Extending valuations to formal completions.

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1 Introduction

This paper is an extended version of the talk given by Miguel Olalla at the International Conference on Valuation Theory in El Escorial in July 2011. Its purpose is to provide an introduction to our joint paper [7] without grinding through all of its technical details. We refer the reader to [7] for details and proofs; only a few proofs are given in the present paper.

Notation. Throughout this paper, we will use the following notation:

(R, m, k) a local noetherian domain
$K = QF(R)$ its field of fractions
R_ν a valuation ring dominating R
$\nu_{|K} : K^* \twoheadrightarrow \Gamma$ the restriction of ν to K.
$\hat{R}$ the $m-$adic completion of R.

The ring $\hat{R}$ is local and its maximal ideal is $m\hat{R}$. Our overall goal is to study extensions $\hat{\nu}$ of ν to $\hat{R}$.

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The ring $\hat{R}$ is not in general an integral domain, so we can only hope to extend ν to a pseudo-valuation $\hat{\nu}$ of $\hat{R}$. Let P be the support of $\hat{\nu}$, that is, the prime ideal which is mapped by $\hat{\nu}$ to ∞.

That means precisely that we want to extend ν to a valuation $\hat{\nu}_-$ of the quotient $\frac{\hat{R}}{P}$.

Such extensions $\hat{\nu}_-$ exists for some minimal prime ideals P of $\hat{R}$.

We shall see that ν determines a unique minimal prime P of $\hat{R}$ when R is **excellent**.

The purpose of our work [7] is to give a systematic description of all such extensions $\hat{\nu}_-$, assuming that R is **excellent**.

Let

$$\Gamma \hookrightarrow \hat{\Gamma} \tag{1}$$

be an extension of ordered groups of the same rank.

Let $(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \cdots \subsetneqq \Delta_0 = \Gamma$ be the isolated subgroups of Γ
and $(0) = \hat{\Delta}_r \subsetneqq \hat{\Delta}_{r-1} \subsetneqq \cdots \subsetneqq \hat{\Delta}_0 = \hat{\Gamma}$ the isolated subgroups of $\hat{\Gamma}$,
so that the inclusion (1) induces inclusions

$$\Delta_\ell \hookrightarrow \hat{\Delta}_\ell \quad \text{and} \tag{2}$$

$$\frac{\Delta_\ell}{\Delta_{\ell+1}} \hookrightarrow \frac{\hat{\Delta}_\ell}{\hat{\Delta}_{\ell+1}}. \tag{3}$$

Let $G \hookrightarrow \hat{G}$ be an extension of graded algebras without zero divisors, such that G is graded by Γ_+ and $\hat{G}$ by $\hat{\Gamma}_+$.

Definition 1.1 *We say that the extension $G \hookrightarrow \hat{G}$ is **scalewise birational** if for any $x \in \hat{G}$ and $\ell \in \{1, \ldots, r\}$ such that $\operatorname{ord} x \in \hat{\Delta}_\ell$ there exists $y \in G$ such that $\operatorname{ord} y \in \Delta_\ell$ and $xy \in G$.*

Of course, scalewise birational implies birational and also that $\hat{\Gamma} = \Gamma$. The main conjecture stated in [7] is the following:

Conjecture 1.1 *There exists an ideal H of $\hat{R}$ with $H \cap R = (0)$ and a valuation $\hat{\nu}_-$, centered at $\frac{\hat{R}}{H}$ and having the following property:*
The graded algebra $gr_{\hat{\nu}_-} \frac{\hat{R}}{H}$ is a scalewise birational extension of $gr_\nu R$.

In §9 we prove some partial results towards Conjecture 1.1 in the special case when R is essentially of finite type over a field. More precisely, we reduce Conjecture 1.1 to two other conjectures. Unfortunately, the latter two conjectures remain open so far.

Acknowledgements

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2 Rank one valuations

Keep the above notation. Assume that rk $\nu = 1$. Then the value group Γ is archimedian.

Let $\Phi = \nu(R \setminus \{0\})$. For $\beta \in \Phi$ let

$$\begin{aligned} \mathcal{P}_\beta = & \quad \{x \in R / \nu(x) \geq \beta\} \\ \mathcal{P}^+_\beta = & \quad \{x \in R / \nu(x) > \beta\} \end{aligned}$$

We now define the main object of study of this section. Let

$$H := \bigcap_{\beta \in \Phi} (\mathcal{P}_\beta \hat{R}). \tag{4}$$

Remark 2.1 *Since the formal completion homomorphism* $R \to \hat{R}$ *is faithfully flat,*

$$\mathcal{P}_\beta \hat{R} \cap R = \mathcal{P}_\beta \quad \textit{for all } \beta \in \Phi. \tag{5}$$

Taking the intersection over all $\beta \in \Phi$*, we obtain* $H \cap R = (0)$*.*

In other words, we have a natural inclusion $R \hookrightarrow \frac{\hat{R}}{H}$*.*

