QUOTIENTS OF MILNOR K-RINGS, ORDERINGS, AND VALUATIONS

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Abstract

We define and study the Milnor K-ring of a field F modulo a subgroup of the multiplicative group of F. We compute it in several arithmetical situations, and study the reflection of orderings and valuations in this ring.

Introduction

Let F be a field and let $F^\times$ be its multiplicative group. The Milnor K-ring $K_*^M(F)$ of F is the tensor (graded) algebra of the $\mathbb{Z}$-module $F^\times$ modulo the homogenous ideal generated by all elements $a_1 \otimes \cdots \otimes a_r$, where $1 = a_i + a_j$ for some $1 \leq i < j \leq r$ [Mi]. Alongside with $K_*^M(F)$, the quotients $K_*^M(F)/m = K_*^M(F)/mK_*^M(F)$, where m is a positive integer, also play an important role in many arithmetical questions. In this paper we study a natural generalization of these two functors. Specifically, we consider a subgroup S of $F^\times$ and define the graded ring $K_*^M(F)/S$ to be the quotient of the tensor algebra over $F^\times/S$ modulo the homogeneous ideal generated by all elements $a_1S \otimes \cdots \otimes a_rS$, where $1 \in a_iS + a_jS$ for some $1 \leq i < j \leq r$. The graded rings $K_*^M(F)$ and $K_*^M(F)/m$ then correspond to $S = \{1\}$ and $S = (F^\times)^m$, respectively.

The ring-theoretic structure of $K_*^M(F)/S$ reflects many of the main arithmetical properties of F, especially those related to orderings and valuations. We illustrate this by computing it in the following situations:

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*) The research has been supported by the Israel Science Foundation grant No. 8008/02–1

2000 Mathematics subject classification: Primary 19F99, Secondary 12J10, 12J15, 12E30

(1) $F^\times/S$ is a finite cyclic group. Here, if F has no orderings containing S then $K_*^M(F)/S$ is trivial in degrees > 1. Otherwise $K_*^M(F)/S$ coincides in degrees > 1 with the tensor algebra over $\{\pm 1\}$ (Theorem 4.1). This includes as a special case the computation of the Milnor K-ring of finite fields, which goes back to Steinberg and Milnor [Mi, Example 1.5].

(2) There is a (Krull) valuation v on F whose 1-units are contained in S. We show that under a mild assumption, $K_*^M(F)/S$ is then obtained from the corresponding K-ring of the residue field and from $v(F^\times)/v(S)$ by means of a natural algebraic construction analogous to the construction of a polynomial ring over a given ring (§5).

(3) $F^\times/S$ is finitely generated, and is generated by the 1-units of a rank-1 valuation v such that S is open in the v-topology on F. We prove that then $K_*^M(F)/S$ is trivial in degrees > 1 (Theorem 6.2).

(4) $F^\times/S$ is finite, and there is a rank-1 valuation v on F with mixed characteristics $(0,p)$ such that $S = (F^\times)^p(1+p^2\mathfrak{m}_v)$, where $\mathfrak{m}_v$ is the valuation ideal (when v is Henselian the latter condition just means that $S = (F^\times)^p$). We show that then $K_*^M(F)/S$ is either the Milnor K-ring of a finite extension of $\mathbb{Q}_p$, or else it is trivial in degrees > 1 and $v(F^\times)$ is p-divisible (Theorem 7.4). The proof is based on the vanishing theorem of (3) above.

These results are mostly of a *local* nature. In a forthcoming paper we compute the functor $K_*^M(F)/S$ in *global* situations, where S is related to a family of orderings and valuations.

Studying Milnor's K-theory modulo a subgroup S by means of the functor $K_*^M(F)/S$ resembles the reduced theory of quadratic forms: there one studies quadratic forms modulo a preordering T on F via the *reduced* Witt ring functor $W_T(F)$, rather than the classical Witt ring – see [Lm], [BK] for details.

Furthermore, the celebrated Bloch–Kato–Milnor conjecture predicts that $K_*^M(F)$ is isomorphic to the Galois cohomology of the absolute Galois group G_F of F with respect to twisted cyclotomic actions [K]. Similarly, when p is a prime number and F contains a primitive pth root of unity, $K_*^M(F)/p$ is related to the Galois cohomology ring of the maximal pro-p Galois group $G_F(p)$ of F with its trivial action on $\mathbb{Z}/p$. From this viewpoint, the generalized functor $K_*^M(F)/S$ serves in some sense as an analog of the Galois cohomology of an arbitrary relative Galois group $\mathrm{Gal}(E/F)$ of F.

1. κ-structures

In this section we define a convenient target category for the Milnor K-ring functor. It is a slight modification of the "κ-Algebras", as defined by Bass and Tate in [BaT].

Denote the tensor algebra of an abelian group Γ by $\mathrm{Tens}(\Gamma)$. We let $\kappa = \bigoplus_{r=0}^\infty \kappa_r = \mathrm{Tens}(\{\pm 1\})$, and denote the nontrivial element of $\kappa_1 \cong \mathbb{Z}/2$ by ε. Thus $\kappa_0 = \mathbb{Z}$ and $\kappa_r = \{0, \varepsilon^r\} \cong \mathbb{Z}/2$ for all $r \geq 1$.

Definition 1.1: A **κ-structure** consists of a graded ring $A = \bigoplus_{r=0}^\infty A_r$ and a graded ring homomorphism $\kappa \to A$ such that:

(i) $A_0 = \mathbb{Z}$ and the homomorphism $\kappa \to A$ is the identity in degree 0;

(ii) A_1 generates A as a ring;

(iii) the image ε_A of ε in A satisfies $a^2 = \varepsilon_A a = a\varepsilon_A$ for all $a \in A_1$.

For every $a, b \in A_1$ we have $ab + ba = (a+b)^2 - a^2 - b^2 = 0$, by (iii). Thus A is anti-commutative. A **morphism** $A \to B$ of κ-structures is a graded ring homomorphism which commutes with the structural homomorphisms $\kappa \to A$, $\kappa \to B$.

The category of κ-structures has direct products. Namely, the direct product $\prod_{i\in I} A_i$ of κ-structures A_i, $i \in I$, is defined by $(\prod_{i\in I} A_i)_0 = \mathbb{Z}$ and $(\prod_{i\in I} A_i)_r = \prod_{i\in I}(A_i)_r$ for $r \geq 1$, with the natural multiplicative structure. The homomorphism $\kappa_r \to \prod_{i\in I}(A_i)_r$ is given by $\varepsilon \mapsto (\varepsilon_{A_i})_{i\in I}$.

