Embedding Henselian fields into power series

Antongiulio Fornasiero[*][†]

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Abstract

Every Henselian field of residue characteristic 0 admits a truncation-closed embedding in a field of generalised power series (possibly, with a factor set). As corollaries we obtain the Ax-Kochen-Ershov theorem and an extension of Mourgues' and Ressayre's theorem: every ordered field which is Henselian in its natural valuation has an integer part. We also give some results for the mixed and the finite characteristic cases.

Key words: Integer part, valuation, Henselian field
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Contents

[*]Università di Pisa, Dipartimento di Matematica “L. Tonelli”, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
[†]fornasiero@mail.dm.unipi.it

1 Introduction

Given an ordered field $\mathbb{K}$, an *integer part* (or I.P. for short) of $\mathbb{K}$ is a subring $R \subseteq \mathbb{K}$ containing 1 and such that for every $x \in \mathbb{K}$ there exists a unique $r \in R$ with $|r - x| < 1$. For instance, $\mathbb{Z}$ is the unique I.P. of any subfield of $\mathbb{R}$. Other ordered fields may have many I.P.s, or none at all.

A field of generalised power series $\mathbf{k}((\Gamma))$ over an ordered field $\mathbf{k} \subseteq \mathbb{R}$ has a standard I.P.

$$R := \left\{ \sum_{\gamma<0} a_\gamma t^\gamma + n : n \in \mathbb{Z} \right\}.$$

In [1] Mourgues and Ressayre proved that every real closed field $\mathbb{K}$ has an I.P.. They proved it by finding a truncation-closed embedding of $\mathbb{K}$ in a power series field $\mathbf{k}((\Gamma))$. The I.P. of $\mathbb{K}$ is given by the intersection of $\mathbb{K}$ with the standard I.P. of $\mathbf{k}((\Gamma))$.

S. Kuhlmann suggested that their proof can be adapted, and their results extended, to Henselian fields. Namely, any ordered field, which is Henselian in its natural valuation, has an integer part. We will prove among other things her conjecture (Corollary 6.3); however, we will have to introduce factor sets into the definition of power series in order to find the truncation-closed embedding.

Let $\mathbb{K}$ be a Henselian valued field, with residue field $\mathbf{k}$ and value group Γ. If $\operatorname{char}\mathbf{k} = 0$, we will construct a truncation-closed embedding of $\mathbb{K}$ in $\mathbf{k}((\Gamma, f))$ (for some factor set f) (Theorem 6.1 and Corollary 6.2). In particular, if $\mathbb{K}$ is an ordered field which is Henselian in its natural valuation, this will prove the conjecture by S. Kuhlmann, in the same way as Mourgues and Ressayre proved their theorem.

In his celebrated paper [2], Kaplansky proved that every field of residue characteristic 0 has some embedding in a power series field. We will prove Theorem 6.1 proceeding in a manner parallel to his: we start with a subfield $\mathbb{F}_0 := \mathbf{k}(\Gamma, f)$ of $\mathbb{K}$, and a truncation-closed embedding $\phi_0 : \mathbb{F}_0 \to \mathbf{k}((\Gamma, f))$. Then, we extend ϕ_0 to a truncation-closed embedding ϕ of all $\mathbb{K}$. The construction of the extension is done step-by-step. At each step, we assume that we have already defined a truncation-closed embedding ϕ_i from some subfield $\mathbb{F}_i$ in $\mathbf{k}((\Gamma, f))$, and we extend ϕ_i to a larger field $\mathbb{F}_{i+1}$. There are two cases:

The algebraic case. If $\mathbb{F}_i$ is not algebraically closed in $\mathbb{K}$, we define $\mathbb{F}_{i+1}$ to be the Henselisation of $\mathbb{F}_i$ inside $\mathbb{K}$. There is a unique extension ϕ_{i+1} of ϕ_i to $\mathbb{F}_{i+1}$. Moreover, since $\operatorname{char}\mathbf{k} = 0$, $\mathbb{F}_{i+1}$ coincides with the relative algebraic closure of $\mathbb{F}_i$ inside $\mathbb{K}$.

The transcendental case. If $\mathbb{F}_i$ is algebraically closed in $\mathbb{K}$, we define $\mathbb{F}_{i+1} := \mathbb{F}_i(x)$, for some $x \in \mathbb{K} \setminus \mathbb{F}_i$; then, we choose a suitable $x' \in \mathbf{k}((\Gamma, f))$ such that the field embedding ϕ_{i+1} extending ϕ_i and sending x to x' preserves the valuation.

In both cases, Theorem 5.16 ensures that ϕ_{i+1} is truncation-closed (if we choose x' wisely in the transcendental case). Moreover, if $\operatorname{tr\,deg}\big(\mathbb{K}/\mathbf{k}(\Gamma, f)\big) \leq \aleph$ for some uncountable cardinal $\aleph$, then it is possible to choose ϕ in such a way that the length of z is less than $\aleph$ for every z in the image of ϕ (Theorem 7.12).

Until now, we have only considered valued fields of residue characteristic 0. If $\operatorname{char}\mathbb{K} = p$, things get more complicated, mainly because in the above construction it is no longer true that the Henselisation of $\mathbb{F}_i$ is equal to its relative algebraic closure. However, if $\mathbb{K}$ (and hence $\mathbb{F}_i$) satisfies Kaplansky's Hypothesis A, the maximal immediate algebraic extension of $\mathbb{F}_i$ is uniquely determined, and in the algebraic case we can define $\mathbb{F}_{i+1}$ to be such extension, and ϕ_{i+1} accordingly. Proposition 9.11 ensures that ϕ_{i+1} is truncation-closed. Hence, we can conclude that if $\mathbb{K}$ is algebraically maximal and satisfies Kaplansky's Hypothesis A, then there is a truncation-closed embedding $\phi : \mathbb{K} \to \mathbf{k}((\Gamma, f))$ (cf. Theorem 9.12 and the paragraph following it). Again, if $\operatorname{tr\,deg}\big(\mathbb{K}/\mathbf{k}(\Gamma, f)\big) < \aleph$ for some uncountable cardinal $\aleph$, then the length of the elements in the image of ϕ can be bounded by $\aleph$ (Theorem 9.14).

It remains to consider the case of mixed characteristic. Here, we need some additional hypotheses on $\mathbb{K}$ (beside being Henselian). The most important is that $\mathbb{K}$ is finitely ramified (for instance, if $v(\operatorname{char}\mathbf{k})$ is the minimum positive element of Γ). Under these assumptions, we will prove that $\mathbb{K}$ can be embedded in $\mathring{\mathrm{K}}((\Lambda, m))$ in a truncation-closed way, where Λ is a certain quotient of Γ and $\mathring{\mathrm{K}}$ is a certain field associated to $\mathbb{K}$ (see §10).

On Conway's field of Surreal numbers, there is a so-called simplicity relation [3]. Under this relation, one can define the initial subsets of **No** and initial embeddings in **No**, in the same way as Mourgues and Ressayre defined truncation-closed subsets and embeddings for $\mathbf{k}((\Gamma))$ starting from the relation "being an initial segment of". We prove that if $\mathbb{K}$ is an ordered field, Henselian in its natural valuation, then a necessary and sufficient condition for $\mathbb{K}$ to have an initial embedding in **No** is that its value group Γ has a initial embedding in **No**, and there exists a cross-section $\Gamma \to \mathbb{K}^\star$ (Lemma 8.3 and Theorem 8.4).

The last section contains some easy counter-examples to some natural conjectures the unwary reader might conceive. S. Boughattas [4] gave some examples of valued field that do not admit integer parts (and hence truncation-closed embeddings). We give, among other things, a simplified version of his example (with regard to the non-existence of truncation-closed embeddings). For a more refined kind of counter-examples, see [5].

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2 Mourgues' and Ressayre's theorem

Here are some of the basic definitions and theorems of general valuation theory (see for instance [6, 7, 2]).

Definition 2.1. A *valued field* $\mathbb{K}$ is a field K together with a surjective homomorphism $v : K^\star \to \Gamma$ (the *valuation*) into some linearly ordered Abelian group Γ (the *value group*) such that

$$v(x+y) \geq \min\{\, v(x), v(y)\,\}\,.$$

The valuation is extended to all K, setting $v(0) = \infty$ (where ∞ is an element outside γ such that $\infty > \gamma$ for every $\gamma \in \Gamma$).
The valuation ring $\mathscr{O}$ is the ring $\{\, x \in \mathbb{K} : v(x) \geq 0\,\}$. Its only maximal ideal is the set of infinitesimal elements $\mathscr{M} := \{\, x \in \mathbb{K} : v(x) > 0\,\}$. The *residue field* $\mathbf{k}$ is the quotient $\mathscr{O}/\mathscr{M}$. Given $x \in \mathscr{O}$, $\overline{x} \in \mathbf{k}$ is its residue.

We will also use the small-o and big-O notation: $x = \mathsf{O}(y)$ iff $v(x) \geq v(y)$, $x = \mathsf{o}(y)$ iff $v(x) > v(y)$. If $v(x) > 0$ then $y = \mathsf{O}(x^2)$ implies that $y = \mathsf{o}(x)$. We will also write $x \gg y$ iff $v(x) < v(y)$.

Definition 2.2 (Henselian). A valued field is *Henselian* iff for each $p \in \mathscr{O}[X]$ and $a \in \mathscr{O}$ with $\overline{p(a)} = 0$ and $\overline{\dot{p}(a)} \neq 0$, there exists an $b \in \mathscr{O}$ such that $p(b) = 0$ and $\overline{b} = \overline{a}$.

If $\mathbb{K}$ is a valued field, it is possible to extend its valuation to $\mathbb{K}[x]$ in many different ways. Unless specified otherwise, we will always use the following definition:

$$v\Big(\sum_{i=0}^{n} a_i X^i\Big) := \min\{\, v(a_i)\,\}\,,$$

called the *Gauss extension*.

Definition 2.3. Let $\mathbb{K} \subseteq \mathbb{F}$ be valued fields, $p(X) \in \mathbb{K}[X]$ be a monic polynomial, and $x \in \mathbb{F}$. We will write $H_{\mathbb{K}}(p,x)$ (or $H(p,x)$ if it is clear which field we are talking about) iff $v(p) = 0$, $v(x) \geq 0$, $p(x) = 0$ and $v(\dot{p}(x)) = 0$.

Note that, by definition, $\mathbb{K}$ is Henselian iff for every $\mathbb{F}$ containing it and for every $p(X) \in \mathbb{K}[X]$ and $x \in \mathbb{F}$, $H(p,x)$ implies $x \in \mathbb{K}$.

The following lemma gives a few equivalent characterisations of Henselianity. On the one hand, they will be used in the discussion; on the other hand, they justify in part the importance of this concept in the study of valued fields.

Lemma 2.4. *Let $\mathbb{K} := (K,v)$ be a valued field. If F/K is a purely inseparable field extension, then v has only one extension to F.*

Moreover, the following are equivalent:

1. *$\mathbb{K}$ is Henselian.*
2. *Let $p \in \mathscr{O}[X]$ with $\deg(p) > 0$ and $a \in \mathscr{O}$ such that $\dot{p}(a) \neq 0$. If $v(p(a)) > 2v(\dot{p}(a))$ then there exists $b \in \mathscr{O}$ such that $p(b) = 0$ and $\overline{b} = \overline{a}$.*
3. *Let $p,q,r \in \mathscr{O}[X]$, with $\deg q > 0$, q monic. Suppose that $\overline{q},\overline{r}$ are non-zero, relatively prime polynomials of $\mathbf{k}[X]$ and $\overline{p} = \overline{qr}$. Then, there exist $q^\star, r^\star \in \mathscr{O}[X]$ such that $\overline{q^\star} = \overline{q}$, $\overline{r^\star} = \overline{r}$ and $f = q^\star r^\star$.*
4. *If*
$$p(X) := X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0 \in \mathscr{O}[X],$$
with $\overline{a_{n-1}} \neq 0$, $\overline{a_0} = \ldots = \overline{a_{n-2}} = 0$, then $p(X)$ has a root $b \in \mathscr{O}$ with $\overline{b} = -\overline{a_{n-1}}$.
5. *v has only one extension to every algebraic extension of $\mathbb{K}$.*

6. *v has only one extension to every separable extension of $\mathbb{K}$.*

For the proof, together with more equivalent forms of Henselianity, see [8].

Definition 2.5. An extension $\mathbb{K} \subseteq \mathbb{F}$ of valued fields is *immediate* iff $\mathbb{K}$ and $\mathbb{F}$ have the same value group and residue field. If $\mathbb{K}$ has two extensions $\mathbb{F}_1$ and $\mathbb{F}_2$, an *analytic embedding* of $\mathbb{F}_1$ in $\mathbb{F}_2$ *over* $\mathbb{K}$ is a homomorphism of valued fields $\phi : \mathbb{F}_1 \to \mathbb{F}_2$ such that $\phi\restriction_{\mathbb{K}} = \mathrm{id}$. Such a homomorphism ϕ is an *analytic isomorphism* (over $\mathbb{K}$) iff it is also an isomorphism of fields.

Definition 2.6 (Henselisation). The *Henselisation* $\mathbb{K}^{\mathrm{H}}$ of $\mathbb{K}$ is the extension of $\mathbb{K}$ such that:

1. $\mathbb{K}^{\mathrm{H}}$ is Henselian
2. If $\mathbb{K} \subseteq \mathbb{F}$ and $\mathbb{F}$ is Henselian, then there exists a unique analytic embedding $\phi : \mathbb{K}^{\mathrm{H}} \to \mathbb{F}$ over $\mathbb{K}$.

The Henselisation of $\mathbb{K}$ always exists, it is unique (up to analytic isomorphisms over $\mathbb{K}$), and it is an algebraic immediate extension of $\mathbb{K}$.

Definition 2.7 (Power series). Let $\mathbf{k}$ be a field, Γ be an linearly ordered Abelian group. The field of generalised power series $\mathbf{k}((\Gamma))$ is the set of formal series

$$\sum_{i<n} a_i t^{\gamma_i},$$

where the $a_i \in \mathbf{k}$ are non-zero, n is an ordinal number, and $\left(\gamma_i\right)_{i<n}$ is a strictly increasing sequence of elements of Γ. Every element of $\mathbf{k}((\Gamma))$ can also be written as

$$x := \sum_{\gamma\in\Gamma} a_\gamma t^\gamma.$$

The *support* of x, in symbols $\operatorname{supp} x$, is the set of $\gamma \in \Gamma$ such that $a_\gamma \neq 0$. Such an x is an element of $\mathbf{k}((\Gamma))$ iff its support is a well-founded subset of Γ. The *length* of x, $\ell(x)$, is the order type of the support of x; namely, $x = \sum_{i<\ell(x)} a_i t^{\gamma_i}$.

Sum and multiplication on $\mathbf{k}((\Gamma))$ are defined by Cauchy sum and product, namely

$$\sum_{\gamma\in\Gamma} a_\gamma t^\gamma \cdot \sum_{\lambda\in\Gamma} b_\lambda t^\lambda := \sum_{\gamma,\lambda\in\Gamma} a_\gamma b_\lambda t^{\gamma+\lambda}.$$ [(1)]

$\mathbf{k}((\Gamma))$ is a field, with a valuation defined by

$$vx := \min(\operatorname{supp} x).$$

$\mathbf{k}(\Gamma)$ is the subfield of $\mathbf{k}((\Gamma))$ generated by $\mathbf{k} \cup \{ t^\gamma : \gamma \in \Gamma \}$.

Our main concern will be the study of Henselian fields and their embeddings in fields of generalised power series.

Definition 2.8. A cross-section is a group homomorphism $s : \Gamma \to \mathbb{K}^\star$ such that $v(s\gamma) = \gamma$ for every $\gamma \in \Gamma$.

[(1)]The fact that the multiplication is well defined on $\mathbf{k}((\Gamma))$, and that $\mathbf{k}((\Gamma))$ is actually a field, is a theorem by Hahn, later extended by B.H. Neumann to division rings.

$\mathbf{k}((\Gamma))$ has a canonical cross-section given by $s\gamma := t^\gamma$. Later, we will see that it is useful to introduce factor sets into the definition of power series fields and cross-sections.

If $\mathbf{k}$ is an ordered field, $\mathbf{k}((\Gamma))$ inherits the order via the rule $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma > 0$ iff $a_\mu > 0$, where $\mu := v(x)$.

On the other hand, on every ordered field K is defined the natural valuation: the value of $x \in K^\star$ is its Archimedean equivalence class: $y \sim x$ iff there exists $n \in \mathbb{N}^\star$ such that $|\frac{y}{n}| < |x| < |ny|$.

Definition 2.9. Let $x := \sum_{i<n} a_i t^{\gamma_i} \in \mathbf{k}((\Gamma))$. An *initial segment* of x is an element of $\mathbf{k}((\Gamma))$ of the form $\sum_{i<m} a_i t^{\gamma_i} \in \mathbf{k}((\Gamma))$ for some $m \leq n$.

A subset $S \subseteq \mathbf{k}((\Gamma))$ is *truncation-closed* iff for every $x \in S$, every initial segment of x is also in S.

An embedding of a valued field $\mathbb{K}$ in $\mathbf{k}((\Gamma))$ is truncation-closed iff its image is truncation-closed.

We are now ready to state a generalisation Mourgues' and Ressayre's theorem.

Theorem 2.10. *Let $\mathbb{K}$ be an ordered field, with natural valuation v, value group Γ and residue field $\mathbf{k}$. Assume that $\mathbb{K}$ is Henselian, and that there is a cross-section $s : \Gamma \to \mathbb{K}^\star$. Then, there is a truncation-closed analytic embedding ϕ from $\mathbb{K}$ to $\mathbf{k}((\Gamma))$.*

Since $\mathbb{K}$ is Henselian and its residue field has characteristic 0, there exists an embedding $\iota : \mathbf{k} \to \mathbb{K}$ such that $v(\iota x) \geq 0$ and $\overline{\iota x} = x$. If we fix such an embedding ι, we will simply say that $\mathbb{K}$ contains its residue field $\mathbf{k}$, and write x instead of ιx.

Then we can find ϕ as in Theorem 2.10 such that $\phi(x) = x$ for every $x \in \mathbf{k}$ and $\phi(s\gamma) = t^\gamma$ for every $\gamma \in \Gamma$.

From Theorem 2.10, reasoning exactly as in [1], we can deduce that every ordered field satisfying the hypothesis of the theorem has an integer part. With Corollary 6.2, we will generalise the theorem, and drop the hypothesis that $\mathbb{K}$ has a cross-section (retaining only the fact that it is Henselian). More precisely, Kaplansky proved that any such field $\mathbb{K}$ admits a section with some factor set f; we will show that $\mathbb{K}$ has a truncation-closed embedding in $\mathbf{k}((\Gamma, f))$.

Note that Theorem 2.10 is already a generalisation of Mourgues' and Ressayre's Theorem, since every real closed field is Henselian and has a cross-section.

In this discussion, all groups are Abelian, and all orders are linear (or total), unless explicitly said otherwise.

The important ingredients in the proof of Mourgues' and Ressayre's Theorem are the following lemmata. In their formulation, I will assume that $\mathbf{k}$ is a real closed field, and Γ is a divisible group. The first is attributed to F. Delon:

Lemma 2.11. *Let $\mathbb{F}$ be a subfield of $\mathbf{k}((\Gamma))$ such that $\mathbf{k} \subseteq \mathbb{F}$ and $v(\mathbb{F}) = \Gamma$; if $\mathbb{F}$ is closed under truncation then so is $\widetilde{\mathbb{F}}$, the real closure of $\mathbb{F}$ inside $\mathbf{k}((\Gamma))$.*

Lemma 2.12. *Let $\mathbb{F}$ be a truncation-closed subfield of $\mathbf{k}((\Gamma))$ and $y \in \mathbf{k}((\Gamma))$ be such that every proper initial segment of y belongs to $\mathbb{F}$. Then $\mathbb{F}(y)$ is also truncation-closed.*

The two lemmata imply that if S is a truncation-closed subfield of $\mathbf{k}((\Gamma))$ containing $\mathbf{k}(\Gamma)$, then the field generated by S and its real closure are also truncation-closed.

Definition 2.13. A valued field is *algebraically maximal* iff it has no proper immediate algebraic extension.

Every algebraically maximal field is also Henselian (since the Henselisation of a field is an algebraic immediate extension). However, the converse is not true; in general, a valued field could have more than one non-isomorphic maximal algebraic immediate extensions. We will see presently a sufficient condition for the converse to hold.

