VALUATION SEMIGROUPS OF NOETHIERIAN LOCAL DOMAINS

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1. Introduction

Suppose that (R, m_R) is a Noetherian local domain, with quotient field K, and ν is a valuation of K which dominates R. Let Γ_ν be the value group of ν. We consider the problem of determining the valuation semigroup

$$S^\nu(R) = \{\nu(f) \mid f \in R \setminus m_R\}.$$

We give some general results and examples, and give a complete discription in the case of regular local rings of dimension 2. This generalizes the classification of valuation semigroups of regular local rings of dimension two with algebraically closed residue fields obtained by Spivakovsky in [46].

This article presents recent results of the author, in joint work with Bernard Teissier, Kia Dalili, Olga Kashcheyeva and Vinh An Pham. It is a write up of a lecture given at the second international valuation theory conference.

2. Value Groups

Suppose that K is a field. A valuation of K is a surjective map $\nu : K^* \to \Gamma_\nu$ where Γ_ν is a totally ordered abelian group such that for $a, b \in K^*$,

1) $\nu(ab) = \nu(a) + \nu(b)$
2) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

Set $\nu(0) = \infty$.

The valuation ring of ν is $V_\nu = \{f \in K \mid \nu(f) \geq 0\}$. The unique maximal ideal of V_ν is $m_\nu = \{f \in K \mid \nu(f) > 0\}$.

We will consider valuations ν which dominate a Noetherian local domain R whose quotient field is K; that is $R \subset V_\nu$ and $m_\nu \cap R = m_R$.

There is a complete description of the groups Γ_ν which are attained.

Theorem 2.1. *(Maclane and Schilling [41], Zariski [53], Kuhlmann [36]) The groups Γ_ν attained by such K and R are the ordered abelian groups of finite rational rank.*

A totally ordered abelian group G has rational rank e if $G\mathbb{Q} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ has dimension e as a rational vector space. A fundamental result is Abhyankar's Inequality ([1], [53]):

$$\text{rat rank}\,\nu + \text{trdeg}_{R/m_R} V_\nu/m_\nu \leq \dim R$$

If equality holds then $\Gamma_\nu \cong \mathbb{Z}^{\text{rat rank}\,\nu}$ (as an unordered group) and V_ν/m_ν is a finitely generated field extension of R/m_R.

The rank of ν is defined as

$$r = \text{rank}\,\nu = \text{length of } \textit{the} \text{ chain of prime ideals in } V_\nu$$

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$$\{0\} = P_{r+1} \subset P_r \subset \cdots \subset P_1 \subset V_\nu.$$

We have an order preserving embedding $\Gamma_\nu \subset (\mathbb{R}^r)_{\text{lex}}$.

The Convex Subgroups of Γ_ν are

$$\Phi_i = \Gamma_\nu \setminus \{\pm\nu(f) \mid f \in P_{i+1}\} = \left(0_{r-i} \times \mathbb{R}^i\right) \cap \Gamma_\nu,$$

where 0_{r-i} is the zero vector of length $r-i$. We thus have a chain

$$\{0\} = \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_r = \Gamma_\nu.$$

3. Extension of valuations to the completion

Theorem 3.1. *(William Heinzer and Judith Sally* HS[32]*) Suppose that ν is a valuation dominating an analytically normal local domain R. Then either ν extends uniquely to a valuation dominating the completion $\hat{R}$ of R or there are infinitely many such extensions, of at least two different ranks.*

Suppose that K is a field, and V is a valuation ring of K. We say that the rank of V increases under completion if there exists an analytically normal local domain T with quotient field N such that V dominates T and there exists an extension of V to a valuation ring of the quotient field of $\hat{T}$ which dominates $\hat{T}$ and which has higher rank than the rank of V.