Recall that the tensor product in the category of *graded rings* is defined by $A \otimes_\mathbb{Z} B = \bigoplus_{r=0}^\infty (\bigoplus_{i+j=r} A_i \otimes_\mathbb{Z} B_j)$, with the product given by

$$(a \otimes b)(a' \otimes b') = (-1)^{i'j} aa' \otimes bb'$$

for $a \in A_i$, $a' \in A_{i'}$, $b \in B_j$, $b' \in B_{j'}$. Given κ-structures A, B, we define their **tensor product** in the category of *κ-structures* to be $A \otimes_\kappa B = (A \otimes_\mathbb{Z} B)/I$, where I is the homogeneous ideal generated by $\varepsilon_A \otimes 1_B - 1_A \otimes \varepsilon_B$. The homomorphism $\kappa \to A \otimes_\kappa B$ is given by $\varepsilon \mapsto \varepsilon_A \otimes 1_B + I = 1_A \otimes \varepsilon_B + I$. Since A, B are anti-commutative, so is $A \otimes_\mathbb{Z} B$. Further, given $a \in A_1$ and $b \in B_1$ we have $(a \otimes 1_B)^2 = (\varepsilon_A \otimes 1_B)(a \otimes 1_B)$ and

$(1_A \otimes b)^2 = (1_A \otimes \varepsilon_B)(1_A \otimes b)$, so by the anti-commutativity,

$$(a \otimes 1_B + 1_A \otimes b)^2 + I = (\varepsilon_A \otimes 1_B)(a \otimes 1_B + 1_A \otimes b) + I$$

in $(A \otimes_\kappa B)_2$. This implies the first equality in (iii) for $A \otimes_\kappa B$. The second is proved similarly, showing that $A \otimes_\kappa B$ is a κ-structure. There are canonical morphisms $\iota\colon A \to A \otimes_\kappa B$, $\iota'\colon B \to A \otimes_\kappa B$ with respect to which $A \otimes_\kappa B$ is the coproduct of A and B in the category of κ-structures (in the sense of e.g. [Ln, Ch. I, §7]). One has $A \cong A \otimes_\kappa \kappa$ and $B \cong \kappa \otimes_\kappa B$ via these morphisms.

Next we construct free objects in this category. Let Γ be an abelian group. We define $\kappa[\Gamma]$ to be the quotient of $\mathrm{Tens}(\kappa_1 \oplus \Gamma)$ by the homogeneous ideal generated by all elements $\varepsilon \otimes \gamma - \gamma \otimes \gamma$, where $\gamma \in \Gamma$. Replacing γ by $\varepsilon + \gamma$ one sees that this ideal also contains $\gamma \otimes \varepsilon - \gamma \otimes \gamma$. The obvious embedding $\kappa_1 \hookrightarrow \kappa_1 \oplus \Gamma$ induces a graded ring homomorphism $\kappa \to \kappa[\Gamma]$. Then $\kappa[\Gamma]$ is a κ-structure satisfying the following universal property (which follows from the universal property of the tensor algebra):

For every κ-structure B and an abelian group homomorphism $\theta\colon \Gamma \to B_1$ there exists a unique morphism $\kappa[\Gamma] \to B$ extending θ.

Given a κ-structure A, we call $A[\Gamma] = A \otimes_\kappa \kappa[\Gamma]$ the **extension** of A by Γ. When $A = \kappa$ it coincides with our previous notation. This extends Serre's construction mentioned in [Mi, p. 323]. We identify $(A[\Gamma])_1 = A_1 \oplus \Gamma$. Let $\iota\colon A \to A[\Gamma]$ be the canonical morphism.

LEMMA 1.2: *Let $\varphi\colon A \to B$ be a morphism of κ-structures and let $\theta\colon \Gamma \to B_1$ be a homomorphism of abelian groups. There exists a unique morphism $A[\Gamma] \to B$ extending θ which commutes with φ and ι.*

Proof: The universal property of $\kappa[\Gamma]$ yields a unique morphism $\kappa[\Gamma] \to B$ extending θ. Now use the fact that the tensor product is a coproduct. □

COROLLARY 1.3: *Given a κ-structure A and abelian groups Γ_1, Γ_2 one has $(A[\Gamma_1])[\Gamma_2] \cong A[\Gamma_1 \oplus \Gamma_2]$.*

EXAMPLE 1.4: Let A be a κ-structure and let Γ be a cyclic group with generator γ. For every $i \geq 1$, we have $\gamma^i = \varepsilon_A^{i-1}\gamma$ in $A[\Gamma]$, by (iii) of Definition 1.1. It follows that $(A[\Gamma])_r = A_r \oplus (A_{r-1} \otimes_{\mathbb{Z}} \Gamma)$ for $r \geq 1$.

2. The functor $K_*^M(F)/S$

Let F be a field and let S be a subgroup of $F^\times$. For $r \geq 0$ let $(F^\times/S)^{\otimes r} = (F^\times/S) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} (F^\times/S)$ (r times). Let $\mathrm{St}_{F,r}(S)$ be the subgroup of $(F^\times/S)^{\otimes r}$ generated by all elements $a_1S \otimes \cdots \otimes a_rS$ such that $1 \in a_iS + a_jS$ for some $i \neq j$. Generalizing standard terminology, we call such elements **Steinberg elements**. Let

$$K_r^M(F)/S = (F^\times/S)^{\otimes r}/\mathrm{St}_{F,r}(S) \quad .$$

In particular, $K_0^M(F)/S = \mathbb{Z}$ and $K_1^M(F)/S = F^\times/S$. For $0 \leq t$ one has $\mathrm{St}_{F,r}(S) \otimes_{\mathbb{Z}} (F^\times/S)^{\otimes t} \subseteq \mathrm{St}_{F,r+t}(S)$ and $(F^\times/S)^{\otimes t} \otimes_{\mathbb{Z}} \mathrm{St}_{F,r}(S) \subseteq \mathrm{St}_{F,r+t}(S)$. Therefore

$$K_*^M(F)/S = \bigoplus_{r=0}^{\infty} K_r^M(F)/S \quad ,$$

is a graded ring respect to the multiplication induced by the tensor product. We call it the **Milnor K-ring of F modulo** S. Given $a_1, \ldots, a_r \in F^\times$ we denote the image of $a_1S \otimes \cdots \otimes a_rS$ in $K_r^M(F)/S$ by $\{a_1, \ldots, a_r\}_S$.

When $S = \{1\}$ we obtain the classical Milnor K-ring $K_*^M(F) = \bigoplus_{r=0}^{\infty} K_r^M(F)$ of F as in [Mi]. In this case we write as usual $\{a_1, \ldots, a_n\}$ for $\{a_1, \ldots, a_n\}_S$. In general, we have graded ring homomorphisms $\mathrm{Tens}(F^\times/S) \to K_*^M(F)/S$ and $K_*^M(F) \to K_*^M(F)/S$.