Example 2.1 *Let* $R = k[u,v]_{(u,v)}$*. Then* $\hat{R} = k[[u,v]]$*. Consider an element*

$$w = u - \sum_{i=1}^{\infty} c_i v^i \in \hat{R}, \textit{ where } c_i \in k^*,$$

transcendental over $k(u,v)$*.*

Consider the injective map $\varphi : k[u,v]_{(u,v)} \to k[[t]]$ *which sends* v *to* t *and* u *to* $\sum\limits_{i=1}^{\infty} c_i t^i$*.*

Let ν *be the valuation induced from the* t*-adic valuation of* $k[[t]]$ *via* φ*.*

The value group of ν *is* $\mathbf{Z}$*. For each* $\beta \in \mathbf{N}$*,* $\mathcal{P}_\beta = \left(v^\beta, u - \sum\limits_{i=1}^{\beta-1} c_i v^i\right)$*.*

Thus $H = (w)$*.*

Theorem 2.1 *1. The ideal* H *is a prime ideal of* $\hat{R}$*.*

2. The valuation ν *extends uniquely to a valuation* $\hat{\nu}_-$*, centered at* $\frac{\hat{R}}{H}$*.*

Proof: Let $\bar{x} \in \frac{\hat{R}}{H} \setminus \{0\}$. Pick a representative x of $\bar{x}$ in $\hat{R}$, so that $\bar{x} = x \bmod H$. Since $x \notin H$, we have $x \notin \mathcal{P}_\alpha \hat{R}$ for some $\alpha \in \Phi$.

Lemma 2.1 (See [15], Appendix 5, lemma 3) *Let* ν *be a valuation of rank one centered in a local noetherian domain* (R, M, k)*. Let*

$$\Phi = \nu(R \setminus (0)) \subset \Gamma.$$

Then Φ *contains no infinite bounded sequences.*

Proof: An infinite ascending sequence $\alpha_1 < \alpha_2 < \dots$ in Φ, bounded above by an element $\beta \in \Phi$, would give rise to an infinite descending chain of ideals in $\frac{R}{\mathcal{P}_\beta}$. Thus it is sufficient to prove that $\frac{R}{\mathcal{P}_\beta}$ has finite length.

Let $\delta := \nu(M) \equiv \min(\Phi \setminus \{0\})$. Since Φ is archimedian, there exists $n \in \mathbf{N}$ such that $\beta \leq n\delta$. Then $M^n \subset \mathcal{P}_\beta$, so that there is a surjective map $\frac{R}{M^n} \twoheadrightarrow \frac{R}{\mathcal{P}_\beta}$. Thus $\frac{R}{\mathcal{P}_\beta}$ has finite length, as desired. $\square$

By Lemma 2.1, the set $\{\beta \in \Phi \mid \beta < \alpha\}$ is finite. Hence there exists a unique $\beta \in \Phi$ such that

$$x \in \mathcal{P}_\beta \hat{R} \setminus \mathcal{P}_\beta^+ \hat{R}. \tag{6}$$

Note that β depends only on $\bar{x}$, but not on the choice of the representative x. Define the function $\hat{\nu}_- : \frac{\hat{R}}{H} \setminus \{0\} \to \Phi$ by

$$\hat{\nu}_-(\bar{x}) = \beta. \tag{7}$$

By (5), if $x \in R \setminus \{0\}$ then

$$\hat{\nu}_-(x) = \nu(x). \tag{8}$$

It is obvious that

$$\hat{\nu}_-(x+y) \geq \min\{\hat{\nu}_-(x), \hat{\nu}_-(y)\} \tag{9}$$

$$\hat{\nu}_-(xy) \geq \hat{\nu}_-(x) + \hat{\nu}_-(y) \tag{10}$$

for all $x, y \in \frac{\hat{R}}{H}$. The point of the next lemma is to show that $\frac{\hat{R}}{H}$ is a domain and that $\hat{\nu}_-$ is, in fact, a valuation (i.e. that the inequality (10) is, in fact, an equality).

Lemma 2.2 *For any non-zero $\bar{x}, \bar{y} \in \frac{\hat{R}}{H}$, we have $\bar{x}\bar{y} \neq 0$ and $\hat{\nu}_-(\bar{x}\bar{y}) = \hat{\nu}_-(\bar{x}) + \hat{\nu}_-(\bar{y})$.*

Proof: Let $\alpha = \hat{\nu}_-(\bar{x})$, $\beta = \hat{\nu}_-(\bar{y})$. Let x and y be representatives in $\hat{R}$ of $\bar{x}$ and $\bar{y}$, respectively. We have $M\mathcal{P}_\alpha \subset \mathcal{P}_\alpha^+$, so that

$$\frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R k \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R \frac{\hat{R}}{M\hat{R}} \cong \frac{\mathcal{P}_\alpha\hat{R}}{(\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha)\hat{R}} \cong \frac{\mathcal{P}_\alpha\hat{R}}{\mathcal{P}_\alpha^+\hat{R}}, \tag{11}$$

and similarly for β. By (11) there exist $z \in \mathcal{P}_\alpha$, $w \in \mathcal{P}_\beta$, such that $z \equiv x \bmod \mathcal{P}_\alpha^+\hat{R}$ and $w \equiv y \bmod \mathcal{P}_\beta^+\hat{R}$. Then

