TOROIDALIZATION OF GENERATING SEQUENCES IN DIMENSION TWO FUNCTION FIELDS

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ABSTRACT. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and let K^*/K be a finite extension of algebraic function fields of transcendence degree 2 over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^*, and let ν be the restriction of ν^* to K. Suppose that $R \to S$ is an extension of algebraic regular local rings with quotient fields K and K^* respectively, such that V^* dominates S and S dominates R. We prove that there exist sequences of quadratic transforms $R \to \bar{R}$ and $S \to \bar{S}$ along ν^* such that $\bar{S}$ dominates $\bar{R}$ and the map between generating sequences of ν and ν^* has a toroidal structure. Our result extends the Strong Monomialization theorem of Cutkosky and Piltant.

1. INTRODUCTION

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, and let K be an algebraic function field over $\mathbf{k}$. Throughout this paper we say that a subring R of K is *algebraic* if R is essentially of finite type over $\mathbf{k}$. We will denote the maximal ideal of a local ring R by m_R.

Let K^*/K be a finite extension of algebraic function fields over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^* and value group Γ^*. Let ν be the restriction of ν^* to K with valuation ring V and value group Γ. Consider an extension of algebraic regular local rings $R \to S$ where R has quotient field K, S has quotient field K^*, R is dominated by S and S is dominated by V^* (i.e., $m_V \cap R = m_R$ and $m_{V^*} \cap S = m_S$).

It has been a classical topic to investigate finite extensions of rings of algebraic integers and mappings between algebraic curves. In these cases K and K^* are of transcendence degree 1 over $\mathbf{k}$, and the homomorphisms of local rings of points are ramified maps $R \to S$ of discrete (rank 1) valuation rings. We have that $R = V$ and $S = V^*$ are local Dedekind domains. Suppose that $(u) = m_R$ and $(x) = m_S$ are the maximal ideals of R and S, respectively, then

$$u = x^e \delta \tag{1.1}$$

where $\delta \in S$ is a unit. The corresponding value groups are $\Gamma \cong \mathbb{Z}$ and $\Gamma^* \cong \mathbb{Z}$ and we have a natural isomorphism $\Gamma^*/\Gamma \cong \mathbb{Z}_e$.

The study of ramification theory, in general, for valuations in arbitrary fields was initiated by Krull and further pursued by many authors (cf. [5] and the literature cited there).

In this paper, we are interested in valuations of two dimensional algebraic function fields, i.e., the situation when K and K^* are of transcendence degree 2 over $\mathbf{k}$. Valuations in dimension two are completely described by a compact set of data called *generating sequence*. Generating sequences provide a very useful tool in the study of algebraic surfaces (cf. [2, 5, 8, 7, 10, 11]).

We shall briefly recall the definition of generating sequences, as in [10]. Let $\Gamma_+ = \nu(R \setminus \{0\})$ be the semigroup of Γ consisting of the values of nonzero elements of R. For $\gamma \in \Gamma_+$, let $I_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}$. A (possibly infinite) sequence $\{Q_i\}$ of elements of R is a generating sequence of ν if for every $\gamma \in \Gamma_+$ the ideal I_γ is generated by the set

$$\{\prod_i Q_i^{a_i} \mid a_i \in \mathbb{N}_0, \ \sum_i a_i \nu(Q_i) \geq \gamma\}.$$

A generating sequence of ν is *minimal* if none of its proper subsequences is a generating sequence of ν. If $\{Q_i\}$ is a minimal generating sequence and $\mathbf{k} = V/m_V$ then $\{\nu(Q_i)\} \subset \Gamma$ forms a minimal set of generators for Γ_+.

Notice that a generating sequence $\{Q_i\}$ in R and the values $\{\nu(Q_i)\}$ completely describe the valuation ν (see [10, Definition 1.1] and [5, Section 7.5] for more detailed discussions). A generating sequence of ν^* in S can be defined similarly.