Definition 2.14 (Finitely ramified). Let $\mathbb{F}$ be a valued field, with value group Γ and residue field $\mathbf{f}$. $\mathbb{F}$ is finitely ramified iff

- either $\mathbf{f}$ has characteristic 0,
- or $\operatorname{char}\mathbb{F} = 0$, $\operatorname{char}\mathbf{f} = p > 0$ and there are only finitely many $\gamma \in \Gamma$ between 0 and $v(p)$.

$\mathbb{F}$ has ramification index 1 iff $\operatorname{char}\mathbf{f} = p$ and $v(p)$ is the minimum positive element of Γ.

For instance, the fields of p-adic numbers have ramification index 1, and their finite algebraic extensions are finitely ramified.

Lemma 2.15. *Let $\mathbb{F}$ be a finitely ramified valued field, and $\mathbb{K}$ be an immediate extension of $\mathbb{F}$. Then for every $n \in \mathbb{N}^\star$ and $x \in \mathbb{K}^\star$ there exists $b \in \mathbb{F}$ such that $v(x-b) > v(x) + v(n)$.*

Proof. Let $c, d \in \mathbb{F}$ such that $v(x) = v(c)$, and $v\left(\frac{x}{c} - d\right) > 0$. If $\operatorname{char}\mathbf{f} = 0$, then $v(n) = 0$, and $b := cd$ satisfies the conclusion.

If $\operatorname{char}\mathbf{f} = p > 0$, let 1 be the minimum of Γ. Let $k \in \mathbb{N}$ such that $k > v(n)$ (it exists, because $\mathbb{F}$ is finitely ramified). Define $b_0, b_1, \ldots \in \mathbb{F}$ as follows:

$$\begin{aligned} b_0 &= cd, \\ b_{n+1} &\text{ such that } v(x - b_n) < v(x - b_{n+1}). \end{aligned}$$

Then, $b := b_k$ satisfies the conclusion. □

The following lemma is [9, Corollary A.3.20]. For the reader's convenience, we will repeat its proof.

Lemma 2.16. *Let $\mathbb{F}$ be a finitely ramified valued field. Then $\mathbb{F}$ is algebraically maximal iff it is Henselian.*

Proof. The only if direction is trivial. Suppose that $\mathbb{K}$ is an algebraic immediate extension of $\mathbb{F}$, and let $x \in \mathbb{K}$.

Let $n > 0 \in \mathbb{N}$ be the degree of x over $\mathbb{F}$, and $c \in \mathbb{F}$ be the trace of x over $\mathbb{F}$. We have to prove that $x \in \mathbb{F}$.

Since $\mathbb{F}$ is finitely ramified, its characteristic is 0, thus $\frac{c}{n}$ is a well defined element of $\mathbb{F}$. By substituting x with $x - \frac{c}{n}$, we can assume that the trace of x is 0. If for contradiction $x \neq 0$, Proposition 2.15 implies that we can find $b \in \mathbb{F}$ such that $v(x-b) > v(x) + v(n)$. Therefore, $v(x) = v(b)$ and

$$v(x-b) > v(b) + v(n). \tag{*}$$

Let $x = x_1, \ldots, x_n$ be the conjugates of x over $\mathbb{F}$. Lemma 2.4 implies that $v(b - x_i) = v(b - x)$, $i = 1, \ldots, n$. Moreover, $\sum_i (x_i - b) = nb$. Therefore,

$$v(n) + v(b) = v(nb) \geq \min\{\, v(x_i - b),\ i = 1, \ldots, n \,\} = v(x - b),$$

in contradiction with (*). □

Finally, if you know what it means, note that a Henselian finitely ramified field is defectless [10, Lemma 2.9]. The importance of Lemma 2.16 in our discussion stems from the following lemma, which is proved in [11] and [10, Proposition 4.10 A].

Lemma 2.17. *Let $\mathbb{F}$ be a valued field which is algebraically maximal. Let $\mathbb{L}_i$ $i = 1,2$ be two immediate extensions of $\mathbb{F}$, and $x_i \in \mathbb{L}_i \setminus \mathbb{F}$. Suppose that for every $y \in \mathbb{F}$ $v(x_1 - y) = v(x_2 - y)$. Then, there is an analytic isomorphism over $\mathbb{F}$ between $\mathbb{F}(x_1)$ and $\mathbb{F}(x_2)$ sending x_1 to x_2.*

We will repeat the proof given in [11], mainly because it is short and elegant, but also because the lemma is not explicitly stated in the paper. An alternative proof can be made using [2, Theorem 2].

Proof. It suffices to prove the following:

Claim 1. For every $p \in \mathbb{F}[X]$, $v\big(p(x_1)\big) = v\big(p(x_2)\big)$.

The proof is by induction on $n := \deg p$. For $n = 0$ there is nothing to prove, for $n = 1$ it is the hypothesis.

Inductive step: the claim is true for every polynomial of degree less than n. We have to prove it for p of degree n. If p is reducible, say $p = qq'$, then the conclusion follows from the inductive hypothesis applied to q and q'. Otherwise, p is irreducible, and w.l.o.g. we can take p monic.

For convenience, call $x := x_1$. If $x \in \mathbb{F}$, the conclusion follows immediately. Thus we can assume that $x \notin \mathbb{F}$.

Let V be the $\mathbb{F}$-linear subspace of $\mathbb{L}_1$ generated by $1, x, \dots, x^{n-1}$:

$$V := \mathbb{F} \oplus x\mathbb{F} + \cdots + x^{n-1}\mathbb{F}.$$

For every $g \in \mathbb{F}[X]$, perform Euclid's division, obtaining $g = sp + r$, with $\deg r < n$. Define $r =: g \bmod p$, the remainder of g (modulo p). Note that $r \in V$ for every remainder r.

Let $\mathbb{V} := \mathbb{F}[X]/(p)$. $\mathbb{V}$ as a $\mathbb{F}$-linear space is canonically isomorphic to V, therefore we can restrict to it the valuation v of $\mathbb{L}_1$. This restriction is a valuation of vector spaces, but not necessarily of fields. Moreover, $\mathbb{V}$ is an algebraic extension of $\mathbb{K}$.

Note that V and $\mathbb{V}$ carry two different multiplication: the one on V has co-domain $\mathbb{L}_1$, but respects the valuation, the one on $\mathbb{V}$ has co-domain $\mathbb{V}$ itself, but does not respect the valuation. Here we use the multiplication on V.

If $v(gh \bmod p) = v(g) + v(h)$ for every $g, h \in V$, then the multiplication on $\mathbb{V}$ respects the valuation, i.e. $(\mathbb{V}, v)$ is a valued field, extending $\mathbb{F}$ and contained in $\mathbb{L}_1$. Therefore, it is an immediate algebraic extension of $\mathbb{F}$. However, $\mathbb{F}$ is algebraically maximal, so either $x \in \mathbb{F}$ or $\deg p = 1$, a contradiction in both cases.

Otherwise, there exist $g, h \in V$ such that $gh = ps + r$, with $\deg r < n$, and $v(r) \neq v(g) + v(h)$. Consequently

$$v(ps(x)) = \min\{\, v(gh(x)),\ v(r(x))\,\},$$

so

$$v(p(x_1) = \min\{\, v(g(x_1)) + v(h(x_1)),\ v(r(x_1))\,\} - v(s(x_1)).$$

But the degree of g, h, r are less than n, hence also the degree of s is less than n, so, by inductive hypothesis, $v(g(x_1)) = v(g(x_2))$, and the same for h, r, s. Therefore,

$$v(p(x_2)) = \min\{\, v(g(x_1)) + v(h(x_1)),\ v(r(x_1))\,\} - v(s(x_1)) = v(p(x_1)). \qquad \square$$

Definition 2.18.
- For us, a *sequence* $(x_i)_{i\in I}$ in a set S is a function from some ordered set I without maximum into S. I will usually be either a limit ordinal or a subset (without maximum) of Γ.
- A sequence $(x_i)_{i\in I}$ in a valued field $\mathbb{F}$ is *pseudo-Cauchy* iff for every $k > j > i \in I$ $v(x_k - x_j) > v(x_j - x_i)$.
- A sequence $(x_i)_{i\in I}$ is *converging* to $x \in \mathbb{F}$ iff for every $j > i \in I$ $v(x_j - x) > v(x_i - x)$.(2)
- A valued field $\mathbb{F}$ is *pseudo-complete* iff every pseudo-Cauchy sequence in $\mathbb{F}$ converges to some $x \in \mathbb{F}$.

Note that a pseudo-Cauchy sequence may converge to many different elements, and that every converging sequence is pseudo-Cauchy. The following is a theorem by Kaplansky [2].

Lemma 2.19. $\mathbf{k}((\Gamma))$ *is pseudo-complete. Every pseudo-complete valued field is Henselian. A valued field is pseudo-complete iff it is maximal (namely, it has no proper immediate extensions).*

We will also need the following well known fact.

Lemma 2.20. *Let $\mathbb{F}$ be a valued field, with value group Γ and residue field $\mathbf{f}$.*

$\mathbb{F}$ *is real closed iff it is Henselian, Γ is divisible and $\mathbf{f}$ is real closed. If* $\operatorname{char}\mathbf{f} = 0$, *then $\mathbb{F}$ is algebraically closed iff it is Henselian, Γ is divisible, and $\mathbf{f}$ is algebraically closed.*

2.1 Proof of Theorem 2.10

For this proof, we will use the following notations.

Notation 2.21.
- $\mathbb{F}^{\mathrm{H}}$ is the Henselisation of a valued field $\mathbb{F}$.
- $\widetilde{\mathbb{L}}$ is the real closure of an ordered field $\mathbb{L}$.
- $\widetilde{\Gamma}$ is the divisible hull of a given ordered Abelian group Γ, with the ordered induced by Γ.
- $\mathbf{k}$ is the residue field of $\mathbb{K}$, and in general given a valued field $\mathbb{F}$, $\mathbf{f}$ is its residue field.
- Given a field $\mathbf{k}$ and a group Γ, $\mathbf{k}((\Gamma))$ is the field of generalised power series in the formal variable t. Note that t is infinitesimal and satisfies $v(t^\gamma) = \gamma$.
- Every ordered field $\mathbb{K}$ is endowed with the natural valuation v.

Let $\mathbb{F} \subseteq \mathbb{K}$ such that:

- $\mathbf{k}(\Gamma) \subseteq \mathbb{F}$

(2)To avoid confusion with the convergence in topological sense (for the valuation topology), we should say that $(x_i)_{i\in I}$ pseudo-converges to x. Every sequence converging topologically to x is eventually pseudo-converging to x, but not conversely. However, we will not be using sequences converging topologically, hence there is no risk of confusion.

- There is a maximal (i.e. non extensible) truncation-closed embedding $\phi : \mathbb{F} \to \mathbf{k}((\Gamma))$ which is the identity on $\mathbf{k}(\Gamma)$.

We have to prove that $\mathbb{F} = \mathbb{K}$. W.l.o.g., we can suppose that ϕ is the identity.

Proposition 2.22. *$\mathbb{F}$ is Henselian.*

Proof. $\widetilde{\mathbf{k}}((\widetilde{\Gamma}))$ is real closed since it is Henselian. Let $\widetilde{\mathbb{F}}$ be the real closure of $\mathbb{F}$, taken inside $\widetilde{\mathbf{k}}((\widetilde{\Gamma}))$. Its residue field is $\widetilde{\mathbf{k}}$, and the value group is $\widetilde{\Gamma}$.

The embedding of $\mathbf{k}$ in $\mathbb{K}$ and the cross-section s from Γ into $\mathbb{K}$ can be extended in a unique way to an embedding of $\widetilde{\mathbf{k}}$ and a cross-section from $\widetilde{\Gamma}$ into $\mathbb{F}$ respectively. In fact, given $\gamma \in \widetilde{\Gamma}$, there exists $n \in \mathbb{N}$ such that $n\gamma \in \Gamma$. Define $s(\gamma) := s(n\gamma)^{1/n}$.

Therefore Lemma 2.11 implies that $\widetilde{\mathbb{F}}$ is truncation-closed.

Claim 1. $\mathbb{F}^{\mathrm{H}}$ is equal to $\widetilde{\mathbb{F}} \cap \mathbf{k}((\Gamma))$.

Fist, we prove that $\mathbb{F}^{\mathrm{H}} \subseteq \widetilde{\mathbb{F}} \cap \mathbf{k}((\Gamma))$. $\mathbf{k}((\Gamma))$ is Henselian and contains $\mathbb{F}$, hence $\mathbb{F}^{\mathrm{H}} \subseteq \mathbf{k}((\Gamma))$. Moreover, $\mathbb{F}^{\mathrm{H}}$ is algebraic over $\mathbb{F}$, and, being a subset of $\mathbf{k}((\Gamma))$, it is also ordered, so $\mathbb{F}^{\mathrm{H}} \subseteq \widetilde{\mathbb{F}}$.

Now, we prove the reverse inclusion. $\widetilde{\mathbb{F}} \cap \mathbf{k}((\Gamma))$ is a field, and it is and algebraic immediate extension of $\mathbb{F}$. Therefore, by Lemma 2.16, it is contained in $\mathbb{F}^{\mathrm{H}}$.

The claim implies that $\mathbb{F}^{\mathrm{H}}$ is a truncation-closed subfield of $\mathbf{k}((\Gamma))$. Moreover, $\mathbb{F}^{\mathrm{H}}$ is a subfield of $\mathbb{K}$, since $\mathbb{K}$ is Henselian. Hence, by maximality of ϕ, $\mathbb{F}^{\mathrm{H}} = \mathbb{F}$. □

Definition 2.23. Let $\mathbb{H}$ be an extension of $\mathbb{F}$. Given $y \in \mathbb{H}$, define

$$\mathrm{I}(y, \mathbb{F}) := \{\, v(y - a) : a \in \mathbb{F} \,\}\,.$$

Proposition 2.24. $\mathbb{F} = \mathbb{K}$.

Proof. If not, let $x \in \mathbb{K} \setminus \mathbb{F}$. By the previous proposition, $\mathbb{F}$ is Henselian. Let $x' \in \mathbf{k}((\Gamma))$ of minimal length in the same cut as x over $\mathbb{F}$. For every $\gamma \in \mathrm{I}(x, \mathbb{F})$ choose a $y_\gamma \in \mathbb{F}$ such that $v(x - y_\gamma) = \gamma$. The sequence (y_γ) is converging to x, therefore it is pseudo-Cauchy, hence, by Lemma 2.19, it has a pseudo-limit in $\mathbf{k}((\Gamma))$. Define $x' \in \mathbf{k}((\Gamma))$ to be a pseudo-limit of (y_γ) in $\mathbf{k}((\Gamma))$ of minimal length.

Claim 1. For every $y \in \mathbb{F}$, $v(x - y) = v(x' - y)$.

In fact, for every $\gamma \in \mathrm{I}(x, \mathbb{F})$, $v(x - y_\gamma) = v(x' - y_\gamma)$. Moreover, if $\gamma := v(x - y)$, we can find $\lambda \in \Lambda$ such that $\lambda > \gamma$. Writing $x - y = (x - y_\lambda) + (y_\lambda - y)$, we obtain $v(y - y_\lambda) = \gamma$, and writing the same for x' we obtain the claim.

We can then apply Lemma 2.17 with $\mathbb{L}_1 := \mathbb{K}$ and $\mathbb{L}_2 := \mathbf{k}((\Gamma))$, and obtain ψ an analytic isomorphism over $\mathbb{F}$ between $\mathbb{F}(x)$ and $\mathbb{F}(x')$. Consequently, ψ is an isomorphism of ordered fields. Finally, Lemma 2.12 implies that $\mathbb{F}(x')$ is truncation-closed, contradicting the maximality of ϕ. □

3 Generalisations

We will now try to generalise Theorem 2.10. We will drop the hypothesis that $\mathbb{K}$ is ordered, and take any valued field $\mathbb{K}$. We need to distinguish three cases:

characteristic 0**:** Both $\mathbb{K}$ and its residue field $\mathbf{k}$ have characteristic 0;

characteristic p**:** Both $\mathbb{K}$ and $\mathbf{k}$ have characteristic $p > 0$;

mixed characteristic: $\mathbb{K}$ has characteristic 0, while $\mathbf{k}$ has characteristic $p > 0$.

These are all the possible cases for a valued field. In the equal characteristic cases, we will try to embed $\mathbb{K}$ in a field of generalised power series. However, we cannot expect that $\mathbb{K}$ can be embedded $\mathbf{k}((\Gamma))$, not even if we drop the requirement that the embedding should be truncation-closed; the main obstacle is the fact that $\mathbb{K}$ could be missing a cross-section. We shall see how to overcome this obstacle, introducing sections with a factor set, and power series field "twisted" by a factor set.

Additional difficulties arise in the characteristic p case: we shall see that further hypotheses are needed.

In the mixed characteristic case, under suitable hypotheses, we will be able to decompose the valuation on $\mathbb{K}$ into a valuation of characteristic 0 and a valuation with value group $\mathbb{Z}$, and embed $\mathbb{K}$ in a field of power series over a field of Witt vectors.

To prove these results, first we need to define factor sets and study their properties. Then, we will generalise Lemmata 2.11 and 2.12.

4 Factor sets and power series

Definition 4.1. Let A and B be two Abelian groups. A 2 *co-cycle* (or co-cycle for short, since we will consider only 2 co-cycles) is a map

$$f : A \times A \to B$$

satisfying the following conditions:

1. $f(\alpha,\beta) = f(\beta,\alpha)$.
2. $f(0,0) = f(0,\alpha) = f(\alpha,0) = 0$.
3. $f(\alpha,\beta+\gamma)f(\beta,\gamma) = f(\alpha+\beta,\gamma)f(\alpha,\beta)$.

Definition 4.2 (Section). Given a valued field $\mathbb{K}$, a *section* is a map $s : \Gamma \to \mathbb{K}^\star$ such that

$$s(0) = 1$$
$$v(s\alpha) = \alpha$$

for every $\alpha \in \Gamma$.

Proposition 4.3. *Given a section s, the map $f : \Gamma \times \Gamma \to \mathbb{K}^\star$ defined by*

$$f(\alpha,\beta) := \frac{s\alpha\, s\beta}{s(\alpha+\beta)}$$

is a 2 co-cycle. Moreover, s is a group homomorphism iff $f = 1$.

Proof.

$$f(\alpha,\beta+\gamma)f(\beta,\gamma) = \frac{s\alpha\, s\beta\, s\gamma}{s(\alpha+\beta+\gamma)} = f(\alpha+\beta,\gamma)f(\alpha,\beta). \qquad \square$$

We could also add to the definition of 2 co-cycle the axiom

4. $f(-\alpha,\alpha) = 1$,

and to the definition of section the corresponding axiom

$$s(-\alpha) = (s\alpha)^{-1}.$$

These additional axioms would not restrict significantly our use of 2 co-cycles and sections, but they do simplify the computations.

The following definition is taken from homological algebra.

Definition 4.4. With the notation of Proposition 4.3, $f := \mathrm{d}s$, the *co-boundary* of s.

Definition 4.5 (Factor set). Let $\mathbb{K}$ be a valued field containing its residue field $\mathbf{k}$. A *factor set* f is a 2 co-cycle whose image is contained in $\mathbf{k}^\star$. If $s : \Gamma \to \mathbb{K}^\star$ is a section and $\mathrm{d}s$ is a factor set, we will say that s is a *good section*, or a section with factor set f.

Definition 4.6 (Power series). Given a 2 co-cycle $f : \Gamma \times \Gamma \to \mathbf{k}^\star$, the field of generalised power series $\mathbf{k}((\Gamma, f))$ with factor set f is the set of formal series

$$\sum_{i<n} a_i t^{\gamma_i},$$

with $a_i \in \mathbf{k}^\star$, n an ordinal number and $\left(\gamma_i\right)_{i<n}$ a strictly increasing sequence of elements of Γ. Sum and multiplication are defined formally, with the condition

$$t^\alpha t^\beta = f(\alpha, \beta) t^{\alpha+\beta}.$$

The axioms on f assure that the multiplication is associative and commutative. $\mathbf{k}((\Gamma, f))$ is actually a field, with valuation given by

$$v\Big(\sum_{i<n} a_i t^{\gamma_i}\Big) := \gamma_0,$$

value group Γ, residue field $\mathbf{k}$ and canonical section $s(\gamma) := t^\gamma$. With this definition, s is a good section, with factor set f.