$$xy \equiv zw \bmod \mathcal{P}_{\alpha+\beta}^+\hat{R}. \tag{12}$$

Since ν is a valuation, $\nu(zw) = \alpha + \beta$, so that $zw \in \mathcal{P}_{\alpha+\beta} \setminus \mathcal{P}_{\alpha+\beta}^+$. By (5) and (12), this proves that $xy \in \mathcal{P}_{\alpha+\beta}\hat{R} \setminus \mathcal{P}_{\alpha+\beta}^+\hat{R}$. Thus $xy \notin H$ (hence $\bar{x}\bar{y} \neq 0$ in $\frac{\hat{R}}{H}$) and $\hat{\nu}_-(\bar{x}\bar{y}) = \alpha + \beta$, as desired. □

By Lemma 2.2, H is a prime ideal of $\hat{R}$. By (9) and Lemma 2.2, $\hat{\nu}_-$ is a valuation, centered at $\frac{\hat{R}}{H}$. To complete the proof of Theorem 2.1, it remains to prove the uniqueness of $\hat{\nu}_-$. Let x, $\bar{x}$, the element $\alpha \in \Phi$ and

$$z \in \mathcal{P}_\alpha \setminus \mathcal{P}_\alpha^+ \tag{13}$$

be as in the proof of Lemma 2.2. Then there exist

$$\begin{array}{ll} u_1, \dots, u_n & \in \mathcal{P}_\alpha^+ \text{ and} \\ v_1, \dots, v_n & \in \hat{R} \end{array} \tag{14}$$

such that $x = z + \sum\limits_{i=1}^n u_i v_i$. Letting $\bar{v}_i := v_i \bmod H$, we obtain $\bar{x} = \bar{z} + \sum\limits_{i=1}^n \bar{u}_i\bar{v}_i$ in $\frac{\hat{R}}{H}$. Therefore, by (13)–(14), for any extension of ν to a valuation $\hat{\nu}'_-$, centered at $\frac{\hat{R}}{H}$, we have

$$\hat{\nu}'_-(\bar{x}) = \alpha = \hat{\nu}_-(\bar{x}), \tag{15}$$

as desired. This completes the proof of Theorem 2.1. □

Definition 2.1 *The ideal H is called the* **implicit prime ideal** *of $\hat{R}$, associated to ν. When dealing with more than one ring at a time, we will sometimes write $H(R, \nu)$ for H.*

Remark 2.2 *We have the following natural isomorphisms of graded algebras:*

$$\begin{array}{ll} gr_\nu R & \cong gr_{\hat{\nu}_-} \frac{\hat{R}}{H} \\ G_\nu & \cong G_{\hat{\nu}_-}. \end{array}$$

We will now study the behaviour of H under local blowings up of R with respect to ν and, more generally, under local homomorphisms.

Let $\pi : (R, m) \to (R', m')$ be a local homomorphism of local noetherian domains. Assume that ν extends to a rank one valuation $\nu' : K' \setminus \{0\} \to \Gamma'$, where $\Gamma \subset \Gamma'$.

The homomorphism π induces a local homomorphism $\hat{\pi} : \hat{R} \to \hat{R}'$ of formal completions. Let $\Phi' = \nu'(R' \setminus \{0\})$. For $\beta \in \Phi'$, let $\mathcal{P}'_\beta$ denote the ν'-ideal of $R_{\nu'}$ of value β, as above. Let $H' = H(R', \nu')$.

Lemma 2.3 *Let $\beta \in \Phi$. Then*

$$\left(\mathcal{P}'_\beta\hat{R}'\right) \cap \hat{R} = \mathcal{P}_\beta\hat{R}. \tag{16}$$

Corollary 2.1 *We have*

$$H' \cap \hat{R} = H.$$

In other words, the implicit ideals behave well under blowings up.

Corollary 2.2 *We have*

$$\text{ht } H' \geq \text{ht } H. \tag{17}$$

In particular,

$$\dim \frac{\hat{R}'}{H'} \leq \dim \frac{\hat{R}}{H}. \tag{18}$$

It may well happen that the inequalities in (17) and (18) are strict. The possibility of strict inequalities in Corollary 2.2 is related to the existence of subanalytic functions, which are not analytic. We illustrate this statement by an example in which ht $H <$ ht H'.