Theorem 3.2. *(C and Olga Kashcheyeva* CK[16], *Spivakovsky* S[46] *in the case when $R/\mathfrak{m}_R$ is algebraically closed) Suppose that V dominates an excellent two dimensional local ring R. Then the rank of V increases under completion if and only if V/m_V is finite over $R/\mathfrak{m}_R$ and V is discrete of rank 1.*

Blowing up may be necessary to obtain an increase in rank.

This gives an interesting case which cannot occur for R with algebraically closed residue field:

If the algebraic closure of the residue field $\mathfrak{k} = R/m_R$ is not finite over $\mathfrak{k}$, then there may exist valuations with value group $\mathbb{Z}$ and an algebraic residue field extension which dominate R such that the rank does not increase under completion.

4. Valuation semigroups

The valuation semigroups

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}$$

are not so well understood, although they contain much information about the singularity type of R and the ideal theory of R.

Question 4.1. *Is it possible to characterize the semigroups which occur as valuation semigroups of a valuation dominating a Noetherian domain?*

5. Zariski's necessary condition

Theorem 5.1. *(Zariski, in Appendix 3 to Volume II of [53]) Suppose that R is a noetherian local domain which is dominated by a valuation ν of the quotient field of R. Then the semigroup $S^R(\nu)$ is a well ordered subset of the positive part of the value group Γ_ν, of ordinal type at most ω^h, where ω is the ordinal type of the well ordered set $\mathbb{N}$, and h is the rank of ν.*

6. Regular local rings of dimension two

We obtain the following necessary and sufficient condition for a semigroup and field extension to be the semigroup and residue field extension of a valuation dominating a complete regular local ring of dimension two in the following theorem.

Theorem 6.1. *(C and Pham An Vinh [22]) Suppose that R is a complete regular local ring of dimension two with residue field $R/\mathfrak{m}_R = \mathfrak{k}$. Let S be a subsemigroup of the positive elements of a totally ordered abelian group and L be a field extension of $\mathfrak{k}$. S is the semigroup of a valuation ν dominating R with residue field $V_\nu/\mathfrak{m}_\nu = L$ if and only if there exist finite or countable sets of elements $\beta_i \in S$ and $\alpha_i \in L$ such that*

1) *The semigroup S is generated by $\{\beta_i\}$ and the field L is generated over $\mathfrak{k}$ by $\{\alpha_i\}$.*
2) *Let*
$$\overline{n}_i = [G(\beta_0, \ldots, \beta_i) : G(\beta_0, \ldots, \beta_{i-1})] \text{ and}$$
$$d_i = [\mathfrak{k}(\alpha_1, \ldots, \alpha_i) : \mathfrak{k}(\alpha_1, \ldots, \alpha_{i-1})].$$
Then there are inequalities
$$\beta_{i+1} > \overline{n}_i d_i \beta_i > \beta_i$$
with $\overline{n}_i < \infty$ and $d_i < \infty$.

Here $G(\beta_0, \ldots, \beta_i)$ is the subgroup generated by $\beta_0, \ldots, \beta_i$.

The assumption that R is complete only appears in the above theorem in the case when the value group is $\mathbb{Z}$ and the residue field extension is finite. This case cannot occur when R is complete, but may appear if R is not complete.

We give a necessary and sufficient condition for a semigroup to be the semigroup of a valuation dominating a regular local ring of dimension two in the following theorem.

Theorem 6.2. *(C and Vinh An Pham [22], Spivakovsky [46] when $R/\mathfrak{m}_R$ is algebraically closed) Suppose that R is a regular local ring of dimension two. Let S be a subsemigroup of the positive elements of a totally ordered abelian group. Then S is the semigroup of a valuation ν dominating R if and only if there exists a finite or countable index set I, of cardinality $\Lambda = |I| - 1 \geq 1$ and elements $\beta_i \in S$ for $i \in I$ such that*

1) *The semigroup S is generated by $\{\beta_i\}_{i \in I}$.*
2) *Let*
$$\overline{n}_i = [G(\beta_0, \ldots, \beta_i) : G(\beta_0, \ldots, \beta_{i-1})].$$
There are inequalities
$$\beta_{i+1} > \overline{n}_i \beta_i$$
with $\overline{n}_i < \infty$ for $1 \leq i < \Lambda$. If $\Lambda < \infty$ then $\overline{n}_\Lambda \leq \infty$.