The aim of this paper is to find structure theorems for generating sequences of ν and ν^*. Our work is inspired by the Strong Monomialization theorem of Cutkosky and Piltant [5, Theorem 4.8], which we recall below.

We first need few definitions. Suppose that R is a local domain. A *monoidal* transform $R \to R'$ is a birational extension of local domains such that $R' = R\left[\frac{I}{x}\right]_m$ where I is a regular prime ideal of R, $0 \neq x \in I$ and m is a prime ideal of $R\left[\frac{I}{x}\right]$ such that $m \cap R = m_R$. If $I = m_R$ then the monoidal transform $R \to R'$ is called a *quadratic* transform. In our situation (dimension two) since R is a regular local ring, any nontrivial monoidal transform $R \to R'$ is a quadratic transform and there exists a regular system of parameters (u, v) of R such that $R' = R\left[\frac{u}{v}\right]_m$, where m is a maximal ideal of $R\left[\frac{u}{v}\right]$. We say that $R \to R'$ is a monoidal transform *along* ν if ν dominates R'.

The celebrated Local Monomialization theorem of Cutkosky [3, Theorem 1.1] states that there exist sequences of monoidal transforms $R \to R_1$ and $S \to S_1$ along ν^* such that ν^* dominates S_1, S_1 dominates R_1, and there are regular parameters (u, v) in R_1 and (x, y) in S_1, units $\delta_1, \delta_2 \in S_1$ and a matrix $A = (a_{ij})$ of nonnegative integers such that $\det A \neq 0$ and

$$\begin{cases} u &=& x^{a_{11}} y^{a_{12}} \delta_1 \\ v &=& x^{a_{21}} y^{a_{22}} \delta_2. \end{cases} \tag{1.2}$$

The existence of R_1 and S_1 such that (1.2) holds follows directly from the standard theorems on resolution of singularities, but in general we will not have the essential condition that $\det A \neq 0$. The difficulty in Cutkosky's work is to achieve the condition $\det A \neq 0$ (we should note that Cutkosky's Local Monomialization theorem is valid in arbitrary dimension).

In our situation, under the additional assumption that Γ^* is a non-discrete subgroup of $\mathbb{Q}$ (which is the essential and subtle case), the Strong Monomialization theorem of Cutkosky and Piltant [5, Theorem 4.8] further assures that A can be taken to have the following special form

$$A = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

Strong Monomialization is an important and useful result. It shows that no matter how complicated the structure of the extension $R \subset S$ is, after blowing-up we obtain a simultaneous resolution, that is, an extension $R_1 \subset S_1$ of regular local rings such that S_1 is the localization of the integral closure of R_1 in K^*, and this extension is very nice and simple, since it is monomial.

Observe that $u, v \in R_1$ (resp. $x, y \in S_1$) are the first two members of a generating sequence of ν (resp. ν^*). Therefore, (1.3) exhibits a toroidal structure of the first two elements of such generating sequences.

The definition of toroidal structures of generating sequences of ν and ν^* is given in Section 2. The goal of our work is to investigate toroidal structures of generating sequences of ν and ν^*. Our main theorem is stated as follows.

Theorem 1.1 (Theorem 8.1)**.** *Let* **k** *be an algebraically closed field of characteristic 0, and let K^*/K be a finite extension of algebraic function fields of transcendence degree 2 over* **k**. *Let ν^* be a* **k**-*valuation of K^* with valuation ring V^*, and let ν be the restriction of ν^* to K. Suppose that $R \to S$ is an extension of algebraic regular local rings with quotient fields K and K^* respectively, such that V^* dominates S and S dominates R. Then there exist sequences of quadratic transforms $R \to \bar{R}$ and $S \to \bar{S}$ along ν^* such that $\bar{S}$ dominates $\bar{R}$ and the map between generating sequences of ν and ν^* in $\bar{R}$ and $\bar{S}$ respectively, has a toroidal structure.*

To prove this theorem, we consider different cases according to Zariski's classification of valuations in two dimensional function fields over an algebraically closed field of characteristic 0 [12]. In most cases, the result follows from a standard application of the Strong Monomialization theorem of Cutkosky and Piltant. These cases are analyzed in Section 3. The bulk of the paper is devoted to the essential case, when Γ^* is a non-discrete subgroup of $\mathbb{Q}$. We shall now briefly describe the main steps of the proof in this case.