Example 2.2 *Let k be a field and let*

$$\begin{aligned} R &= k[x,y,z]_{(x,y,z)}, \\ R' &= k[x',y',z']_{(x',y',z')}, \end{aligned}$$

where $x' = x$, $y' = \frac{y}{x}$ and $z' = z$. We have $K = k(x,y,z)$, $\hat{R} = k[[x,y,z]]$, $\hat{R}' = k[[x',y',z']]$. Let t_1, t_2 be auxiliary variables and let $\sum\limits_{i=1}^{\infty} c_i t_1^i$ (with $c_i \in k$) be an element of $k[[t_1]]$, transcendental over $k(t_1)$. Let θ denote the valuation, centered at $k[[t_1,t_2]]$, defined by $\theta(t_1) = 1$, $\theta(t_2) = \sqrt{2}$ (the value group of θ is the additive subgroup of $\mathbf{R}$, generated by 1 and $\sqrt{2}$). Let $\iota : R' \hookrightarrow k[[t_1,t_2]]$ denote the injective map defined by $\iota(x') = t_2$, $\iota(y') = t_1$, $\iota(z') = \sum\limits_{i=1}^{\infty} c_i t_1^i$. Let ν denote the restriction of θ to K, where we view K as a subfield of $k((t_1,t_2))$ via ι. Let $\Phi = \nu(R \setminus \{0\})$; $\Phi' = \nu(R' \setminus \{0\})$. For $\beta \in \Phi'$, P'_β is generated by all the monomials of the form ${x'}^\alpha {y'}^\gamma$ such that $\sqrt{2}\alpha + \gamma \geq \beta$, together with $z' - \sum\limits_{j=1}^{i} c_j {y'}^j$, where i is the greatest non-negative integer such that $i < \beta$.

Let $w' := z' - \sum\limits_{i=1}^{\infty} c_i {y'}^i$. Then $H' = (w')$, but $H = H' \cap \hat{R} = (0)$, so that ht $H = 0 < 1 =$ ht H'.

3 Introduction to the general case

Keep the above notation. Let $r = \text{rk } \nu$.

Let

$$\Gamma = \Delta_0 \supsetneqq \cdots \supsetneqq \Delta_{r-1} \supsetneqq \Delta_r = (0)$$

be the isolated subgroups of Γ and

$$(0) = P_0 \subsetneqq P_1 \subseteqq \cdots \subseteqq P_{r-1} \subseteqq P_r = m$$

the prime ν-ideals of R.

For a prime ideal P in R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$.

Let

$$(0) = \mathbf{m}_0 \subsetneqq \mathbf{m}_1 \subsetneqq \cdots \subsetneqq \mathbf{m}_r = \mathbf{m}_\nu$$

be the prime ideals of the valuation ring R_ν.

By definitions, our valuation ν is a composition of r rank one valuations $\nu = \nu_1 \circ \nu_2 \cdots \circ \nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$.

3.1 Local blowings up and trees

We consider *extensions* $R \to R'$ of local rings, that is, injective morphisms such that R' is an R-algebra essentially of finite type and $m' \cap R = m$. We suppose that both R and R' are contained in a fixed valuation ring R_ν.

Such extensions form a direct system $\{R'\}$.

We will assume that

$$\lim_{\overrightarrow{R'}} R' = R_\nu. \tag{19}$$

Definition 3.1 *A **tree** of R'-algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R'-algebra. Note that we do not require the maps in the direct system $\{S'\}$ to be injective. A morphism $\{S'\} \to \{T'\}$ of trees is the datum of a map of R'-algebras $S' \to T'$ for each R' commuting with the tree morphisms for each map $R' \to R''$.*

Lemma 3.1 *Let $R \to R'$ be an extension of local rings. We have:*
1) The ideal $N := m\hat{R} \otimes_R 1 + 1 \otimes_R m'$ is maximal in the R-algebra $\hat{R} \otimes_R R'$.
2) The natural map of completions $\hat{R} \to \hat{R}'$ is injective.

Definition 3.2 *Let $\{S'\}$ be a tree of R'-algebras. For each S', let I' be an ideal of S'. We say that $\{I'\}$ is a **tree of ideals** if for any arrow $b_{S'S''} \colon S' \to S''$ in our direct system, we have $b^{-1}_{S'S''} I'' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.*

Example 3.1

- *The maximal ideals of the local rings of our system $\{R'\}$ form a tree of ideals.*
- *For any non-negative element $\beta \in \Gamma$, the valuation ideals $\mathcal{P}'_\beta \subset R'$ of value β form a tree of ideals of $\{R'\}$.*
- *Similarly, the i-th prime valuation ideals $P'_i \subset R'$ form a tree.*
- *If rk $\nu = r$, the prime valuation ideals P'_i give rise to a chain*

$$P'_0 = (0) \subsetneqq P'_1 \subseteq \cdots \subseteq P'_r = m' \tag{20}$$

of trees of prime ideals of $\{R'\}$.

Taking the limit in (20), we obtain a chain

$$(0) = \mathbf{m}_0 \subsetneqq \mathbf{m}_1 \subsetneqq \cdots \subsetneqq \mathbf{m}_r = \mathbf{m}_\nu \tag{21}$$

of prime ideals of the valuation ring R_ν.

For each $1 \le \ell \le r$ one has the equality

$$\lim_{\overrightarrow{R'}} \frac{R'}{P'_\ell} = \frac{R_\nu}{\mathbf{m}_\ell}.$$

Then **specifying the valuation** ν is equivalent to specifying valuations $\nu_0, \nu_1, \ldots, \nu_r$, where ν_0 is the trivial valuation of K and, for $1 \le \ell \le r$, ν_ℓ is a valuation of the residue field $k_{\nu_{\ell-1}} = \kappa(\mathbf{m}_{\ell-1})$, centered at the local ring $\lim\limits_{\longrightarrow} \frac{R'_{P'_\ell}}{P'_{\ell-1}R'_{P'_\ell}} = \frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$ and taking its values in the totally ordered group $\frac{\Delta_{\ell-1}}{\Delta_\ell}$.

