_i \rightarrow \hat{R}_j$ if $i < j$, and $\{\hat{R}_i \mid i \in I\}$ is a directed system of

normal local rings (Scholie 7.8.3 [11]). Let

$$T = \cup_{i \in I} \hat{R}_i$$

and $K_\infty = \cup K_i$. T is a normal domain with quotient field K_∞ and maximal ideal $m_T = \cup m_{\hat{R}_i}$.

Given a valuation ring V as above, there exists a directed system of normal local rings $\{R_i\}$ whose union is V . A particular construction is as follows. We take R_0 to be any normal local ring which is dominated by V . If $m \in \mathbf{N}$ and $f_1, \dots, f_m \in V$ we set $i = (f_1, \dots, f_m)$ and let R_i be the localization of the normalization of $R_0[f_1, \dots, f_m]$ which is dominated by V .

Lemma 4.1. *The ring $T = \cup_{i \in I} \hat{R}_i$ is independent of choice of directed system $\{R_i \mid i \in I\}$ satisfying (a),(b),(c), and (d).*

Proof. Let J be a partially ordered set, and let $\{S_j \mid j \in J\}$ be a collection of algebraic local rings with quotient field K , such that $\{S_j\}$ satisfies (a),(b),(c) and (d). We show that $\cup_{i \in I} \hat{R}_i = \cup_{j \in J} \hat{S}_j$.

Let $i \in I$. Since R_i is essentially of finite type over k and dominated by V , there exist $f_1, \dots, f_m \in V$ such that $R_i = k[f_1, \dots, f_m]_{m_V \cap k[f_1, \dots, f_m]}$. Since $V = \cup_{j \in J} S_j$ and J is directed, there exists $j \in J$ such that $f_1, \dots, f_m \in S_j$, and so $k[f_1, \dots, f_m]_{m_V \cap k[f_1, \dots, f_m]} \subset S_j$, since S_j is dominated by V . Hence $R_i \subset S_j$. There is then a natural inclusion $\hat{R}_i \subset \hat{S}_j$, and thus $\cup_{i \in I} \hat{R}_i \subset \cup_{j \in J} \hat{S}_j$. The other inclusion is proven in the same way. \square

Proposition 4.2. *The ring $T = \cup_{i \in I} \hat{R}_i$ is Henselian.*

Proof. Let $F \in T[x]$ and $\phi_1, \phi_2 \in T/m_T[x]$ be monic polynomials such that ϕ_1 and ϕ_2 are relatively prime and $\bar{F} = \pi(F) = \phi_1 \phi_2$, where $\pi : T \rightarrow T/m_T$ is the natural projection. We need to show that there exist monic polynomials $F_1, F_2 \in T[x]$ such that $\pi(F_1) = \phi_1$, $\pi(F_2) = \phi_2$, and $F = F_1 F_2$.

Since both $T = \cup \hat{R}_i$ and $T/m_T = \cup \hat{R}_i/m_{\hat{R}_i}$ are directed unions, there exists $c \in I$ such that $F \in \hat{R}_c[x]$ and $\phi_1, \phi_2 \in \hat{R}_c/m_{\hat{R}_c}$.

Since \hat{R}_c is complete, there exist monic polynomials $F_1, F_2 \in \hat{R}_c[x]$ such that $\pi'(F_1) = \phi_1$, $\pi'(F_2) = \phi_2$, and $F = F_1 F_2$. Since $F_1, F_2 \in T[x]$, F_1 and F_2 are the desired polynomials. \square

Example 4.3. *In general, T is not a valuation ring.*

Proof. Let k be a field and $R = k[x, y, z]_{(x, y, z)}$. We will define a valuation ν on $K = Q(R)$ which dominates R . Let $p(t) = \sum_{i=1}^{\infty} a_i t^i \in k[[t]]$ be a transcendental series. If $f \in k[x, y, z]$ write $f = z^r g(x, y, z)$ where $g(x, y, 0) \neq 0$. We define

$$\nu(f) = (r, \text{ord } g(t, p(t), 0)).$$

The value group of ν is $\mathbf{Z} \oplus \mathbf{Z}$ with the lexicographic order. Let V be the valuation ring of ν and let R_i be a directed system of regular local rings satisfying (a), (b), (c) and (d) with $R_0 = R$. Such a system exists by Theorem 1.1. In particular, $V = \cup R_i$.

We will suppose that T is a valuation ring and derive a contradiction. Let $\bar{\nu}$ be an extension of ν to $Q(T)$ such that T is the valuation ring of $\bar{\nu}$. There is a natural embedding of value groups

$$\Gamma_\nu = \mathbf{Z} \oplus \mathbf{Z} \subset \Gamma_{\bar{\nu}}.$$

$\frac{z}{y-p(x)} \in Q(T)$. If $\bar{\nu}(\frac{z}{y-p(x)}) \geq 0$ let $f = \frac{z}{y-p(x)}$. If $\bar{\nu}(\frac{z}{y-p(x)}) < 0$ let $f = \frac{y-p(x)}{z}$. By construction $\bar{\nu}(f) \geq 0$ so that $f \in T$ (by our assumption that T is a valuation ring). Thus there exists \hat{R}_i such that $f \in \hat{R}_i$. By our hypothesis, R_i dominates R . Since

R_i is essentially of finite type over R , $Q(R) = Q(R_i)$ and R is a UFD, there exists an ideal $I \subset R$ of height ≥ 2 and $a \in I$ with

$$\nu(a) = \min\{\nu(h) \mid h \in I\}$$

such that $R[\frac{I}{a}] \subset V$ and

$$R_i = R[\frac{I}{a}]_{m_V \cap R[\frac{I}{a}]}$$

Since $I \not\subset (z)$, we have that $\nu(a) \in \{0\} \oplus \mathbf{Z}$. Let $R' = \hat{R}[\frac{I}{a}]_{m_T \cap \hat{R}[\frac{I}{a}]}$.

$$f \in Q(\hat{R}) \cap \hat{R}_i = Q(R') \cap \hat{R}' = R'.$$

For the last equality, c.f. Lemma 2.1 [6].

We can thus write $f = \frac{g}{h}$ with $g, h \in \hat{R}[\frac{I}{a}]$ and $h \notin m_T \cap \hat{R}[\frac{I}{a}]$. Thus $\bar{\nu}(h) = 0$. There exists $n \in \mathbf{N}$ such that $a^n g = g_0$, $a^n h = h_0$ and $g_0, h_0 \in \hat{R}$. Thus $\bar{\nu}(h_0) = n\nu(a) + \bar{\nu}(h) = (0, m)$ for some $m \in \mathbf{N}$.

If $f = \frac{z}{y-p(x)}$, we have $zh_0 = (y-p(x))g_0$ implies $y-p(x)$ divides h_0 in \hat{R} . But

$$y-p(x) = (y - \sum_{i=1}^{m+1} a_i x^i) - x^{m+2} \left(\sum_{i=m+2}^{\infty} a_i x^{i-m-2} \right).$$

Thus $\bar{\nu}(y-p(x)) > (0, m)$, which is a contradiction.

If $f = \frac{y-p(x)}{z}$, then $(y-p(x))h_0 = zg_0$ implies z divides h_0 in \hat{R} .

This is a contradiction since $\nu(z) = (1, 0) > (0, m) = \bar{\nu}(h_0)$. □

As an extension of the above example, we construct valuations ν_1, ν_2 of $Q(\hat{R})$ which extend ν and dominate \hat{R} such that ν_1 is an immediate extensions of ν ($k(\nu) = k(\nu_1)$ and $\Gamma_\nu = \Gamma_{\nu_1}$) but ν_2 is a rank 3 valuation, of higher rank than ν .

We first define ν_1 . For $f \in \hat{R}$, write

$$f(x, y, z) = \sum a_{ij}(y-p(x))^i z^j \tag{1}$$

with $a_{ij} \in k[[x]]$. Let

$$\begin{aligned} a &= \min\{i+j \mid a_{ij}(x) \neq 0\}, \\ b &= \min\{\text{ord}(a_{ij}(x)) \mid i+j = a\}. \end{aligned}$$

Set $\nu_1(f) = (a, b) \in \mathbf{Z} \oplus \mathbf{Z}$. ν_1 defines a valuation on $Q(\hat{R})$ which dominates \hat{R} . We will now verify that ν_1 extends ν . For $f \in k[x, y]$, write

$$f = \sum b_j(x, y) z^j$$

with each $b_j \in k[x, y]$. Suppose that $\nu(f) = (r, s)$. Comparing with (1), we see that for all j ,

$$b_j(x, y) = \sum_{i=0}^{\infty} a_{ij}(x)(y-p(x))^i.$$

Thus we either have that $b_j = 0$ or $a_{0j} \neq 0$, since $y-p(x) \nmid b_j(x, y)$. Thus

$$a = \min\{j \mid b_j(x, y) \neq 0\} = r.$$

Set $g = \frac{f}{z^r}$. We have

$$b = \text{ord}(a_{0r}(x)) = \text{ord}(g(x, p(x))) = s.$$

Thus $\nu_1(f) = \nu(f)$. It follows that ν_1 is an immediate extension of ν .