Next we define a graded ring homomorphism $\kappa \to K_*^M(F)/S$ by setting $\varepsilon \mapsto -S \in F^\times/S$. Since the identities $\{a, a\}_S = \{-1, a\}_S = \{a, -1\}_S$ of Definition 1.1(iii) are well-known to hold when $S = \{1\}$ [Mi, §1], they also hold in $K_*^M(F)/S$. Hence $K_*^M(F)/S$ is a κ-structure.

Proposition 2.1: *For positive integers m, r and for $S = (F^\times)^m$ we have $K_r^M(F)/S = K_r^M(F)/m$.*

Proof: There is an obvious graded ring homomorphism φ: $K_r^M(F)/m \to K_r^M(F)/S$ which commutes with the canonical projections from $(F^\times)^{\otimes r}$. Conversely, suppose $a_1, \ldots, a_r \in F^\times$ and $a_1S \otimes \cdots \otimes a_rS \in \mathrm{St}_{F,r}(S)$, i.e., $1 = a_i\alpha^m + a_j\beta^m$ for some $i < j$ and $\alpha, \beta \in F^\times$. Then

$$\{a_1, \ldots, a_r\} \in \{a_1, \ldots, a_i\alpha^m, \ldots, a_j\beta^m, \ldots, a_r\} + mK_r^M(F) = mK_r^M(F) \quad .$$

We obtain a projection ψ: $K_r^M(F)/S \to K_r^M(F)/m$ which also commutes with the projections from $(F^\times)^{\otimes r}$. Thus φ and ψ are converse maps, whence isomorphisms. □

We consider the class of all pairs (F,S) where F is a field and $S \le F^\times$ as category, in which morphisms $(F,S) \to (F_1,S_1)$ are pairs of compatible embeddings $F \hookrightarrow F_1$, $S \hookrightarrow S_1$. For such a pair and for $r \ge 0$ we have a group homomorphism $(F^\times/S)^{\otimes r} \to (F_1^\times/S_1)^{\otimes r}$ mapping $\mathrm{St}_{F,r}(S)$ to $\mathrm{St}_{F_1,r}(S_1)$. It therefore induces a κ-structure morphism Res: $K_*^M(F)/S \to K_*^M(F_1)/S_1$, which we call the **restriction** morphism. The map $(F,S) \mapsto K_*^M(F)/S$ is thus a covariant functor from the category of pairs (F,S) to the category of κ-structures.

A topology on a field F is called a **ring topology** if the addition and multiplication maps $F \times F \to F$ are continuous. We will need:

PROPOSITION 2.2: *Let $\mathcal{T}$ be a ring topology on a field F_1 and let F be a subfield of F_1 which is $\mathcal{T}$-dense in F_1. Let S be a subgroup of $F^\times$ and let S_1 be a $\mathcal{T}$-open subgroup of $F_1^\times$ containing S. Then* Res: $K_*^M(F)/S \to K_*^M(F_1)/S_1$ *is an epimorphism. When $S = F \cap S_1$, it is an isomorphism.*

Proof: For every $a \in F_1^\times$ we have $F \cap aS_1 \ne \emptyset$ by the density assumption. Hence the natural homomorphism $F^\times/S \to F_1^\times/S_1$ is surjective. Consequently, so is Res: $K_*^M(F)/S \to K_*^M(F_1)/S_1$.

Suppose that $S = F \cap S_1$. For each r the induced map $(F^\times/S)^{\otimes r} \to (F_1^\times/S_1)^{\otimes r}$ is an isomorphism. Therefore the injectivity of Res would follow by a snake lemma argument once we show that the induced map $\mathrm{St}_{F,r}(S) \to \mathrm{St}_{F_1,r}(S_1)$ is surjective. To this end we take a generator $a_1S_1 \otimes \cdots \otimes a_rS_1 \in \mathrm{St}_{F_1,r}(S_1)$, where $a_1,\dots,a_r \in F_1^\times$ and $1 \in a_iS_1 + a_jS_1$ for some distinct i,j. By continuity, there exist nonempty $\mathcal{T}$-open subsets V,W of S_1 such that $a_iV + a_jW \subseteq S_1$. Using the density assumption we find $x_1,\dots,x_r \in F$ with $x_i \in a_iV$, $x_j \in a_jW$, and $x_l \in a_lS_1$ for all $l \ne i,j$. Then $x_i + x_j \in S_1 \cap F = S$, so $x_1S \otimes \cdots \otimes x_rS \in \mathrm{St}_{F,r}(S)$. Furthermore, $x_1S \otimes \cdots \otimes x_rS$ maps to $a_1S_1 \otimes \cdots \otimes a_rS_1$ under the homomorphism above, as required. $\square$

3. Orderings

Let again F be a field, and let S be a subgroup of $F^\times$. Following standard terminology (see, e.g., [NSW, p. 191]), we call the map $\mathrm{Bock}_{F,S}\colon F^\times/S \to K_2^M(F)/S$, $\{x\}_S \mapsto \{x\}_S^2 = \{x,-1\}_S$, the **Bockstein operator** of the subgroup S of F. It is clearly a group homomorphism.

Lemma 3.1: *If $\mathrm{Bock}_{F,S}$ is injective then S is additively closed.*

Proof: It suffices to show that $1+S\subseteq S$. To this end take $s\in S$. Then

$$\mathrm{Bock}_{F,S}(\{1+s\}_S) = \{1+s,-1\}_S = \{1+s,-s\}_S = 0 \quad .$$

By the injectivity, $\{1+s\}_S = 0$, so $1+s\in S$. $\square$

By an **ordering** on F we mean an additively closed subgroup P of $F^\times$ such that $F^\times = P \cupdot -P$. Recall that a ring is reduced if it has no nilpotent elements $\neq 0$. The following fact is a variant of [BaT, I, Th. (3.1)].

Proposition 3.2: *The following conditions are equivalent:*

(a) $K_*^M(F)/S \cong \kappa$ *as κ-structures;*

(b) $F^\times = S \cupdot -S$ *and* $K_*^M(F)/S$ *is reduced;*

(c) $F^\times = S \cupdot -S$ *and* $\{-1,-1\}_S \neq 0$*;*

(d) S *is an ordering on* F.

Proof: (a)⇒(b)⇒(c): Immediate.
(c)⇒(d): By Lemma 3.1 and the assumptions, S is additively closed. The rest is clear.
(d)⇒(a): We first show that $\mathrm{St}_{F,r}(S)$ is trivial for all $r\geq 2$. Indeed, take $a_1,\dots,a_r\in F^\times$ with $1\in a_iS+a_jS$ for some distinct $1\leq i,j\leq r$. If a_i,a_j were both in $-S$ then we would get $-1\in S+S\subseteq S$, a contradiction. Hence at least one of a_i,a_j must be in S. It follows that $a_1S\otimes\cdots\otimes a_rS = 1$ in $(F^\times/S)^{\otimes r}$, as claimed.