We have the following natural generalization of Conjecture 1.1:

Conjecture 3.1 *Assume that* dim $R' =$ dim R *for all $R' \in \mathcal{T}$. Then there exists a tree of prime ideals H' of $\hat{R}'$ with $H' \cap R' = (0)$ and a valuation $\hat{\nu}_-$, centered at $\lim\limits_{\rightarrow} \frac{\hat{R}'}{H'}$ and having the following property: For any $R' \in \mathcal{T}$ the graded algebra $gr_{\hat{\nu}_-} \frac{\hat{R}'}{H'}$ is a scalewise birational extension of $gr_\nu R'$.*

4 Implicit ideals

From now on we fix a valuation ring R_ν dominating R. A local homomorphisms $R' \to R''$ of local domains dominated by R_ν will be called a **ν-extension** of local domains. Fix a tree $\mathcal{T} = \{R'\}$ of noetherian local $R-$subalgebras of R_ν having the following property: "for each ring $R' \in \mathcal{T}$, all the birational ν-extensions of R' belong to $\mathcal{T}$".

Moreover, we assume that $QF(R_\nu)$ equals $\lim\limits_{\overrightarrow{R'}} K'$, where $K' = QF(R')$.

We will define a chain of $2r+1$ prime ideals

$$H_0 \subset H_1 \subset \cdots \subset H_{2r} = H_{2r+1} = m\hat{R},$$

satisfying $H_{2\ell} \cap R = H_{2\ell+1} \cap R = P_\ell$ for $0 \le \ell \le r$.

We will show that these ideals H_i behave well under blowings up, that is, $H'_i \cap \hat{R} = H_i$.

4.1 Odd implicit ideals

Definition 4.1 *Let $0 \le \ell < r$. We define our main object of study, the $(2\ell+1)$-st implicit prime ideal $H_{2\ell+1} \subset \hat{R}$, by*

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta \hat{R}' \right) \bigcap \hat{R} \right), \tag{22}$$

where R' ranges over $\mathcal{T}$. We put $H_{2r+1} = m\hat{R}$.

We think of (22) as a tree equation: if we replace R by any other $R'' \in \mathcal{T}$ in (22), it defines the corresponding ideal $H''_{2\ell+1} \subset \hat{R}''$.

Example 4.1 *Let $R = k[x,y,z]_{(x,y,z)}$.*

Let ν be the valuation with value group $\Gamma = \mathbf{Z}^2_{lex}$, defined as follows.

Take a transcendental power series $\sum\limits_{j=1}^{\infty} c_j u^j$ in a variable u over k.

Consider the homomorphism $R \hookrightarrow k[[u,v]]$ which sends x to v, y to u and z to $\sum\limits_{j=1}^{\infty} c_j u^j$.

Consider the valuation ν, centered at $k[[u,v]]$, defined by $\nu(v) = (0,1)$ and $\nu(u) = (1,0)$; its restriction to R will also be denoted by ν, by abuse of notation.

We have $\nu(x) = (0,1)$, $\nu(y) = (1,0)$ and $\nu(z) = (1,0)$.

Given $\beta = (a,b) \in \mathbf{Z}^2_{lex}$, we have

$$\mathcal{P}_\beta = x^b \left(y^a, z - c_1 y - \cdots - c_{a-1} y^{a-1}\right).$$

Then

$$\bigcap_{\beta \in (0) \oplus \mathbf{Z}} \left(\mathcal{P}_\beta \hat{R}\right) = (y,z) \text{ and}$$

$$\bigcap_{\beta \in \Gamma = \Delta_0} \left(\mathcal{P}_\beta \hat{R}\right) = \left(z - \sum_{j=1}^{\infty} c_j y^j\right).$$

It is not hard to show that

$$H_1 = \left(z - \sum_{j=1}^{\infty} c_j y^j\right) \hat{R} \text{ and}$$

$$H_3 = (y,z)\hat{R}.$$

An extension $\hat{\nu}$ of ν has value group $\hat{\Gamma} = \mathbf{Z}^3_{lex}$ and is defined by $\hat{\nu}(x) = (0,0,1)$, $\hat{\nu}(y) = (0,1,0)$ and

$$\hat{\nu}\left(z - \sum_{j=1}^{\infty} c_j y^j\right) = (1,0,0).$$

The ideal H_1 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus \mathbf{Z}^2_{lex}$ of $\hat{\Gamma}$ and H_3 is the one corresponding to $(0) \oplus (0) \oplus \mathbf{Z}$.

The prime ideal $H := H_1$ and the valuation $\hat{\nu}_-$, induced by $\hat{\nu}$ on $\frac{\hat{R}}{H}$ (that is, the valuation centered at $\frac{\hat{R}}{H}$ with which $\hat{\nu}$ is composed) satisfy the conclusion of Conjecture 1.1.