$\mathbf{k}(\Gamma, f)$ is the subfield of $\mathbf{k}((\Gamma, f))$ generated by $\mathbf{k} \cup \{t^\gamma : \gamma \in \Gamma\}$.

If we do not specify a factor set f, we will always mean that f is the constant map 1 (agreeing with the notation $\mathbf{k}((\Gamma))$). Similar definitions can be given for $\mathbf{k}$ only a ring, or Γ only an ordered semi-group.

The following are well-known facts.

Lemma 4.7.
- *If $\mathbf{k}$ is a ring and Γ an ordered semi-group, then $\mathbf{k}((\Gamma, f))$ is a ring.*
- *$\mathbf{k}((\Gamma, f))$ is a field iff Γ is actually a group and $\mathbf{k}$ is a field. In this case, $\mathbf{k}((\Gamma, f))$ is maximal, hence it is Henselian.*
- *An ordering on $\mathbf{k}$ induces an ordering on $\mathbf{k}((\Gamma, f))$. With this ordering, $\mathbf{k}((\Gamma, f))$ is a real closed field iff $\mathbf{k}$ is a real closed field and Γ is divisible.*
- *$\mathbf{k}((\Gamma, f))$ is algebraically closed iff $\mathbf{k}$ is algebraically closed and Γ is divisible.*

In the sequel, we will try to embed in a truncation-closed way a Henselian field $\mathbb{K}$ of equal characteristic in $\mathbf{k}((\Gamma, f))$, for a suitable f.

First, we need to embed $\mathbf{k}$ in $\mathbb{K}$.

Lemma 4.8 (Kaplansky). *Let $\mathbb{K}$ be a valued field, with residue field $\mathbf{k}$ and valuation ring $\mathscr{O}$. Suppose that $\mathbb{K}$ has the same characteristic as $\mathbf{k}$ and it is Henselian and perfect. Then, there is a field embedding $\iota : \mathbf{k} \to \mathscr{O}$ such that $\overline{\iota a} = a$ for every $a \in \mathbf{k}$.*

Assume now that $\mathbf{f}_0$ is a subfield of $\mathbf{k}$, and $\iota_0 : \mathbf{f}_0 \to \mathscr{O}$ is a field embedding such that $\overline{\iota_0 a} = a$ for every $a \in \mathbf{f}_0$. Then, ι_0 can be extended to a field embedding $\iota : \mathbf{k} \to \mathscr{O}$ such that $\overline{\iota a} = a$ for every $a \in \mathbf{k}$.

Proof. In the first case, let $\mathbf{f} \subseteq \mathbf{k}$ be a subfield of $\mathbf{k}$ with a maximal embedding $\iota : \mathbf{f} \to \mathscr{O}$. $\mathbf{f}$ exists, because the same prime field is in both $\mathbf{k}$ and $\mathbb{K}$. In the second case, let $\mathbf{f} \subseteq \mathbf{k}$ be a subfield of $\mathbf{k}$ containing $\mathbf{f}_0$, with a maximal embedding $\iota : \mathbf{f} \to \mathscr{O}$ extending ι_0.

W.l.o.g., we can suppose that ι is the identity.

Suppose for contradiction that there exists $a \in \mathbf{k} \setminus \mathbf{f}$, and let $\mathbf{h} := \mathbf{f}(a) \subseteq \mathbf{k}$.

If a is transcendental over $\mathbf{f}$, let x be any element of $\mathbb{K}$ such that $\bar{x} = a$.

If a is algebraic, we can reduce to the case when either $\mathbf{h}/\mathbf{f}$ is purely inseparable, or it is separable.

In the inseparable case, $a^{p^d} \subseteq \mathbf{h}$ for some $d > 0$. Let $x \in \mathbb{K}$ such that $x^{p^d} = a$ (it exists, because $\mathbb{K}$ is perfect).

In the separable case, let $q(X) \in \mathbf{f}[X]$ be the minimum polynomial of a. It is a separable polynomial, hence, by Hensel's lemma, there exists $x \in \mathscr{O}$ such that $q(x) = 0$ and $\bar{x} = a$.

In all three cases, we can extend ι to $\mathbf{h}(a)$ by fixing $\mathbf{f}$ and sending a to x, a contradiction. $\square$

If we fix once and for all an embedding $\iota : \mathbf{k} \to \mathbb{K}$, we will say that $\mathbb{K}$ contains $\mathbf{k}$ and take ι the identity. Moreover, by saying that $\mathbb{K}$ contains $\mathbf{k}$, we imply that we are in the equal characteristic case.

Now, we give some sufficient conditions for the existence of a good section.

Definition 4.9. Let $\mathbb{K}$ be a valued field, with value group Γ and residue field $\mathbf{k}$. The *characteristic exponent* of $\mathbb{K}$ is either 1 if $\operatorname{char} \mathbf{k} = 0$, or p if $\operatorname{char} \mathbf{k} = p > 0$. A good section $s : \Gamma \to \mathbb{K}$ is *p-good* iff it satisfies the following conditions for every $\gamma \in \Gamma$:

1. $s(-\gamma) = s(\gamma)^{-1}$;
2. $s(p\gamma) = (s\gamma)^p$, where p is the characteristic exponent of $\mathbb{K}$.

Note that the second condition is empty if $\operatorname{char} \mathbf{k} = 0$.

The following lemma is a slightly improved version of Lemma 13 in [2].

Lemma 4.10. *Let $\mathbb{K}$ be a valued field, containing its residue field $\mathbf{k}$. Suppose that $\mathbb{K}$ is Henselian and perfect. Let Γ be its value group, and p be its characteristic exponent. Then, there exists a p-good section $s : \Gamma \to \mathbb{K}^\star$ (with some factor set f).*

Assume now that Θ is a subgroup of Γ, and $s_0 : \Theta \to \mathbb{K}^\star$ is a map such that, for every $\alpha, \beta \in \Theta$,

1. *$v(s_0\alpha) = \alpha$;*
2. *$\mathrm{d}s_0(\alpha, \beta) \in \mathbf{k}^\star$;*
3. *$s_0(p\alpha) = (s_0\alpha)^p$ and $s_0(-\alpha) = (s_0\alpha)^{-1}$, where p is the characteristic exponent of $\mathbb{K}$.*

Then, there exists a p-good section $s : \Gamma \to \mathbb{K}^\star$ extending s_0. If s_0 satisfies only 1) and 2), then it can be extended to a good section s.

Proof. In the first case, define $\Theta := \{0\}$, and $s_0(0) := 1$. Let $R := \mathbb{Z}/p^\infty \subset \mathbb{Q}$. $\mathbb{K}$ is perfect, hence $\mathbb{K}^\star$ and Γ are in a natural way R-modules. Let $B \subseteq \Gamma$ be a maximal subset of Γ which is $\mathbb{Q}$-linearly independent over Θ. For every $\lambda \in B$, choose $x_\lambda \in \mathbb{K}$ such that $v(x_\lambda) = \lambda$. Let $\Upsilon \subseteq \Gamma$ be the R-submodule of Γ generated by $\Theta \cup B$. For every $\gamma = \theta + r_1\lambda_1 + \cdots + r_n\lambda_n \in \Upsilon$, define

$$s\gamma := (s_0\theta)x_{\lambda_1}^{r_1} \cdots x_{\lambda_n}^{r_n}.$$

Let Λ be a R-submodule of Γ containing Υ and admitting a maximal p-good section s extending the one on Υ. If, for contradiction, $\Lambda \neq \Gamma$, let $\gamma \in \Gamma \setminus \Lambda$. There exists $n \in \mathbb{N}^\star$ such that $(p,n) = 1$ and $n\gamma \in \Lambda$; let m be the smallest such n. Let $y := s(m\gamma)$, and let $z \in \mathbb{K}^\star$ such that $v(z) = \gamma$. Let $w := y/z^m$. Hence, $v(w) = 0$. Let $a = \overline{w} \in \mathbf{k}^\star$.

By Hensel's lemma, there exists $x \in \mathscr{O}$ such that $x^m = w/a$. Define $s\gamma := zx$.

For every $\alpha \in \Lambda + R\gamma$, choose a representation

$$\alpha = \lambda + r\gamma,$$

for some $r \in R$ and $\lambda \in \Lambda$, such that the representation chosen for $p\alpha$ is $p\lambda + pr\gamma$ and the one for $-\alpha$ is $-\lambda - r\gamma$. We can extend s to $\Lambda + R\gamma$ by defining

$$s(\alpha) := s\lambda \cdot (s\gamma)^r.$$

It is easy to verify that, modulo $\mathbf{k}^\star$, $s(\alpha)$ is independent from the chosen representation of α, namely the extension of s is a good section, and that s is actually p-good if s_0 is, and we reached a contradiction. ☐

5 Extending subfields of $\mathbf{k}((\Gamma))$

We will now study $\mathbf{k}((\Gamma, f))$ more in detail. In this section, $\mathbf{k}$ is a field, Γ an ordered group, $f : \Gamma \times \Gamma \to \mathbf{k}^\star$ is a 2 co-cycle, and $\mathbb{K} := \mathbf{k}((\Gamma, f))$ is the field of generalised power series with factor set f.

Notation 5.1.

- $x \trianglelefteq y$ means that x is an initial segment of y.
- $x \triangleleft y$ iff x is a *proper* initial segment of y.
- $\mathrm{supp}(x)$ is the *support* of x.

$(\mathbb{K}, \trianglelefteq)$ is a *tree*: namely, it satisfies the following definition.

Definition 5.2 (Tree). A structure $(T, \triangleleft)$ is a *tree* iff the following conditions are satisfied:

- It is a partial order.
- It is well-founded.
- Every non-empty subset of T has a greatest lower bound (in T).
- For every $a \in T$, the set $\mathscr{P}(a) := \{y \in \mathbb{K} : y \triangleleft x\}$ is linearly ordered.

Therefore, we can do induction on $\triangleleft$. As usual, with abuse of notation we will say that T is a tree.

Remark 5.3. Every tree T has a minimum, the *root* of the tree. Moreover, every chain bounded above has a least upper bound (in T).

Proof. The g.l.b. of T itself is the root. Given a chain C, the g.l.b. of the set of the upper bounds of C is the l.u.b. of C. □

Note that 0 is the root of $\mathbb{K}$. Moreover, the following stronger condition holds for $\mathbb{K}$:

Remark 5.4. Any upper bound of a chain C in $\mathbb{K}$ without a maximum is a pseudo-limit of C, and conversely. Therefore, every chain in $\mathbb{K}$ has a l.u.b..

Definition 5.5. Given $A \subseteq T$, we will write $A \succ x$ iff for every $y \triangleleft x$ there exists $z \in A$ such that $y \trianglelefteq z$.

Remark 5.6. If $R \subseteq T$ is truncation-closed, $A \subseteq R$ and $A \succ x$, then $R \cup \{x\}$ is also truncation-closed.

Remark 5.7. If $x \in T$, then $\mathscr{P}(x) \succ x$.

Remark 5.8. Let $x \in \mathbb{K}$ and $A \succ x$. If $c \in \mathbf{k}$ and $\gamma \in \Gamma$, then $ct^{\gamma}A \succ ct^{\gamma}x$.

Remark 5.9. If $x, y, z \in \mathbb{K}$, $y \trianglelefteq x$, $\operatorname{supp}(y) < \gamma$ and $v(z - x) \geq \gamma$, then $y \trianglelefteq z$.

First, we will show how to perform some computations in $\mathbb{K}$. The following facts are well known.

Lemma 5.10. *Let $x, y \in \mathbb{K}$, $A := \operatorname{supp}(x)$ and $B := \operatorname{supp}(y)$. Then,*

1) $\operatorname{supp}(x + y) \subseteq A \cup B$.

2) The support of xy is contained in the subgroup of Γ generated by $A \cup B$.

3) If $x \neq 0$, then the support of $1/x$ is contained in the subgroup generated by A.

Let $a_0, \dots, a_{n-1} \in \mathbb{K}$, with $A_i := \operatorname{supp}(a_i)$,

$$p(X) := X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{K}[X],$$

and $x \in \mathbb{K}$ such that $p(x) = 0$. Let Λ be the subgroup of Γ generated by $A_1 \cup \ldots \cup A_n$, and $\widetilde{\Lambda}$ be its divisible hull.

4) $\operatorname{supp}(x)$ *is contained in* $\widetilde{\Lambda}$.

5) If $H(p, x)$, then $\operatorname{supp}(x)$ *is contained in Λ (see Definition 2.3).*

Proof. The first two assertions can be proved by direct computation.

3) Let Θ be the subgroup generated by A, f' be restriction of f to $\Theta \times \Theta$, and $\mathbb{L} := \mathbf{k}((\Theta, f'))$. Since $\mathbb{L}$ is a field and $x \in \mathbb{L}$, $1/x \in \mathbb{L}$, proving the assertion.

4) Define:

- $\widetilde{\Gamma}$ be the divisible hull of Γ.
- $\widetilde{\mathbf{k}}$ and $\widetilde{\mathbb{K}}$ be the algebraic closures of $\mathbf{k}$ and $\mathbb{K}$ respectively.
- $\widetilde{s} : \widetilde{\Gamma} \to \widetilde{\mathbb{K}}^{\star}$ be some good extension of s (it exists by Lemma 4.10).
- $\widetilde{f} : \widetilde{\Gamma} \times \widetilde{\Gamma} \to \widetilde{\mathbf{k}}^{\star}$ be the co-boundary of $\widetilde{s}$.

- $f' : \widetilde{\Lambda} \times \widetilde{\Lambda} \to \widetilde{\mathbf{k}}^{\star}$ be the restriction of $\widetilde{\mathbf{f}}$.

Define also:

$$\begin{aligned}\mathbb{M} &:= \widetilde{\mathbf{k}}((\widetilde{\Gamma}, \tilde{f})) \supseteq \mathbb{K}\\ \mathbb{L} &:= \widetilde{\mathbf{k}}((\widetilde{\Lambda}, f')) \subseteq \mathbb{M}\\ \mathbb{F} &:= \mathbb{K} \cap \mathbb{L}.\end{aligned}$$

They all are maximal, and the tree-structures on $\mathbb{K}$, $\mathbb{F}$, and $\mathbb{L}$ are the ones induced by $\mathbb{M}$. Moreover, $\mathbb{L}$ is algebraically closed, hence it contains x, proving that the support of x is contained in $\widetilde{\Lambda}$.

If $f = 1$, a similar method proves that the coefficients (of the power series representation) of x are in the algebraic closure of the field generated by the coefficients of the a_i.

5) This assertion too can be done by direct computation, but we prefer a different, computations-free, approach.
Let f' be the restriction of f to $\Lambda \times \Lambda$. Since $\mathbb{L} := \mathbf{k}((\Lambda, f'))$ is maximal, it is Henselian. Moreover, $p(X) \in \mathbb{L}[X]$ and $H_{\mathbb{L}}(p, x)$, hence $x \in \mathbb{L}$, proving the conclusion. □

For related computations about the coefficients of an element of $\mathbf{k}((\Gamma))$, see for instance [12, Theorem 6.1].

The following proposition is a version for valued fields of the implicit function theorem.

Proposition 5.11. *Let $\mathbb{F}$ be a valued field, $\mathscr{O}$ be its valuation ring, $p(X) \in \mathscr{O}[X]$, $a, b \in \mathscr{O}$ with $\delta := v(a - b) > 0$. Assume that $p(a) = 0$ and $v(\dot{p}(a)) = \alpha < \infty$. If $\delta > \alpha$, then $v(p(b)) = \alpha + \delta$. In particular, if $v(\dot{p}(a)) = 0$, then $v(p(b)) = \delta$.*

If $q(X) \in \mathscr{O}[X]$ is a polynomial such that $q(b) = 0$ and $\gamma := v(p - q) > \alpha$, then $\delta + \alpha = v(q(a)) \geq \gamma$.

Proof. Since $v(p) \geq 0$,

$$p(b) = p(a) + \dot{p}(a)(b - a) + \mathsf{O}((b - a)^2) = \dot{p}(a)(b - a) + \mathsf{O}((b - a)^2).$$

Moreover, $v(b - a) > v(\dot{p}(a))$ implies that $\mathsf{O}((b - a)^2) = \mathsf{o}(\dot{p}(a)(b - a))$, hence

$$v(p(b)) = v(\dot{p}(a)(b - a)) = \alpha + \delta.$$

$v(p - q) > \alpha$ implies that $v(\dot{q}(b)) = \alpha$, so, exchanging p and q, we get $v(q(a)) = \alpha + \delta$. Finally, let p_i and q_i be the coefficients of $p(X)$ and $q(X)$ respectively. Then,

$$v(p(b)) = v(p(b) - q(b)) \geq \min\{ v(p_i - q_i) + iv(b) \} \geq v(p - q) = \gamma. \qquad \square$$

Lemma 5.12 (Ostrowski). *If $(x_i)_{i \in I}$ is a pseudo-Cauchy sequence in some valued field $\mathbb{F}$ and $p(X) \in \mathbb{F}[X]$, then $(p(x_i))_{i \in I}$ is eventually pseudo-Cauchy. If moreover $(x_i)_{i \in I}$ converges to x, then $(p(x_i))_{i \in I}$ converges to $p(x)$.*

Proof. See for instance [13, Lemma 9 Chapter 2]. □

Hence, if $(x_i)_{i \in I}$ is pseudo-Cauchy sequence in $\mathbb{F}$, and $p(X) \in \mathbb{F}[X]$, there are two cases: either

$$v\big(p(a_i)\big) = v\big(p(a_j)\big) \tag{5.1}$$

for all sufficiently large $i, j \in I$, or

$$v\big(p(a_i)\big) < v\big(p(a_j)\big) \tag{5.2}$$

for all sufficiently large $i < j \in I$.

Definition 5.13 (Type of a sequence). A pseudo-Cauchy sequence $\big(x_i\big)_{i\in I}$ in $\mathbb{F}$ is of *transcendental type* (with respect to $\mathbb{F}$) iff (5.1) holds for every polynomial $p(X) \in \mathbb{F}[X]$. If, on the other hand, (5.2) holds for at least one polynomial $p(X)$, we say that $\big(x_i\big)_{i\in I}$ is of *algebraic type*.

The distinction plays a fundamental role in [2].

Proposition 5.14. *If $\mathbb{H}$ is an extension of $\mathbb{F}$ with the same value group, let $\mathrm{I}(y,\mathbb{F})$ be as in Definition 2.23. Then,*

1. $\mathrm{I}(y,\mathbb{H})$ *is an initial segment of* Γ.
2. *If $\mathbb{H}$ is an immediate extension of $\mathbb{F}$, then $\mathrm{I}(y,\mathbb{F})$ has no maximum. In this case, if $y \in \mathbb{H}\setminus\mathbb{F}$, then there is a pseudo-Cauchy sequence $\big(x_i\big)_{i\in I}$ in $\mathbb{F}$ converging to y and without pseudo-limit in $\mathbb{F}$.*

Proof. If $y \in \mathbb{F}$, then $\mathrm{I}(y,\mathbb{F}) = \Gamma$, and the conclusions are obvious. Otherwise, $y \in \mathbb{H}\setminus\mathbb{F}$.

1) Let $\alpha = v(y-a) \in \mathrm{I}(y,\mathbb{F})$, and $\beta < \alpha \in \Gamma$. Let $b \in \mathbb{F}$ such that $v(a-b) = \beta$ (it exists, because $\mathbb{F}$ has the same value group as $\mathbb{H}$). Since $y-b = (y-a)+(a-b)$, $v(y-b) = \beta$.

2) Suppose, for contradiction, that $v(x-y)$ is the maximum of $\mathrm{I}(y,\mathbb{F}) = \Gamma$. Let $a \in \mathbb{F}$ such that $v(x-y) = v(a)$. Let $b \in \mathbb{F}$ such that $v(b) = 0$ and $\overline{\frac{x-y}{a}} = \overline{b}$. Then, $v\big(x-(y+ab)\big) > v(a) = v(x-y)$, a contradiction.
Choose for every $\gamma \in \mathrm{I}(y,\mathbb{F})$ $x_\gamma \in \mathbb{F}$ such that $v(y-x_\gamma) = \gamma$. The sequence $\big(x_\gamma\big)_{\gamma\in\mathrm{I}(y,\mathbb{F})}$ satisfies the conclusion. □

Proposition 5.15. *Let $\mathbb{F}$ be a valued field. $\mathbb{F}$ is algebraically maximal iff every pseudo-Cauchy sequence $\big(x_i\big)_{i\in I}$ in $\mathbb{F}$ of algebraic type has a pseudo-limit (in $\mathbb{F}$).*

Proof. $\Leftarrow$) Suppose that, for contradiction, $\big(x_i\big)_{i\in I}$ is a pseudo-Cauchy sequence of algebraic type without pseudo-limit in $\mathbb{F}$. [2, Theorem 3] implies there exists an immediate algebraic extension $\mathbb{L}$ of $\mathbb{F}$ where $\big(x_i\big)_{i\in I}$ has a pseudo-limit. Hence, $\mathbb{L}$ is a proper extension of $\mathbb{F}$, contradicting the fact that $\mathbb{F}$ is algebraically maximal.