Consequently, $K_*^M(F)/S = \mathrm{Tens}(F^\times/S) \cong \mathrm{Tens}(\{\pm 1\}) = \kappa$ as graded rings. Further, this is a κ-structure isomorphism. $\square$

A **preordering** on F is an additively closed subgroup S of $F^\times$ containing $(F^\times)^2$ but not -1. Preorderings can be characterized K-theoretically as follows.

PROPOSITION 3.3: *Suppose that $(F^\times)^2 \le S < F^\times$. The following conditions are equivalent:*

(a) *S is a preordering on F;*

(b) $\mathrm{Bock}_{F,S}$ *is injective.*

Proof: (a)⇒(b): Let $x \in F^\times$ satisfy $\{x\}_S^2 = 0$ and let P be an ordering on F containing S. Then $\{x\}_P^2 = 0$, and since $K_*^M(F)/P \cong \kappa$ is reduced (Proposition 3.2), $\{x\}_P = 0$, i.e., $x \in P$. Being a preordering, S is the intersection of all the orderings P containing it [Lm, Th. 1.6]. Consequently, $x \in S$, as desired.

(b)⇒(a): In light of Lemma 3.1, S is additively closed. By assumption, there exists $x \in F^\times \setminus S$. By the injectivity, $\{x, -1\}_S \neq 0$. Hence $-1 \notin S$, so S is a preordering. □

4. The cyclic case

Using the K-theoretic analysis of orderings obtained in the previous section, we can now completely describe $K_*^M(F)/S$ when $F^\times/S$ is a finite cyclic group.

THEOREM 4.1: *Let F be a field and let S be a subgroup of $F^\times$ such that $F^\times/S$ is finite and cyclic. Then one of following holds:*

(a) *$K_r^M(F)/S = 0$ for all $r \ge 2$;*

(b) *$(F^\times : S) = 2m$ with m odd, and there exists a unique ordering P on F containing S, and furthermore,* Res: *$K_*^M(F)/S \to K_*^M(F)/P$ ($\cong \kappa$) is an isomorphism in all degrees $r \ge 2$.*

Proof: Let $p_1^{d_1} \cdots p_n^{d_n}$ be the primary decomposition of $(F^\times : S)$. For each $1 \le i \le n$ choose $a_i \in F^\times$ such that the coset $\{a_i\}_S$ generates the p_i-primary part of $F^\times/S$. Let $a = a_1 \cdots a_n$. Then the coset $\{a\}_S$ generates $F^\times/S$, and one has $\{a, a\}_S = \{a, -1\}_S = \sum_{i=1}^n \{a_i, -1\}_S$.

Assume that (a) does not hold, i.e., $K_r^M(F)/S \neq 0$ for some $r \ge 2$. Since the canonical map $(F^\times/S)^r \to K_r^M(F)/S$ is multi-linear, $\{a, \ldots, a\}_S$ generates $K_r^M(F)/S$. Hence $\{a, \ldots, a\}_S \neq 0$, and therefore $\{a, a\}_S \neq 0$. It follows that $\{a_i, -1\}_S \neq 0$ for some $1 \le i \le n$. We obtain that the orders of $\{a_i, -1\}_S$ and of $\{-1\}_S$ are precisely 2. Furthermore, $p_i^{d_i}\{a_i, -1\}_S = 0$, so we must have $p_i = 2$. Therefore $2^{d_i - 1}\{a_i\}_S = \{-1\}_S$, and we get

$$2^{d_i-1}\{a_i,-1\}_S=2^{d_i-1}\{a_i,a_i\}_S=\{a_i,-1\}_S\neq 0 \quad .$$

This implies that $d_i=1$. Consequently, $(F^\times:S)=2m$, with m odd.

Let P be the unique subgroup of $F^\times$ of index 2 which contains S. Then P/S is cyclic of order m, and is generated by $\{a^2\}_S$. Since $\{-1\}_S$ has order 2 in $F^\times/S$, it is not in P/S. Therefore $F^\times=P\mathbin{\dot\cup}-P$.

Next we claim that $1+P\subseteq P$. Indeed, suppose that $x\in P$. In particular, $x\neq -1$. Take s,t with $-x\in a^sS$ and $1+x\in a^tS$. Then

$$0=\{-x,1+x\}_S=\{a^s,a^t\}_S=st\{a,a\}_S \quad .$$

Now $-x\notin P$, so s is odd. But $\{a,a\}_S=\{a,-1\}_S$ has order 2. It follows that t must be even, i.e., $1+x\in P$. Conclude that P is additively closed, whence an ordering.

Finally, for every r the functorial map $K_r^M(F)/S\to K_r^M(F)/P$ is clearly surjective. When $2\leq r$ the group $K_r^M(F)/S$ is generated by $\{a,a,\dots,a\}_S=\{a,-1,\dots,-1\}_S$, whence has order at most 2. By Proposition 3.2, $K_r^M(F)/P$ has order 2. Consequently, the above map is an isomorphism, and (b) holds.

For the uniqueness part of (b), assume that $S\leq P'<F^\times$ is another ordering on F. Then $4|(F^\times:P\cap P')|(F^\times:S)=2m$, contrary to the fact that m is odd. □

Corollary 4.2: *Let S be a subgroup of $F^\times$ with $F^\times/S$ cyclic of prime power order. Then either $K_r^M(F)/S=0$ for all $r\geq 2$, or S is an ordering (whence $K_*^M(F)/S\cong\kappa$).*

As mentioned in the introduction, Theorem 4.1 generalizes the well-known fact that $K_2^M(F)=0$ for a finite field F ([Mi, Example 1.5], [FV, Ch. IX, Prop. 1.3]). Indeed, $F^\times$ is cyclic [Ln, Ch. VII, §5, Th. 11] and since $\operatorname{char}F>0$, there are no orderings on F.

5. S-compatible valuations

Recall that a (Krull) **valuation** on a field F is a group homomorphism v from $F^\times$ into an ordered abelian group $(\Gamma,\leq)$ such that $v(x+y)\geq\min\{v(x),v(y)\}$ for all $x,y\in F$ with $x\neq -y$. One defines $v(0)$ to be a formal value $+\infty$ which is strictly larger than every value in Γ. Let O_v be the valuation ring of v, and $\mathfrak{m}_v$ its maximal ideal. Thus $x\in F$ lies in O_v (resp., $\mathfrak{m}_v$) if and only if $v(x)\geq 0$ (resp., $v(x)>0$). Let $O_v^\times$ be the unit group of O_v, let $G_v=1+\mathfrak{m}_v$ be the group of principal units of v, let $\bar F_v=O_v/\mathfrak{m}_v$ be the residue field of v, and $\pi_v\colon O_v\to\bar F_v$, $a\mapsto\bar a$, the canonical projection.