Example 4.2 *Let* $S = \frac{k[x,y]_{(x,y)}}{(y^2-x^2-x^3)}$. *There are two distinct valuations centered in* (x,y).

Let $a_i \in k,\ i \geq 2$ *be such that*

$$\underbrace{\left(y + x + \sum_{i\geq 2} a_i x^i\right)}_{f} \underbrace{\left(y - x - \sum_{i\geq 2} a_i x^i\right)}_{g} = y^2 - x^2 - x^3.$$

We shall denote by ν_+ *the rank one discrete valuation defined by*

$$\nu_+(x) = \nu_+(y) = 1,$$

$$\nu_+(y+x) = 2,$$

$$\nu_+\left(y + x + \sum_{i\geq 2}^{b-1} a_i x^i\right) = b.$$

Now let $R = \frac{k[x,y,z]_{(x,y,z)}}{(y^2-x^2-x^3)}$. *Let* $\Gamma = \mathbf{Z}^2_{lex}$.

Let ν *be the composite valuation of the* (z)*-adic one with* ν_+.

The point of this example is to show that

$$H^*_{2\ell+1} = \bigcap_{\beta\in\Delta_\ell} \mathcal{P}_\beta \hat{R}$$

does not work as the definition of the $(2\ell+1)$*-st implicit prime ideal because the resulting ideal* $H^*_{2\ell+1}$ *is not prime.*

Indeed, as $\mathcal{P}_{(a,0)} = (z^a)$, *we have*

$$H^*_1 = \bigcap_{(a,b)\in\mathbf{Z}^2} \mathcal{P}_{(a,b)}\hat{R} = (0).$$

Clearly $f, g \notin H^*_1 = (0)$, *but* $f \cdot g = (0)$, *so the ideal* H^*_1 *is not prime.*

In fact we have $H_1 = (f)$ *and* $H_3 = (z, f)$.

Let $H := H_1$ *and let* $\hat{\nu}_-$ *be the valuation of* $\frac{\hat{R}}{H} \cong k[[x,z]]$ *with value group* $\mathbf{Z}^2_{lex}$, *defined by* $\hat{\nu}_-(x) = (0,1)$, $\hat{\nu}_-(z) = (1,0)$. *then* H *and* $\hat{\nu}_-$ *satisfy the conclusion of Conjecture 1.1.*

Despite what one might think from this and the previous example, there are situations when one cannot take $H = H_1$ *in Conjecture 1.1. An explicit example of this are given in [7].*

Proposition 4.1 *We have* $H_{2\ell+1} \cap R = P_\ell$.

Proposition 4.2 *The ideals* $H'_{2\ell+1}$ *behave well under* ν*-extensions* $R \to R'$ *in* $\mathcal{T}$:

$$H_{2\ell+1} = H'_{2\ell+1} \cap \hat{R}.$$

The main result of [7] is

Theorem 4.1 *[Odd implicit ideals] The implicit ideal* $H_{2\ell+1}$ *is prime.*

Next, we discuss one of the main notions used in the proof of Theorem 4.1 — that of **stable** rings in $\mathcal{T}$.

Let the notation be as above. Take an $R' \in \mathcal{T}$ and $\overline{\beta} \in \frac{\Delta_\ell}{\Delta_{\ell+1}}$. Let

$$\mathcal{P}_{\overline{\beta}} = \left\{ x \in R \;\middle|\; \nu(x) \mod \Delta_{\ell+1} \geq \overline{\beta} \right\}. \tag{23}$$

If

$$\beta(\ell) = \min\{\gamma \in \Phi \mid \beta - \gamma \in \Delta_{\ell+1}\}$$

(this makes sense because Φ is well ordered, since R is noetherian — see [ZS], Appendix 4, Proposition 2) then $\mathcal{P}_{\overline{\beta}} = \mathcal{P}_{\beta(l)}$.

We have the obvious inclusion of ideals

$$\mathcal{P}_{\overline{\beta}}\hat{R} \subset \mathcal{P}_{\overline{\beta}}\hat{R}' \cap \hat{R}. \tag{24}$$

A useful subtree of $\mathcal{T}$ is formed by the ℓ-stable rings, which we now define. An important property of stable rings, proved in [7], is that the inclusion (24) is an equality whenever R' is stable.

Definition 4.2 *A ring $R' \in \mathcal{T}(R)$ is said to be ℓ-**stable** if the following two conditions hold:*

(1) the ring

$$\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R \hat{R}\right)_{M'} \tag{25}$$

is an integral domain and

(2) there do not exist an $R'' \in \mathcal{T}(R')$ and a non-trivial algebraic extension L of $\kappa(P'_\ell)$ which embeds both into $\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R \hat{R}\right)_{M'}$ and $\kappa(P''_\ell)$.