$\Rightarrow$) Suppose not. Let $\mathbb{E}$ be an immediate algebraic extension of $\mathbb{F}$, $x \in \mathbb{E}\setminus\mathbb{F}$, and $p(X) \in \mathbb{F}[X]$ be the minimum polynomial of x. Proposition 5.14 implies that there is a sequence $\big(x_i\big)_{i\in I}$ in $\mathbb{F}$ converging to x and without limit in $\mathbb{F}$, and Lemma 5.12 implies that $\big(p(x_i)\big)_{i\in I}$ converges to 0. Therefore, $\big(x_i\big)_{i\in I}$ is of algebraic type. □

The following Theorem is a generalisation of Lemmata 2.11 and 2.12.

Theorem 5.16. *Let $S \subseteq \mathbb{K}$ be a truncation-closed subset of $\mathbb{K}$. Then, the following sets are also truncation-closed:*

1. *The group generated by S.*
2. *The ring generated by S.*

3. *The field generated by S.*

4. *The Henselisation of the field generated by S.*

In particular, if $\mathbb{F} \subseteq \mathbb{K}$ is truncation-closed, then also $\mathbb{F}^{\mathrm{H}}$ is truncation-closed.

The proof of the theorem is in subsection 5.1.

Let $\mathbb{F}$ be a subfield of $\mathbb{K}$, and $\mathbb{L}$ be the relative algebraic closure of $\mathbb{F}$ inside $\mathbb{K}$. If $\operatorname{char}\mathbf{k} = 0$, then $\mathbb{L}$ coincides with the Henselisation of $\mathbb{F}$. Otherwise, it will in general be bigger. Later, we will prove that if $\mathbb{F}$ is truncation-closed, then also $\mathbb{L}$ is truncation-closed, under the condition that the factor set is p-good (namely, that the canonical section on $\mathbb{K}$ is p-good).

Corollary 5.17. *Let $\mathbb{F}$ be a truncation-closed subfield of $\mathbb{K}$, and $\left(x_i\right)_{i\in I}$ be a pseudo-Cauchy sequence in $\mathbb{F}$ without pseudo-limit in $\mathbb{F}$. Then, there exists $x \in \mathbb{K}$ which is a pseudo-limit of $\left(x_i\right)_{i\in I}$ and such that $\mathbb{F}(x)$ is also truncation-closed.*

If $\left(x_i\right)_{i\in I}$ is of transcendental type and y is any pseudo-limit of it (in some extension $\mathbb{L}$ of $\mathbb{F}$), then there is an analytic isomorphism between $\mathbb{F}(x)$ and $\mathbb{F}(y)$ over $\mathbb{F}$ and sending x to y.

Proof. By Theorem 5.16, it suffices to find a pseudo-limit x such that $\mathbb{F} \cup \{x\}$ is truncation-closed. Denote $v(x_i - x_j)$ $i < j$ by γ_i. Let $x \in \mathbb{K}$ be some pseudo-limit of $\left(x_i\right)_{i\in I}$ (it exists, because $\mathbb{K}$ is maximal), $y_i \in \mathbb{K}$ be the truncation of x at γ_i. $y_i \trianglelefteq y_j$ iff $i < j$; let x be the l.u.b. of $\left(y_i\right)_{i\in I}$ (with respect to the order $\trianglelefteq$).

Claim 1. x is a pseudo-limit of $\left(x_i\right)_{i\in I}$.

$v(x - x_i) = \gamma_i$, $v(x - y_i) \geq \gamma_i$, hence $v(x_i - x) \geq \gamma_i$, proving the claim.

Claim 2. $y_i \in \mathbb{F}$ for every $i \in I$.

In fact, $v(x_i - y_i) \geq \gamma_i$ and $\operatorname{supp} y_i < \gamma_i$, thus $y_i \trianglelefteq x_i$. Besides, $\mathbb{F}$ is truncation-closed, and $x_i \in \mathbb{F}$, therefore $y_i \in \mathbb{F}$.

Finally, $\mathbb{F} \cup \{x\}$ is truncation-closed, proving the first part of the corollary.

The second part of the corollary is [2, Theorem 2]. □

5.1 Proof of Theorem 5.16

We will make use of ideas from surreal numbers [14]. Assertion 1 is obvious. Note that an arbitrary union of truncation-closed subsets is also initial. Therefore, for every $R \subseteq \mathbf{k}((\Gamma, f))$, we can define *the* maximal truncation-closed subset of R, namely the union of all truncation-closed subsets of R.

Ring: Let $x, y \in \mathbb{K}$. We will write

$$x = \sum_{i<n} a_i t^{\gamma_i}$$
$$y = \sum_{j<m} b_j t^{\lambda_i}$$

for some ordinal numbers m, n, with the a_i and b_j all different from 0. Given $n' < n$, x' will be the truncation of x at n:

$$x' = \sum_{i<n'} a_i t^{\gamma_i},$$

and similarly for y'.

Let R be the maximal truncation-closed subset of the ring generated by S. By Assertion 1, R is a subgroup of $\mathbb{K}$. Let $x, y \in R$. By definition of product,

$$\operatorname{supp}(xy) \subseteq \left\{ a_i + b_j : i < n, j < m \right\}.$$

If $z \triangleleft xy$ there exist $n' < n$ and $m' < m$ such that

$$\delta := \gamma_{n'} + \lambda_{m'} = \min\bigl(\operatorname{supp}(xy) \setminus \operatorname{supp}(z)\bigr) = v(xy - z).$$

Therefore,

$$(x - x')(y - y') = \sum_{\substack{n' \leq i < n \\ m' \leq j < m}} a_i b_j f(\gamma_i, \lambda_j) t^{\gamma_i + \lambda_j}.$$

Let $w := xy - (x - x')(y - y') = xy' + x'y - x'y'$. Hence,

$$v(xy - w) = \delta$$

and $z \trianglelefteq w$.

Remark 5.6 implies that for the second assertion it suffices to show that $R \succ xy$. This can be done by induction: we can suppose that we have proved the claim for every $(\tilde{x}, \tilde{y})$ such that $\tilde{x} \trianglelefteq x$, $\tilde{y} \trianglelefteq y$, and at least one of the inequalities is strict. However, w is a sum of products of this kind, hence Assertion 1 implies that $w \in R$ and consequently Assertion 2.

Field: Let us prove Assertion 3. Let R be the maximal truncation-closed subset of the field generated by S. By Assertion 2, R is a subring of $\mathbb{K}$. By Remark 5.6, it is enough to prove that if $0 \neq x \in R$, then there exists $A \subseteq R$ such that $A \succ 1/x$. The construction of A is done inductively.

Claim 5.18. We can reduce to the case when $v(x) = 0$.

In fact, let $b := a_0 t^{\gamma_0}$ be the leading term of x. Let $z := x/b$. Note that $b \in R$, $R \cup \{ 1/b \}$ is initial (and therefore Assertion 2 implies that $1/b \in R$), and $v(z) = 0$. If we can find $A \subseteq R$ such that $A \succ 1/z$, then, by Remark 5.8, $\frac{1}{b}A \succ x$, and $\frac{1}{b}A \subseteq R$, proving the assertion.

So, we can suppose that $v(x) = 0$. Let $y := 1/x$. Start with $0 \in A$. Suppose that we have already constructed $a \in A$ and let $0 \neq x' \triangleleft x$. We add to A the element

$$a' := \frac{1 + (x' - x)a}{x'}. \tag{5.3}$$

By induction on x, $1/x' \in R$, and therefore $a' \in R$. $v(x) = 0$, hence $v(x') = 0$.

$$v(y - a') = v\bigl(1 + (x' - x)a - x'y\bigr) - v(x') = v(x - x') + v(y - a) - v(x') = v(x - x') + v(y - a).$$

The support of y is a subset of the group generated by $\operatorname{supp}(x)$. Therefore, if $y' \triangleleft y$, then there exists $l \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_l \in \operatorname{supp}(x)$ such that

$$\gamma_1 + \cdots + \gamma_l = \min\bigl(\operatorname{supp}(y) \setminus \operatorname{supp}(y')\bigr) = v(y - y').$$

It suffices to prove the following

Claim 5.19. For every $i \leq l$ there is $a \in A$ such that

$$v(y - a) \geq \gamma_1 + \cdots + \gamma_i.$$

By induction on i, we can suppose we have already found $a \in A$ such that

$$v(y-a) \geq \gamma_1 + \cdots + \gamma_{i-1}.$$

Let x' be the truncation of x at γ_i, and let a' as defined in Equation (5.3). Then,

$$v(y-a') = v(y-a) + v(x-x') \geq (\gamma_1 + \cdots + \gamma_{i-1}) + \gamma_i,$$

proving the claim. If we apply the claim to $i = l$, we get $y' \trianglelefteq a$, proving Assertion 3.

Here is an example: $x = 1-t$, with $f = 1$. Consequently, $x' = 1$, and $x' - x = t$. The elements in A are given by the sequence

$$a_0 = 0$$
$$a_{n+1} = 1 + t a_n,$$

i.e. $a_n = 1 + t + t^2 + \cdots + t^n$, and

$$1/t = 1 + t + t^2 + \cdots.$$

Henselisation: For Assertion 4, let $\mathbb{F}$ be the maximal truncation-closed subset of the Henselisation of the field generated by S. By Assertion 3, $\mathbb{F}$ is a subfield of $\mathbb{K}$. We have to prove that it is Henselian.

Definition 5.20. Let $p, q \in \mathbb{K}[X]$, $p = \sum_{i \leq n} a_i X^i$, $q = \sum_{i \leq m} b_i X^i$, $\deg p = n$. $q \trianglelefteq p$ iff $m \leq n$ and there exists $l \leq n$ such that $a_i = b_i$ for every $i > l$, while $b_l \triangleleft a_l$.

$(\mathbb{K}[x], \trianglelefteq)$ is a well-founded partial order, therefore we can do induction on it.

Let $p[X] \in \mathbb{F}[X]$ and $b \in \mathbb{K}$. I remind that $H(p,b)$ means that $p(X)$ is monic, $v(p) = 0$, $v(b) \geq 0$, $p(b) = 0$ and $v(\dot{p}(b)) = 0$. We have to prove that if $H(p,b)$, then $b \in \mathbb{F}$. Since $\mathbb{F}$ is maximal initial, it suffices to prove that $\mathbb{F} \succ b$.

We will proceed by induction on p. Assume that

$$p(X) = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n.$$

Let Λ be the group generated by $\operatorname{supp}(a_0) \cup \cdots \cup \operatorname{supp}(a_n)$. Then, $\operatorname{supp}(b) \subseteq \Lambda$. Let $c \triangleleft b$, and

$$\delta := v(b-c) = \min\bigl(\operatorname{supp}(b) \setminus \operatorname{supp}(c)\bigr).$$

Therefore, $\delta = \gamma_1 + \cdots + \gamma_l$ for some $\gamma_i \in \operatorname{supp}(a_{j_i})$. Since we can suppose that $\delta > 0$(3), there exist $0 < \gamma \in \operatorname{supp} a_m$ such that $k\gamma > \operatorname{supp}(b)$ for some $m, k \in \mathbb{N}$.

Let a' be the truncation of a_m at γ (namely, $a' \triangleleft a_m$ and $v(a' - a_m) = \gamma$), and

$$q(X) := p(X) + (a' - a_m)X^m = X^n + a_{n-1}X^{n-1} + \cdots + a_{m+1}X^{m+1} + a'X^m + \cdots.$$

Then, $q \triangleleft p$ and $v(q-p) = \gamma$. By Hensel's lemma, there exists $d \in \mathbb{K}$ such that $v(b-d) > 0$ and $q(d) = 0$. $H(q,b)$ is true, therefore, by inductive hypothesis, $d \in \mathbb{F}$. Moreover, Proposition 5.11 implies that $v(d-b) \geq \gamma$.

Now proceed using Newton's algorithm and define

$$d_0 := d$$
$$d_{i+1} := d_i - \frac{p(d_i)}{\dot{p}(d_i)}$$

to find $d_i \in \mathbb{F}$ such that $v(d_i - b) \geq i\gamma$. Therefore, $v(d_{k+1} - b) > k\gamma$, so $c \trianglelefteq d_{k+1}$, proving that $\mathbb{F} \succ b$.

(3) As opposed to $\delta = 0$.

Proposition 5.21. *Let* $\operatorname{char}\mathbf{k} = p > 0$. *Suppose that* $a, b \in \mathbb{K}$. *Then* $a \lhd b$ *iff* $b^p \lhd a^p$. *In general, if*

$$a = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma,$$

then

$$a^p = \sum_{\gamma} a_\gamma{}^p c_\gamma t^{p\gamma}, \tag{5.4}$$

where $c_\gamma := f(\gamma,\gamma) f(\gamma, 2\gamma) \cdots f(\gamma, (p-1)\gamma) \in \mathbf{k}$.

The factor set f is called p-good if $c_\gamma = 1$ for every $\gamma \in \Gamma$.

Proof. The first assertion is an immediate consequence of the second one. By definition,

$$a^p = \underbrace{\big(\sum_\gamma a_\gamma c_\gamma t^\gamma\big) \cdots \big(\sum_\gamma a_\gamma c_\gamma t^\gamma\big)}_{p \text{ times}}$$

Hence, the γ-monomial of a^p is

$$\big(a^p\big)_\gamma := \sum_{\gamma_1+\cdots+\gamma_p=\gamma} a_{\gamma_1}\cdots a_{\gamma_p} t^{\gamma_1}\cdots t^{\gamma_p} = \sum_{\gamma_1+\cdots+\gamma_p=\gamma} a_{\gamma_1}\cdots a_{\gamma_p} f(\gamma_1,\ldots,\gamma_p) t^\gamma, \tag{5.5}$$

where

$$f(\gamma_1,\ldots,\gamma_p) := \frac{t^{\gamma_1}\cdots t^{\gamma_p}}{t^{\gamma_1+\cdots+\gamma_p}}.$$

Therefore,

$$a = \underbrace{\sum_{\gamma\in\Gamma} a_\gamma^p f(\gamma,\ldots,\gamma) t^{p\gamma}}_{=(5.4)} + \sum_{(\gamma_1,\ldots,\gamma_i)\in\Gamma^p\setminus\Delta} a_{\gamma_1}\cdots a_{\gamma_p} f(\gamma_1,\ldots,\gamma_p) t^{\gamma_1+\cdots+\gamma_p}, \tag{5.6}$$

where Δ is the diagonal of Γ^p. We have to show that the second summand in the previous expression is 0. Fix $\gamma_1 \ldots, \gamma_p \in \Gamma$ not all equal to each other, say

$$\begin{aligned}
\gamma_1 &= \ldots = \gamma_{n_1}\\
\gamma_{n_1+1} &= \ldots = \gamma_{n_1+n_2}\\
&\cdots\\
\gamma_{n_1+\cdots+n_{k-1}+1} &= \ldots = \gamma_p,
\end{aligned}$$

with $n_1 + \cdots + n_k = p$ and $\gamma_{n_1}, \gamma_{n_1+n_2}, \ldots, \gamma_p$ all distinct. The monomial

$$a_{\gamma_1}\cdots a_{\gamma_p} f(\gamma_1,\ldots,\gamma_p) t^{\gamma_1+\cdots+\gamma_p}$$

appears in equation (5.6) as many times as the number of ways of distributing p objects among k boxes of capacity $n_1, \ldots, n_k$ respectively. The latter is equal to

$$m := \binom{p}{n_1}\binom{p-n_1}{n_2}\cdots\underbrace{\binom{p-n_1-\cdots-n_{k-1}}{n_k}}_{=1}.$$

However, $k > 1$ and all the n_i are non-zero, thus $1 \le n_1 < p$. Moreover, p is prime, hence $p \mid m$, and the conclusion follows.

An alternative proof:

1. Using the formula $(x+y)^p = x^p + y^p$, prove equation (5.4) in the case $\operatorname{supp}(a)$ is finite.
2. For the general case, observe that in equation (5.5) only a finite number of monomials of a is involved and apply the above particular case. $\square$

An immediate consequence of the previous proposition is the following lemma.

Lemma 5.22. *Suppose that* $\operatorname{char}\mathbf{k} = p > 0$. *Let* $\mathbb{F}$ *be a truncation-closed subfield of* $\mathbb{K}$, *and* $\widetilde{\mathbb{F}} := \{a \in \mathbb{K} : a^p \in \mathbb{F}\}$. *Then,* $\widetilde{\mathbb{F}}$ *is also truncation-closed.*

6 Truncation-closed embeddings in characteristic 0

We are now ready to state and prove the embedding theorem in characteristic 0. Let $\mathbb{K}$ be a valued field, with value group Γ and residue field $\mathbf{k}$ of characteristic 0. Assume that $\mathbb{K}$ contains its residue field, and that there exists a good section $s : \Gamma \to \mathbb{K}^\star$; let f be its factor set. Note that if $\mathbb{K}$ is Henselian, these two assumptions hold, by Lemmata 4.8 and 4.10.

Let $\mathbf{k}((\Gamma, f))$ be the field of generalised power series with factor set f, and $s' : \Gamma \to \mathbf{k}((\Gamma, f))$ be its canonical section. Note also that $\mathbf{k}(\Gamma, f)$ is a common subfield of $\mathbf{k}((\Gamma, f))$ and of $\mathbb{K}$.

Theorem 6.1. *With the above notation, if* $\mathbb{K}$ *is Henselian, then it has a truncation-closed analytic embedding* ϕ *in* $\mathbf{k}((\Gamma, f))$, *which is over* $\mathbf{k}$ *and commutes with* s, s', *namely* $s' \circ \phi = s$.

Proof. Let ϕ be a maximal truncation-closed analytic embedding of a subfield $\mathbb{F} \subseteq \mathbb{K}$ in $\mathbf{k}((\Gamma, f))$ over $\mathbf{k}(\Gamma, f)$. W.l.o.g., ϕ is the identity. Theorem 5.16 implies that $\mathbb{F}$ is Henselian, and Lemma 2.16 implies that it is algebraically maximal.

Suppose, for contradiction, that $\mathbb{F} \neq \mathbb{K}$. Let $x \in \mathbb{K} \setminus \mathbb{F}$. Proposition 5.14 implies that there exists a sequence $(x_i)_{i \in I}$ in $\mathbb{F}$ converging to x and without pseudo-limit in $\mathbb{F}$.

Since $\mathbb{F}$ is algebraically maximal, by Proposition 5.15, $(x_i)_{i \in I}$ is of transcendental type. Corollary 5.17 implies that there exists $x' \in \mathbf{k}((\Gamma, f))$ such that x' is a pseudo-limit of $(x_i)_{i \in I}$ and $\mathbb{F}(x')$ is truncation-closed and analytically isomorphic to $\mathbb{F}(x)$ over $\mathbb{F}$, a contradiction. $\square$

Corollary 6.2. *Let* $\mathbb{K}$ *be a Henselian field such that* $\operatorname{char}\mathbb{K} = \operatorname{char}\mathbf{k} = 0$. *Then,* $\mathbb{K}$ *has a truncation-closed embedding in* $\mathbf{k}((\Gamma, f))$ *for some factor set* f.

Proof. We have proved that every such field contains its residue field and admits a good section with some factor set f. Apply Theorem 6.1. $\square$

Corollary 6.3. *If* $\mathbb{K}$ *is an ordered Henselian field, then* $\mathbb{K}$ *has a truncation-closed embedding in* $\mathbf{k}((\Gamma, f))$ *for some factor set* f. *Consequently, every ordered Henselian* $\mathbb{K}$ *has integer part.*

Proof. Every such field contains its residue field and admits a good section with some factor set f. Therefore, the first part of the corollary follows immediately from Theorem 6.1.