Let S a subgroup of $F^\times$. Its push-down $\bar S_v=\pi_v(S\cap O_v^\times)$ under v is a subgroup of $\bar F_v^\times$. The maps v and π_v induce short exact sequences of abelian groups:

$$1\ \to\ S\cap O_v^\times\ \to\ S\xrightarrow{v}v(S)\ \to\ 0\quad,\quad 1\ \to\ S\cap G_v\ \to\ S\cap O_v^\times\xrightarrow{\pi_v}\bar S_v\ \to\ 1\quad. \tag{5.1}$$

In particular, this holds for $S=F^\times$. The snake lemma therefore gives rise to canonical exact sequences

$$1\ \to\ O_v^\times/(S\cap O_v^\times)\ \to\ F^\times/S\xrightarrow{v^*}v(F^\times)/v(S)\ \to\ 0 \tag{5.2}$$

$$1\ \to\ G_v/(S\cap G_v)\ \to\ O_v^\times/(S\cap O_v^\times)\xrightarrow{\pi_v^*}\bar F_v^\times/\bar S_v\ \to\ 1\quad. \tag{5.3}$$

Following [AEJ], we say that the valuation v is **S-compatible** if $G_v\leq S$ (when $S=(F^\times)^p$ for p prime and char $\bar F_v\neq p$, this is a weak form of Hensel's lemma [Wd, Prop. 1.2]). Then the sequences (5.2)–(5.3) combine to a single canonical short exact sequence

$$1\ \to\ \bar F_v^\times/\bar S_v\xrightarrow{\eta}F^\times/S\xrightarrow{v^*}v(F^\times)/v(S)\ \to\ 0\quad, \tag{5.4}$$

where for $a\in O_v^\times$ with residue $\bar a$ we set $\eta(\{\bar a\}_{\bar S_v})=\{a\}_S$.

We will be interested in situations where (5.2) splits. For example, this is so in the following cases:

(1) $v(F^\times)\cong\mathbb{Z}$ and $S=\{1\}$. Then a section of v^* corresponds to a choice of a uniformizer for v.

(2) $(F^\times)^p\leq S$ for some prime number p. In fact, then $F^\times/S$ and $v(F^\times)/v(S)$ are free $\mathbb{Z}/p$-modules.

(3) $(F^\times)^q \le S \le (F^\times)^q O_v^\times$, where $q = p^s$ is a prime power. Indeed, the group $v(F^\times)$ is torsion-free, whence a flat $\mathbb{Z}$-module. Therefore $v(F^\times)/v(S) = v(F^\times)/q$ is a flat $\mathbb{Z}/q$-module. Since $\mathbb{Z}/q$ is a nilpotent local ring, it is a consequence of the Nakayama lemma [Ma, 3.G] that $v(F^\times)/q$ is a free $\mathbb{Z}/q$-module.

We now obtain a connection between valuations and extensions of κ-structures, in the sense of §1.

THEOREM 5.1: *Let F be a field and let S be a subgroup of $F^\times$. Every section of (5.2) induces canonically an epimorphism of κ-structures*

$$K_*^M(F)/S \longrightarrow (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \quad .$$

Moreover, this morphism is injective if and only if v is S-compatible.

Proof: Let θ: $v(F^\times)/v(S) \to F^\times/S$ be a section of v^*. Take $S \le \Delta \le F^\times$ with $\Delta/S = \mathrm{Im}(\theta)$. Then $F^\times/S = (SO_v^\times/S) \times (\Delta/S)$. Thus every $x \in F^\times$ can be written as $x = ab$ with $a \in O_v^\times$ and $b \in \Delta$. We set $\bar{a} = \pi_v(a)$ and write $[v(b)]_S$ for the coset of $v(b)$ in $v(F^\times)/v(S)$. We obtain a well-defined group epimorphism

$$F^\times/S \to (\bar{F}_v^\times/\bar{S}_v) \oplus (v(F^\times)/v(S)) \quad , \quad \{x\}_S \mapsto \{\bar{a}\}_{\bar{S}_v} + [v(b)]_S \quad . \tag{5.5}$$

This abelian group epimorphism uniquely extends to a graded ring epimorphism

$$\lambda\colon \mathrm{Tens}(F^\times/S) \to (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \quad .$$

We claim that λ is trivial on $\mathrm{St}_{F,r}(S)$ for all r. It suffices to show that $\lambda(\{x\}_S \otimes \{y\}_S) = 0$ when $x, y \in F^\times$ and $1 \in xS + yS$. We may assume that $1 = x + y$. Write $x = ab$ and $y = cd$, with $a, c \in O_v^\times$ and $b, d \in \Delta$. Then

$$\begin{aligned}\lambda(xS \otimes yS) &= \big(\{\bar{a}\}_{\bar{S}_v} + [v(b)]_S\big) \cdot \big(\{\bar{c}\}_{\bar{S}_v} + [v(d)]_S\big) \\ &= \{\bar{a}, \bar{c}\}_{\bar{S}_v} + \big(\{\bar{a}\}_{\bar{S}_v} \cdot [v(d)]_S - \{\bar{c}\}_{\bar{S}_v} \cdot [v(b)]_S\big) + [v(b)]_S \cdot [v(d)]_S \quad .\end{aligned}$$

To show that this expression vanishes we distinguish between four cases:

CASE I: $x \in G_v$. Here we can take $a = x$ and $b = 1$. Then $\{\bar{a}\}_{\bar{S}_v} = 0$ and $[v(b)]_S = 0$, so the assertion is clear.

CASE II: $x \in \mathfrak{m}_v$. Then $y \in G_v$, so we can take $c = y$ and $d = 1$. Hence $\{\bar{c}\}_{\bar{S}_v} = 0$ and $[v(d)]_S = 0$, and we are done again.

CASE III: $x \in O_v^\times \setminus G_v$. Then $y = 1 - x \in O_v^\times$, so we can take $a = x$, $b = 1$, $c = y$, and $d = 1$. Hence $\lambda(xS \otimes yS) = \{\bar{x}, \overline{1-x}\}_{\bar{S}_v} = 0$ once again.