*We say that R is **stable** if it is ℓ-stable for each $\ell \in \{0, \dots, r\}$.*

Remark 4.1 *(1) Rings of the form (25) are a basic object of study in [7]. Another way of looking at the same ring, which we often use, comes from interchanging the order of tensor product and localization. Namely, let T' denote the image of the multiplicative system $\left(R' \otimes_R \hat{R}\right) \setminus M'$ under the natural map $R' \otimes_R \hat{R} \to \kappa\left(P'_\ell\right) \otimes_R \hat{R}$. Then the ring (25) equals the localization $(T')^{-1}\left(\kappa\left(P'_\ell\right) \otimes_R \hat{R}\right)$.*

(2) In the special case $R' = R$ in Definition 4.2, we have

$$\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R \hat{R}\right)_{M'} = \kappa\left(P_\ell\right) \otimes_R \hat{R}.$$

If, moreover, $\frac{R}{P_\ell}$ is analytically irreducible then the hypothesis that $\kappa\left(P_\ell\right) \otimes_R \hat{R}$ is a domain holds automatically; in fact, this hypothesis is equivalent to analytic irreducibility of $\frac{R}{P_\ell}$.

(3) Consider the special case when R' is Henselian. Excellent Henselian rings are algebraically closed inside their formal completions, so both (1) and (2) of Definition 4.2 hold automatically for this R'. Thus excellent Henselian local rings are always stable.

The existence of stable rings $R' \in \mathcal{T}$ is shown in §§7–8 of [7]. More precisely, in [7] parallel theories (implicit ideals, stability, etc.) are constructed not only for formal completions, but also for henselizations and for finite local étale extensions. In §7 of [7] we prove the existence of stable rings for henselization. After that, the existence of stable rings for completion follows as an easy corollary and is proved in §8 of [7].

The following Proposition justifies the name "stable".

Proposition 4.3 *Fix an integer ℓ, $0 \le \ell \le r$. Assume that R' is ℓ-stable and let $R'' \in \mathcal{T}(R')$. Then R'' is ℓ-stable.*

The next Proposition is a technical result on which much of [7] is based. For $\overline{\beta} \in \frac{\Gamma}{\Delta_{\ell+1}}$, let

$$\mathcal{P}_{\overline{\beta}+} = \left\{x \in R \;\middle|\; \nu(x) \mod \Delta_{\ell+1} > \overline{\beta}\right\}. \tag{26}$$

As usual, $\mathcal{P}'_{\overline{\beta}+}$ will stand for the analogous notion, but with R replaced by R', etc.

Proposition 4.4 *Assume that R itself is $(\ell+1)$-stable and let $R' \in \mathcal{T}(R)$.*

1. *For any $\overline{\beta} \in \frac{\Delta_\ell}{\Delta_{\ell+1}}$*

$$\mathcal{P}'_{\overline{\beta}}\hat{R}' \cap \hat{R} = \mathcal{P}_{\overline{\beta}}\hat{R}. \tag{27}$$

2. *For any* $\overline{\beta} \in \frac{\Gamma}{\Delta_{\ell+1}}$ *the natural map*

$$\frac{\mathcal{P}_{\overline{\beta}}\hat{R}}{\mathcal{P}_{\overline{\beta}+}\hat{R}} \to \frac{\mathcal{P}'_{\overline{\beta}}\hat{R}'}{\mathcal{P}'_{\overline{\beta}+}\hat{R}'} \tag{28}$$

is injective.

Corollary 4.1 *Take an integer* $\ell \in \{0, \dots, r-1\}$ *and assume that* R *is* $(\ell+1)$*-stable. Then*

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}. \tag{29}$$

Proof: By Lemma 4 of Appendix 4 of [15], the ideals $\mathcal{P}_{\overline{\beta}}$ are cofinal among the ideals $\mathcal{P}_\beta$ for $\beta \in \Delta_\ell$. $\Box$

Corollary 4.2 *Assume that* R *is stable. Take an element* $\beta \in \Gamma$. *Then* $\mathcal{P}'_\beta \hat{R}' \cap \hat{R} = \mathcal{P}_\beta$.

Once Proposition 4.4 and its Corollaries are established, the proof of the primality of the odd implicit prime ideals proceeds similarly to the case of rank one valuations, with the additional ingredient of considering the local blowing up $R \to R'$ along certain ν-ideals.

4.2 Even implicit ideals

Proposition 4.5 *There exists a unique minimal prime ideal* $H_{2\ell}$ *of* $P_\ell \hat{R}$*, contained in* $H_{2\ell+1}$.

Proof: Since $H_{2\ell+1} \cap R = P_\ell$, $H_{2\ell+1}$ belongs to the fiber of the map $Spec\ \hat{R} \to Spec\ R$ over P_ℓ. Since R was assumed to be excellent, $S := \hat{R} \otimes_R \kappa(P_\ell)$ is a regular ring. Hence its localization $\bar{S} := S_{H_{2\ell+1}S} \cong \frac{\hat{R}_{H_{2\ell+1}}}{P_\ell \hat{R}_{H_{2\ell+1}}}$ is a regular *local* ring. In particular, $\bar{S}$ is an integral domain, so (0) is its unique minimal prime ideal. The set of minimal prime ideals of $\bar{S}$ is in one-to-one correspondence with the set of minimal primes of P_ℓ, contained in $H_{2\ell+1}$, which shows that such a minimal prime $H_{2\ell}$ is unique, as desired. $\Box$

Corollary 4.3 $H_{2\ell} \cap R = P_\ell$.