For the second part, we can reason as in [1]. $\square$

Proposition 6.4. *Assume that $\mathbb{K}$ is Henselian and contains $\mathbf{k}$ (but not that* $\operatorname{char}\mathbf{k} = 0$*). Suppose that its value group is the integers. Then, $\mathbb{K}$ has a truncation-closed analytically embedding ϕ in $\mathbf{k}((\mathbb{Z}))$, which is over $\mathbf{k}$ and commutes with s, s'.*

Proof. Since the value group is $\mathbb{Z}$, we can find a cross-section $s : \mathbb{Z} \to \mathbf{k}^\star$. Moreover, the completion(4) of $\mathbf{k}(\mathbb{Z})$ is $\mathbf{k}((\mathbb{Z}))$ and contains $\mathbb{K}$. Finally, any subfield of $\mathbf{k}((\mathbb{Z}))$ containing $\mathbf{k}[t]$ is truncation-closed. ☐

7 Field families and subfields of $\mathbf{k}((\Gamma))$ of bounded length

Before examining Henselian fields of finite and mixed characteristic, we will study further the truncation-closed subfields of $\mathbf{k}((\Gamma))$.

Let $\mathbf{k}$ be a field, Γ be an Abelian ordered group, and $\Gamma^+ := \{\gamma \in \Gamma : \gamma \geq 0\}$. Fix once and for all a 2-co-cycle $f : \Gamma \times \Gamma \to \mathbf{k}^\star$.

7.1 Field families

For every $A \subseteq \Gamma$ and $\gamma \in \Gamma$, define $A + \gamma := \{\alpha + \gamma : \alpha \in A\}$ and $[A]$ to be the semigroup generated by A (namely, the set of finite sums of elements from A). Let $\mathfrak{A}$ be a family of subsets of Γ.

Definition 7.1. $\mathfrak{A}$ is a *field-family* in Γ if

1. Every $A \in \mathfrak{A}$ is well-ordered.
2. For every $\gamma \in \Gamma$, the singleton $\{\gamma\}$ is in $\mathfrak{A}$.
3. For every $A, B \in \mathfrak{A}$, $A \cup B \in \mathfrak{A}$.
4. For $A \in \mathfrak{A}$ and $B \subseteq A$, $B \in \mathfrak{A}$.
5. For every $A \in \mathfrak{A}$ and $\gamma \in \Gamma$, $A + \gamma \in \mathfrak{A}$,
6. For every $A \in \mathfrak{A}$ such that $A \subseteq \Gamma^+$, $[A] \in \mathfrak{A}$.

$\mathbf{k}((\mathfrak{A}, f))$ is the subset of $\mathbf{k}((\Gamma, f))$ of power series with support in $\mathfrak{A}$. Field families were introduced by Rayner [15].

Lemma 7.2. *If $\mathfrak{A}$ is a field family in Γ, then $\mathbf{k}((\mathfrak{A}, f))$ is a Henselian subfield of $\mathbf{k}((\Gamma, f))$.*

The proof is a modification of the one given in [15].

Proof.

$$\operatorname{supp}(x+y) \subseteq \operatorname{supp}(x) \cup \operatorname{supp}(y) \text{ and}$$
$$\operatorname{supp}(xy) \subseteq \operatorname{supp}(x) + \operatorname{supp}(y).$$

Thus, $\mathbf{k}((\mathfrak{A}, f))$ is a ring.

If $v(x) = 0$, then, by B.H. Neumann's lemma, $\operatorname{supp}(1/x) \subseteq [\operatorname{supp}(x)]$, hence $\mathbf{k}((\mathfrak{A}, f))$ is a field.

Let $p(X) \in \mathbf{k}((\mathfrak{A}, f))[X]$, $x \in \mathbf{k}((\Gamma, f))$ such that $v(p) = 0$, $v(x) \geq 0$, $p(x) = 0$ and $v(\dot{p}(x)) = 0$. We have to show that $x \in \mathbf{k}((\mathfrak{A}, f))$.

(4)As a metric space

Define $\operatorname{supp}\bigl(p(X)\bigr)$ as the union of the supports of the coefficients of $p(X)$. Let $Q := [\operatorname{supp}\bigl(p(X)\bigr)] \in \mathfrak{A}$, and $\mathfrak{T}$ be the set of $x_i \in \mathbf{k}((\Gamma))$ such that $\operatorname{supp}(x_i) \subseteq Q$, and $0 < v(x_i - x) \in \Gamma \cup \{\infty\}$. Define a partial order on $\mathfrak{T}$, $x_i < x_j$, to mean $v(x_i - x) < v(x_j - x)$. Any chain in $\mathfrak{T}$ has an upper bound in $\mathfrak{T}$, for if $\{x_i : i \in I\}$ is a chain, and, for every $i \in I$, $\gamma_i := v(x_i - x)$, then we can define $y := \sum_\lambda a_\lambda t^\lambda$, where $a_\lambda = a_\lambda^{(i)}$ if $\lambda < \gamma_i$ for some $i \in I$ (where $x_i = \sum_\lambda a_\lambda^{(i)} t^\lambda$), and $a_\lambda = 0$ otherwise. Note that $\operatorname{supp}(y) \subseteq Q$ by construction. It is then clear that $y \in \mathfrak{T}$.

By Zorn's lemma, $\mathfrak{T}$ has a maximal element, say y_0. If $y_0 \neq x$, let

$$y_1 := y_0 - \frac{p(y_0)}{\dot{p}(y_0)}.$$

Then $y_1 \in \mathfrak{T}$, and $v(y_1 - x) > v(y_0 - x)$, contradicting the maximality of y_0. □

7.2 Examples

7.2.1 $\mathbf{k}((\Gamma, f))_\varepsilon$

The family of all well-ordered subsets of Γ is a field-family (by B.H. Neumann's lemma). If $A \subseteq \Gamma$ is well-ordered, $\mathrm{o}(A)$ is the order-type of A.

We remind that an epsilon number is an ordinal ε such that $\omega^\varepsilon = \varepsilon$ (ordinal exponentiation).

Lemma 7.3. *Let ε be an epsilon number and $\mathfrak{A}$ be the family of well-ordered subsets of Γ of order-type less than ε. Then, $\mathfrak{A}$ is a field-family.*

In this case, $\mathbf{k}((\mathfrak{A}, f))$ is denoted by $\mathbf{k}((\Gamma, f))_\varepsilon$. (5)

Proof. The only difficult point is 6. However, [16] prove that if $A \subseteq \Gamma^+$ is well-ordered, then $\mathrm{o}([A]) \leq \omega^{\omega \mathrm{o}(A)}$, and this concludes the proof. □

Corollary 7.4. *If ε is an epsilon number, then $\mathbf{k}((\Gamma, f))_\varepsilon$ is a Henselian field.*

7.2.2 Algebraically closed fields

The following is a well-known fact.

Lemma 7.5. *Let $\mathbb{K}$ be a valued field, with value group Γ and residue field $\mathbf{k}$. Assume that* $\operatorname{char}\mathbf{k} = 0$. *Then, $\mathbb{K}$ is algebraically closed iff it is Henselian, $\mathbf{k}$ is algebraically closed and Γ is divisible.*

Let $\mathbf{k}$ be an algebraically closed field, and p be either $\operatorname{char}\mathbf{k}$ if it is > 0, or 1 otherwise. Suppose that f is a p-good factor set. Let $\widetilde{\Gamma}$ be the divisible hull of Γ.

Define $\mathfrak{A}$ to be the family of well-ordered subsets of $\widetilde{\Gamma}$ contained in some subset of the form

$$\frac{1}{d} \bigcup_{k \in \mathbb{N}} \frac{\Gamma}{p^k},$$

as d varies in $\mathbb{N}^\star$. $\mathfrak{A}$ is a field family. For the following theorem we need some facts that we will prove later.

(5) The usual definition of $\mathbf{k}((\Gamma, f))_\varepsilon$, e.g. in [7], is only for ε a cardinal number, and asks that the cardinality of the support is less or equal to ε. I hope that the present lemma justifies our departure from that convention.

Theorem 7.6 (Rayner). $\mathbf{k}((\mathfrak{A}))$ *defined as above is an algebraically closed field.*

$\mathbf{k}((\mathfrak{A}))$ is a Henselian valued field, with algebraically closed residue field and divisible value group. If $\operatorname{char}\mathbf{k} = 0$, $\mathbf{k}((\mathfrak{A}))$ is algebraically closed, by Lemma 7.5. If $\operatorname{char}\mathbf{k} = p$, the conclusion is immediate from Lemma 9.9 and Proposition 9.3.

7.3 Ax-Kochen-Ershov theorem

Let $\aleph$ be an uncountable cardinal number.

Definition 7.7 ($\aleph$-pseudo-complete). Let $\mathbb{K}$ be a valued field. $\mathbb{K}$ is *$\aleph$-pseudo-complete* iff every pseudo-Cauchy sequence of length strictly less than $\aleph$ has a pseudo-limit in $\mathbb{K}$.(6)

Remark 7.8. Let $\mathbb{K}$ be a valued field. If $\mathbb{K}$ is $\aleph$-saturated (in the sense of model theory), then it is $\aleph$-pseudo-complete.

Remark 7.9. $\mathbf{k}((\Gamma, f))_{\aleph}$ is $\aleph$-pseudo-complete.

Remark 7.10. Let ε_n be the n^{th} epsilon number, for n an ordinal, and $\aleph$ an uncountable cardinal number. Then, $\varepsilon_{\aleph} = \aleph$.

Proposition 7.11. *Let m be an ordinal number, $\mathbb{F}$ a truncation-closed subfield of $\mathbf{k}((\Gamma, f))_{\varepsilon_m}$ and $(x_i)_{i\in I}$ a pseudo-Cauchy sequence in $\mathbb{F}$. Then, there exists $x \in \mathbf{k}((\Gamma, f))_{\varepsilon_{m+1}}$ which is a pseudo-limit of $(x_i)_{i\in I}$ and such that $\mathbb{F}(x)$ is a truncation-closed subfield of $\mathbf{k}((\Gamma, f))_{\varepsilon_{m+1}}$.*

If $(x_i)_{i\in I}$ is of transcendental type and y is any pseudo-limit of it (in some extension $\mathbb{L}$ of $\mathbb{F}$), then there is an analytic isomorphism from $\mathbb{F}(x)$ to $\mathbb{F}(y)$ over $\mathbb{F}$ and sending x to y.

Proof. Define x as in Corollary 5.17. Since x is a supremum of a sequence in $\mathbf{k}((\Gamma, f))_{\varepsilon_m}$, its length is less than ε_{m+1}. Hence, $\mathbb{F}(x) \subseteq \mathbf{k}((\Gamma, f))_{\varepsilon_{m+1}}$.

The second part is the same as in Corollary 5.17. □

If $F \subseteq K$ is a field extension, $\operatorname{tr}\deg(K/F)$ is the transcendence degree of K over F.

Theorem 7.12. *Let $\mathbb{K}$ be a Henselian valued field with residue field $\mathbf{k}$ of characteristic 0 and value group Γ. Suppose that $\mathbb{K}$ contains $\mathbf{k}$ and that there is a section $s : \Gamma \to \mathbb{K}^\star$ with factor set f. Let $\aleph$ be an uncountable cardinal number such that $\operatorname{tr}\deg(\mathbb{K}/\mathbf{k}(\Gamma, f)) \leq \aleph$.*

1. *There is a truncation-closed embedding of $\mathbb{K}$ in $\mathbf{k}((\Gamma, f))_{\aleph}$, preserving the section.*
2. *If moreover $\mathbb{K}$ is $\aleph$-pseudo-complete, then every such embedding is onto.*

The second assertion is a classical theorem by Ax, Kochen and Ershov (see for instance [17, 18, 19, 7, 20]).

Proof. Let $(c_i)_{i<\aleph}$ be a transcendence basis of $\mathbb{K}/\mathbf{k}(\Gamma, f)$. Define $\mathbb{K}_0 := \mathbf{k}(\Gamma, f)$, $\mathbb{K}_n := \mathbb{K}_0(c_i : i < n)$. Then, $\mathbb{K} = \mathbb{K}_{\aleph}^{\mathrm{H}}$. Define, by induction on n, a truncation-closed embedding $\phi_n : \mathbb{K}_n^{\mathrm{H}} \to \mathbf{k}((\Gamma, f))_{\varepsilon_n}$.

If n is a limit ordinal, then $\phi_n = \bigcup_{i<n} \phi_i$.

(6)Note that the definition of $\aleph$-pseudo-complete is different from the one given in [17] (where they ask that the length of the sequence is exactly $\aleph$). The reason is apparent in the following discussion.

If $n = m + 1$, apply Proposition 7.11 to extend ϕ_m to ϕ_n.

$\phi_{\aleph}$ is the embedding we were looking for.

For the second part, suppose for contradiction, that, once we have embedded $\mathbb{K}$ in a truncation-closed way in $\mathbf{k}((\Gamma, f))_{\aleph}$, $x \in \mathbf{k}((\Gamma, f))_{\aleph} \setminus \mathbb{K}$ is of minimal length. Therefore, x is a pseudo-limit of the sequence $\left(x_i\right)_{i \in I}$ of its proper initial segments. However, the length of such a sequence is less than $\aleph$, hence $\left(x_i\right)_{i \in I}$ has a pseudo-limit y in $\mathbb{K}$. Moreover $x \trianglelefteq y$, contradicting the fact that $\mathbb{K}$ is truncation-closed. $\square$

There is a kind of converse to Theorem 7.12.

Lemma 7.13. *Let* $\mathbf{k}$ *be a field of characteristic* 0, $\mathbb{K}$ *a Henselian valued field with residue field* $\mathbf{k}$ *and value group* Γ, *and* $\aleph$ *an uncountable cardinal number. Assume that* $\mathbb{K}$ *contains* $\mathbf{k}$, *and let* $s : \Gamma \to \mathbb{K}^{\star}$ *be a good section with* $\mathrm{d}s = f$. *If* $\mathbb{K}$ *is* $\aleph$*-pseudo-complete, then there is an analytic embedding of* $\mathbf{k}((\Gamma, f))_{\aleph}$ *in* $\mathbb{K}$ *over* $\mathbf{k}(\Gamma, f)$ *and preserving the section.*

Proof. The proof is fairly routine: we will build the embedding inductively. $\mathbf{k}(\Gamma, f)$ is a common subfield of both $\mathbb{K}$ and $\mathbf{k}((\Gamma, f))_{\aleph}$. Let ψ be a maximal (namely, non-extendable) embedding from a truncation-closed subfield $\mathbb{F}$ of $\mathbf{k}((\Gamma, f))_{\aleph}$ containing $\mathbf{k}(\Gamma, f)$ in $\mathbb{K}$. We have to prove that $\mathbb{F} = \mathbf{k}((\Gamma, f))_{\aleph}$.

Suppose not, and assume that ψ is the identity. $\mathbb{F}$ is Henselian, otherwise, using Corollary 7.4 and Theorem 5.16, we could extend ψ to the Henselisation of $\mathbb{F}$. Let $x \in \mathbf{k}((\Gamma, f))_{\aleph} \setminus \mathbb{F}$ of minimal length; define $\left(x_i\right)_{i<\alpha}$ to be the sequence of truncations of x; since $x \notin \mathbb{F}$, α is a limit ordinal and $\left(x_i\right)_{i<\alpha}$ has no pseudo-limit in $\mathbb{F}$. Obviously, $\alpha < \aleph$ and $\left(x_i\right)_{i<\alpha}$ is in $\mathbb{F}$ and converges to x. Since $\mathbb{K}$ is $\aleph$-pseudo-complete, $\left(x_i\right)_{i<\alpha}$ has also a pseudo-limit $x' \in \mathbb{K}$. $\mathbb{F}$ is of residue characteristic 0 and Henselian, therefore Lemma 2.16 implies that it is algebraically maximal. By Proposition 5.15, $\left(x_i\right)_{i<\alpha}$ is of transcendental type. [2, Theorem 2] implies that $\mathbb{F}(x)$ is analytically isomorphic to $\mathbb{F}(x')$ over $\mathbb{F}$, a contradiction. $\square$

Using the same proof, Theorem 7.12 can be strengthened in the following way. With the same hypothesis on $\mathbb{K}$, let $\kappa < \aleph$ be some epsilon number, let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$ containing $\mathbf{k}(\Gamma, f)$, and ϕ_0 be a truncation-closed embedding from $\mathbb{K}_0$ in $\mathbf{k}((\Gamma, f))_{\kappa}$ (analytic and over $\mathbf{k}(\Gamma, f)$). Then, there is a truncation-closed embedding ϕ of $\mathbb{K}$ in $\mathbf{k}((\Gamma, f))_{\aleph}$ extending ϕ_0.

Similarly, in Lemma 7.13 we can also suppose to have a field $\mathbb{F}$ containing $\mathbf{k}(\Gamma, f)$, and an analytic isomorphism ψ_0 from a truncation-closed subfield of $\mathbf{k}((\Gamma, f))_{\aleph}$ in $\mathbb{F}$ over $\mathbf{k}(\Gamma, f)$. Then there exists an embedding ψ of $\mathbf{k}((\Gamma, f))_{\aleph}$ in $\mathbb{K}$ extending ψ_0 and preserving the section.

Note that the use of the factor set f did not add any additional difficulty to the proofs (except the notational burden of, for instance, writing $\mathbf{k}((\Gamma, f))$ instead of $\mathbf{k}((\Gamma))$).

N. Alling [21] proved that if $\aleph$ is a regular cardinal, such that $\sum_{\kappa<\aleph} 2^{\kappa} \leq \aleph$, and Γ is the ordered divisible Abelian group which is saturated and of power $\aleph$, then $\mathbb{R}((\Gamma))_{\aleph}$ (resp. $\mathbb{C}((\Gamma))_{\aleph}$) is the real closed (algebraically closed) field, which is saturated and of power $\aleph$.

8 Surreal numbers

Let $\mathbb{K}$ be an ordered Henselian field. In this section, we will investigate the existence of an initial embedding from $\mathbb{K}$ in **No**, the field of surreal numbers **No**. The results stated here will not be used in the rest of the article. Therefore, if you are not interested in surreal numbers, you can skip it (if you do not know what surreal numbers are, you should read [14]).

Definition 8.1. Let a, b be surreal numbers. a is *simpler* than b, in symbols $a \preceq b$, iff there exists sets of surreal numbers L, L' and R, R' such that $L \subseteq L'$, $R \subseteq R'$, $L' < R'$, $a = \{ L \mid R \}$ and $b = \{ L' \mid R' \}$.

A subset $S \subseteq \mathbf{No}$ is *initial* iff for every $b \in S$ and every $a \in \mathbf{No}$ such that $a \preceq b$, $a \in S$.

It is easy to see that $(\mathbf{No}, \preceq)$ is a tree. The fundamental relation between the linear order $<$ and the partial order $\preceq$ on $\mathbf{No}$ is that in every $<$-convex subset[(7)] $S \subseteq \mathbf{No}$ there is a $\preceq$-minimum.

On $\mathbf{No}$ is also defined a power-series structure; more precisely, $\mathbf{No}$ is isomorphic to the ordered field $\mathbb{R}((\mathbf{No}))$ (the group of exponents is $\mathbf{No}$ itself). The image of $x \in \mathbf{No}$ under the canonical cross-section is denoted with ω^{-x} (therefore, $\omega := \omega^1$ is infinite).

Remark 8.2. If $x \preceq y \in \mathbf{No}$ then $\omega^x \preceq \omega^y$.

Proof. If $x = \{ x^L \mid x^R \}$ is the canonical representation of x, then

$$\omega^x = \{ 0, r\omega^{x^L} : r > 0 \in \mathbb{R} \mid s\omega^{x^R} : s > 0 \in \mathbb{R} \},$$

and similarly for y. But $x \preceq y$, therefore $x' \prec y$ for every $x' \prec x$, so $r\omega^{x'}$ is an option in the formula for ω^y. □

Lemma 8.3. *Let $\mathbb{K}$ be an arbitrary initial subfield of* $\mathbf{No}$. *Then, $\mathbb{K}$ is also truncation-closed, and therefore admits a cross-section* (*with respect to its natural valuation v*).

Moreover, its value group Γ is also initial. Finally, $\mathbb{K}$ contains its residue field as an initial subfield.

Proof. The first assertion is obvious. Let $\gamma \in \Gamma$, and let $\gamma' \prec \gamma$. We have to prove that $\gamma' \in \Gamma$. Set $\gamma = -v(a)$ for some $a \in \mathbb{K}$. Write the normal form of a

$$a = a_0\omega^\gamma + \cdots$$

Then, $\omega^\gamma \preceq a$, therefore $\omega^\gamma \in \mathbb{K}$. By Remark 8.2, $\omega^{\gamma'} \in \mathbb{K}$, hence $\gamma' \in \Gamma$, and we have proved that Γ is initial.