Now we define the extension ν_2 . For $f \in \hat{R}$, write $f = z^\alpha g(x, y, z)$ where $z \nmid g(x, y, z)$. Write $g(x, y, 0) = (y-p(x))^\beta h(x, y)$ where $y-p(x) \nmid g(x, y, 0)$. Set

$\gamma = \text{ord}(h(x, p(x)))$. Define $\nu_2(f) = (\alpha, \beta, \gamma) \in \mathbf{Z}^3$, where \mathbf{Z}^3 has the lexicographic order. ν_2 extends to a valuation of $Q(\hat{R})$ which dominates \hat{R} , and such that ν_2 extends ν .

Theorem 4.4. *T is a valuation ring if and only if for all i, there exists a unique valuation ring V_i with quotient field K_i which dominates V and \hat{R}_i .*

Proof. Suppose that for all i , there exists a unique valuation ring V_i with quotient field K_i which dominates V and \hat{R}_i . By Proposition 4.1 [14]

$$V_i = \hat{R}_i[V]_{m_V \hat{R}_i[V]} \subset K_i$$

is a valuation ring for all i . Let

$$V_\infty = \cup V_i = (\cup \hat{R}_i)[V]_{m_V(\cup \hat{R}_i)[V]} \subset K_\infty$$

where K_∞ is the quotient field of T . $f \in K_\infty$ implies $f \in K_i$ for some i which implies $f \in V_i$ or $\frac{1}{f} \in V_i$. Thus $f \in V_\infty$ or $\frac{1}{f} \in V_\infty$. Thus V_∞ is a valuation ring. $V \subset \cup \hat{R}_i$ implies $V_\infty = \cup \hat{R}_i = T$, and we conclude that T is a valuation ring.

Now suppose that T is a valuation ring. Let $m = \cup m_{R_i} \hat{R}_i$ be the maximal ideal of T . Without loss of generality, we may assume that V/m_V is algebraic over k . For if this is not the case, we can replace k with a rational function field k' over k contained in all of the R_i such that V/m_V is algebraic over k' .

Suppose that for some index i , V_1 is a valuation ring with quotient field K_i which dominates \hat{R}_i and V . We will show that there exists a valuation ring W_1 with quotient field K_∞ which dominates T and such that $W_1 \cap K_i = V_1$.

Consider the domain $A = T[V_1] \subset K_\infty$. Let $I \subset A$ be the ideal $I = (m + m_{V_1})A$. We will first establish that $1 \notin I$. If it were true that $1 \in I$, then there would exist an index $j > i$ such that $1 \in (m_{R_j} + m_{V_1})\hat{R}_j[V_1]$. Since $R_i \rightarrow R_j$ is birational, there exists an ideal $b \subset R_i$ and $x \in b$ such that $\frac{b}{x} \subset V$ and R_j is a localization of $R_i[\frac{b}{x}]$ at a maximal ideal. Thus $\hat{R}_i[\frac{b}{x}] \subset V_1$ since $V \subset V_1$. Let $\bar{\nu}$ be a valuation of K_i which has V_1 for its valuation ring. We have $\bar{\nu}(\hat{R}_i[\frac{b}{x}]) \geq 0$ and $\bar{\nu}(m_{R_j}) > 0$ since $m_{R_j} = m_{V_1} \cap R_j$. Thus $m_{R_j} \hat{R}_i[\frac{b}{x}] \subset m_{V_1}$, and V_1 dominates $\hat{R}_i[\frac{b}{x}]_{m_{R_j} \hat{R}_i[\frac{b}{x}]}$. Thus there exists a valuation ring U_1 of K_j such that $U_1 \cap K_i = V_1$ and U_1 dominates \hat{R}_j (as follows from page 177 of [14]). $(m_{R_j} + m_{V_1})\hat{R}_j[V_1] \subset m_{U_1}$ implies $1 \notin (m_{R_j} + m_{V_1})\hat{R}_j[V_1]$. Thus we have a contradiction, and $1 \notin I$.

Let a be a prime ideal in A which contains I . Suppose that $h \in A/a$. There exists an index j such that we can write h as a class $h = [\sum f_r g_r]$ with $f_r \in V_1$ and $g_r \in \hat{R}_j$. We have natural inclusions $\hat{R}_j/m_{\hat{R}_j} \rightarrow A/a$ and $V_1/m_{V_1} \rightarrow A/a$ such that h is the image of the induced map $\hat{R}_j/m_{\hat{R}_j} \hat{\otimes}_k V_1/m_{V_1} \rightarrow A/a$. Thus h is algebraic over V_1/m_{V_1} , since $\hat{R}_j/m_{\hat{R}_j}$ is finite over k . We conclude that A/a is algebraic over V_1/m_{V_1} .

There exists a valuation ring W_1 which contains A such that $m_{W_1} \cap A = a$ and W_1/m_{W_1} is algebraic over A/a , by Corollary 3 to Theorem 5' of Section 4, Chapter VI [24]. Let $\bar{W}_1 = W_1 \cap K_i$. \bar{W}_1 contains V_1 and $m_{V_1} \subset m_{\bar{W}_1}$. But $\bar{W}_1/m_{\bar{W}_1}$ is algebraic over V_1/m_{V_1} since W_1/m_{W_1} is algebraic over V_1/m_{V_1} . Thus $V_1 = \bar{W}_1$ by Theorem 2 of Section 3, Chapter VI [24] and Corollary 1 to Theorem 5 of Section 4, Chapter VI [24].

We have thus proved the existence of an extension W_1 of V_1 to K_∞ which restricts to V_1 and dominates T .

Continuing with the proof of the theorem, suppose that for some index i , the extension of V to K_i which dominates \hat{R}_i is not unique. There are then extensions V_1 and V_2 of V to K_i which dominate \hat{R}_i such that $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$.

We have shown that there then exist valuation rings W_1 and W_2 of K_∞ such that $T \subset W_1 \cap W_2$, $W_1 \cap K_i = V_1$ and $W_2 \cap K_i = V_2$. Thus $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. But this is impossible since T is a valuation ring of K_∞ , by Theorem 3, Section 3, Chapter VI [24]. \square

Corollary 4.5. *T is a valuation ring if for all i there does not exist an extension of V to K_i which dominates \hat{R}_i of higher rank than the rank of V .*

Proof. This follows from Theorem 4.4 and the remark on page 181 of [14] which shows that if the extension of V to K_i which dominates \hat{R}_i is not unique then there must be an extension of higher rank. \square

The converse to the above corollary is false, as is seen by the following simple example. Let $p(t) \in k[[t]]$ be a transcendental power series with constant term zero. Consider the rank 1 discrete valuation ν on $k(x, y)$ defined by the embedding of k -algebras

$$k(x, y) \rightarrow k \ll t \gg$$

generated by $x = t, y = p(t)$ where $k \ll t \gg$ denotes the quotient field of $k[[t]]$. ν dominates $R = k[x, y]_{(x, y)}$. The valuation ring V of ν extends uniquely to a rank 2 valuation ring of the quotient field of $k[[x, y]]$ which dominates $k[[x, y]]$. Furthermore, the construction gives a unique extension of V to a rank 2 valuation ring which dominates $Q(\hat{S})$ for any algebraic normal local ring S of $k(x, y)$ such that V dominates S and S dominates R .

It follows from Theorem 4.4 that the examples in [14] of valuation rings dominating regular local rings R which do not have unique extensions in $Q(\hat{R})$ dominating \hat{R} generate examples where T is not a valuation ring.

5. THE PRIME IDEAL OF ELEMENTS OF INFINITE VALUE

We will assume in this section that V has rank 1, that is, the value group of V is a (possibly nondiscrete) subgroup Γ of \mathbf{R} . Other notations and assumptions will be as in Section 4.

Lemma 5.1. *Suppose that V has rank 1 and that R is an algebraic normal local ring of K such that V dominates R and $f \in \hat{R}$. Then one of the following must hold.*

- (1) *There exists $\rho \in \Gamma$ such that if $\{f_n\}$ is any Cauchy sequence in R which converges to f , then $\nu(f_n) = \rho$ for all $n \gg 0$.*
- (2) *If $\rho \in \Gamma$ and if $\{f_n\}$ is any Cauchy sequence in R which converges to f , then $\nu(f_n) > \rho$ for $n \gg 0$.*

Proof. We first argue that (1) or (2) must hold for a fixed Cauchy sequence $\{f_n\}$ in R which converges to f . Suppose that (2) doesn't hold. Then there exists $\bar{\rho} \in \Gamma$ such that given $n_0 \in \mathbf{N}$, there exists $\bar{n} > n_0$ such that $\nu(f_{\bar{n}}) \leq \bar{\rho}$. Let $i \in \mathbf{N}$ be such that $i\nu(m_R) > \bar{\rho}$, and let n_0 be such that $f_m - f_n \in m_R^i$ if $m, n > n_0$. There exists $\bar{n} > n_0$ such that $\nu(f_{\bar{n}}) \leq \bar{\rho}$. Then $\nu(f_n) = \nu(f_{\bar{n}})$ if $n > n_0$, so (1) holds for $\{f_n\}$.