Let (x, y) be a regular system of parameters in S. We begin in Section 4 by constructing a sequence of *jumping polynomials* $\{T_i\}_{i \geq 0}$ in S with $T_0 = x$ and $T_1 = y$, which contains as a subsequence a minimal generating sequence of ν^*. Our notion of jumping polynomials is very similar to Favre and Jonsson's notion of *key polynomials*

[7]. The idea of key polynomials is originally due to MacLane [9]. By normalizing, we may assume that $\nu^*(x) = 1$. Let $\nu^*(y) = \frac{p_1}{q_1}$, where p_1 and q_1 are coprime positive integers. For each $i \geq 1$, we define T_{i+1} recursively. Let p_{i+1} and q_{i+1} be coprime positive integers defined by

$$\nu^*(T_{i+1}) = q_i \nu^*(T_i) + \frac{1}{q_1 \ldots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Our proof of Theorem 1.1 proceeds in the following line of arguments.

(1) We observe that the collection of jumping polynomials $\{T_i\}_{i \geq 0}$ forms a generating sequence of ν^* in S (Theorem 4.6).
(2) Let $d = d(R, S)$ be the greatest common divisor of $\{p_i \mid i \geq 1\}$. We show that the powers of x that appear in T_i are multiple of d for all $i \geq 2$ (Corollary 7.2). In other words T_i, for $i \geq 2$, is a polynomial in x^d and y.
(3) Let us denote by $t(R, S)$ the power t in (1.3) obtained from the Strong Monomialization theorem. For simplicity, assume that the constant δ_1 of (1.2) is equal to 1. We observe that if $t(R, S)$ divides d then T_i, for $i \geq 2$, is a polynomial in u and v. This shows that u, together with the collection $\{T_i\}_{i \geq 1}$, form a generating sequence of ν in R. Therefore we obtain the desired toroidal structure.
(4) The core of our argument is to show that if $t(R, S)$ does not divides $d(R, S)$ then we can find sequences of quadratic transforms $R \to R'$ and $S \to S'$ so that $t(R', S') < t(R, S)$. More precisely, let $M = \min\{i > 0 \mid t \nmid p_i\}$. Then $t(R', S')$ is the greatest common divisor of t and p_M. Lemma 8.2 is crucial in the proof of this step.

Finally, starting with a sequence of jumping polynomials in S' and repeating the above process, after a finite number of iterations we end up with the situation where $t(R', S')$ divides $d(R', S')$. Then we conclude as in (3).

We remark that in order to make our arguments work we need a very explicit description of the quadratic transforms that we perform. There are several preparatory lemmas to this end.

2. Statement of the result

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 and let K^*/K be a finite extension of algebraic function fields of transcendence degree 2 over $\mathbf{k}$. Let ν^* be a $\mathbf{k}$-valuation of K^* with valuation ring V^* and value group Γ^* and let ν be the restriction of ν^* to K with valuation ring V and value group Γ.

Suppose that S is an algebraic regular local ring with quotient field K^* which is dominated by V^* and R is an algebraic regular local ring with quotient field K which is dominated by S. We will show that there exist sequences of quadratic transforms $R \to R'$ and $S \to S'$ along ν^* such that S' dominates R' and the map between generating sequences of S' and R' has the following toroidal structure.

(1) If ν^* is divisorial then $R' = V$ and $S' = V^*$ with regular parameters $u \in R'$ and $x \in S'$ such that $u = x^a \gamma$ for some unit $\gamma \in S'$. We also have that $\{x\}$ is a minimal generating sequence of ν^* and $\{u\}$ is a minimal generating sequence of ν.