Proposition 4.6 *We have* $H_{2\ell-1} \subset H_{2\ell}$.

Proposition 4.7 *The ideals* $H'_{2\ell}$ *behave well under* ν*-extensions* $R \to R'$ *in* $\mathcal{T}$*:*

$$H_{2\ell} = H'_{2\ell} \cap \hat{R}.$$

5 A clasification of extensions of ν to $\hat{R}$

Definition 5.1 *A chain of trees* $\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m'\hat{R}'$ *of prime ideals of* $\hat{R}'$ *is said to be* ***admissible*** *if:*

1. $H'_i \subset \tilde{H}'_i$.
2. $\tilde{H}'_{2\ell} \cap R' = \tilde{H}'_{2\ell+1} \cap R' = P'_\ell$.
3. $$\bigcap_{\overline{\beta} \in \left(\frac{\Delta_\ell}{\Delta_{\ell+1}}\right)_+} \lim_{\overrightarrow{R'}} \left(\mathcal{P}'_{\overline{\beta}}\hat{R}' + \tilde{H}'_{2\ell+1}\right) \hat{R}'_{\tilde{H}'_{2\ell+2}} \cap \hat{R} \subset \tilde{H}_{2\ell+1}$$

   where $\mathcal{P}'_{\overline{\beta}}$ *denote the preimage in* R' *of the* $\nu_{\ell+1}$*-ideal of* $\frac{R'}{P_\ell}$ *of value greater than or equal to* $\overline{\beta}$.

Theorem 5.1 *Specifying the valuation $\hat{\nu}_-$ is equivalent to specifying the following data (described recursively in i):*

(1) An admissible chain of trees $\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m'\hat{R}'$ of prime ideals of $\hat{R}'$.

(2) For each i, $1 \leq i \leq 2r$, a valuation $\hat{\nu}_i$ of $k_{\hat{\nu}_{i-1}}$ ($\hat{\nu}_0$ is the trivial valuation), whose restriction to $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim\limits_{\overrightarrow{R'}} \frac{\hat{R}'_{\tilde{H}'_i}}{\tilde{H}'_{i-1}\hat{R}'_{\tilde{H}'_i}}$.

The data $\{\hat{\nu}_i\}_{1\leq i\leq 2r}$ is subject to the following additional condition: if $i = 2\ell$ is even then $rk\ \hat{\nu}_i = 1$ and $\hat{\nu}_i$ is an extension of ν_ℓ to $k_{\hat{\nu}_{i-1}}$ (which is naturally an extension of $k_{\nu_{\ell-1}}$).

§6 of [7] discusses uniqueness properties of $\hat{\nu}_-$. Since the statements of these results are quite technical, we omit them in the present exposition.

References

[1] S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math., 63 (1956) 491–526.

[2] S. D. Cutkosky and S. El Hitti, *Formal prime ideals of infinite value and their algebraic resolution*, Ann. Fac. Sci. Toulouse Math. (6) 19 (2010), no. 3–4, 635–649.

[3] S. Cutkosky and L. Ghezzi, *Completions of valuation rings*, Contemp. Math. 386 (2005), 13–34.

[4] A. Grothendieck, J. Dieudonné, Eléments de Géométrie algébrique, Chap. IV, Pub. Math. IHES, No. 24, 1965.

[5] W. Heinzer and J. Sally, *Extensions of valuations to the completion of a local domain.* J. Pure Appl. Algebra, Vol. 71, no 2–3, pp. 175–186, (1991).

[6] J. Herrera, M.A. Olalla, M. Spivakovsky, *Valuations in algebraic field extensions*, J. Algebra 312 (2007), no. 2, 1033–1074.

[7] J. Herrera, M.A. Olalla, M. Spivakovsky, B. Teissier *Extending a valuation centered in a local domain to the formal completion*, to appear in the Proceedings of the LMS (DOI: 10.1112/plms/pds002)

[8] H. Matsumura, *Commutative Algebra.* Benjamin/Cummings Publishing Co., Reading, Mass., 1970.

[9] M. Nagata, *Local Rings.*, Interscience Publishers, 1960.

[10] M. Spivakovsky, *Valuations in function fields of surfaces.* Amer. J. Math 112, 1, 107–156 (1990).

[11] B. Teissier, *Valuations, deformations, and toric geometry*, Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume), F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, **33**, 2003, 361-459.

[12] M. Vaquié, *Valuations*, *in* "Resolution of Singularities, a research textbook in tribute to Oscar Zariski", Birkhäuser, Progress in Math. No. 181, 2000.

[13] M. Vaquié, *Famille admissible de valuations et défaut d'une extension*, J. Algebra 311 (2007), no. 2, 859–876.

[14] O. Zariski, *Local uniformization theorem on algebraic varieties*, Ann. of Math., 41 (1940), 852–896.

[15] O. Zariski, P. Samuel *Commutative Algebra, Vol. II*, Springer-Verlag (1960).