We deduce from the above theorem a generalization of a result of Noh [45].

Corollary 6.3. *Suppose that R is a regular local ring of dimension two and ν is a valuation dominating R such that ν is discrete of rank 1. Then $S^R(\nu)$ is symmetric.*

Example 6.4. *(C and Vinh An Pham* CV [22]*) There exists a semigroup S which satisfies the sufficient conditions 1) and 2) of the above theorem, such that if $(R, \mathfrak{m}_R)$ is a 2-dimensional regular local ring dominated by a valuation ν such that $S^R(\nu) = S$, then $R/\mathfrak{m}_R = V_\nu/\mathfrak{m}_\nu$; that is, there can be no residue field extension.*

The proof of the above theorem gives an algorithm to construct a generating sequence of ν in the two dimensional regular local ring R, and it gives an algorithm to expand a given element of R in terms of the generating sequence, and thus compute it's value.

Suppose that ν is a valuation dominating a noetherian local ring R. For $\varphi \in \Gamma_\nu$, define valuation ideals

$$\mathcal{P}_\varphi(R) = \{f \in R \mid \nu(f) \geq \varphi\},$$

and

$$\mathcal{P}^+_\varphi(R) = \{f \in R \mid \nu(f) > \varphi\}.$$

The associated graded ring of ν on R is

$$\mathrm{gr}_\nu(R) = \bigoplus_{\varphi \in \Gamma_\nu} \mathcal{P}_\varphi(R)/\mathcal{P}^+_\varphi(R).$$

Suppose that $f \in R$ and $\nu(f) = \varphi$. Then the initial form of f in $\mathrm{gr}_\nu(R)$ is

$$\mathrm{in}_\nu(f) = f + \mathcal{P}^+_\varphi(R) \in [\mathrm{gr}_\nu(R)]_\varphi = \mathcal{P}_\varphi(R)/\mathcal{P}^+_\varphi(R).$$

A set of elements $\{F_i\}$ such that $\{\mathrm{in}_\nu(F_i)\}$ generates $\mathrm{gr}_\nu(R)$ as a $\mathfrak{k}$-algebra is called a generating sequence of ν in R.

Corollary 6.5. *(C, Kia Dalili and Olga Kashcheyeva* CDK [13]*, C and Vinh An Pham) Suppose that R is a regular local ring of dimension two and ν is a rank 1 valuation dominating R. Embed the value group of ν in $\mathbb{R}_+$ so that 1 is the smallest nonzero element of $S^R(\nu)$. Let $\varphi(n) = |S^R(\nu) \cap (0, n)|$ for $n \in \mathbb{Z}_+$. Then*

$$\lim_{n \to \infty} \frac{\varphi(n)}{n^2}$$

exists. The set of limits which are obtained by such valuations ν dominating R is the real half open interval $[0, \frac{1}{2})$.

7. Normal surface singularities

At this point we ask if some variation of our necessary and sufficient conditions 1) and 2) for a semigroup to be the valuation semigroup of a regular local ring of dimension two holds for the local ring R of a normal surface singularity. For instance, if

$$a_0 < a_1 < \cdots < a_i <$$

is the minimal set of generators of the semigroup $S^R(\nu)$ of a rational valuation ν, is

$$a_{i+1} > 2a_i \text{ for } i \gg 0?$$

Do the minimal generators (at least) become further apart as i increases?

Example 7.1. *(C and Vinh An Pham [22]) Suppose that $\mathfrak{k}$ is a field and R is the localization of $\mathfrak{k}[u,v,w]/uv-w^2$ at the maximal ideal (u,v,w). Then there exists a rational valuation ν dominating R such that if*

$$a_0 < a_1 < \cdots$$

is the sequence of minimal generators of $S^R(\nu)$, then given $n \in \mathbb{N}$, there exists $i > n$ such that

$$a_{i+1} = a_i + \frac{a_0}{3}$$

and a_{i+1} is in the group generated by $a_0, a_1, \ldots, a_i$.