(2) If ν^* has rank 2 then there exist regular parameters (x, y) in S' and (u, v) in R' such that $\{x, y\}$ is a minimal generating sequence of ν^*, $\{u, v\}$ is a minimal generating sequence of ν and

$$u = x^a y^b \delta$$
$$v = y^d \gamma$$

for some units $\delta, \gamma \in S'$, and for some nonnegative integers a, b, d such that $ad \neq 0$.

(3) If ν^* has rank 1 and rational rank 2 then there exist regular parameters (x, y) in S' and (u, v) in R' such that $\{x, y\}$ is a minimal generating sequence of ν^*, $\{u, v\}$ is a minimal generating sequence of ν and

$$u = x^a y^b \delta$$
$$v = x^c y^d \gamma$$

for some units $\delta, \gamma \in S'$, and for some nonnegative integers a, b, c, d such that $ad - bc \neq 0$.

(4) If Γ and Γ^* are non-discrete subgroups of $\mathbb{Q}$ then there exist a minimal generating sequence $\{H_i\}_{i \geq 0}$ of ν^* in S' and regular parameters (u, v) in R' such that

$$u = H_0^a \gamma$$
$$v = H_1$$

for some unit $\gamma \in S'$, and $H_i \in R'$ for all $i > 1$. Furthermore, $\{u, \{H_i\}_{i>0}\}$ is a generating sequence of ν in R'.

(5) If ν is discrete but not divisorial then there exist regular parameters (x, y) in S' and (u, v) in R' such that Γ^* is generated by $\nu^*(x)$, Γ is generated by $\nu(u)$ and $u = x^a \gamma$ for some unit $\gamma \in S'$. Moreover, S' has a non-minimal generating sequence $\{x, \{T_i\}_{i>0}\}$ such that $\{u, \{T_i\}_{i>0}\}$ form a non-minimal generating sequence in R'.

3. Valuations in 2 dimensional function fields

Zariski in [12] gave a classification of valuations in two dimensional function fields over an algebraically closed field of characteristic zero. We refer to [6] (Chapter 8, Section 1) for a modern treatment of the subject and for the definitions and background needed in this section.

We will prove our main theorem by analyzing the different types of valuations of K^*. Notations are as in Section 2.

3.1. **One dimensional valuations.** By definition, ν^* is divisorial. In this case ν and ν^* are discrete, and V and V^* are iterated quadratic transforms of R and S respectively (see [1, Proposition 4.4]).

Let u be a regular parameter of V and let x be a regular parameter of V^*. Then there is a relation

$$u = x^a \gamma$$

where $\gamma \in V^*$ is a unit, and $a \geq 1$. Since $\{u\}$ is a mimimal generating sequence for V, and $\{x\}$ is a minimal generating sequence for V^* the theorem is proved.

3.2. **Zero dimensional valuations of rational rank 2.** By Strong Monomialization [5, Theorem 4.8] there exist sequences of quadratic transforms $R \to R'$ and $S \to S'$ along ν^* such that R' has regular parameters (u, v), S' has regular parameters (x, y), and

$$u = x^a y^b \delta$$
$$v = x^c y^d \gamma$$

for some units $\delta, \gamma \in S'$ and for some nonnegative integers a, b, c, d such that $ad - bc \neq 0$. Further, $c = 0$ if ν^* has rank two. We also have that $\{\nu^*(x), \nu^*(y)\}$ is a rational basis of $\Gamma^* \otimes \mathbb{Q}$, and $\{\nu(u), \nu(v)\}$ is a rational basis of $\Gamma \otimes \mathbb{Q}$.