If $\{f_n\}$ and $\{g_l\}$ are two distinct Cauchy sequences in R which converge to f , then for all $i \in \mathbf{N}$, there exists $n(i)$ such that $f_n - g_l \in m_R^i$ if $n, l \geq n(i)$. Thus (1)(or (2)) holds for $\{f_n\}$ if and only if (1) (or (2)) holds for $\{g_l\}$. \square

Definition 5.2. Let R be as in the statement of Lemma 5.1. Let

$$p(\hat{R})_\infty = \{f \in \hat{R} \mid (2) \text{ of Lemma 5.1 holds for a Cauchy sequence } \{f_n\} \text{ in } R \text{ which converges to } f\}$$

Lemma 5.3. Let R be as in the statement of Lemma 5.1. Then

- (1) $p(\hat{R})_\infty$ is a prime ideal of \hat{R} such that $p(\hat{R})_\infty \cap R = (0)$.
- (2) There exists a unique extension $\bar{\nu}$ of ν to the quotient field \bar{K} of $\hat{R}/p(\hat{R})_\infty$ which dominates \hat{R} . Let \bar{V} be the valuation ring of $\bar{\nu}$. Then (\bar{V}, \bar{K}) is an immediate extension of (V, K) . That is, $\Gamma_{\bar{V}} = \Gamma_V$ and $k(\bar{V}) = k(V)$.

Proof. The facts that $p(\hat{R})_\infty$ is prime and $p(\hat{R})_\infty \cap R = (0)$ are immediate from Lemma 5.1. Suppose that $0 \neq f + p(\hat{R})_\infty \in \hat{R}/p(\hat{R})_\infty$. We can find a Cauchy sequence $\{f_n\}$ in R such that $\{f_n\}$ satisfies (1) of Lemma 5.1 and $\{f_n\}$ converges to f . Let $\rho = \nu(f_n)$ for $n \gg 0$. We necessarily have that $\bar{\nu}(g) = \rho$ if $g = f + p(\hat{R})_\infty$. \square

By a classical abuse of notation, we will say that $\nu(f) = \infty$ if $f \in p(\hat{R})_\infty$.

6. SIMULTANEOUS RESOLUTION OF p_∞

Definition 6.1. Suppose that R is a normal local ring which is essentially of finite type over a field k of characteristic zero, with quotient field K . A normal uniformizing transformation sequence (NUTS) is a sequence of ring homomorphisms

$$\begin{array}{ccccccc}
 R & \rightarrow & \bar{T}'_0 & \rightarrow & \bar{T}_0 & & \\
 & & \downarrow & & \searrow & & \\
 & & \bar{T}'_1 & \rightarrow & \bar{T}''_1 & \rightarrow & \bar{T}_1 \\
 & & & & \downarrow & & \searrow \\
 & & & & \bar{T}'_2 & \rightarrow & \bar{T}''_2 & \rightarrow & \bar{T}_2 \\
 & & & & & & \downarrow & & \searrow \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & \bar{T}'_n & \rightarrow & \bar{T}''_n & \rightarrow & \bar{T}_n
 \end{array} \tag{2}$$

such that $\bar{T}_0 = \hat{R}$, the completion of R with respect to its maximal ideal, and for all i , \bar{T}_i is the completion with respect to its maximal ideal of a birational extension \bar{T}'_i of \bar{T}''_{i-1} . For all i , \bar{T}'_i is a normal local ring, \bar{T}''_i is a normal local ring, essentially of finite type over \bar{T}'_i with quotient field K_i such that $\bar{T}'_i \subset \bar{T}''_i \subset \bar{T}_i$ and K_0 is a finite extension of K , K_{i+1} is a finite extension of K_i for all $i \geq 0$.

Definition 6.1 is the extension of the definition of a UTS in Chapter 3 of [6] to normal local rings.

To simplify notation, we will often denote the NUTS (2) by $(R, \bar{T}'_n, \bar{T}_n)$ or by

$$R \rightarrow \bar{T}_0 \rightarrow \bar{T}_1 \rightarrow \bar{T}_n.$$

We will denote the NUTS consisting of the maps

$$\begin{array}{ccccccc}
 \bar{T}'_{n-1} & \rightarrow & \bar{T}''_{n-1} & \rightarrow & \bar{T}_{n-1} & & \\
 & & \downarrow & & \searrow & & \\
 & & \bar{T}'_n & \rightarrow & \bar{T}''_n & \rightarrow & \bar{T}_n
 \end{array}$$

by $\bar{T}_{n-1} \rightarrow \bar{T}_n$.

Suppose that ν is a rank 1 k -valuation of K , and R is dominated by ν . Suppose that $\bar{\nu}$ is an extension of ν to the quotient field of \bar{T}_n which dominates \bar{T}_n . Then we

will say that $\bar{T}_0 \rightarrow \bar{T}_n$ is a NUTS along ν . When there is no danger of confusion, we will denote $\bar{\nu}$ by ν .

We define

$$\begin{aligned} p(\bar{T}_n)_\infty &= \{f \in \bar{T}_n \mid \bar{\nu}(f) = \infty\}, \\ \lambda(\bar{T}_n'') &= \lambda(\bar{T}_n) = \dim \bar{T}_n / p(\bar{T}_n)_\infty. \end{aligned}$$

We define

$$\lambda(R) = \lambda(\bar{T}_0) = \dim \hat{R} / p(\hat{R})_\infty.$$

Lemma 6.2. *Suppose that*

$$\bar{T} \rightarrow \bar{T}(1) \rightarrow \cdots \rightarrow \bar{T}(t)$$

is a NUTS along ν . Then $\lambda(\bar{T}) \geq \lambda(\bar{T}(t))$.

Lemma 6.2 is the generalization of Lemma 6.3 of [8] to a NUTS. The proof is the same.

Let V be the valuation ring (in K) of ν and let

$$\lambda_V = \min \{\lambda(R) \mid R \text{ is a normal algebraic local ring of } K \text{ which is dominated by } V\}.$$

Theorem 6.3. *There exists an algebraic regular local ring \bar{R} of K such that if $(R_1, \bar{T}''(t), \bar{T}(t))$ is a NUTS along ν with R_1 an algebraic normal local ring of K such that R_1 dominates \bar{R} , then*

$$\lambda(\bar{T}(t)) = \lambda_V.$$

Proof. Suppose that R is an algebraic normal local ring such that $\lambda(R) = \lambda_V$. Let \bar{R} be an algebraic regular local ring of K such that \bar{R} dominates R and V dominates \bar{R} . $\lambda(\bar{R}) = \lambda_V$ by Lemma 6.2. Suppose that $(R_1, \bar{T}''(t), \bar{T}(t))$ is a NUTS along ν with, R_1 an algebraic normal local ring of K such that R_1 dominates \bar{R} . We have $\lambda(R_1) = \lambda_V$ and $\lambda(\bar{T}(t)) \leq \lambda_V$ by Lemma 6.2. Let L be the quotient field of $\bar{T}''(t)$. L is a finite extension of K . By Theorem 4.2 [9] there exists an algebraic normal local ring R_3 of K and an algebraic regular local ring T_2 of L such that our extension of V to L dominates T_2 , T_2 dominates $\bar{T}''(t)$, T_2 dominates R_3 and R_3 dominates R_1 , with the property that \hat{T}_2 is finite over \hat{R}_3 . Since $p(\hat{T}_2)_\infty \cap \hat{R}_3 = p(\hat{R}_3)_\infty$, we have that $\lambda(R_3) = \lambda(T_2)$. By Lemma 6.2,

$$\lambda_V = \lambda(R_1) \geq \lambda(\bar{T}''(t)) \geq \lambda(T_2) = \lambda(R_3) = \lambda_V.$$

□

Theorem 6.4. *Suppose that K^* is a finite field extension of K and ν^* is an extension of ν to K^* . Then $\lambda_{V^*} = \lambda_V$ and there exists an algebraic regular local ring \tilde{R} of K such that the conclusions of Theorem 6.3 hold with $\bar{R} = \tilde{R}$ and if S is an algebraic normal local ring of K^* which is dominated by V^* and dominates \tilde{R} , and $(S, \bar{T}''(t), \bar{T}(t))$ is a NUTS along ν^* , then $\lambda(\bar{T}(t)) = \lambda_V$.*

Proof. Let \bar{R} be the regular local ring of the conclusions of Theorem 6.3. Let S_1 be a normal algebraic local ring of K^* such that V^* dominates S_1 and $\lambda(S_1) = \lambda_{V^*}$. Let S_2 be an algebraic regular local ring such that V^* dominates S_2 , S_2 dominates S_1 and S_2 dominates \bar{R} . By Lemma 6.2, $\lambda(S_2) = \lambda_{V^*}$. By Theorem 4.2 [9] there exists an algebraic normal local ring R_1 of K and an algebraic regular local ring S_3 of K^* such that V^* dominates S_3 , S_3 dominates S_2 , S_3 dominates R_1 and R_1 dominates \bar{R} , with the property that \hat{S}_3 is finite over \hat{R}_1 . Since $p(\hat{S}_3)_\infty \cap \hat{R}_1 = p(\hat{R}_1)_\infty$, $\lambda(R_1) = \lambda(S_3)$. By Theorem 6.3 and Lemma 6.2,

$$\lambda_V = \lambda(R_1) = \lambda(S_3) = \lambda(S_2) = \lambda_{V^*}.$$

By Lemma 5.3 [9] there exists an algebraic regular local ring \tilde{R} of K such that V dominates \tilde{R} , \tilde{R} dominates \bar{R} and if S is an algebraic normal local ring of K^* which is dominated by V^* and which contains \tilde{R} , then S dominates S_2 .

By Theorem 6.3 applied to $S \subset K^*$ which dominates S_2 , the conclusions of Theorem 6.4 hold. \square

We now state a generalization of Theorem 5.1 [6] which resolves the prime ideal of infinite value terms.