CASE IV: $x^{-1} \in \mathfrak{m}_v$. For any a, b as above, $y = a(x^{-1} - 1) \cdot b$, with $a(x^{-1} - 1) \in O_v^\times$. Thus we may take $c = a(x^{-1} - 1)$ and $d = b$. Then $\{\bar{c}\}_{\bar{S}_v} = \{-\bar{a}\}_{\bar{S}_v}$. Further, $\{\bar{a}\}_{\bar{S}_v} - \{-\bar{a}\}_{\bar{S}_v} = \{-\bar{1}\}_{\bar{S}_v}$ and $\{\bar{a}, -\bar{a}\}_{\bar{S}_v} = 0$. It follows that

$$\lambda(xS \otimes yS) = \{-\bar{1}\}_{\bar{S}_v} \cdot [v(b)]_S + [v(b)]_S \cdot [v(b)]_S = 0 \quad ,$$

using property (iii) of Definition 1.1.

This proves the claim. Consequently, λ induces an epimorphism of κ-structures

$$\bar{\lambda} \colon K_*^M(F)/S \longrightarrow (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \quad ,$$

as desired.

For the second assertion of the theorem, suppose that v is S-compatible. Then (5.4) is exact. The abelian group monomorphism η of (5.4) induces a morphism $\mathrm{Tens}(\bar{F}_v^\times/\bar{S}_v) \to \mathrm{Tens}(F^\times/S)$ of graded rings. Since $G_v \leq S$, it maps $\mathrm{St}_{\bar{F}_v,r}(\bar{S}_v)$ into $\mathrm{St}_{F,r}(S)$ for every $r \geq 1$. Hence it induces a κ-structure morphism $K_*^M(\bar{F}_v)/\bar{S}_v \to K_*^M(F)/S$. By the universal property of extensions (Lemma 1.2), there exists a unique κ-structure morphism $\bar{\nu}$ which extends the section θ and for which the following diagram commutes (where ι is the canonical morphism as in §1):

$$\begin{array}{ccc} K_*^M(\bar{F}_v)/\bar{S}_v & \xrightarrow{\iota} & (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \\ & \searrow & \downarrow{\scriptstyle \bar{\nu}} \\ & & K_*^M(F)/S \quad . \end{array}$$

In degree 1, $\bar{\nu}$ coincides with the isomorphism $\eta \oplus \theta$. Hence it is surjective in all degrees. By construction, $\bar{\lambda}$ is given in degree 1 by the map (5.5). It follows that $\bar{\lambda} \circ \bar{\nu} = \mathrm{id}$ in degree 1, and therefore in all degrees. This proves that $\bar{\nu}$ is injective. Therefore both $\bar{\nu}$ and $\bar{\lambda}$ are isomorphisms.

Conversely, suppose that $\bar{\lambda}$ is an isomorphism. Its definition in degree 1 shows that it maps G_vS/S trivially. Hence $G_v \leq S$, as required. $\square$

Remark 5.2: When v is a discrete valuation and $S=\{1\}$, the first part of Theorem 5.1 is due to Bass and Tate [BaT, I, Prop. 4.3]. They also prove its second part when (F,v) is a complete discretely valued field with positive residue characteristic prime to m and when $S=(F^\times)^m$ [BaT, I, Cor. 4.7]. Note that in the latter case v is S-compatible by Hensel's lemma. Wadsworth [Wd, §2] proves Theorem 5.1 for any valued field (F,v) when $S=(F^\times)^q G_v$ and q is a prime power.

Remark 5.3: The epimorphism of Theorem 5.1 is functorial in the following sense: suppose that (F_1,v_1) is a valued field extension of (F,v), that $S\leq F^\times$, $S_1\leq F_1^\times$, $S\leq S_1$, and that there exist homomorphic sections θ,θ_1 of the projections $v^*\colon F^\times/S\to v(F^\times)/v(S)$, $v_1^*\colon F_1^\times/S_1\to v_1(F_1^\times)/v_1(S_1)$ induced by v,v_1, respectively. Moreover, suppose that the following square commutes:

$$\begin{array}{ccc} v(F^\times)/v(S) & \xrightarrow{\theta} & F^\times/S \\ \downarrow & & \downarrow \\ v_1(F_1^\times)/v_1(S_1) & \xrightarrow{\theta_1} & F_1^\times/S_1\ . \end{array}$$

Then the epimorphisms given in Theorem 5.2 and the restriction morphisms induce a square:

$$\begin{array}{ccc} K_*^M(F)/S & \longrightarrow & (K_*^M(\bar F_v)/\bar S_v)[v(F^\times)/v(S)] \\ \downarrow & & \downarrow \\ K_*^M(F_1)/S_1 & \longrightarrow & (K_*^M((\overline{F_1})_{v_1})/(\overline{S_1})_{v_1})[v_1(F_1^\times)/v_1(S_1)]\ . \end{array}$$

This square commutes in degree 1, hence in all degrees.

Remark 5.4: There are partially converse results to Theorem 5.1. Namely, if $S=(F^\times)^p$ for a prime number p and if $K_*^M(F)/S$ is an extension of some κ-structure by $(\mathbb{Z}/p)^d$ then (apart from some well-understood exceptional cases) F is equipped with an S-compatible valuation v with $v(F^\times)/pv(F^\times)\cong(\mathbb{Z}/p)^d$. Indeed, this follows from results of Arason, Elman, Hwang, Jacob, and Ware ([J], [Wr], [AEJ], [HJ]); see [E2] for a K-theoretic formulation of this line of results.

6. A vanishing theorem

Recall that a valuation v on F induces a ring topology $\mathcal{T}_v$ on F, with basis consisting of all sets $a+bO_v$, where $a,b\in F$ and $b\neq 0$. For $0<\gamma\in v(F^\times)$ the set

$$W_\gamma=\Big\{x\in F^\times \mid v(1-x)\geq\gamma\Big\}$$

is a $\mathcal{T}_v$-open subgroup of $G_v=1+\mathfrak{m}_v$.

LEMMA 6.1: *Let v be a valuation on the field F. Let S be a subgroup of $F^\times$ such that $G_v/(S\cap G_v)$ is a finitely generated group. Then there exists $0<\gamma\in v(F^\times)$ such that:*

(i) $SG_v=SW_\gamma$;

(ii) *if* $\operatorname{char}\bar{F}_v=p$ *then* $1+pO_v\leq W_\gamma$.

Proof: We choose $a_1,\ldots,a_n\in\mathfrak{m}_v$ such that the cosets of $1-a_i$, $i=1,\ldots,n$, generate $G_v/(S\cap G_v)$. Hence $(1-a_i)S$, $i=1,\ldots,n$, generate SG_v/S. Take any $0<\gamma\leq\min\{v(a_1),\ldots,v(a_n)\}$. Then $1-a_i\in W_\gamma$, $i=1,\ldots,n$. Combined with $W_\gamma\leq G_v$, this shows that $SW_\gamma/S=SG_v/S$. When $\operatorname{char}\bar{F}_v=p$ we take

$$\gamma=\min\{v(p),v(a_1),\ldots,v(a_n)\} \quad . \qquad \square$$

One says that the valuation v on F has **rank** 1 if $v(F^\times)$ embeds in $\mathbb{R}$ as an ordered abelian group.