The valuation on $\mathbb{K}$ is the natural valuation, therefore the residue field $\mathbf{k}$ is, in a canonical way, a subfield of $\mathbb{R}$, and every subfield of $\mathbb{R}$ is initial. We have to prove that $\mathbf{k} \subseteq \mathbb{K}$. Let $a \in \mathbf{k}$. Then, there exists $x \in \mathbb{K}$ such that $x = a + \varepsilon$, with $v(\varepsilon) > 0$. Consequently, $a \preceq x$, so $a \in \mathbb{K}$. □

Theorem 8.4. *Let $\mathbb{K}$ be an ordered field, v be its natural valuation,* $\mathbf{k}$ *the residue field and Γ the value group. Assume that:*

- $\mathbb{K}$ *is Henselian*
- Γ *has an initial embedding in* $\mathbf{No}$ *as an ordered group.*
- $\mathbb{K}$ *admits a cross-section.*

Then, there is an initial embedding of $\mathbb{K}$ in $\mathbf{No}$ *as an ordered field.*

The previous theorem is also a consequence of Theorem 2.10, together with [3, Theorem 18].

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(7) Namely, if $a < b \in S$, $c \in \mathbf{No}$ and $a \leq c \leq b$, then $c \in S$.

8.1 Proof of Theorem 8.4

Any ordered field $\mathbb{F}$ is endowed with its natural valuation. In this proof, $\widetilde{\mathbb{F}}$ will denote its real closure. Proceed as in the proof of Theorem 2.10. Instead of Lemmata 2.11 and 2.12, we will use the following two lemmata, proved in [22].

Lemma 8.5. *Let $\mathbb{F}$ be an initial subfield of* **No***. Then $\widetilde{\mathbb{F}}$ is also initial.*

Lemma 8.6. *Let S be an initial subset of* **No***. Then, the field generated by S is also initial.*

The initial embedding of Γ in **No** induces in a canonical way an initial embedding of $\mathbf{k}((\Gamma))$. Let $\mathbb{F} \subseteq \mathbb{K}$ such that:

- $\mathbf{k}(\Gamma) \subseteq \mathbb{F}$.
- There is a maximal initial embedding ϕ of $\mathbb{F}$.

We have to prove that $\mathbb{K} = \mathbb{F}$. W.l.o.g., we can suppose that ϕ is the identity.

Lemma 8.7. *$\mathbb{F}$ is Henselian.*

Proof. Lemma 8.5 implies that $\widetilde{\mathbb{F}}$ is an initial subfield. Moreover, $\mathbb{F}^{\mathrm{H}} = \widetilde{\mathbb{F}} \cap \mathbf{k}((\Gamma))$. But $\widetilde{\mathbb{F}}$ and $\mathbf{k}((\Gamma))$ are both initial, so $\mathbb{F}^{\mathrm{H}}$ is. □

Lemma 8.8. $\mathbb{F} = \mathbb{K}$.

Proof. If not, let $x \in \mathbb{K} \setminus \mathbb{F}$. By the previous proposition, $\mathbb{F}$ is Henselian. Let x' in **No** be the simplest element satisfying the same cut as x over $\mathbb{F}$ (it exists by definition of **No**). Therefore, $\mathbb{F} \cup \{x'\}$ is an initial subset of **No**. Lemma 8.6 implies that $\mathbb{F}(x)$ is an initial subfield of **No**, contradicting the maximality of ϕ. □

9 Additive polynomials and power series fields

In this section, we will study the finite characteristic case.

When we will say group, we will always mean Abelian group (unless specified otherwise). Let

- $\mathbf{k}$ be a perfect field of characteristic $p > 0$, and $\widetilde{\mathbf{k}}$ its algebraic closure;[8]
- Γ be an ordered p-divisible Abelian group, and $\widetilde{\Gamma}$ its divisible hull;
- $f : \Gamma \times \Gamma \to \mathbf{k}^\star$ be a p-good 2-co-cycle, and $\widetilde{f} : \widetilde{\Gamma} \times \widetilde{\Gamma} \to \widetilde{\mathbf{k}}$ an extension of f to $\widetilde{\Gamma}$;
- $\mathbb{K} := \mathbf{k}((\Gamma, f))$, and $\widetilde{\mathbb{K}} := \widetilde{\mathbf{k}}((\widetilde{\Gamma}, \widetilde{f}))$ be the corresponding power series fields.

Definition 9.1. Let $\mathbb{F}$ be a field of characteristic $p > 0$. An *additive polynomial* is a polynomial $q(X) \in \mathbb{F}[X]$ such that $q(x+y) = q(x) + q(y)$ for every x and y in the algebraic closure of $\mathbb{F}$.
A *p-polynomial* is a polynomial of the form $q(X) - c$, where c is a constant term and $q(X)$ is an additive polynomial (cf. [23]).

An additive polynomial is of the form $a_0 X + a_1 X^p + \cdots + a_n X^{p^n}$.

[8] p-closure is also enough.

9.1 Artin-Schreier polynomials

Fix $d \in \mathbb{N}^\star$ and $b \in \mathbf{k}^\star$. Let $q := p^d$, and $\mathbf{k}[X] \ni q(X) := X^q - bX$. Note that $q(X)$ is an additive polynomial.

We wish to study the solutions in $\widetilde{\mathbb{K}}$ of the equation

$$q(X) = a \tag{9.1}$$

with $a \in \mathbb{K}$; more precisely, we will give the power series expansion of these solutions.

Remark 9.2. Let x be a solution (in $\widetilde{\mathbb{K}}$) of (9.1). Then, $y \in \widetilde{\mathbb{K}}$ is a solution of (9.1) iff $x - y$ is a solution of $q(X) = 0$.

Let $\Lambda := \mathbb{Z}/p^\infty = \left\{ \frac{n}{p^m} : n \in \mathbb{Z}, m \in \mathbb{N} \right\}$ and $\mathbb{F} = \mathbf{k}((\Lambda))$ be the power series field with coefficient in $\mathbf{k}$, exponents in Λ, factor set 1, and where we use s instead of t[9].

Proposition 9.3. *A solution in $\mathbb{F}$ of the equation $X = bX^q + s$ is*

$$f(s;b) := \sum_{n=0}^{\infty} b^{1+q+q^2+q^3+\cdots+q^{n-1}} s^{q^n}.^{(10)} \tag{9.2}$$

A solution of $q(X) = s$ is

$$g^+(s;b) := -\sum_{n=0}^{\infty} \frac{s^{q^n}}{b^{1+q+q^2+q^3+\cdots+q^n}}.^{(11)} \tag{9.3}$$

A solution of $q(X) = s^{-1}$ is

$$g^-(s^{-1};b) := \sum_{n=1}^{\infty} b^{1/q+1/q^2+1/q^3+\cdots+1/q^{n-1}} s^{-1/q^n}.^{(12)} \tag{9.4}$$

The proof is by direct computation, using Proposition 5.21. We can see that the coefficients of $g^\pm$ are in $\mathbf{k}$, and their exponents are in Λ.

Let $\mathscr{M}$ be the ideal of infinitesimal elements of $\mathbb{K}$. For a fixed $b \in \mathbf{k}$, $g^+(s;b)$ defines an analytic function from $\mathscr{M}$ into itself, which we will denote with the same name.

Remark 9.4. If $z \in \mathbb{K}$ is purely infinite (namely, $\operatorname{supp} z < 0$), then $g^-(z;b)$ converges, hence it defines a function of z.

Proof. Given a sequence $b_1, b_2, \ldots \in \widetilde{\mathbf{k}}$, let

$$g(z) := \sum_{n \in \mathbb{N}^\star} b_n z^{\frac{1}{q^n}}.$$

We need to show that if $z \in \widetilde{\mathbb{K}}$ is purely infinite, then $x := g(z)$ is a well defined element of $\widetilde{\mathbb{K}}$. Namely, we have to check that

1. $\forall \lambda \in \widetilde{\Gamma}$ there are only finitely many $(n, \gamma) \in \mathbb{N}^\star \times \operatorname{supp} z$ such that $\lambda = \frac{\gamma}{q^n}$, and

(9) Namely, the elements of $\mathbb{F}$ are formal sums $\sum_{\gamma \in \Lambda} a_\gamma s^\gamma$ with well-ordered support.
(10) The 0-term of the summation is s.
(11) The 0-term of the summation is s/b.
(12) The first term of the summation is $bs^{1/q}$.

2. $\operatorname{supp} x$ is well-ordered.

We will prove only the second point (the first is done in a similar way). Assume, for contradiction, that there exists an infinite sequence $\lambda_1 > \lambda_2 > \lambda_3 > \cdots \in \operatorname{supp} x$. Then, for every $i \in \mathbb{N}^\star$, there exists $\gamma_i \in \operatorname{supp} z$ and $n_i \in \mathbb{N}^\star$ such that $\lambda_i = \frac{\gamma_i}{q^{n_i}}$. Since $\operatorname{supp} z$ and $\mathbb{N}^\star$ are well-founded, after taking a subsequence, we can assume that $\gamma_1 \leq \gamma_2 \dots$ and $n_1 \leq n_2 \dots$. Since all the γ_i are negative, this is a contradiction. □

By direct computation, it is now easy to see that:

- if $v(a) > 0$, a solution of Equation (9.1) is $g^+(a; b)$;
- if a is purely infinite, a solution is $g^-(a; b)$;
- if $a \in \mathbf{k}$, a solution is a certain element $c_0 \in \widetilde{\mathbf{k}}$.

Hence, in general a solution of Equation (9.1) is

$$\sum_{\gamma<0} g^-(a_\gamma t^\gamma; b) + c_0 + \sum_{\gamma>0} g^+(a_\gamma t^\gamma; b) \in \widetilde{\mathbf{k}}((\frac{\Gamma}{p^\infty}, \widetilde{f})).$$

In particular, a solution of the equation

$$X^q = X + a \tag{9.5}$$

is

$$x := \sum_{\gamma<0} \sum_{n=1}^{\infty} (a_\gamma t^\gamma)^{1/q^n} + c_0 - \sum_{\gamma>0} \sum_{n=0}^{\infty} (a_\gamma t^\gamma)^{q^n}, \tag{9.6}$$

where $c_0 \in \widetilde{\mathbf{k}}$ is a solution of $X^q = X + a_0$.

Note also that if $v(a) > 0$, then $(X^q - bX - a)(0) = a$, which is infinitesimal, and $v(\dot{q}(x)) = v(b) = 0$ for every x, thus the existence of an infinitesimal solution $x \in \mathbb{K}$ is also implied by Hensel's lemma.

Every $x \in \mathbb{K}$ can be written uniquely as $x = x^- + x_0 + x^+$, where x^- is purely infinite, $x_0 \in \mathbf{k}$, and $v(x^+) > 0$. Therefore, if we define $g^\pm(y) := g^\pm(y; 1)$, (9.6) becomes $x = g^-(a^-) + c_0 + g^+(a^+)$.

Hence, the support of any solution x of Equation (9.5) is contained in

$$\bigcup_{n\geq 1} \frac{\operatorname{supp}(a^-)}{q^n} \dot{\cup} \{0\} \dot{\cup} \bigcup_{n\geq 0} q^n \operatorname{supp}(a^+).$$

Given $m \in \mathbb{N}$, define

$$\begin{aligned} g_m^+(a) &:= -\sum_{n=0}^{m} \sum_{\gamma>0} (a_\gamma t^\gamma)^{q^n} = -\sum_{n=0}^{m} (a^+)^{q^n}, \\ g_m^-(a) &:= \sum_{n=1}^{m} \sum_{\gamma<0} (a_\gamma t^\gamma)^{1/q^n} = \sum_{n=1}^{m} (a^-)^{1/q^n}. \end{aligned} \tag{9.7}$$

Note that $g_m^\pm$ and $g^\pm$ are additive functions.

Lemma 9.5. *Let $a \in \mathbb{K}$, x as in Equation* (9.6)*. If $v(a) > 0$, then*

$$\left\{ g^+(b) + g_m^+(a-b) : b \triangleleft a, 0 \leq m \in \mathbb{N} \right\} \succ x$$

If a is purely infinite, then

$$\left\{ g^-(b) + g_m^-(a-b) : b \triangleleft a, 0 \leq m \in \mathbb{N} \right\} \succ x$$

Proof. Assume $v(a) > 0$. Let $y \triangleleft x$ and $\alpha := v(y-x)$. Then, $\alpha = q^{m-1}\lambda$ for some $m \geq 1$ and $0 < \lambda \in \operatorname{supp}(a)$. Let b be the truncation of a at λ (namely, $b \triangleleft a$ and $v(b-a) = \lambda$) and

$$z := g^{+}(b) + g^{+}_{m}(a-b).$$

It remains to show that $v(z-x) > \alpha$.

$$x - z = g^{+}(a) - g^{+}(b) - g^{+}_{m}(a-b) = (g^{+} - g^{+}_{m})(a-b) = -\sum_{n \geq m} \sum_{\gamma \geq \lambda} (a_\gamma t^\gamma)^{q^n}.$$

Therefore, $v(x-z) \geq q^m \lambda > q^{m-1}\lambda = \alpha$.

If a is purely infinite, let $y \triangleleft x$ and $\alpha := v(y-x)$. Then, $\alpha = \frac{\lambda}{q^{m-1}}$ for some $m \geq 2$ and $0 > \lambda \in \operatorname{supp}(a)$. Define b to be the truncation of a at λ, and

$$z := g^{-}(b) + g^{-}_{m}(a-b).$$

Then, $v(x-z) \geq \frac{\lambda}{q^m} > \frac{\lambda}{q^{m-1}} = \alpha$. □

Hypothesis A. Let $\mathbb{K}$ be a valued field, Γ be its value group and $\mathbf{k}$ its residue field, with $\operatorname{char} \mathbf{k} = p$. If $p = 0$, the hypothesis is vacuous. If $p > 0$, then

1. Any polynomial of the form

$$X^{p^n} + a_{n-1}X^{p^{n-1}} + \cdots + a_1 X^p + a_0 x + b,$$

   with coefficients in $\mathbf{k}$ has a root in $\mathbf{k}$.

2. $\Gamma = p\Gamma$.

Kaplansky introduced the Hypothesis A in [2], and G. Whaples in [24] proved that a field $\mathbf{k}$ of characteristic $p > 0$ satisfies the condition A-1 iff it has no algebraic extension of degree divisible by p (cf. also [25] for an elementary proof of this fact).

Theorem 9.6 (Kaplansky[(13)]). *Let $\mathbb{K}$ be a valued field, Γ be its value group and $\mathbf{k}$ its residue field, with $\operatorname{char} \mathbf{k} = p$. $\mathbb{K}$ is maximal iff it contains a pseudo-limit for each of its pseudo-convergent sequences.*

If $\mathbb{K}$ is maximal, $\mathbf{k}$ is perfect, and $\Gamma = p\Gamma$, then $\mathbb{K}$ is perfect.

If $\mathbf{k}$ and Γ satisfy Hypothesis A, then the maximal immediate extension $\mathbb{L}$ of $\mathbb{K}$ is uniquely determined up to analytic isomorphism over $\mathbb{K}$. Moreover, $\mathbb{L}$ is perfect and isomorphic to $\mathbf{k}((\Gamma, f))$ for some factor set f.

Let $\mathbb{F}$ and $\mathbb{F}'$ be two maximal extensions of $\mathbb{K}$, with the same value group Λ and residue field $\mathbf{f}$. If $\mathbf{f}$ and Λ satisfy Hypothesis A, and if every element of $\mathbf{f}$ has an n^{th} root for every n, then $\mathbb{F}$ and $\mathbb{F}'$ are analytically isomorphic over $\mathbb{K}$.

If $\operatorname{char} \mathbb{K} = \operatorname{char} \mathbf{k}$, then $\mathbb{K}$ is isomorphic to a subfield of a power series field.

An immediate consequence of Lemma 9.5 is the following corollary.

Corollary 9.7. *Assume that $\mathbb{K} := \mathbf{k}((\Gamma, f))$ satisfies Kaplansky's Hypothesis A. Let $\mathbb{F}$ be a truncation-closed subfield of $\mathbb{K}$.*

Fix q a power of p, and let $\mathbb{E}$ be the smallest perfect subfield of $\mathbb{K}$ containing $\mathbb{F}$ and such that every equation of the form $X^q = X + a$, with a in $\mathbb{E}$, has a solution in $\mathbb{E}$.

Then, $\mathbb{E}$ is also truncation-closed.

(13) See [2] and [13, Chapter 7]

Sketch of proof. Let S be the maximal truncation-closed subset of $\mathbb{E}$. Then, S is a perfect subfield of $\mathbb{K}$ containing $\mathbb{F}$. Suppose, for contradiction, that $a \in S$ is of minimal length such that, if x is the solution of $X^q = X + a$ given by (9.6), then $x \notin S$. Since $x = g^-(a^-) + c_0 + g^+(a^+)$, we can assume w.l.o.g. that either $a = a^-$ or $a = a^+$. We will deal with the case $a = a^-$ (the other one is similar). By minimality of a, $g^-(b) \in S$ for every $b \lhd a$. Moreover, $g^-_m(a-b) \in S$ for every $m \in \mathbb{N}$ and every $b \in S$, since g^-_m is a finite sum. Hence, by Lemma 9.5, $S \succ x$, contradicting the maximality of S.

Note that $\mathbb{E}$ is built by successive extensions by purely inseparable elements and roots of polynomials of the form $X^q - X - a$. □

The following lemma is a consequence of Ostrowski's theorem [6, Theorem 2 pag. 236].

Lemma 9.8. *Let $\mathbb{F}$ be a Henselian valued field, with residue characteristic $p > 0$. Let $\mathbb{H}$ be an immediate algebraic extension of $\mathbb{F}$ such that $n := [\mathbb{H} : \mathbb{F}]$ is finite. Then, n is a power of p.*

Lemma 9.9. *Let $\mathbb{F}$ be a valued field with residue field $\mathbf{k}$ and value group Γ. Assume that:*

1. *$\mathbb{F}$ is Henselian, perfect and of characteristic $p > 0$.*
2. *$\mathbf{k}$ is algebraically closed.*
3. *Γ is divisible.*
4. *Every polynomial $X^p - X - a \in \mathbb{F}[X]$ has a solution in $\mathbb{F}$.*

Then, $\mathbb{F}$ is algebraically closed.

Proof. The proof proceeds as in [15]. Note that Condition 4 is equivalent to

4'. *Every polynomial $X^p - X - a \in \mathbb{F}[X]$ has* all *solutions in $\mathbb{F}$.*

Let $\mathbb{L}$ be a finite extension of $\mathbb{F}$, and $n := [\mathbb{L} : \mathbb{F}]$. We must prove that $n = 1$. W.l.o.g., $\mathbb{L}/\mathbb{F}$ is normal, and since $\mathbb{F}$ is perfect, it is a Galois extension, with Galois group G. $\mathbb{L}$ must be an immediate extension of $\mathbb{F}$, and Lemma 9.8 implies that $n = p^k$. If, by absurd, $k > 0$, then G contains a normal subgroup H of power p^{k-1} [26, Corollary 6.6]. Let $\mathbb{L}'$ be the fixed field of H: it is a Galois extension of $\mathbb{F}$ of degree p. By [26, Theorem 6.4], any such extension is generated by a zero of $X^p - X - a$ for some $a \in \mathbb{F}$, which, by 4', is already in $\mathbb{F}$, a contradiction. □

Lemma 9.10. *Assume that $\mathbf{k}$ is algebraically closed and Γ is divisible. Let $\mathbb{F}$ is a truncation-closed subfield of $\mathbf{k}((\Gamma, f))$ containing $\mathbf{k}(\Gamma, f)$, and $\mathbb{E}$ be its algebraic closure (inside $\mathbf{k}((\Gamma, f))$). Then $\mathbb{E}$ is also truncation-closed.*

Proof. Lemma 9.9 implies that $\mathbb{E}$ is the closure of the Henselisation of $\mathbb{F}$ under solutions of equations $X^p = X + a$, hence Corollary 9.7 implies that $\mathbb{E}$ is truncation-closed. □

Proposition 9.11. *Let $\mathbb{H}$ be a truncation-closed subfield of $\mathbf{k}((\Gamma, f))$ containing $\mathbf{k}(\Gamma, f)$, and $\mathbb{E} \subseteq \mathbf{k}((\Gamma, f))$ be its relative algebraic closure. Then, $\mathbb{E}$ is also truncation-closed.*

Moreover, $\mathbb{E}$ is algebraically maximal.