Theorem 6.5. *Let k be a field of characteristic zero, K an algebraic function field, K^* a finite algebraic extension of K , ν^* a k -valuation of K^* , $\nu = \nu^* \upharpoonright K$, such that $\text{rank } \nu = 1$, $\text{rat rank } \nu = s$ and*

$$s \leq \lambda = \lambda_V \leq n,$$

where V is the valuation ring of ν and $n = \text{trdeg}_k K - \text{trdeg}_k V/m_V$. Suppose that S^* is an algebraic local ring with quotient field K^* which is dominated by ν^* and R^* is an algebraic local ring with quotient field K which is dominated by S^* . Let V^* be the valuation ring of ν^* . Then there exists a commutative diagram

$$\begin{array}{ccc} R_0 & \rightarrow & S \subset V^* \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

where $S^* \rightarrow S$ and $R^* \rightarrow R_0$ are sequences of monoidal transforms along ν^* such that R_0 has regular parameters (x_1, \dots, x_n) and S has regular parameters (y_1, \dots, y_n) such that there are units $\delta_1, \dots, \delta_s \in S$ and a $s \times s$ matrix $A = (a_{ij})$ of natural numbers such that $\det(A) \neq 0$,

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots y_s^{a_{1s}} \delta_1 \\ &\vdots \\ x_s &= y_1^{a_{s1}} \cdots y_s^{a_{ss}} \delta_s \\ x_{s+1} &= y_{s+1} \\ &\vdots \\ x_n &= y_n \end{aligned} \tag{3}$$

and $\{\nu(x_1), \dots, \nu(x_s)\}, \{\nu(y_1), \dots, \nu(y_s)\}$ are rational bases of $\Gamma_\nu \otimes \mathbf{Q} = \Gamma_{\nu^*} \otimes \mathbf{Q}$. Furthermore,

$$p(\hat{R}_0)_\infty = (g_1, \dots, g_{n-\lambda})$$

with

$$g_i \equiv x_{s+i} \pmod{m(\hat{R}_0)^2}$$

for $1 \leq i \leq n - \lambda$, and

$$p(\hat{S})_\infty = p(\hat{R}_0)_\infty \hat{S}$$

are regular primes.

Remark 6.6. *Suppose that in the hypothesis of Theorem 6.5 we further assume that $R^* \rightarrow S^*$ is such that R^* and S^* have regular parameters $(x_1^*, \dots, x_n^*), (y_1^*, \dots, y_n^*)$ satisfying (3) and such that $\{\nu(x_1^*), \dots, \nu(x_n^*)\}, \{\nu(y_1^*), \dots, \nu(y_n^*)\}$ are rational bases of $\Gamma_\nu \otimes \mathbf{Q} = \Gamma_{\nu^*} \otimes \mathbf{Q}$. Then in the conclusions of Theorem 6.5 we further have that there exist $b_j(i) \in \mathbf{N}$, units $\alpha_i \in R_0$ such that*

$$x_i^* = x_1^{b_1(j)} \cdots x_s^{b_s(j)} \alpha_i$$

for $1 \leq i \leq s$ and there exist $c_j(i) \in \mathbf{N}$, units $\beta_i \in S$ such that

$$y_i^* = y_1^{c_1(j)} \cdots y_s^{c_s(j)} \beta_i$$

for $1 \leq i \leq s$.

This follows since all transformations in the proof of Theorem 6.5 are “CUTS in the first n variables” (page 49 [6]).

Proof. (of Theorem 6.5). Let \tilde{R} be the regular local ring of the conclusions of Theorem 6.4. We first construct a commutative diagram

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \subset V^* \\ \uparrow & & \uparrow \\ R^* & \rightarrow & S^* \end{array}$$

such that the conclusions of Theorem 5.1 [6] hold, and R_1 dominates \tilde{R} . Let $R = R_1$, $\bar{T}'' = R_1$ and $\bar{T} = \hat{R}_1$. We will now show that we can construct a CUTS $\bar{T} \rightarrow \bar{T}(t)$ along ν , which is in the first n variables (with the notation of Theorem 4.7 of [6]), such that $p(\bar{T}(t))_\infty$ has the form of (53) of page 49 of [6],

$$p(\bar{T}(t))_\infty = (\bar{z}_{r(1)}(t) - Q_{r(1)}(\bar{z}_1(t), \dots, \bar{z}_{r(1)-1}(t)), \dots, \bar{z}_{r(n-\lambda)}(t) - Q_{r(n-\lambda)}(\bar{z}_1(t), \dots, \bar{z}_{r(n-\lambda)-1}(t))) \quad (4)$$

with $s < r(1) < r(2) < \dots < r(n-\lambda) \leq n$ and such that for $1 \leq i \leq n-\lambda$,

$$Q_{r(i)} = \bar{z}_1(t)^{a_1(r(i))} \dots \bar{z}_s(t)^{a_s(r(i))} u_{r(i)}$$

where $u_{r(i)}$ is a unit series in $\bar{z}_1(t), \dots, \bar{z}_{r(i)}(t)$ with coefficients in $k(c_0, \dots, c_t)$ (with the notation of (53) of page 49 of [6]).

The construction of $\bar{T} \rightarrow \bar{T}(t)$ follows from the proof of (53) of [6], with the insertion of the following at the bottom of page 54. “Since $\nu(Q_{\bar{m}}) = \nu(\bar{z}_{\bar{m}}(t)) < \infty$ we can perform by (54) [6] a UTS in the first $\bar{m}-1$ variables to get $Q_{\bar{m}} = \bar{z}_1(t')^{a_1} \dots \bar{z}_s(t')^{a_s} u_{\bar{m}}$ where $u_{\bar{m}} \in k(c_0, \dots, c_t)[[\bar{z}_1(t'), \dots, \bar{z}_{\bar{m}-1}(t')]]$ is a unit series”.

Set $S = S_1$, $\bar{U}'' = S_1$, $\bar{U} = \hat{S}$. We can now construct a CUTS $\bar{U} \rightarrow \bar{U}(t')$ so that $(R_1, \bar{T}''(t'), \bar{T}(t'))$ and $(S_1, \bar{U}''(t'), \bar{U}(t'))$ is a CUTS along ν^* , by Lemma 4.3 and Lemma 4.4 [6].

Set $\tilde{g}_j(\bar{z}_1(t'), \dots, \bar{z}_{r(j)}(t')) = \bar{z}_{r(j)}(t') - Q_{r(j)}(\bar{z}_1(t'), \dots, \bar{z}_{r(j)-1}(t'))$ for $1 \leq j \leq n-\lambda$.

We will now show that the strict transform of $p(\hat{R}_1)_\infty$ in $\bar{T}(t')$ is $p(\bar{T}(t'))_\infty$. It suffices to show that the strict transform of $p(\hat{R}_1)_\infty$ in $\bar{T}(1)$ is $p(\bar{T}(1))_\infty$. Then the result follows by induction on t' . Let $p = p(\hat{R}_1)_\infty$. There exists an ideal I in R_1 , $f \in I$, and a maximal ideal n in $\hat{R}_1[\frac{I}{f}]$ such that $\bar{T}(1) = \widehat{R_1[\frac{I}{f}]_n}$. Let

$$\bar{p} = \cup_{j=1}^{\infty} \left(p\hat{R}_1[\frac{I}{f}]_n : I^j \hat{R}_1[\frac{I}{f}]_n \right)$$

be the strict transform of p in $\hat{R}_1[\frac{I}{f}]_n$. $\bar{p} \neq \hat{R}_1[\frac{I}{f}]_n$ since the strict transform in $\hat{R}_1[\frac{I}{f}]_n$ of an element of infinite value must have infinite value. \bar{p} is a prime ideal in $\hat{R}_1[\frac{I}{f}]_n$, and

$$\hat{R}_1/p \rightarrow \hat{R}_1[\frac{I}{f}]_n/\bar{p}$$

is birational (Section 0.2 [13], Corollary II.7.15 [12]). Thus $\dim \bar{T}(1)/\bar{p}\bar{T}(1) = \lambda$. $\bar{p}\bar{T}(1)$ is a prime contained in $p(\bar{T}(1))_\infty$ and $\dim \bar{T}(1)/p(\bar{T}(1))_\infty = \lambda$ by Theorem 6.4 (since R_1 contains \hat{R}). Thus $\bar{p}\bar{T}(1) = p(\bar{T}(1))_\infty$.