Let $z \in S'$. Then $z \in \hat{S}' = S'/m_{S'}[[x, y]] = \mathbf{k}[[x, y]]$, since ν^* is zero dimensional and $\mathbf{k}$ is algebraically closed. Observe that z has an expansion $z = \sum_{i \geq 1} a_i x^{b_i} y^{c_i}$, where $a_i \in \mathbf{k}$, b_i and c_i are non negative integers, and the terms have increasing value, since $\nu^*(x)$ and $\nu^*(y)$ are rationally independent. It follows that $\nu^*(z) = b_1 \nu^*(x) + c_1 \nu^*(y)$. Hence $\{x, y\}$ is a minimal generating sequence of ν^* in S', and similarly $\{u, v\}$ is a minimal generating sequence of ν in R', and the theorem is proved.

The rest of the paper will be devoted to studying the remaining cases, that is zero dimensional valuations of rational rank 1.

3.3. **Non-discrete zero dimensional valuations of rational rank 1.** We can normalize Γ^* so that it is an ordered subgroup of $\mathbb{Q}$, whose denominators are not bounded, as Γ^* is not discrete. In Example 3, Section 15, Chapter VI of [13], examples are given of two-dimensional algebraic function fields with value group equal to any given subgroup of the rational numbers. This case is much more subtle.

3.4. **Discrete zero dimensional valuations of rational rank 1.** If ν^* is discrete, then ν is also discrete. This case will be handled in the same way as the case of non-discrete zero dimensional valuations of rational rank 1, but the generating sequences of ν^* and ν will not be minimal.

4. CONSTRUCTION OF JUMPING POLYNOMIALS

From now on we work under the assumption that the value group of ν^* is a subgroup of $\mathbb{Q}$ and $\mathrm{trdeg}_{\mathbf{k}}(V^*/m_{V^*}) = 0$. Let (x, y) be a system of regular parameters in S. We normalize the value group Γ^* of K^* so that $\nu^*(x) = 1$.

We shall now construct a sequence of polynomials $\{T_i\}_{i \geq 0}$ in S. Let

$$\begin{cases} T_0 &= x \\ T_1 &= y. \end{cases}$$

Set $q_0 = \infty$ and choose a pair of coprime positive integers (p_1, q_1) so that $\nu^*(y) = \dfrac{p_1}{q_1}$. For $i \geq 1$, T_{i+1} is defined recursively as follows. Let

$$T_{i+1} = T_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} T_j^{n_{i,j}},$$

where $n_{i,j} < q_j$ is a nonnegative integer such that $q_i \nu^*(T_i) = \nu^*(\prod_{j=0}^{i-1} T_j^{n_{i,j}})$, that is $q_i \nu^*(T_i) = \sum_{j=0}^{i-1} n_{i,j} \nu^*(T_j)$, and $\lambda_i \in \mathbf{k}$ is the residue of $\dfrac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$.

Finally, choose positive integers p_{i+1} and q_{i+1} so that $(p_{i+1}, q_{i+1}) = 1$ and

$$\nu^*(T_{i+1}) = q_i \nu^*(T_i) + \frac{1}{q_1 \cdots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Definition 4.1. The polynomial T_i will be called the i-th *jumping polynomial* and the value $\nu^*(T_i)$ will be called the i-th j-*value*. We will denote the i-th j-value by β_i. We say that β_i is an *independent* j-value if $q_i \neq 1$.

Remark 4.2. For $i > 0$ let $Q_i = q_1 \cdots q_i$. Observe that $Q_i \beta_i$ is an integer number, $\beta_{i+1} = q_i \beta_i + \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$ and $q_{i+1} \beta_{i+1} \geq \beta_{i+1} > q_i \beta_i$.