Proof. Let $\widetilde{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$, $\widetilde{\Gamma}$ be the divisible hull of Γ. Extend the p-good co-cycle f to a p-good co-cycle

$$\widetilde{f} : \widetilde{\Gamma} \times \widetilde{\Gamma} \to \mathbf{k}^{\star}.^{(14)}$$

Let $\widetilde{\mathbf{k}}((\widetilde{\Gamma}, \widetilde{f}))$ be the corresponding power series field, $\mathbb{L}$ be the field generated by $\mathbb{H}$, $\widetilde{\mathbf{k}}$ and $\widetilde{\Gamma}$ in $\widetilde{\mathbf{k}}((\widetilde{\Gamma}, \widetilde{f}))$ and $\widetilde{\mathbb{L}}$ be its algebraic closure (embedded in $\widetilde{\mathbf{k}}((\widetilde{\Gamma}, \widetilde{f}))$). To simplify the notation, we will assume that $\widetilde{f} = 1$, and drop it. Lemma 9.10 implies that $\widetilde{\mathbb{L}}$ is truncation-closed. Moreover, $\mathbb{E} = \mathbf{k}((\Gamma)) \cap \widetilde{\mathbb{L}}$. Therefore, $\mathbb{E}$ is truncation-closed.

Let $\mathbb{F}$ be an immediate algebraic extension of $\mathbb{E}$. $\mathbb{E}$ is Henselian (since it is a relatively algebraically closed subfield of the Henselian field $\mathbf{k}((\Gamma))$), so, by the uniqueness extension property of Henselian fields, there exists a unique embedding of $\mathbb{F}$ in $\widetilde{\mathbb{L}}$ analytic over $\mathbb{E}$. We have to prove that $\mathbb{F} = \mathbb{E}$. It is enough to show that $\mathbb{F} \subseteq \mathbf{k}((\Gamma))$. Suppose not. Let $x \in \mathbb{F} \setminus \mathbf{k}((\Gamma))$ of minimal length. We can choose $\mathbb{F}$ such that $\ell(x)$ is minimal.

Therefore, for every $y \triangleleft x$, either $\mathbb{E}(y)$ is not immediate algebraic over $\mathbb{E}$, or $y \in \mathbb{E}$ (otherwise, $\big(\mathbb{E}(y), y\big)$ would satisfy the same condition as $\big(\mathbb{F}, x\big)$, contradicting the minimality of $\ell(x)$).

Suppose that every $y \triangleleft x$ is in $\mathbb{E}$. In this case, we must have that $\ell(x)$ is a successor ordinal (otherwise, $x \in \mathbf{k}((\Gamma))$). Hence, $x = y + a_\gamma t^\gamma$, for a unique $y \triangleleft x$ such that $\operatorname{supp}(y) < \gamma$. However, $\mathbb{E}(x)$ is an immediate extension of $\mathbb{E}$, thus $\gamma \in \Gamma$ and $a_\gamma \in \mathbf{k}$, whence $x \in \mathbf{k}((\Gamma))$, a contradiction.

Therefore, there exists $y \triangleleft x$ such that $\mathbb{E}(y)$ is not an immediate algebraic extension of $\mathbb{E}$. Choose y to be of minimal length. However, $\widetilde{\mathbb{L}}$ is a truncation-closed subfield of $\widetilde{\mathbf{k}}((\widetilde{\Gamma}))$, and $x \in \widetilde{\mathbb{L}}$, so $y \in \widetilde{\mathbb{L}}$, hence y is algebraic over $\mathbb{E}$. We conclude that $\mathbb{E}(y)$ is not an immediate extension of $\mathbb{E}$, thus $y \notin \mathbf{k}((\Gamma))$. Again, the length of y must be a successor ordinal (otherwise, $y \in \mathbf{k}((\Gamma))$, because all $z \triangleleft y$ are in $\mathbf{k}((\Gamma))$ by minimality of $\ell(y)$). Hence, $y = z + a_\gamma t^\gamma$, for a unique $z \triangleleft y$ such that $\operatorname{supp}(z) < \gamma$. $\gamma \in \widetilde{\Gamma}$, $a_\gamma \in \widetilde{\mathbf{k}}$ and, by minimality, $z \in \mathbb{E}$. However, $v(x - z) = v(x - y) = \gamma$, thus $\gamma \in \Gamma$. Moreover, $\overline{\left(\frac{x-z}{t^\gamma}\right)} = a_\gamma$, thus $a_\gamma \in \mathbf{k}$. Therefore, $\mathbb{E}(y)$ is an immediate extension of $\mathbb{E}$, and we have a contradiction. $\square$

For a (relatively) long time I tried to prove Proposition 9.11 directly, and failed, until I saw that I could enlarge the original field $\mathbb{H}$ to $\mathbb{L}$, prove the lemma for it, and then restrict back to $\mathbb{H}$. This kind of "enlargement" trick is quite useful, and I use it also in other places.

F. Delon gives an example of a maximal valued field $\mathbb{M}$ and a subfield $\mathbb{E}$ such that $\mathbb{E}$ is relatively algebraically closed in $\mathbb{M}$, and $\mathbb{M}/\mathbb{E}$ is immediate, and yet $\mathbb{E}$ is not algebraically maximal [27, Example 1.12 pag. 14]. Hence, the somewhat lengthy proof of the second part of Proposition 9.11 is really needed, and we face a phenomenon peculiar to truncation-closed subfields, which I think deserves further investigation.

Moreover, I think that a direct proof of Proposition 9.11 would give a better insight of the structure of algebraically maximal fields, in the same way as Lemma 9.9 clarifies the structure of algebraically closed fields.

Theorem 9.12. *Let $\mathbb{F}$ be an algebraically closed valued field of characteristic $p > 0$, $\mathbf{k}$ be its residue field and Γ be its value group. Then, for every embedding of $\mathbf{k}$ in $\mathbb{F}$ and p-good section $s : \Gamma \to \mathbb{F}^\star$, there is an analytic truncation-closed embedding of $\mathbb{F}$ in $\mathbf{k}((\Gamma, f))$ over $\mathbf{k}$ and commuting with s, where $f := \mathrm{d}s$.*

(14) $\widetilde{f}$ exists by Lemma 4.10.

Note that there is always at least one embedding of $\mathbf{k}$ and one p-good section s.

Proof. Proceed as in the proof of Theorem 6.1, using Lemma 9.10 instead of Theorem 5.16. □

An additional problem in finite characteristic is that a valued field $\mathbb{F}$ (even a Henselian one) may have many non-isomorphic maximal algebraic immediate extensions. However, if $\mathbb{F}$ satisfies Kaplansky's Hypothesis A, then it has only one such extension. Therefore, using Proposition 9.11 and [2, Theorem 5], the previous theorem can be extended to algebraically maximal valued fields satisfying Kaplansky's Hypothesis A.

9.2 Subfields of $\mathbf{k}((\Gamma))$ of bounded length in finite characteristic.

Lemma 9.13. *If Γ is divisible and $\mathbf{k}$ is algebraically closed, then $\mathbf{k}((\Gamma, f))_\varepsilon$ is algebraically closed.*

In general, $\mathbf{k}((\Gamma, f))_\varepsilon$ is algebraically maximal.

Proof. Call $\mathbb{F} := \mathbf{k}((\Gamma, f))_\varepsilon$. Corollary 7.4 implies that $\mathbb{F}$ is a Henselian field.

If Γ is divisible and $\mathbf{k}$ algebraically closed, propositions 5.21 and 9.3 imply that $\mathbb{F}$ is also perfect and closed under solutions of polynomials $X^p - X - c$, therefore, by Lemma 9.9, it is algebraically closed.

For the general case, suppose for contradiction that $\mathbb{E}$ is some immediate extension of $\mathbb{F}$, $p(X) \in \mathbb{F}[X]$ is monic irreducible, and $x \in \mathbb{E}$ is some root of $p(X)$. Since $\mathbb{L} := \widetilde{\mathbf{k}}((\widetilde{\Gamma}, \widetilde{\mathbf{f}}))_\varepsilon$ is algebraically closed, all roots of $p(X)$ are in $\mathbb{L}$. Let $y \in \mathbb{L}$ be one of these roots. $\mathbb{F}$ is Henselian, therefore $\mathbb{F}(y)$ is analytically isomorphic to $\mathbb{F}(x)$ over $\mathbb{F}$ (Lemma 2.4), hence they are both proper immediate extensions of $\mathbb{F}$.

Let $y = \sum_{i<\alpha} a_i t^{\gamma_i}$, where $\alpha < \varepsilon$, and $a_\gamma t^\gamma$ be the first monomial not in $\mathbb{F}$: therefore, either $a_\gamma \notin \mathbf{k}$, or $\gamma \notin \Gamma$. Let z be the truncation of y at γ:

$$y = z + a_\gamma t^\gamma + \mathrm{o}(t^\gamma).$$

However, $z \in \mathbb{F}$, hence $\mathbb{F}(y)$ cannot be an immediate extension of $\mathbb{F}$ (because if $a_\gamma \notin \mathbf{k}$ it would extend the residue field, if $\gamma \notin \Gamma$ the value group). □

Putting together Lemma 9.13 and Proposition 9.11, one can proceed as in the proof of Theorem 7.12 and prove the following analogue of Theorem 7.12 in the finite characteristic case.

Theorem 9.14. *Let $\operatorname{char}\mathbf{k} > 0$, and $\mathbb{F}$ be an algebraically maximal valued field. Assume that $\mathbb{F}$ contains its residue field $\mathbf{k}$ and that there is a p-good section s from its value group Γ into $\mathbb{F}^\star$, with $f := \mathrm{d}s$. Assume moreover that $\mathbb{F}$ satisfies Kaplansky's Hypothesis A. Let $\aleph$ be an uncountable cardinal such that $\operatorname{tr\,deg}\big(\mathbb{F}/\mathbf{k}(\Gamma, f)\big) \leq \aleph$. Then,*

1. *There exists a truncation-closed embedding of $\mathbb{F}$ in $\mathbf{k}((\Gamma, f))_\aleph$.*
2. *If moreover $\mathbb{F}$ is $\aleph$-pseudo-complete, then every such embedding is onto.*

10 The mixed characteristic case

We will now treat the case of fields of mixed characteristic. In particular, we will re-prove the Ax-Kochen isomorphism theorem for formally p-adic fields.

10.1 Decomposition of valuations

Let Γ be an Abelian ordered group, $\Delta \subseteq \Gamma$ be a *convex* subgroup of Γ, Λ be the quotient Γ/Δ, and $\rho : \Gamma \to \Lambda$ be the corresponding projection. Note that the ordering on Γ induces an ordering $<$ on Λ, and with this ordering Λ is also an ordered group.

Let $s : \Lambda \to \Gamma$ be a map such that $\rho s = \mathrm{id}_\Lambda$, and $m = ds : \Gamma \times \Gamma \to \Delta$ be is co-boundary (namely, $m(\alpha,\beta) = s\alpha + s\beta - s(\alpha+\beta)$). Define $\Lambda \times_m \Delta$ as the set of pairs (λ,δ) with sum twisted by m:

$$(\lambda,\delta) +_m (\lambda',\delta') = \bigl(\lambda+\lambda',\ \delta+\delta'+m(\lambda,\lambda')\bigr),$$

and lexicographic order.

Lemma 10.1. *The map*

$$\begin{aligned}\Phi_m :&\Lambda \times_m \Delta \to \Gamma\\ &(\lambda,\delta) \mapsto s\lambda + \delta\end{aligned}$$

is an isomorphism of ordered groups.

Proof.

$$\begin{aligned}\Phi_m(\lambda,\delta) + \Phi_m(\lambda',\delta') = s\lambda + s\lambda' + \delta + \delta' = s(\lambda+\lambda') + m(\lambda,\lambda') + \delta + \delta' =\\ = \Phi_m\bigl(\lambda+\lambda', m(\lambda,\lambda') + \delta + \delta'\bigr) = \Phi_m\bigl((\lambda,\delta) +_m (\lambda',\delta')\bigr).\end{aligned}$$

Moreover, Φ_m preserves the order because s preserves the order. $\square$

In the future, we will be interested in the case where Γ has a minimum positive element 1, and Δ is the subgroup generated by 1. The concepts and notations of this section are taken from [10, 7].(15)

Let $\mathbb{K}$ be a valued field with value group Γ, and residue field $\mathbf{k}$.

Let $\dot{v} : \mathbb{K}^\star \to \Lambda$ be the composition $\rho \circ v$; it is a valuation, the *coarsening* of v. Its valuation ring is

$$\dot{\mathscr{O}} := \{\, x \in \mathbb{K} : \dot{v}x \geq 0 \,\} = \{\, x \in \mathbb{K} : vx \geq \Delta \,\},$$

its maximal ideal is

$$\dot{\mathscr{M}} := \{\, x \in \mathbb{K} : \dot{v}x > 0 \,\} = \{\, x \in \mathbb{K} : vx > \Delta \,\},$$

and its residue field is $\mathring{\mathrm{K}} := \dot{\mathscr{O}}/\dot{\mathscr{M}}$. Note that

$$\dot{\mathscr{O}} \supseteq \mathscr{O} \supseteq \mathscr{M} \supseteq \dot{\mathscr{M}}.$$

If $x + \dot{\mathscr{M}} \in \mathring{\mathrm{K}}$, define $\mathring{v}(x + \dot{\mathscr{M}}) := v(x)$. It is easy to check that $\mathring{v}$ does not depend on the choice of x, takes values in Δ and is a valuation with residue field $\mathbf{k}$.

Lemma 10.2. $(\mathbb{K}, v)$ *is Henselian iff both* $(\mathbb{K}, \dot{v})$ *and* $(\mathring{\mathrm{K}}, \mathring{v})$ *are Henselian.*

(15) Except that Kochen defines $\mathbf{k}((\Gamma))_\aleph$ as the set of formal sums whose support have cardinality less or equal to $\aleph$, while we impose that the order type (or equivalently, the cardinality) of the support is strictly less than $\aleph$.
Similarly, Kochen defines that $\mathbb{K}$ is $\aleph$-pseudo-complete iff every pseudo-converging sequence of length $\aleph$ has a pseudo-limit, while we ask that the length is strictly less than $\aleph$.

Fix a cross-section $\pi : \Delta \to \mathring{K}^\star$ (if such a cross-section exists[16]). Define $\mathring{K}((\Lambda, m))$ as the set of formal sums

$$\sum_{\lambda\in\Lambda} a_\lambda t^\lambda,$$

with multiplication defined by

$$t^\alpha t^\beta = \pi\big(m(\alpha,\beta)\big)t^{\alpha+\beta}.$$

Define the valuations $\dot{v} : \mathring{K}((\Lambda,m))^\star \to \Lambda$ and $v : \mathring{K}((\Lambda,m))^\star \to \Gamma$ by

$$\dot{v}\Big(\sum_{\lambda\in\Lambda} a_\lambda t^\lambda\Big) = \lambda_0$$
$$v\Big(\sum_{\lambda\in\Lambda} a_\lambda t^\lambda\Big) = s(\lambda_0) + \mathring{v}(a_{\lambda_0}),$$

where λ_0 is the minimum of the support.

Lemma 10.3. *There is a cross-section* $r : \Gamma \to \mathring{K}((\Lambda,m))^\star$.

Proof. As we said before, Γ is isomorphic to $\Lambda \times_m \Delta$ via Φ_m. We can suppose that Φ_m is the identity. Given $(\lambda,\delta) \in \Lambda \times_m \Delta$, define

$$r(\lambda,\delta) := \pi(\delta)t^\lambda.$$

Thus,

$$\begin{aligned} r(\lambda,\delta)r(\lambda',\delta') &= t^\lambda t^{\lambda'}\pi(\delta)\pi(\delta') = \\ &= t^{\lambda+\lambda'}\pi\big(m(\lambda,\lambda')\big)\pi(\delta)\pi(\delta') = t^{\lambda+\lambda'}\pi\big(\delta+\delta'+m(\lambda,\lambda')\big). \end{aligned}$$

Moreover,

$$r\big((\lambda,\delta) +_m (\lambda',\delta')\big) = r\big(\lambda+\lambda', \delta+\delta'+m(\lambda,\lambda')\big) = t^{\lambda+\lambda'}\pi\big(\delta+\delta'+m(\lambda,\lambda')\big).$$

□

Remark 10.4. The map $\Lambda \to \mathring{K}((\Lambda,m))^\star$ sending λ to t^λ is a section with factor set $\pi \circ m$.

Let $s' : \Lambda \to \Gamma$ be another section, m' be its co-boundary, and $f := s - s'$. Since $\rho f = 0$, the image of f is contained in the kernel of ρ, which is Δ.

Remark 10.5.

$$f\alpha + f\beta - f(\alpha+\beta) = m(\alpha,\beta) - m'(\alpha,\beta).$$

Proof. The differential operator d is linear, hence $\mathrm{d}f = \mathrm{d}s - \mathrm{d}s' = m - m'$. □

Lemma 10.6. *The map* $\Psi : \mathring{K}((\Lambda,m)) \to \mathring{K}((\Lambda,m'))$ *that fixes* $\mathring{K}$ *and sends* t^λ *into* $\pi(f\lambda)t'^\lambda$ *is an isomorphism of valued fields (with respect to the valuation* v*, where* t' *is the canonical section of* $\mathring{K}((\Lambda,m'))$*), and preserves the tree structure.*

[16] For instance, if $\Delta = \mathbb{Z}$.

Proof.
$$v\big(\Psi(t^\lambda)\big) = v\big(\pi(f\lambda)t'^\lambda\big) = f(\lambda) + s'\lambda = s\lambda = v(t^\lambda),$$
therefore Ψ preserves the valuation.

It remains to prove that $\Psi(t^\alpha t^\beta) = \Psi(t^\alpha)\Psi(t^\beta)$.
$$\Psi(t^\alpha t^\beta) = \Psi\big(t^{\alpha+\beta}\pi m(\alpha,\beta)\big) = t'^{\alpha+\beta}\pi\big(m(\alpha,\beta) + f(\alpha+\beta)\big).$$
$$\Psi(t^\alpha)\Psi(t^\beta) = t'^\alpha t'^\beta \pi(f\alpha)\pi(f\beta) = t'^{\alpha+\beta}\pi\big(m'(\alpha,\beta) + f\alpha + f\beta\big).$$
Remark 10.5 implies the conclusion. ☐

We have proved that $\mathring{\mathrm{K}}((\Lambda,m))$ does not depend on the particular choice of the section s; equivalently, it does not depend on the particular co-cycle m, but only on its equivalence class in $\mathrm{Ext}^1(\Lambda,\Delta)$.

For the rest of this section, assume that there is an embedding of $\mathring{\mathrm{K}}$ in $\mathbb{K}$, and a cross-section $r : \Gamma \to \mathbb{K}^\star$ such that $r{\restriction}_\Delta = \pi$.

Lemma 10.7. *The map $t : \Lambda \to \mathbb{K}^\star$, $t := r \circ s$, is a section with factor set $\pi \circ m$.*

Proof. r is a group homomorphism coinciding with π on Δ, hence
$$\mathrm{d}t = \mathrm{d}(r \circ s) = r \circ (\mathrm{d}s) = \pi \circ m.$$ ☐

See Diagram 1.

Diagram 1: Global picture.

$$\begin{tikzcd}
\mathring{\mathrm{K}}^\star \arrow[r, tail] \arrow[d, shift left, two heads, "\mathring{v}"] & \mathring{\mathrm{K}}((\Lambda,m))^\star \arrow[d, shift left, two heads, "v"] \arrow[dr, two heads, "\dot{v}"'] & \\
\Delta \arrow[r, tail] \arrow[u, shift left, tail, "\pi"] \arrow[d, shift right, tail, "\pi"'] & \Gamma \arrow[u, shift left, tail, "r"] \arrow[d, shift right, tail, "r"'] \arrow[r, shift right, two heads, "\rho"'] & \Lambda \arrow[l, shift right, dotted, "s"'] \arrow[ul, dotted, bend right, "t"'] \arrow[dl, dotted, bend left, "t"] \\
\mathring{\mathrm{K}}^\star \arrow[r, tail] \arrow[u, shift right, two heads, "\mathring{v}"'] & \mathbb{K}^\star \arrow[u, shift right, two heads, "v"'] \arrow[ur, two heads, "\dot{v}"] &
\end{tikzcd}$$

The continuous arrows are group homomorphisms, the dotted ones are sections.