Hence there exist $f_1, \dots, f_{n-\lambda} \in p(\hat{R}_1)_\infty$, $c_1(i), \dots, c_s(i) \in \mathbf{N}$ for $1 \leq i \leq n-\lambda$ and $b_{ij} \in \bar{T}(t')$ such that $\det(b_{ij})$ is a unit in $\bar{T}(t')$, $f_i = M_i(\sum_{j=1}^{n-\lambda} b_{ij}\tilde{g}_j)$ where

$M_i = \bar{z}_1(t')^{c_1(i)} \cdots \bar{z}_s(t')^{c_s(i)}$ for $1 \leq i \leq n - \lambda$ and

$$p(\bar{T}(t'))_\infty = \left(\frac{f_1}{M_1}, \dots, \frac{f_{n-\lambda}}{M_{n-\lambda}} \right).$$

Let m be a positive integer such that

$$m > \frac{(\max_{1 \leq j \leq n-\lambda} \nu(\bar{z}_{r(j)}(t'))) + (\max_{1 \leq i \leq n-\lambda} \nu(M_i))}{\nu(m(\bar{T}(t')))} \quad (5)$$

By Theorem 4.8 [6] (with $l = n$) there exists a CRUTS along ν , $(R_1, R_1, T(t'))$ and $(S_1, S_1, U(t'))$ with associated MTSs

$$\begin{array}{ccc} S & \rightarrow & S(t') \\ \uparrow & & \uparrow \\ R & \rightarrow & R(t') \end{array}$$

such that (with the notation of Theorem 4.8 [6])

$$f_i = \bar{M}_i \left(\sum_{j=1}^{n-\lambda} b_{ij}(\tilde{x}_1(t'), \dots, \tilde{x}_n(t')) \tilde{g}_j(\tilde{x}_1(t'), \dots, \tilde{x}_{r(j)}(t')) + h_i \in p(\hat{R}(t'))_\infty \right)$$

with $\bar{M}_i = \tilde{x}_1(t')^{c_1(i)} \cdots \tilde{x}_n(t')^{c_s(i)}$, $h_i \in m(T(t'))^m$ (where m is the integer of (5)) for $1 \leq i \leq n - \lambda$, and such that (by (A3) of page 83 of [6]) $\nu(\bar{z}_i(t')) = \nu(\tilde{x}_i(t'))$ for $1 \leq i \leq n$, and

$$\begin{aligned} \nu(\bar{z}_{r(i)}(t')) &= \nu(Q_{r(i)}(\bar{z}_1(t'), \dots, \bar{z}_{r(i)-1}(t'))) \\ &= \nu(Q_{r(i)}(\tilde{x}_1(t'), \dots, \tilde{x}_{r(i)-1}(t'))) \\ &= \nu(\tilde{x}_{r(i)}(t')) \end{aligned}$$

for $1 \leq i \leq n - \lambda$.

Now we perform the MTS

$$\begin{array}{ccc} S(t') & \rightarrow & S(t'') \\ \uparrow & & \uparrow \\ R(t') & \rightarrow & R(t'') \end{array}$$

of the proof of Theorem 4.9 [6] (with $l = n$). Because of the form of the \tilde{g}_j , we have for $1 \leq j \leq n - \lambda$,

$$\tilde{g}_j = \bar{x}_1(t'')^{b_1(j)} \cdots \bar{x}_s(t'')^{b_s(j)} \bar{g}_j$$

where

$$\bar{g}_j \equiv \tilde{d}_j \bar{x}_{r(j)}(t'') \pmod{(\bar{x}_1(t''), \dots, \bar{x}_{r(j)-1}(t'')) \hat{R}(t'') + m(\hat{R}(t''))^2}$$

for some nonzero $\tilde{d}_j \in R(t'')/m(R(t''))$, where

$$\nu(\bar{x}_1(t'')^{b_1(j)} \cdots \bar{x}_s(t'')^{b_s(j)}) = \nu(\tilde{x}_{r(j)}(t')) = \nu(\bar{z}_{r(j)}(t')). \quad (6)$$

We further have

$$\bar{h}_i = \frac{h_i}{\bar{M}_i} = \bar{x}_1(t'')^{e_1(i)} \cdots \bar{x}_s(t'')^{e_s(i)} \tilde{h}_i$$

for $1 \leq i \leq n - \lambda$ with $\tilde{h}_i \in \hat{R}(t')$ and

$$\nu(\bar{x}_1(t'')^{e_1(i)} \cdots \bar{x}_s(t'')^{e_s(i)}) > \max \nu(\bar{z}_{r(j)}(t)) \quad (7)$$

for $1 \leq j \leq n - \lambda$ by (5). By Lemma 4.2 [6], (6) and (7), we can further choose the final CUTS of type (M1) (on the top of page 89 of [6]) so that

$$e_j(i) > \max_{1 \leq \alpha \leq n-\lambda} b_j(\alpha)$$

for $1 \leq j \leq s$ and $1 \leq i \leq n - \lambda$.

Let $A = (b_{ij})^{-1}$, a matrix with coefficients in $\hat{R}(t'')$.

$$A \begin{pmatrix} \frac{f_1}{M_1} \\ \vdots \\ \frac{f_{n-\lambda}}{M_{n-\lambda}} \end{pmatrix} = \begin{pmatrix} \tilde{g}_1 + d_1 \\ \vdots \\ \tilde{g}_{n-\lambda} + d_{n-\lambda} \end{pmatrix}$$

where for $1 \leq i \leq n - \lambda$,

$$d_i = \bar{x}_1(t'')^{b_1(i)+1} \dots \bar{x}_s(t'')^{b_s(i)+1} \bar{d}_i$$

for some $\bar{d}_i \in \hat{R}(t'')$. Thus

$$g_i = \bar{g}_i + \bar{d}_i \in p(\hat{R}(t''))_\infty$$

and

$$g_i \equiv \tilde{d}_i \bar{x}_{r(i)}(t'') \pmod{(\bar{x}_1(t''), \dots, \bar{x}_{r(i)-1}(t'')) \hat{R}(t'') + m(\hat{R}(t''))^2}.$$

Thus $(g_1, \dots, g_{n-\lambda})$ is a complete intersection and a regular prime ideal in $\hat{R}(t'')$. Since $\lambda(\hat{R}(t'')) = \lambda$ (by Theorem 6.3), we have that $(g_1, \dots, g_{n-\lambda})$ is a basis of $p(\hat{R}(t''))_\infty$. Since $(g_1, \dots, g_{n-\lambda}) \hat{S}(t'')$ is a prime ideal and $\lambda(S(t'')) = \lambda$ (by Theorem 6.4), it follows that $p(\hat{R}(t''))_\infty \hat{S}(t'') = p(\hat{S}(t''))_\infty$.

We can now make a change of variables in the regular parameters $(x_1(t''), \dots, x_n(t''))$ and $(y_1(t''), \dots, y_n(t''))$ of the proof of Theorem 4.9 [6] to get the desired forms of the g_i .

$\{\nu(x_1), \dots, \nu(x_s)\}$ and $\{\nu^*(y_1), \dots, \nu^*(y_s)\}$ are rational bases of $\Gamma^* \otimes \mathbf{Q}$ by the construction of the sequence $R^* \rightarrow R_0$ and $S^* \rightarrow S$. \square

7. RANK 1 VALUATIONS

Let notations be as in Section 4. Further assume that V has rank 1. Consider our directed set $\{R_i \mid i \in I\}$ satisfying (a), (b), (c) and (d). For $i \in I$, we define

$$p_i = p(\hat{R}_i)_\infty = \{f \in \hat{R}_i \mid \nu(f) = \infty\}.$$

For $i < j$, the natural inclusions $\hat{R}_i \rightarrow \hat{R}_j$ induce inclusions $\hat{R}_i/p_i \rightarrow \hat{R}_j/p_j$. Thus $\{\hat{R}_i/p_i \mid i \in I\}$ is a directed system, and we have a local domain

$$\bar{T} = \lim_{\rightarrow} \hat{R}_i/p_i = \cup \hat{R}_i/p_i.$$

Let \bar{K}_∞ be the quotient field of \bar{T} .

Lemma 7.1. *Suppose that V has rank 1. Then the ring $\bar{T} = \cup_{i \in I} \hat{R}_i/p_i$ does not depend on the directed system of rings $\{R_i \mid i \in I\}$ satisfying (a), (b), (c) and (d).*

Proof. The proof is essentially the same as that of Lemma 4.1. We must observe that the inclusion $\hat{R}_i \rightarrow \hat{S}_j$ of the proof of Lemma 4.1 induces a natural inclusion $\hat{R}_i/p_i \rightarrow \hat{S}_j/q_j$, where $q_j = \{f \in \hat{S}_j \mid \nu(f) = \infty\}$. \square

Theorem 7.2 is a generalization of Zariski's local uniformization theorem [23]. Our proof is an extension in rank 1 of the proof for general rank in [9, 6.2]. We incorporate the conclusions of Theorem 6.5 which resolves the prime ideal of infinite value terms.

Theorem 7.2. *Let k be a field of characteristic zero, K an algebraic function field over k , and let ν be a rank 1 k -valuation of K , of rational rank s , with valuation ring V . Let*

$$n = \text{trdeg}_k K - \text{trdeg}_k V/m_V,$$

$\lambda = \lambda_V$ (defined before Theorem 6.3).

Then there exists a partially ordered set I and algebraic regular local rings $\{R_i \mid i \in I\}$ with quotient field K which are dominated by V such that

$$V = \varinjlim R_j = \cup_{j \in I} R_j$$

and R_j has regular parameters $(x_1(j), \dots, x_n(j))$ such that

$$(1) \quad \{\nu(x_1(j)), \dots, \nu(x_s(j))\}$$

is a rational basis of $\Gamma \otimes \mathbf{Q}$.

(2) If $j < k \in I$ then there are relations

$$x_i(j) = \prod_{l=1}^s x_l(k)^{d_{il}} \delta_i \quad (8)$$

for $1 \leq i \leq s$ where $\delta_i \in R_k$ are units. The $s \times s$ matrix $D(j, k) = (d_{il})$ of (8) has nonzero determinant.

(3) The prime ideal

$$p_j = p(\hat{R}_j)_\infty = \{f \in \hat{R}_j \mid \nu(f) = \infty\} = (g_1(j), \dots, g_{n-\lambda}(j))$$

with

$$g_i(j) \equiv x_{s+i}(j) \pmod{m(\hat{R}_j)^2}.$$

In particular, p_j is a regular prime.