THEOREM 6.2: *Let v be a valuation of rank 1 on the field F. Let S be a $\mathcal{T}_v$-open subgroup of $F^\times$ such that $F^\times/S$ is finitely generated and $F^\times=SG_v$. Then $K_r^M(F)/S=0$ for all $r\geq 2$.*

Proof: It suffices to show that $aS\otimes bS\in\mathrm{St}_{F,2}(S)$ for $a,b\in G_v$. Suppose that this is not the case. In particular, $a,b\notin S$. Lemma 6.1 yields $0<\gamma\in v(F^\times)$ such that $F^\times=SG_v=SW_\gamma$.

We define inductively a sequence $c_1,c_2,\ldots\in G_v$ such that for each i,

$$1-c_i\in(1-b)(1-W_\gamma)^{i-1} \quad , \quad aS\otimes bc_i^{-1}S\in\mathrm{St}_{F,2}(S) \quad .$$

We can take $c_1=b$. Next suppose that c_i has already been constructed. Since $aS\otimes bS\notin\mathrm{St}_{F,2}(S)$ we have $c_i\neq 1$. We choose $y_i\in S$ such that $a/(1-c_i^{-1})\in y_iW_\gamma$. As $a\notin S$ and

$y_i \in S$, we may define $c_{i+1} = c_i(1-y_i^{-1}a)$. Since $c_i \in G_v$ we have $y_i^{-1}a \in (1-c_i^{-1})W_\gamma \subseteq \mathfrak{m}_v$. Hence $c_{i+1} \in G_v$. Now

$$\frac{1-c_{i+1}}{1-c_i} = 1 - \frac{y_i^{-1}a}{1-c_i^{-1}} \in 1 - W_\gamma \quad ,$$

so by the induction hypothesis, $1 - c_{i+1} \in (1-b)(1-W_\gamma)^i$. Furthermore,

$$\begin{aligned} aS \otimes bc_{i+1}^{-1}S &= aS \otimes bc_i^{-1}S - aS \otimes (1-y_i^{-1}a)S \\ &= aS \otimes bc_i^{-1}S - y_i^{-1}aS \otimes (1-y_i^{-1}a)S \in \mathrm{St}_{F,2}(S) \quad . \end{aligned}$$

This completes the inductive construction.

Since v has rank 1, the sets $(1-W_\gamma)^s$, $s = 1, 2, 3, \ldots$, form a local basis for $\mathcal{T}_v$ at 0. As $b \neq 1$, the set $(1-b)^{-1}(1-S)$ is a $\mathcal{T}_v$-open neighborhood of 0. Hence there exists a positive integer t such that $(1-W_\gamma)^t \subseteq (1-b)^{-1}(1-S)$. Then $1-c_{t+1} \in (1-b)(1-W_\gamma)^t \subseteq 1-S$, so $c_{t+1} \in S$. We conclude that $aS \otimes bS = aS \otimes bc_{t+1}^{-1}S \in \mathrm{St}_{F,2}(S)$, a contradiction. □

7. Wild valuations of rank 1

In this section we study $K_*^M(F)$ when F is a field of characteristic 0 equipped with a valuation v with $\mathrm{char}\,\bar{F}_v = p > 0$. First we assume that v is a discrete valuation. Thus $\mathfrak{m}_v = aO_v$ for some $a \in \mathfrak{m}_v$. For $i \geq 1$ the map $1 + \mathfrak{m}_v^i \to \bar{F}_v$, $1 + a^i b \mapsto \pi_v(b)$, is a group homomorphism with kernel $1 + \mathfrak{m}_v^{i+1}$.

Lemma 7.1: *Let $(E,u)/(F,v)$ be an extension of discrete valued fields with the same value group and residue field. Then:*

(a) *$(1+\mathfrak{m}_u^i)/(1+\mathfrak{m}_v^i) \cong E^\times/F^\times$ canonically for all $i \geq 1$;*

(b) *for every $\mathcal{T}_u$-open subgroup S of $E^\times$ one has $E^\times = F^\times S$.*

Proof: (a) For $i = 1$ this follows from the exact sequences (5.2)–(5.3) (for the subgroup $F^\times$ of $E^\times$). For $1 \leq i$ the preceding remark gives a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & 1+\mathfrak{m}_v^{i+1} & \to & 1+\mathfrak{m}_v^i & \to & \bar{F}_v & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & 1+\mathfrak{m}_u^{i+1} & \to & 1+\mathfrak{m}_u^i & \to & \bar{E}_u & \to & 1 \quad . \end{array}$$

The snake lemma gives rise to a canonical isomorphism

$$(1+\mathfrak{m}_u^{i+1})/(1+\mathfrak{m}_v^{i+1}) \xrightarrow{\sim} (1+\mathfrak{m}_u^{i})/(1+\mathfrak{m}_v^{i}) \quad ,$$

so we are done by induction.

(b) Since u is discrete, the subgroups $1+\mathfrak{m}_u^i$, $i=1,2,3,\dots$, form a local basis for $\mathcal{T}_u$ at 1. Hence there exists i with $1+\mathfrak{m}_u^i \le S$. By (a), $E^\times = F^\times(1+\mathfrak{m}_u^i)$, so $E^\times = F^\times S$. □

Now let p be a prime number and let $q=p^d$ be a p-power, $d \ge 1$.

PROPOSITION 7.2: *Let v be a discrete valuation on a field F such that* $\operatorname{char} F = 0$ *and* $\operatorname{char} \bar{F}_v = p$. *Let (E,u) be the completion of (F,v) and let $S=(F^\times)^q(1+q^2\mathfrak{m}_v)$. Then* Res: $K_*^M(F)/S \to K_*^M(E)/q$ *is an isomorphism.*

Proof: By the Hensel–Rychlik lemma [FV, Ch. II, (1.3), Cor. 2], $1+q^2\mathfrak{m}_u \le (E^\times)^q$. In particular, $(E^\times)^q$ is $\mathcal{T}_u$-open in E. By Lemma 7.1(b), $E^\times = F^\times(1+q\mathfrak{m}_u)$. Hence $(E^\times)^q = (F^\times)^q(1+q^2\mathfrak{m}_u)$. It follows that $F \cap (E^\times)^q = (F^\times)^q(1+q^2\mathfrak{m}_v) = S$.