So the minimal generators can be close together, although they must differ by at least $\frac{a_0}{3}$

Lemma 7.2. *Let $\mathfrak{k}$ be an algebraically closed field, and let $A = \mathfrak{k}[x^2, xy, y^2]$, a subring of the polynomial ring $B = \mathfrak{k}[x,y]$. Let $\mathfrak{m} = (x^2, xy, y^2)A$ and $\mathfrak{n} = (x,y)B$. Suppose that ν is a rational valuation dominating $B_{\mathfrak{n}}$, such that ν has a generating sequence*

$$P_0 = x, P_1 = y, P_2 \ldots$$

in $\mathfrak{k}[x,y]$ such that each P_i is a $\mathfrak{k}$-linear combinations of monomials in x and y of odd degree, and

$$\beta_0 = \nu(x), \beta_1 = \nu(y), \beta_2 = \nu(P_2), \ldots$$

is the increasing sequence of minimal generators of $S^\nu(B_{\mathfrak{n}})$, with $\beta_{i+1} > n_i\beta_i$ for $i \geq 1$, where $n_i = [G(\beta_0, \ldots, \beta_i) : G(\beta_0, \ldots, \beta_{i-1})]$. Then

$$S^\nu(A_{\mathfrak{m}}) = \left\{ \begin{array}{l} a_0\beta_0 + a_1\beta_1 + \cdots + a_i\beta_i \mid i \in \mathbb{N}, a_0, \ldots, a_i \in \mathbb{N} \\ \text{and } a_0 + a_1 \cdots + a_i \equiv 0 \text{ mod } 2 \end{array} \right\}.$$

We define the desired valuation ν on $B = \mathfrak{k}[x,y]$ by constructing a generating sequence:

$$P_0 = x, P_1 = y, P_2 = y^3 - x^5, P_3 = P_2^3 - x^{18}y, \cdots$$

where

$$P_{i+1} = P_i^3 - x^{a_i}P_{i-1}$$

with a_i an even positive integer, and $\beta_0 = \nu(x) = 1$, $\beta_1 = \nu(y) = \frac{5}{3}$, $\beta_i = \nu(P_i) = b_i + \frac{5}{3^i}$ with $b_i \in \mathbb{Z}_+$, for $i \geq 2$, by requiring that 3 divides $a_i + b_{i-1}$ and

$$b_i = \frac{a_i + b_{i-1}}{3} > 3b_{i-1}$$

for $i \geq 2$. a_i, b_i satisfying these relations can be constructed inductively from b_{i-1}.

8. Extension of valuation semigroups under a finite extension

In a finite field extension, the quotient of the valuation group of an extension of a valuation by the value group is always a finite group. This raises the following question: Suppose that $R \to T$ is a finite extension of regular local rings, and ν is a valuation which dominates R. Is $S^T(\nu)$ a finitely generated module over the semigroup $S^R(\nu)$?

The answer is no.

Example 8.1. *(C and Vinh An Pham [22]) There exists a finite extension $R \to T$ of two dimensional regular local rings and a valuation ν dominating T such that $S^T(\nu)$ is not a finitely generated $S^R(\nu)$ module.*

9. Extension of valuation semigroups under a quadratic transform

Finite generation also fails under a quadratic transform.

Example 9.1. *(C and Bernard Teissier CT1 [20]) There exists a quadratic transform $A \to B$ of regular local rings of dimension two and a valuation ν dominating B such that $S^B(\nu)$ is not a finitely generated $S^A(\nu)$ module.*

10. Regular local rings of dimension 3

The semigroups attainable on a regular local ring of dimension 3 are even more complicated.