(4) For $j \in I$, let Λ_j be the free \mathbf{Z} -module $\Lambda_j = \sum_{i=1}^s \nu(x_i(j))\mathbf{Z}$. Then

$$\Gamma = \varinjlim \Lambda_j = \cup_{j \in I} \Lambda_j.$$

Proof. Let R^* be an algebraic regular local ring such that V dominates R^* . By Theorem 6.5 (with $K = K^*$ and $R^* = S^*$), there exists a sequence of monoidal transforms $R^* \rightarrow R_0$ along V such that (1) and (3) of this theorem hold on R_0 .

Suppose that m is a positive integer and $f = (f_1, \dots, f_m) \in V^m$. We will construct a sequence of monoidal transforms $R_0 \rightarrow R_f$ along V such that $f_1, \dots, f_m \in R_f$, (1) and (3) of this theorem hold for R_f and (2) of this theorem holds for $R_0 \rightarrow R_f$. We will further have $\nu(f_1), \dots, \nu(f_m) \in \Lambda_f$.

By Theorem 4.9 [9] with the R^* , S^* of the statement of Theorem 4.9 set as $R^* = S^* = R_0$, and $v_i = x_i(0)$ if $1 \leq i \leq s$, and $v_{s+1} = f_1, \dots, v_{s+m} = f_m$, there exists a sequence of monoidal transforms $R_0 \rightarrow R_1$ along V such that (1) of this theorem holds for R_1 , (2) of this theorem holds for $R_0 \rightarrow R_1$, $f_1, \dots, f_m \in R_1$ and $\nu(x_1), \dots, \nu(x_s) \in \Lambda_1$. By Theorem 6.5 (with $K = K^*$, $R^* = S^*$) and Remark 6.6, there exists a sequence of monoidal transforms $R_1 \rightarrow R_f$ along V such that (1), (2) and (3) of this theorem hold for $R_0 \rightarrow R_f$ and $\nu(f_1), \dots, \nu(f_m) \in \Lambda_f$. We have $\det D(0, f) \neq 0$ since $\{\nu(x_i(0)) \mid 1 \leq i \leq s\}$ and $\{\nu(x_i(f)) \mid 1 \leq i \leq s\}$ are two bases of $\Gamma \otimes \mathbf{Q}$.

Let $I = \sqcup_{m \in \mathbf{N}_+} V^m$ be the disjoint union. For $f \in I$ we construct R_f as above. If $f = 0$ we let R_0 be the R_0 constructed above. Define a partial order on I by $f \leq g$ if $R_f \subset R_g$.

Suppose that $R_\alpha \subset R_\beta$. We have $R_0 \subset R_\alpha \subset R_\beta$.

$$x_i(0) = \prod_{j=1}^s x_j(\alpha)^{c_{ij}} \delta_i$$

for $1 \leq i \leq s$ with δ_i a unit in R_α and

$$x_i(0) = \prod_{j=1}^s x_j(\beta)^{d_{ij}} \epsilon_i$$

for $1 \leq i \leq s$ with ϵ_i a unit in R_β . Thus in R_β there are factorizations

$$x_i(\alpha) = \prod_{j=1}^s x_j(\beta)^{\epsilon_{ij}} \lambda_i$$

for $1 \leq i \leq s$ and λ_i a unit in R_β . We have $\det(D(\alpha, \beta)) \neq 0$ since (1) holds for R_α and R_β . Thus (2) holds for $R_\alpha \rightarrow R_\beta$. To show that $V = \lim_{\rightarrow} R_j$, we must verify that I is a directed set. That is, for $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $R_\alpha \subset R_\gamma$ and $R_\beta \subset R_\gamma$.

There exists $f_1, \dots, f_t \in V$ such that if $A = k[f_1, \dots, f_t]$, $m = A \cap m_V$ then $R_\alpha = A_m$. There exists $g_1, \dots, g_n \in V$ such that if $B = k[g_1, \dots, g_n]$, $n = B \cap m_V$ then $R_\beta = B_n$. Set $\gamma = (f_1, \dots, f_t, g_1, \dots, g_n)$. By construction, $A, B \subset R_\gamma$. Since $m_V \cap R_\gamma = m_\gamma$ is the maximal ideal of R_γ , we have $R_\alpha, R_\beta \subset R_\gamma$.

(4) holds by our construction, since $\nu(f) \in \Lambda_f$ if $f \in V$. \square

Theorem 7.3. *Suppose that V has rank 1 and k has characteristic zero. Then the ring $\overline{T} = \cup_{i \in I} (\hat{R}_i/p_i)$ is a valuation ring.*

Proof. Let s denote the rational rank of ν . By Lemma 7.1 we can assume that the rings R_i 's are as in Theorem 7.2. R_i has regular parameters $x_1(i), \dots, x_n(i)$ and $\hat{R}_i = R_i/m_{R_i}[[x_1(i), \dots, x_n(i)]]$.

Let $f \in \hat{R}_i$. We recall that if $\nu(f) < \infty$, then by Theorem 4.8 and Theorem 4.10 [6] after a MTS $R_i \rightarrow \overline{R}(1)$ along ν

$$f = \overline{x}_1(1)^{d_1} \cdots \overline{x}_s(1)^{d_s} u(\overline{x}_1(1), \dots, \overline{x}_n(1))$$

where $\overline{x}(1), \dots, \overline{x}_n(1)$ are regular parameters in the ring $\overline{R}(1)$, $\nu(\overline{x}(1)), \dots, \nu(\overline{x}_s(1))$ are rationally independent and $u \in \widehat{\overline{R}(1)}$ is a unit power series. Further, there exist units $\alpha_j \in \overline{R}(1)$ such that

$$x_j(i) = \overline{x}_1(1)^{\beta_1(j)} \cdots \overline{x}_s(1)^{\beta_s(j)} \alpha_j$$

for $1 \leq j \leq s$.

Let $h \in Q(\overline{T})$. We want to show that either $h \in \overline{T}$, or $1/h \in \overline{T}$. So it suffices to show that if $\nu(h) \geq 0$, then $h \in \overline{T}$. Write $h = a/b$ where $a \in \cup_{i \in I} \hat{R}_i/p_i$ and $0 \neq b \in \cup_{i \in I} \hat{R}_i/p_i$. Then $a \in \hat{R}_j/p_j$ for some $j \in I$, and $b \in \hat{R}_k/p_k$ for some $k \in I$. After a MTS $R_j \rightarrow \overline{R}(1)$ along ν we have

$$a = \overline{x}_1(1)^{d_1} \cdots \overline{x}_s(1)^{d_s} u(\overline{x}_1(1), \dots, \overline{x}_n(1))$$

where $\overline{x}_1(1), \dots, \overline{x}_n(1)$ are regular parameters in the ring $\overline{R}(1)$ and $u \in \widehat{\overline{R}(1)}$ is a unit. Further, there exist units $\alpha_j \in \overline{R}(1)$ such that

$$x_j(i) = \overline{x}_1(1)^{\beta_1(j)} \cdots \overline{x}_s(1)^{\beta_s(j)} \alpha_j \tag{9}$$

for $1 \leq j \leq s$.

After another MTS $R_k \rightarrow \overline{R}(2)$ along ν we have

$$b = \overline{x}_1(2)^{e_1} \cdots \overline{x}_s(2)^{e_s} u'(\overline{x}_1(2), \dots, \overline{x}_n(2))$$

where $\overline{x}_1(2), \dots, \overline{x}_n(2)$ are regular parameters in the ring $\overline{R}(2)$ and $u' \in \widehat{\overline{R}(2)}$ is a unit. Further, there exist units $\gamma_j \in \overline{R}(2)$ such that

$$x_j(k) = \overline{x}_1(2)^{\delta_1(j)} \cdots \overline{x}_s(2)^{\delta_s(j)} \gamma_j \tag{10}$$

for $1 \leq j \leq s$.

We have that $\overline{R}(1) = k[f_1, \dots, f_m]_{m_V \cap k[f_1, \dots, f_m]}$ for some $f_1, \dots, f_m \in V$, and $\overline{R}(2) = k[g_1, \dots, g_n]_{m_V \cap k[g_1, \dots, g_n]}$ for some $g_1, \dots, g_n \in V$.

Let $c = (f_1, \dots, f_m, g_1, \dots, g_n) \in V^{m+n}$. Let R_c be constructed as in the proof of Theorem 7.2. Then $R_c \in \cup_{i \in I} R_i$ and $\overline{R}(1) \subset R_c$, $\overline{R}(2) \subset R_c$. The ring R_c has regular parameters $x_1(c), \dots, x_n(c)$ and by (2) of Theorem 7.2 and (9) and (10) the ‘‘good form’’ of a and b is preserved in R_c :

$$a = x_1(c)^{a_1} \dots x_s(c)^{a_s} \bar{u}(x_1(c), \dots, x_n(c))$$

$$b = x_1(c)^{b_1} \dots x_s(c)^{b_s} \bar{u}'(x_1(c), \dots, x_n(c))$$

where \bar{u} and \bar{u}' are units in \hat{R}_c .