Since F is $\mathcal{T}_u$-dense in E, the assertion now follows from Proposition 2.2. □

Note that here the field E is a complete discrete valued field of characteristic 0 and finite residue field of characteristic p. Therefore it is a finite extension of $\mathbb{Q}_p$. For a detailed analysis of the Milnor K-ring of such fields we refer to [FV, Ch. IX].

The following theorem extends arguments of Pop, which are implicit in the proof of [P, Kor. 2.7]. In Theorem 7.4 below we use it in conjunction with Theorem 6.2 to compute the functor $K_*^M(F)/S$ in another mixed characteristic situation.

THEOREM 7.3: *Let v be a valuation of rank* 1 *on a field F such that* $\operatorname{char} F = 0$ *and* $\operatorname{char} \bar{F}_v = p$. *Suppose that $F^\times/(F^\times)^p(1+p\mathfrak{m}_v)$ is finite. Then either:*

(a) *$v(F^\times)$ is discrete and $\bar{F}_v$ is finite; or*

(b) *$v(F^\times)$ is p-divisible and $\bar{F}_v$ is perfect.*

Proof: Let $S=(F^\times)^p(1+p\mathfrak{m}_v)$. We break the argument into five steps.

PART I: *$\bar{F}_v$ is perfect.* Indeed, $\bar{S}_v = (\bar{F}_v^\times)^p$. By the exact sequences (5.2)–(5.3), $\bar{F}_v^\times/(\bar{F}_v^\times)^p$ is finite. Since $\operatorname{char} \bar{F}_v = p$, this quotient must be trivial [E3, Cor. 1.6], as desired.

Part II: $S \cap G_v = G_v^p(1+p\mathfrak{m}_v)$. Consider the commutative diagram of exponentiations by p:

$$\begin{array}{ccccccccc} 1 & \rightarrow & G_v & \rightarrow & O_v^\times & \rightarrow & \bar{F}_v^\times & \rightarrow & 1 \\ & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \\ 1 & \rightarrow & G_v & \rightarrow & O_v^\times & \rightarrow & \bar{F}_v^\times & \rightarrow & 1 \end{array} .$$

Since $\operatorname{char} \bar{F}_v = p$, the right vertical map is injective. By the snake lemma, $(O_v^\times)^p \cap G_v = G_v^p$. Hence also $(F^\times)^p \cap G_v = G_v^p$. Since $1 + p\mathfrak{m}_v \leq G_v$ we obtain

$$S \cap G_v = ((F^\times)^p \cap G_v)(1 + p\mathfrak{m}_v) = G_v^p(1 + p\mathfrak{m}_v) \quad .$$

Part III: $S \cap G_v \subseteq (1-S)(1+p\mathfrak{m}_v)$. Recall that $p|\binom{p}{i}$ for $i = 1, \ldots, p-1$. Hence for every $a \in \mathfrak{m}_v$ we have

$$(1-a)^p \in 1 - a^p + p\mathfrak{m}_v = (1-a^p)(1+p\mathfrak{m}_v) \subseteq (1-S)(1+p\mathfrak{m}_v) \quad .$$

Thus $G_v^p \subseteq (1-S)(1+p\mathfrak{m}_v)$. Now use Part II.

Part IV: *$v(F^\times)$ is either discrete or p-divisible.* In view of the structure of the ordered group $\mathbb{R}$, it suffices to find $0 < \gamma \in v(F^\times)$ such that for every $b \in F$ with $0 < v(b) < \gamma$ one has $v(b) \in pv(F^\times)$. Since $F^\times/S$ is finite, the sequences (5.2)–(5.3) imply that so is $G_v/(S \cap G_v)$. Hence we may take γ as in Lemma 6.1. By property (i) of W_γ, and since $W_\gamma \leq G_v$ we have $1 - b \in G_v = (SW_\gamma) \cap G_v = (S \cap G_v)W_\gamma$. It therefore follows from part III and from property (ii) of W_γ that $1 - b \in (1-S)W_\gamma$. So choose $s \in S$ with $1 - b \in (1-s)W_\gamma$. As $W_\gamma \leq G_v$ we get $1 - s \in G_v$. Hence

$$v(b-s) = v\left(\frac{b-s}{1-s}\right) = v\left(1 - \frac{1-b}{1-s}\right) \geq \gamma \quad .$$

Since $v(b) < \gamma$, necessarily $v(b) = v(s) \in v(S) = pv(F^\times)$, as desired.

Part V: *When $v(F^\times)$ is discrete, $\bar{F}_v$ is finite.* As we have observed, in this case $G_v/(1+\mathfrak{m}_v^2) = (1+\mathfrak{m}_v)/(1+\mathfrak{m}_v^2) \cong \bar{F}_v$. Using again that $p\,|\binom{p}{i}$ for $1 \leq i \leq p-1$, we get $G_v^p(1+p\mathfrak{m}_v) \leq 1 + \mathfrak{m}_v^2$. In light of Part II, this gives rise to a group epimorphism $G_v/(S \cap G_v) \to \bar{F}_v$. We have already noted that $G_v/(S \cap G_v)$ is finite. Conclude that so is $\bar{F}_v$. □

THEOREM 7.4: *Let v be a valuation of rank 1 on a field F such that* $\mathrm{char}\, F = 0$ *and* $\mathrm{char}\, \bar{F}_v = p$. *Let $S = (F^\times)^q(1 + q^2\mathfrak{m}_v)$ and suppose that $(F^\times : S) < \infty$. Then one of the following holds:*

(a) *$v(F^\times)$ is discrete, $\bar{F}_v$ is finite, and $K_*^M(F)/S \cong K_*^M(E)/q$ for the completion E of F with respect to v;*

(b) *$v(F^\times)$ is p-divisible and $K_r^M(F)/S = 0$ for all $r \geq 2$.*

Proof: We have $v(S) = qv(F^\times)$ and $\bar{S}_v = (\bar{F}_v^\times)^q$. Since $(F^\times)^q(1+q^2\mathfrak{m}_v) \leq (F^\times)^p(1+p\mathfrak{m}_v)$, the finiteness assumption implies that $(F^\times : (F^\times)^p(1 + p\mathfrak{m}_v)) < \infty$. By Theorem 7.3, one of the following cases occurs:

CASE (I): *$v(F^\times)$ is discrete and $\bar{F}_v$ is finite.* Then we apply Proposition 7.2.

CASE (II): *$v(F^\times)$ is p-divisible and $\bar{F}_v$ is perfect.* Then $v(S) = v(F^\times)$ and $\bar{S}_v = \bar{F}_v^\times$. The exact sequences (5.2)–(5.3) therefore show that $F^\times = SG_v$. Since S is $\mathcal{T}_v$-open in F, Theorem 6.2 implies that $K_r^M(F)/S = 0$ for $r \geq 2$. □

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