Example 10.1. *(C, Kia Dalili and Olga Kashcheyeva CDK [13]) There exists a rational valuation ν dominating a regular local ring R of dimension 3 such that if*

$$a_0 < a_1 < \cdots$$

is the sequence of minimal generators of $S^R(\nu)$, then given $\varepsilon > 0$, there exists $i > n$ such that

$$a_{i+1} - a_i < \varepsilon.$$

In this example new generators get closer and closer together.

11. An upper bound for growth of real (rank 1) valuations

Theorem 11.1. *(C, Olga Kashcheyeva and Kia Dalili CDK [13], C and Bernard Teissier CT2 [21]) Suppose that R is a local domain which is dominated by a real valuation, and suppose that a_0 is the smallest element of $S^R(\nu)$. Then for $n \in \mathbb{Z}_+$,*

$$|S^R(\nu) \cap (0, na_0)| < \ell(R/m_R^n).$$

In particular, for $n \gg 0$,

$$|S^R(\nu) \cap (0, na_0)| < p_R(n)$$

where $p_R(n)$ is the Hilbert polynomial of R. Thus growth is bounded above by a polynomial of degree $d = \dim R$.

12. Zariski's necessary condition is not sufficient

Corollary 12.1. *(C, Kia Dalili and Olga Kashcheyeva CDK [13], C and Bernard Teissier CT2 [21]) There exists a well ordered subsemigroup U of $\mathbb{Q}_+$ such that U has ordinal type ω and $U \neq S^R(\nu)$ for any valuation ν dominating a local domain R.*

Proof. Take any subset T of $\mathbb{Q}_+$ such that $a_0 = 1$ is the smallest element of T and $n^n \leq |T \cap (0, n)| < \infty$ for all $n \in \mathbb{N}$. For all positive integers r, let

$$rT = \{a_1 + \cdots + a_r \mid a_1, \ldots, a_r \in T\}.$$

let $U = \omega T = \cup_{i=1}^{\infty} rT$ be the semigroup generated by T. U is well ordered of ordinal type ω. By the bound on the previous slide, U cannot be the semigroup of a valuation dominating a local domain. □

Question 12.2. *Suppose that $S \subset \mathbb{R}_+$ is a semigroup which contains a smallest element a_0. Suppose there exist $c > 0$ and $d \in \mathbb{N}_+$ such that*

$$|S \cap (0, na_0)| < cn^d$$

for all $n \in \mathbb{N}$. Is S the semigroup of a valuation dominating a noetherian local ring?

Of course the dimension of R could be much larger than d.

A semigroup S satisfying the above condition satisfies Zariski's necessary condition: S is well ordered of ordinal type ω and has rational rank $\leq d$.

Question 12.3. *Suppose that ν is a real valuation which dominates a local ring R of dimension d. Let a_0 be the smallest element of $S^R(\nu)$. Does the limit*

$$\lim_{n\to\infty}\frac{|S^R(\nu)\cap(0,na_0)|}{n^d}\in\mathbb{R} \tag{1}$$

exist?

The limit exists if R is a regular local ring of dimension 2 by [13] and [22]. In fact, the limit exists quite generally for a domain which is a regular local ring or an excellent local domain with an algebraically closed residue field, by a very recent result in [12].

13. Higher Rank Valuations

Suppose that R is a regular local ring, and ν is a rank 2 valuation dominating R. Let $\Phi_1\subset\Gamma_\nu$ be the nontrivial convex subgroup of Γ_ν, and let $P_1\subset V_\nu$ be the nontrivial, non maximal prime ideal of V_ν. Let ν_1 be the induced composite valuation with valuation ring $V_{\nu_1}=(V_\nu)_{P_1}$, and valuation group $\Gamma_{\nu_1}\cong\Gamma_\nu/\Phi_1$.