If $(\mathbb{K},\dot{v})$ is Henselian and $\operatorname{char}\mathring{\mathrm{K}}((\Lambda,m)) = 0$, Theorem 6.1 implies that $\mathbb{K}$ admits a a truncation-closed embedding ϕ in $\mathring{\mathrm{K}}((\Lambda,m))$, which is analytic (with respect to the valuation $\dot{v}$) and over $\mathring{\mathrm{K}}$ and preserves the cross-section r. If moreover $\aleph$ is an uncountable cardinal such that $\operatorname{tr}\deg\big(\mathbb{K}/\mathring{\mathrm{K}}(\Lambda,m)\big) \le \aleph$, then we can suppose that the image of ϕ is contained in $\mathring{\mathrm{K}}((\Lambda,m))_\aleph$.

Lemma 10.8. *In the above situation, ϕ is analytic also with respect to v. Namely, $v(\phi x) = vx$ for every $x \in \mathbb{K}$.*

Proof. Let $x \in \mathbb{K}^\star$, $\lambda := \dot{v}x$, $y := \dfrac{x}{t^\lambda}$. It suffices to prove that $vy = v(\phi y)$.

$\dot{v}y = 0$, hence there exists a (unique) $z \in \mathring{\mathrm{K}}^\star$ such that $\dot{v}(y - z) > 0$, and therefore $\dot{v}(\phi y - \phi z) > 0$. Thus, $vy = vz$, and $v(\phi y) = v(\phi z)$, However, $\phi z = z$ by hypothesis. $\square$

In conclusion, we have a truncation-closed embedding of $\mathbb{K}$ into $\mathring{\mathrm{K}}((\Lambda, m))$, which is analytic with respect to the valuation v.

10.1.1 Aside

Suppose that we are given, instead of a cross-section $r : \Gamma \to \mathbb{K}^\star$, a section $t : \Lambda \to \mathbb{K}^\star$ such that $t(0) = 1$. For every $(\lambda, \delta) \in \Lambda \times_m \Delta = \Gamma$, define $r(s\lambda + \delta) := t(\lambda)\pi(\delta)$. Again, we have $t = r \circ s$ and $r\restriction_\Delta = \pi$. See Diagram 2.

Diagram 2: Nine elements.

$$\begin{tikzcd}
\mathring{\mathscr{O}}^\star \arrow[r, tail] \arrow[d, tail] & \mathring{\mathrm{K}}^\star \arrow[r, two heads, "\mathring{v}"] \arrow[d, tail] & \Delta \arrow[d, tail] & \text{(a)} \\
\mathscr{O}^\star \arrow[r, tail] \arrow[d, two heads] & \mathbb{K}^\star \arrow[r, two heads, "v"] \arrow[d, two heads] & \Gamma \arrow[d, two heads, "\rho"'] & \text{(b)} \\
\dot{\mathrm{U}}_1 \arrow[r, tail] & \mathbb{K}^\star/\mathring{\mathrm{K}}^\star \arrow[r, two heads, "\dot{v}"] & \Lambda & \text{(c)} \\
\text{(i)} & \text{(ii)} & \text{(iii)} &
\end{tikzcd}$$

The diagram is commutative and should be completed with zeros, in such a way that its rows and columns are short exact sequences.

- $\dot{\mathrm{U}}_1 := \{ x \in \mathbb{K}^\star : \dot{v}(x - 1) > 0 \}$.
- The row (a) is split by π.
- If $(\mathbb{K}, \dot{v})$ is Henselian and perfect, and moreover $\operatorname{char}\mathbb{K} = \operatorname{char}\mathring{\mathrm{K}}$, then $\dot{\mathrm{U}}_1$ is divisible and (c) splits.
- (b) splits iff there exists a cross-section r.
- The column (i) splits.
- $\dot{\mathscr{O}}^\star = \mathscr{O}^\star \cdot \pi(\Delta) \simeq \mathscr{O}^\star\Delta$.

Lemma 10.9.

$$\mathrm{d}r = \frac{\mathrm{d}t \circ \rho}{\pi \circ \mathrm{d}s \circ \rho}.$$

Proof. Given $\gamma, \gamma' \in \Gamma$, define $\lambda := \rho\gamma$, $\delta := \gamma - s\lambda$, and similarly λ' and δ'.

$$\frac{r(s\lambda,\delta)r(s\lambda',\delta')}{r(s\lambda+\delta+s\lambda'+\delta')} = \frac{r(s\lambda+\delta)r(s\lambda'+\delta')}{r\big(s(\lambda+\lambda')+\mathrm{d}s(\lambda,\lambda')+\delta+\delta'\big)} =$$
$$= \frac{t(\lambda)\pi(\delta)t(\lambda')\pi(\delta')}{t(\lambda+\lambda')\pi\big(\delta+\delta'+s(\lambda,\lambda')\big)} = \frac{\mathrm{d}t(\lambda,\lambda')}{\pi s(\lambda,\lambda')}. \quad \square$$

In particular, if r is a group homomorphism, then $\mathrm{d}r = 1$, hence $\mathrm{d}t \circ \rho = \pi \circ \mathrm{d}s \circ \rho$. Since ρ is surjective, this is true iff $\mathrm{d}t = \pi \circ \mathrm{d}s$, and we recover Lemma 10.7.

However, if r is *not* a group homomorphism, then $\mathrm{d}t \neq \pi \circ m$, hence we cannot apply Theorem 6.1.

10.2 Application to the mixed characteristic case

The results of this subsection are classical theorems by Ax and Kochen [7, Theorems 1 and 5].

Let $\mathbb{K}$ be a valued field, with value group Γ and residue field $\mathbf{k}$. Assume that:

1. $\operatorname{char}\mathbb{K} = 0$, $\operatorname{char}\mathbf{k} = p > 0$;
2. $\mathbb{K}$ is Henselian;
3. Γ has a minimum positive element 1 and $v(p) = 1$;
4. there is a cross-section $r : \Gamma \to \mathbb{K}^\star$ such that $r(1) = p$.

Let $\Delta \simeq \mathbb{Z}$ be the subgroup of Γ generated by 1. It is a convex subgroup, hence $\Lambda := \Gamma/\Delta$ is an ordered Abelian group. The core field $\overset{\circ}{\mathrm{K}}$ has characteristic 0 and value group Δ. Assume moreover that $\overset{\circ}{\mathrm{K}}$ embeds into $\mathbb{K}$. Let $\pi : \Delta \to \overset{\circ}{\mathrm{K}}{}^\star$ be the map sending n into p^n, and $s : \Gamma \to \Lambda$ any section.

By the results of § 10.1, $\mathbb{K}$ has a truncation-closed embedding in $\overset{\circ}{\mathrm{K}}((\Lambda, m))$. If, moreover, $\aleph$ is an uncountable cardinal such that $\operatorname{tr deg}\big(\mathbb{K}/\overset{\circ}{\mathrm{K}}(\Lambda, m)\big) \leq \aleph$, we can suppose that the image of such an embedding is contained in $\overset{\circ}{\mathrm{K}}((\Lambda, m))_\aleph$. Hence, if $\mathbb{K}$ is also $\aleph$-pseudo-complete, then $\mathbb{K}$ is isomorphic to $\overset{\circ}{\mathrm{K}}((\Lambda, m))_\aleph$.

If $\mathbb{K}$ is $\aleph_1$-pseudo-complete, then $\overset{\circ}{\mathrm{K}}$ is also $\aleph_1$-pseudo-complete. However, the value group of $\overset{\circ}{\mathrm{K}}$ is isomorphic to the integers, hence in this case $\aleph_1$-pseudo-complete is the same as complete. Moreover, there is only one (up to analytic isomorphisms) complete valued field of mixed characteristic with residue field $\mathbf{k}$ and value group $\mathbb{Z}$ and satisfying 3, the field of Witt vectors (see [28]). We have seen that $\overset{\circ}{\mathrm{K}}((\Lambda, m))$ does not depend on m, thus in this case $\overset{\circ}{\mathrm{K}}((\Lambda, m))$ is uniquely determined by $\mathbf{k}$ and Γ.

Assume that $\mathbb{K}$ is a non-principal ultra-product of a countable family of valued Henselian fields with ramification index 1 (see Definition 2.14). Then, there is a cross-section r satisfying Assumption 4 (see [7, Proposition 5(b)]). Besides, $(\mathbb{K}, v)$ is $\aleph_1$-pseudo-complete, and contains $\overset{\circ}{\mathrm{K}}$. Suppose that each of the fields in the family has cardinality $\leq 2^{\aleph_0}$, and that the Continuum hypothesis holds (namely, $2^{\aleph_0} = \aleph_1$). In this case, the cardinality of $\mathbb{K}$ is $\aleph_1$, thus $\mathbb{K}$ is isomorphic to $\overset{\circ}{\mathrm{K}}((\Lambda, m))_{\aleph_1}$.

All the results in this section could have been done for finitely ramified fields (of mixed characteristic) containing a suitable root of p, instead than fields with ramification index 1.

11 Examples

Here we will collect some counter-examples. We will not aim for maximal generality, only for a sufficiently representative set of easy cases.

We will also give some generalisation of Boughattas' counter-examples, but only from the point of view of the (lack of) existence of truncation-closed embeddings, not of integer parts.

Definition 11.1. An ordered group Γ has *rank* 1 iff for every $\alpha,\beta \in \Gamma$ there exists $n \in \mathbb{N}$ such that $|\frac{\alpha}{n}| < \beta < n|\alpha|$. Equivalently, iff it can be embedded as an ordered subgroup in $\mathbb{R}$.

Definition 11.2 (Complete). Let $\mathbb{F}$ be a valued field with value group Γ. A sequence $\left(x_i\right)_{i\in I}$ in $\mathbb{F}$ is *Cauchy* iff for every $\gamma \in \Gamma$ there exists $n \in I$ such that for every $i, j > n$ $v(x_i - x_j) > \gamma$. $\mathbb{F}$ is *complete* iff every Cauchy sequence in $\mathbb{F}$ has a limit.

Definition 11.3. An extension $\mathbb{K}/\mathbb{F}$ of valued fields is *unramified* iff the induced map between the value groups is an isomorphism, namely $v(\mathbb{F}) = v(\mathbb{K})$.

11.1 Purely inseparable extension

Let Γ be a non-trivial ordered Abelian group, $\mathbf{h}$ a field of characteristic $p > 0$, and $u_0, u_1, \dots, u_i, \dots$ $i \in \mathbb{N}$ algebraically independent elements over $\mathbf{h}$. We will produce a valued field of characteristic p and value group Γ which is Henselian, but admits an immediate purely inseparable exension.

Define
$$\begin{aligned}\mathbf{f} &:= \mathbf{h}(u_0^p, u_1^p, \dots),\\ \mathbb{F} &:= \mathbf{f}((\Gamma))\\ \mathbb{K} &:= \mathbb{F}(u_0, u_1, \dots) \subseteq \widetilde{\mathbb{F}},\end{aligned}$$

where $\widetilde{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. Let $\left(\gamma_i\right)_{i\in\mathbb{N}}$ be a strictly increasing sequence of elements of Γ, and
$$x := \sum_{i\in\mathbb{N}} u_i t^{\gamma_i}.$$

We have the following facts:

- The characteristic of $\mathbb{K}$ is p, its residue field is $\mathbf{k} := \mathbf{h}(u_0, u_1, \dots)$ and its value group is Γ.
- $\mathbb{K}$ is an unramified extension of the maximal field $\mathbb{F}$, hence it is Henselian.
- $x^p \in \mathbb{K}$, but $x \notin \mathbb{K}$. Moreover, $\mathbb{K}(x)$ is an immediate algebraic purely inseparable extension of $\mathbb{K}$. Hence, $\mathbb{K}$ is *not* algebraically maximal.
- $\mathbb{K}$ is Henselian, because it is an algebraic extension of the Henselian field $\mathbb{F}$.

If Γ has rank 1, $\mathbb{K}$ is not complete, because we can take $\gamma_i \to \infty$ as $i \to \infty$, and in that case x is in the completion of $\mathbb{K}$.

If $\Gamma \neq \mathbb{Z}$, it is possible to find a sequence $\left(\gamma_i\right)_{i\in\mathbb{N}}$ as above and a $\mu \in \Gamma$ such that for every $i \in \mathbb{N}$ $\gamma_i < \mu$. Let $x' := x + t^\mu$. Then, for every $y \in \mathbb{K}$
$$v(x - y) = v(x' - y).$$

However, there is no isomorphism between $\mathbb{K}(x)$ and $\mathbb{K}(x')$ fixing $\mathbb{K}$ and sending x into x'.

Moreover, x is not even in $\widehat{\mathbb{K}}$, the completion of $\mathbb{K}$. Since $\mathbb{K}$ is Henselian, its completion $\widehat{\mathbb{K}}$ is also Henselian.

Hence, $\widehat{\mathbb{K}}$ is a field of characteristic p and value group Γ that is Henselian, complete but not inseparably maximal (namely, it has a proper immediate purely inseparable algebraic extension).

Delon observed that if $\mathbb{H}$ is a separably maximal valued field (namely, it has no proper immediate separable algebraic extension), then its completion is algebraically maximal [29, Corollary6.8].

$\mathbb{K}$ has a natural truncation-closed embedding ι into $\mathbf{k}((\Gamma))$. Let $\mathbb{L} := \mathbb{K}(x')$. There is no truncation-closed embedding of $\mathbb{L}$ into $\mathbf{k}((\Gamma))$ extending ι. In fact, if ι' were such an embedding, then $x \prec \iota' x'$, but x is not in the image of ι', a contradiction. Cf. also Proposition 6.4.

Examples of valued fields which are Henselian, but admit proper immediate separable algebraic extensions are well known: see for instance [6, Example 2 pag. 246].

11.2 Boughattas' counterexample

Definition 11.4 (*n*-real closed fields). An ordered field $\mathbf{k}$ is *n-real closed* if every polynomial $p(X) \in \mathbf{k}[X]$ of degree $\leq n$ admits a zero in $\mathbf{k}$ as soon as it has a zero in the real closure of $\mathbf{k}$.

In an analogous way, one can define *n*-algebraically closed fields.

S. Boughattas [4] gave an example of a *n*-real closed field which does not admit an integer part, and, *a fortiori*, a truncation-closed embedding in a power series field.

We will give an easy generalisation of his counterexample to *n*-algebraically closed fields, also in characteristic p; we will treat the ordered field case and the unordered one at the same time, because the constructions are very similar.

We will show that the fields we are going to produce are not Henselian, so they do not contradict our theorem (in fact, they are as far from being Henselian as possible, given the constraints of being *n*-real closed).

In [5], F.V. Kuhlmann gives more examples of valued fields with no weak complements to the valuation ring (see his article for the definition). In particular, these fields do not admit a truncation-closed embedding in a power series field (and they are not Henselian either).

Let $\mathbf{f}$ be either $\mathbb{Q}$ or the field of p elements $\mathrm{GF}(p)$, for some prime p.

Given a field $\mathbb{F}$, let $\widetilde{\mathbb{F}}$ be either:

- the real closure of $\mathbb{F}$, if $\mathbb{F}$ is an ordered field, or
- the algebraic closure of $\mathbb{F}$, otherwise.

Let $\mathbb{F}^{[n]}$ be the *n*-closure of $\mathbb{F}$, namely either the *n*-real closure or the *n*-algebraic closure.

Let Γ be a divisible ordered Abelian group, $\mathbb{L} := \widetilde{\mathbf{f}}((\Gamma))$ and $\mathbb{K}$ be a (real or algebraically) closed subfield of $\mathbb{L}$. Let $\Lambda \subset \Gamma$ be the convex hull in Γ of $v(\mathbb{K})$. For every $c \in \mathbb{K}$ choose $\gamma_c \in \Gamma$ such that $\gamma_c > \Lambda$ and if $c \neq d$, then γ_c and γ_d are $\mathbb{Q}$-linearly independent over Λ (we suppose that Γ has been chosen large enough to allow this). Define $c' := c + t^{\gamma_c}$.

Let

$$\mathbb{F} := \mathbf{f}\big(c' : c \in \mathbb{F}\big),$$

the valued subfield of $\mathbb{L}$ generated by the c'. Finally, given $n > 0 \in \mathbb{N}$, let $\mathbb{F}^{[n]} \subset \mathbb{L}$ be the n-closure of $\mathbb{F}$.

Remark 11.5. The c' are algebraically independent over $\mathbb{K}$.

Lemma 11.6. $\mathbb{F}^{[n]}$

1. *has residue field* $\widetilde{\mathbf{f}}$;
2. *is not Henselian*;
3. *does not admit a truncation-closed embedding in* $\widetilde{\mathbf{f}}((\Gamma, f))$ *for any* f.

Proof. 1) $\mathbb{F}^{[n]}$ is a subfield of $\mathbb{L}$, so its residue field is contained in $\widetilde{\mathbf{f}}$. Conversely, every $c \in \widetilde{\mathbf{f}}$ is infinitesimally near to $c' \in \mathbb{F}$.

2) Let $a = b' \in \mathbb{F}$ for some $b \in \widetilde{\mathbf{f}}$. Fix a prime q larger than both the characteristic of $\mathbf{f}$ and n. Let $p(X) := X^q - a \in \mathbb{F}[X]$, and $\overline{p}(X) := X^q - b \in \widetilde{\mathbf{f}}[X]$ be its residue. $\overline{p}(X)$ has a simple root in $\widetilde{\mathbf{f}}$, however $p(X)$ has no root in $\mathbb{F}^{[n]}$, hence the latter is not a Henselian field.

3) Every truncation-closed subfield of $\widetilde{\mathbf{f}}((\Gamma, f))$ with residue field $\widetilde{\mathbf{f}}$ contains $\widetilde{\mathbf{f}}$. However, $2^{1/q}$ is not in $\mathbb{F}^{[n]}$, therefore the latter does not admit a truncation-closed embedding in $\widetilde{\mathbf{f}}((\Gamma, f))$. □

We can see that an immediate obstacle to the existence of a truncation-closed embedding of $\mathbb{F}^{[n]}$ in $\widetilde{\mathbf{f}}((\Gamma, f))$ is that $\mathbb{F}^{[n]}$ does not contain its residue field $\widetilde{\mathbf{f}}$.

Question 11.7. Can we find a valued field $\mathbb{F}$ which contains its residue field, is n-closed, but does not admit a truncation-closed embedding in $\mathbf{f}((\Gamma, f))$?

Take a field $\mathbb{K}$ as before containing t. Let $\mathbb{E} := \widetilde{\mathbf{f}}(\mathbb{F})$, with $\mathbb{F}$ as before. Its residue field is $\widetilde{\mathbf{f}}$; let Ψ be its value group.

Claim 11.8. There is no good section $s : \Psi \to \mathbb{E}^{[n]}$.

Consequently, $\mathbb{E}^{[n]}$ does not admit a truncation-closed embedding in $\mathbf{f}((\Psi, f))$.

In fact, if s were such a section, fix $q > 0 \in \mathbb{N}$. Let $x := s(1)$ and $y := s(1/q)$. Therefore, $x = cy^q = (c^{1/q}y)^q$ for some $c \in \widetilde{\mathbf{f}}$. Thus, $x^{1/q} \in \mathbb{E}^{[n]}$ for every $q \in \mathbb{N}$. This implies that $x \in \widetilde{\mathbf{f}}$, contradicting the fact that $v(x) = 1$.

Question 11.9. Can we find a valued field $\mathbb{F}$ which contains its residue field, has a good section, is n-closed, but still does not admit a truncation-closed embedding in $\mathbf{f}((\Gamma, f))$?

Let $\mathbb{K}$ as above, and

$$\mathbb{H} := \widetilde{\mathbf{f}}(t^{\mathbb{Q}})\big(c' : c \in \mathbb{K} \setminus (t^{\mathbb{Q}} \cup \widetilde{\mathbf{f}})\big).$$

Claim 11.10. There is no truncation-closed embedding of $\mathbb{H}^{[n]}$ in $\mathbf{f}((\Gamma))$ extending the canonical embedding of $\widetilde{\mathbf{f}}(t^{\mathbb{Q}})$.

Fix $q > 0 \in \mathbb{N}$ and let

$$d := \big((t+1)^{1/q}\big)' \in \mathbb{H},$$
$$a := (1+t)^{1/q} \in \mathbb{L}.$$

Then, $d = a + \mathrm{o}(t^{\mathbb{Q}})$. Therefore, $a \triangleleft d$, so $a \in \mathbb{H}^{[n]}$. Hence, $(1+t)^{1/q} \in \mathbb{H}^{[n]}$ for every $q \in \mathbb{N}$, a contradiction.

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