Let $g = x_1(c)^{a_1} \dots x_s(c)^{a_s} / x_1(c)^{b_1} \dots x_s(c)^{b_s}$. Since $g \in K = Q(R_c)$ and $\nu(g) = \nu(h) \geq 0$, we have that $g \in V$ and $g \in R_g$, which is in the directed system I . We have $g = x_1(g)^{f_1} \dots x_s(g)^{f_s} w(x_1(g), \dots, x_n(g))$ with $f_i \geq 0$ for every i and w a unit in R_g . There exists $\gamma \in I$ such that $R_c \subset R_\gamma$ and $R_g \subset R_\gamma$. Then

$$h = x_1(\gamma)^{t_1} \dots x_s(\gamma)^{t_s} \bar{w}(x_1(\gamma), \dots, x_n(\gamma)),$$

with $t_i \geq 0$ for every i and \bar{w} a unit in \hat{R}_γ . Hence $h \in \hat{R}_\gamma$ and so $h \in \overline{T}$. \square

Theorem 7.4. *Suppose that V has rank 1 and k has characteristic zero. Then*

- (1) $(\overline{T}, Q(\overline{T}))$ is an Henselian immediate extension of (V, K) .
- (2) There exists a directed system of regular local rings $\{R_i\}$ satisfying (a), (b), (c), and (d) such that each \hat{R}_i/p_i is a regular local ring, and $\overline{T} = \cup \hat{R}_i/p_i$.

Proof. Let \overline{K}_i be the quotient field of \hat{R}_i/p_i . Then for all $i \in I$, $(\overline{T} \cap \overline{K}_i, \overline{K}_i)$ is an immediate extension of (V, K) by Lemma 5.3. Thus $(\overline{T}, Q(\overline{T}))$ is an immediate extension of (V, K) .

By an extension of Proposition 4.2, \overline{T} is Henselian.

Statement (2) follows from the construction of Theorem 7.2. \square

Suppose that W is a rank 1 valuation ring, with valuation ω . Let $\phi(x) = e^{-\omega(x)}$ for $x \in L = Q(W)$. A sequence $(x_i)_{i \in \mathbf{N}}$ of elements of L is ϕ -Cauchy if given $\epsilon > 0$, there exists n_0 such that $\phi(x_n - x_m) < \epsilon$ for all $m, n \geq n_0$ (Section 2, [10]). W is said to be complete if all ϕ -Cauchy sequences (x_i) converge to an $x \in L$.

Example 7.5. *Even if V has rank 1 and $T = \overline{T}$ is a valuation ring, (which necessarily has residue field $k(T) = k(V)$ and value group $\Gamma_V = \Gamma_T$), (T, K_∞) is not in general complete. In particular, it is not a maximal immediate extension (in the sense of Krull [18] and Kaplansky [16]).*

Proof. Let $K = k(x, y)$ be a rational function field in two variables over a field k of characteristic zero. Let $R = k[x, y]_{(x, y)}$. Let ν be the rank one valuation of K with nondiscrete value group which we can take to be \mathbf{Q} and residue field k which dominates R constructed in Example 3, page 102 of [24]. Let

$$R \rightarrow R_1 \rightarrow R_2 \rightarrow \dots \rightarrow R_i \rightarrow \dots$$

be the system of regular local rings for $i \in \mathbf{N}$ of the construction such that $\cup R_i = V$ is the valuation ring of ν .

We will first establish that $T = \cup \hat{R}_i$ is a valuation ring with residue field $k(V)$ and value group \mathbf{Q} . For any fixed i , let $q_i = \{f \in R_i \mid \nu(f) = \infty\}$. By Lemma 5.3 ν extends uniquely to a valuation of $Q(\hat{R}_i/q_i)$ which dominates \hat{R}_i/q_i and has residue field k and value group \mathbf{Q} . If $q_i \neq (0)$, then \hat{R}_i/q_i is a 1 dimensional excellent local

ring, so the only valuation rings of $Q(\hat{R}_i/q_i)$ which dominate \hat{R}_i/q_i are discrete, which is a contradiction. Thus $q_i = 0$. By Theorems 7.3 and 7.4, $T = \overline{T}$ is a valuation ring with value group \mathbf{Q} and residue field k .

In each regular local ring \hat{R}_i there is the sequence of all valuation ideals

$$\cdots \subset I_n(i) \subset \cdots \subset I_1(i) \subset I_0(i) = m(\hat{R}_i) \subset \hat{R}_i.$$

Let $p_j(i) = \nu(I_j(i))$. For fixed j , $\lim_{j \rightarrow \infty} p_j(i) = \infty$ (c.f. Lemma 2.3 [6]). Notice that $\cup_{i \in \mathbf{N}} \cup_{j \in \mathbf{N}} \{p_j(i)\} = \mathbf{Q}_+$, but arbitrarily large elements of \mathbf{Q}_+ are not in $\cup_{j \in \mathbf{N}} \{p_j(i)\}$ for a fixed i since $\cup_{j \in \mathbf{N}} \{p_j(i)\}$ is not discrete.

We can inductively construct for all $i \in \mathbf{N}$, $\sigma(i), \lambda(i) \in \mathbf{N}$ and $a_i \in I_{\lambda(i)}(\sigma(i))$ such that $i < \sigma(i)$, $\sigma(i-1) < \sigma(i)$, $\nu(a_i) = p_{\lambda(i)}(\sigma(i))$,

$$\nu(a_i) > \max \{p_{\lambda(0)}(\sigma(0)), \dots, p_{\lambda(i-1)}(\sigma(i-1)), i\},$$

and $p_{\lambda(i)}(\sigma(i)) \notin \cup_{k < \sigma(i)} \cup_{j \in \mathbf{N}} \{p_j(k)\}$ for every i .

For $i \in \mathbf{N}$, set $\alpha_i = a_1 + \cdots + a_i$. For $i < j$, we have

$$\nu(\alpha_j - \alpha_i) = \nu(a_{i+1}) > i.$$

Thus $\{\alpha_i\}$ is a ϕ -Cauchy sequence. Suppose that there exists a limit $\tau \in K_\infty$ of $\{\alpha_i\}$. Then

$$\nu(\tau - \alpha_i) = p_{\lambda(i+1)}(\sigma(i+1)) \quad (11)$$

for all i , by the definition of a limit. We have $\tau \in T$ so that $\tau \in \hat{R}_{\sigma(i)}$ for some i . Thus $\tau - \alpha_i \in \hat{R}_{\sigma(i)}$. But by (11) we have that $\nu(\tau - \alpha_i)$ is not the value of an element of $\hat{R}_{\sigma(i)}$, a contradiction. \square

8. RAMIFICATION OF COMPLETIONS OF RANK 1 VALUATION RINGS

Theorem 8.1 is a generalization of Theorem 6.3 [9], which resolves the prime ideal of infinite value terms.

Theorem 8.1. *Let k be a field of characteristic zero, K an algebraic function field over k , K^* a finite algebraic extension of K , V^* a rank 1 k -valuation ring of K^* of rational rank s , $V = V^* \cap K$. Let*

$$e = [\Gamma^* : \Gamma]$$

be the ramification index of V^ relative to V ,*

$$f = [V^*/m_{V^*} : V/m_V]$$

be the residue degree of V^ relative to V , and let τ be a primitive element of V^*/m_{V^*} over V/m_V . Let*

$$n = \text{trdeg}_k K^* - \text{trdeg}_k V^*/m_{V^*} = \text{trdeg}_k K - \text{trdeg}_k V/m_V,$$

$\lambda = \lambda_V = \lambda_{V^*}$ (as shown in Theorem 6.4).

Then there exists a partially ordered set I and algebraic regular local rings $\{S_i \mid i \in I\}$ with quotient field K^ which are dominated by V^* where S_j has regular parameters $(y_1(j), \dots, y_n(j))$ such that*

(1)

$$\{\nu^*(y_1(j)), \dots, \nu^*(y_s(j))\}$$

is a rational basis of $\Gamma^ \otimes \mathbf{Q}$.*

- (2) For all $k \in I$ there exist algebraic regular local rings $R_0(k)$ with quotient field K which are dominated by V such that there exist factorizations

$$R_0(k) \rightarrow R_k \rightarrow S_k$$

so that there are regular parameters $(x_1(k), \dots, x_n(k))$ in $R_0(k)$, units $\delta_1(k), \dots, \delta_s(k) \in S_k$ and a $s \times s$ matrix $A(k) = (a_{ij}(k))$ of nonnegative integers such that $\det(A(k)) \neq 0$ and

$$\begin{aligned} x_1(k) &= y_1(k)^{a_{11}(k)} \dots y_s(k)^{a_{1s}(k)} \delta_1(k) \\ &\vdots \\ x_s(k) &= y_1(k)^{a_{s1}(k)} \dots y_s(k)^{a_{ss}(k)} \delta_s(k) \\ x_{s+1}(k) &= y_{s+1}(k) \\ &\vdots \\ x_n(k) &= y_n(k). \end{aligned} \tag{12}$$

R_k is a normal local ring with quotient field K (which is obtained by a toric blowup of $R_0(k)$) such that S_k is a localization at a maximal ideal of the integral closure of R_k in K^* . The prime ideals

$$p_j = p(\hat{R}_j)_\infty = \{f \in \hat{R}_j \mid \nu(f) = \infty\} = (g_1(j), \dots, g_{n-\lambda}(j))$$

with

$$g_i(j) \equiv x_{s+i}(j) \pmod{m(\hat{R}_j)^2}$$

and

$$q_j = p(\hat{S}_j)_\infty = p(\hat{R}_j)_\infty \hat{S}_j = (g_1(j), \dots, g_{n-\lambda}(j))$$

with

$$g_i(j) \equiv y_{s+i}(j) \pmod{m(\hat{S}_j)^2}.$$

Furthermore, there are isomorphisms of abelian groups

$$\Gamma^*/\Gamma \cong \mathbf{Z}^s / A(k)\mathbf{Z}^s,$$

$$[S_k/m_{S_k} : R_k/m_{R_k}] = f, \quad |\det(A(k))| = e, \quad [QF(\hat{S}_k) : QF(\hat{R}_k)] = ef$$

and $S_k/m_{S_k} = R_k/m_{R_k}[\tau]$.