We have a natural surjection $\lambda:\Gamma_\nu\to\Gamma_{\nu_1}$. Given $\varphi\in S^{\nu_1}(R)$, define

$$\tilde\varphi=\min\{\nu(f)\mid f\in R\text{ and }\nu_1(f)=\varphi\}.$$

Suppose that R is a regular local ring of dimension 2, dominated by a rank 2 valuation ν. By Abhyankar's Theorem, we have that $\Gamma_\nu\cong(\mathbb{Z}^2)_{\rm lex}$. We have that $S^R(\nu)$ is a finitely generated semigroup, and the function $\tilde\varphi$ is eventually linear.

However, the situation is much more complicated when R has higher dimension.

Example 13.1. *(C and Bernard Teissier [20]) There exists an example of a valuation ν dominating a regular local ring of dimension three, whose value group is $\Gamma_\nu=(\mathbb{Z}^2)_{\rm lex}$, and the semigroup $S^R(\nu)$ is not a finitely generated semigroup.*

Further, the function $\tilde\varphi$ can be extremely wild, as shown in the following example.

Example 13.2. *(C and Bernard Teissier [21]) Suppose that $f:\mathbb{N}\to\mathbb{Z}$ is a decreasing function, $g:\mathbb{N}\to\mathbb{Z}$ is an increasing function, and K is a field. Then there exists a rank 2 valuation ν of the five dimensional rational function field $K(x,y,u,v,z)$ with value group $(H\times\mathbb{Z})_{lex}$, where $H=(\frac{1}{2^\infty}\mathbb{Z}+\frac{1}{2^\infty}\mathbb{Z}\sqrt{2})\subset\mathbb{R}$, which dominates the regular local ring $R=K[x,y,u,v,z]_{(x,y,u,v,z)}$, such that for any valuation ω equivalent to ν with value group $(H\times\mathbb{Z})_{lex}$, for all sufficiently large $n\in\mathbb{N}$, there exists $\lambda_1\in H\cap[0,n[$ such that $\pi_2(\tilde\lambda_1)<f(n)$ and there exists $\lambda_2\in H\cap[0,n[$ such that $\pi_2(\tilde\lambda_2)>g(n)$, where $\pi_2:H\times\mathbb{Z}\to\mathbb{Z}$ is the second projection.*

14. A polynomial bound for growth of valuations of rank > 1.

Define prime ideals p_i in R by

$$p_i=P_i\cap R$$

and for $\varphi\in\Gamma_\nu$, define valuation ideals in R

$$\mathcal{P}_\varphi=\{f\in R\mid\nu(f)\geq\varphi\}$$

and

$$\mathcal{P}_{\varphi}^{+} = \{f \in R \mid \nu(f) > \varphi\}.$$

For $a < b \in \Gamma_\nu$, define

$$[a, b[= \{x \in \Gamma_\nu \mid a \leq x < b\}.$$

Theorem 14.1. *(C and Bernard Teissier* CT2 [21]*) Let R be a local domain and ν a valuation of R which is of rank n. There exist functions $s_n(\varepsilon)$ and $s_i(\varepsilon, y_{i+1}, y_{i+2}, \ldots, y_n)$ for $1 \leq i \leq n-1$, such that*

$$\sum_{\varphi_n \in [0, t_n y_n[} \sum_{\varphi_{n-1} \in [\tilde{\varphi}_n, \tilde{\varphi}_n + t_{n-1} y_{n-1}[} \cdots \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + t_1 y_1[} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^{+}_{\varphi_1})_{p_0})$$

$$\leq (1+\varepsilon) \frac{\prod_{i=0}^{n} e_{m_i}((R/p_{i+1})_{p_i})}{\prod_{i=1}^{n} (\dim (R/p_{i+1})_{p_i})!} \prod_{i=1}^{n} y_i^{\dim (R/p_{i+1})_{p_i}}$$

for $y_n, y_{n-1}, \ldots, y_1 \in \mathbb{N}$ satisfying

$$y_n \geq s_n(\varepsilon), y_{n-1} \geq s_{n-1}(\varepsilon, y_n), \ldots, y_1 \geq s_1(\varepsilon, y_2, \ldots, y_n).$$

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