- (3) Let k' be an algebraic closure of V^*/m_{V^*} . Suppose that $j < k \in I$.
- (a) There are relations

$$y_i(j) = \prod_{c=1}^s y_c(k)^{d_{ic}} \epsilon_i \tag{13}$$

where d_{ic} are natural numbers and $\epsilon_i \in S_k$ is a unit for $1 \leq i \leq s$. Let $D(j, k)$ be the $s \times s$ matrix of (13). Then $\det(D(j, k)) \neq 0$.

- (b) There exists a commutative diagram

$$\begin{array}{ccc} R_k & \rightarrow & S_k \\ \uparrow & & \uparrow \\ R_j & \rightarrow & S_j \end{array} \tag{14}$$

- (c) We have actions of Γ^*/Γ on $\hat{S}_j \hat{\otimes}_{S_j/m_{S_j}} k'$ such that

$$(\hat{S}_j \hat{\otimes}_{S_j/m_{S_j}} k')^{\Gamma^*/\Gamma} \cong \hat{R}_j \hat{\otimes}_{R_j/m_{R_j}} k'$$

for all j , and this action is compatible with restriction.

We have an isomorphism

$$\hat{S}_j \hat{\otimes}_{S_j/m_{S_j}} k' \cong k' [[\bar{y}_1(j), \dots, \bar{y}_n(j)]]$$

where $\bar{y}_1(j), \dots, \bar{y}_n(j)$ are defined by

$$x_i(j) = \begin{cases} \bar{y}_1(j)^{a_{i1}(j)} \dots \bar{y}_s(j)^{a_{is}(j)} & \text{if } 1 \leq i \leq s \\ \bar{y}_i(j) & \text{if } s+1 \leq i \leq n \end{cases}$$

Let $(b_{\alpha,\beta}(j)) = \text{adj}A(j)$ and ω be a primitive e -th root of unity. The action of Γ^*/Γ on $\hat{S}_j \hat{\otimes}_{S_j/m_{S_j}} k' \cong k' [[\bar{y}_1(j), \dots, \bar{y}_n(j)]]$ is defined for

$$c \in \mathbf{Z}^s / A(j)\mathbf{Z}^s \cong \Gamma^*/\Gamma$$

by

$$(4) \quad \sigma_c(\bar{y}_\alpha(j)) = \begin{cases} \omega^{\sum_{\beta=1}^s b_{\alpha,\beta}(j)c_\beta} \bar{y}_\alpha(j) & \text{if } 1 \leq \alpha \leq s \\ \bar{y}_\alpha(j) & \text{if } s+1 \leq \alpha \leq n. \end{cases}$$

$$V^* = \varinjlim S_j = \cup_{j \in I} S_j$$

and

$$V = \varinjlim R_j = \cup_{j \in I} R_j.$$

For $j \in I$, let Λ_j be the free \mathbf{Z} module $\Lambda_j = \sum_{i=1}^s \nu(x_i(j))\mathbf{Z}$, and let Ω_j be the free \mathbf{Z} module $\Omega_j = \sum_{i=1}^s \nu^*(y_i(j))\mathbf{Z}$. Then

$$\Gamma = \varinjlim \Lambda_j = \cup_{j \in I} \Lambda_j$$

and

$$\Gamma^* = \varinjlim \Omega_j = \cup_{j \in I} \Omega_j.$$

Proof. Suppose that R' is the regular local ring of Theorem 6.1 [9], and \tilde{R} is the regular local ring of Theorem 6.4. By Theorem 6.1 [9], there exists a sequence of local rings

$$R_0(0) \rightarrow R_0 \rightarrow S_0$$

such that $R' \subset R_0(0)$, $\tilde{R} \subset R_0(0)$ and the conclusions of Theorem 6.3 [9] and Theorem 6.5 hold for this sequence. In particular, (1) and (2) of the theorem hold for $R_0(0) \rightarrow R_0 \rightarrow S_0$ and $p(\hat{S}_0)_\infty, p(\hat{R}_0)_\infty$ have the desired form.

Suppose that m is a positive integer, $f = (f_1, \dots, f_m) \in (V^*)^m$. Set $u_i = y_i(0)$, $1 \leq i \leq n$. Set $u_{n+i} = f_i$ for $1 \leq i \leq m$. If $f_i \in V^* \cap K = V$, also set $v_i = f_i$.

By Theorem 4.9 [9] and Theorem 6.1 [9], with the R^*, S^* in the assumptions of Theorem 4.9 [9] set as $R^* = R_0(0)$, $S^* = S_0$, and with the $\{u_i\}$ and $\{v_i\}$ defined as above, and then applying Theorem 6.5 (and Remark 6.6), there exists a commutative diagram

$$\begin{array}{ccccc} R_0(f) & \rightarrow & R_f & \rightarrow & S_f \\ \uparrow & & & & \uparrow \\ R_0(0) & \rightarrow & & & S_0 \end{array}$$

such that the vertical arrows are sequences of monoidal transforms along V^* , (1) and (2) of this theorem hold for

$$R_0(f) \rightarrow R_f \rightarrow S_f$$

and (3)(a) of this theorem holds for

$$\begin{array}{ccc} R_f & \rightarrow & S_f \\ \uparrow & & \uparrow \\ R_0 & \rightarrow & S_0 \end{array} .$$

We have that $\det(D(0, f)) \neq 0$ since (1) holds for S_0 and S_f . Define a partial ordering on $I = \sqcup_{m \in \mathbf{N}_+} (V^*)^m$ by $f \leq g$ if $S_f \subset S_g$. We will associate to $0 \in V^*$ the sequence $R_0(0) \rightarrow R_0 \rightarrow S_0$ constructed in the beginning of the proof. Suppose that $\alpha \leq \beta$. We have

$$S_0 \subset S_\alpha \subset S_\beta$$

so the proof of (2) of Theorem 7.2 shows that (3)(a) of this Theorem holds for α, β . (3)(b) holds since

$$R_\alpha = S_\alpha \cap K \subset S_\beta \cap K = R_\beta.$$

(3)(c) is immediate, since the conclusions of Theorem 6.1 [9] hold. In particular, (11) of Theorem 4.7 [9] holds.

Finally, we will establish (4) of the Theorem. By construction, $V^* = \cup_{j \in I} S_j$. If $f \in V$, we have $f \in S_f \cap K = R_f$, thus $V = \cup_{j \in I} R_j$. By construction, $\cup_{j \in I} \Omega_j = \Gamma^*$, since $\nu^*(f) \in \Omega_f$ for $f \in V^*$. We also have $\cup_{j \in I} \Lambda_j = \Gamma$, since $\nu(f) \in \Lambda_f$ for $f \in V$. I is a directed set as shown in the proof of Theorem 7.2. \square

Theorem 8.2. *Let assumptions be as in Theorem 8.1. There exists a partially ordered set I and algebraic regular local rings $\{S_i \mid i \in I\}$ with quotient field K^* which are dominated by V^* and algebraic local rings with toric singularities $\{R_i \mid i \in I\}$ such that*

(1)

$$V^* = \varinjlim S_i = \cup_{i \in I} S_i, V = \varinjlim R_i = \cup_{i \in I} R_i$$

and each S_i is a localization at a maximal ideal of the integral closure of R_i in K^* .

(2) $\hat{S}_j/p(\hat{S}_j)_\infty$ are regular local rings for all j and

$$T^* = \varinjlim \hat{S}_j/p(\hat{S}_j)_\infty$$

is a Henselian valuation ring such that $(T^*, Q(T^*))$ is an immediate extension of (V^*, K^*) .

(3) $\hat{R}_j/p(\hat{R}_j)_\infty$ has normal toric singularities for all j and

$$T = \varinjlim \hat{R}_j/p(\hat{R}_j)_\infty$$

is a Henselian valuation ring such that $(T, Q(T))$ is an immediate extension of (V, K) .

(4) Further suppose that $k = V^*/m_{V^*}$ is algebraically closed (of characteristic zero). Then the action of Γ^*/Γ on \hat{S}_j by k -algebra isomorphisms extends to an action of Γ^*/Γ on $\hat{S}_j/p(\hat{S}_j)_\infty$, and an action of Γ^*/Γ on T^* such that $\hat{R}_j/p(\hat{R}_j)_\infty \cong (\hat{S}_j/p(\hat{S}_j)_\infty)^{\Gamma^*/\Gamma}$ and $T = (T^*)^{\Gamma^*/\Gamma}$.

Proof. The theorem follows from Theorems 8.1, 7.3 and 7.4. The fact that $p(\hat{S}_j)_\infty$ is fixed by Γ^*/Γ follows from (2) of Theorem 8.1. \square

Now the proof of Theorem 1.3 follows from Theorem 8.2. The proof that $Q(U^*)$ is Galois over $Q(U)$ is as in Theorem 1.2.

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