Consider the subsequence $\{\beta_{i_l}\}_{l \geq 0}$ of all independent j-values. Let $\bar{\beta}_l = \beta_{i_l}$ denote the l-th independent j-value, $\bar{q}_l = q_{i_l}$ and $H_l = T_{i_l}$. Since

$$T_{i+1} = T_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} T_j^{n_{i,j}}, \text{ where } 0 \leq n_{i,j} < q_j,$$

it follows that $n_{i,j} = 0$ whenever $q_j = 1$. Therefore only the H_l's with $i_l < i$ will appear in the product $\prod_{j=0}^{i-1} T_j^{n_{i,j}}$. Thus, if $0 < i = i_l$ then

$$T_{i+1} = H_l^{\bar{q}_l} - \lambda_i \prod_{j=0}^{l-1} H_j^{n_{i,i_j}}.$$

If $i + 1 < i_{l+1}$ then $q_{i+1} = 1$ and

$$T_{i+2} = T_{i+1} - \lambda_{i+1} \prod_{j=0}^{l} H_j^{n_{i+1,i_j}} = H_l^{\bar{q}_l} - \lambda_i \prod_{j=0}^{l-1} H_j^{n_{i,i_j}} - \lambda_{i+1} \prod_{j=0}^{l} H_j^{n_{i+1,i_j}}.$$

In general, the recursive formula for H_{l+1} with $l > 0$ will be

$$H_{l+1} = H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}} - \lambda_{i_l+1} \prod_{j=0}^{l} H_j^{n_{i_l+1,i_j}} - \lambda_{i_l+2} \prod_{j=0}^{l} H_j^{n_{i_l+2,i_j}} - \cdots$$

$$\cdots - \lambda_{i_{l+1}-1} \prod_{j=0}^{l} H_j^{n_{i_{l+1}-1,i_j}} = H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}} - \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'} \prod_{j=0}^{l} H_j^{n_{i',i_j}}.$$

We also notice that the sequence of independent jumping polynomials $\{H_l\}_{l \geq 0}$ starts with $H_0 = x$ and $H_1 = y - \sum_{j=1}^{i_1-1} \lambda_j x^{\beta_j}$.

Independent j-values furthermore have a number of basic properties. If $l > 0$ and $i_l \leq i < i_{l+1}$ then $q_1 \cdots q_i = \bar{q}_1 \cdots \bar{q}_l$ and the following equalities hold

$$\bar{\beta}_0 = \beta_0 = 1, \ \bar{q}_0 = q_0 = \infty,$$

$$\bar{\beta}_1 = p_1 + \cdots + p_{i_1-1} + \frac{p_{i_1}}{q_{i_1}},$$

$$\bar{\beta}_{l+1} = \bar{q}_l \bar{\beta}_l + \frac{p_{i_l+1} + p_{i_l+2} + \cdots + p_{i_{l+1}-1}}{\bar{q}_1 \cdots \bar{q}_l} + \frac{1}{\bar{q}_1 \cdots \bar{q}_l} \cdot \frac{p_{i_{l+1}}}{\bar{q}_{l+1}}.$$

Remark 4.3. For all $l > 0$ denote by $\bar{Q}_l = \bar{q}_1 \cdots \bar{q}_l$ and $\bar{p}_l = (p_{i_{l-1}+1} + \cdots + p_{i_l-1})\bar{q}_l + p_{i_l}$. Then $(\bar{p}_l, \bar{q}_l) = (p_{i_l}, q_{i_l}) = 1$, $\bar{\beta}_1 = \frac{\bar{p}_1}{\bar{q}_1}$ and $\bar{\beta}_{l+1} = \bar{q}_l \bar{\beta}_l + \frac{1}{\bar{Q}_l} \cdot \frac{\bar{p}_{l+1}}{\bar{q}_{l+1}}$. In particular, $\bar{q}_{l+1} \bar{\beta}_{l+1} > \bar{\beta}_{l+1} > \bar{q}_l \bar{\beta}_l$.

Remark 4.4. In general, if (x, y) is a system of regular parameters in S we may not necessarily have $\nu^*(x) = 1$. Then in order to define a sequence of jumping polynomials $\{T_i\}_{i \geq 0}$ such that $T_0 = x$ and $T_1 = y$, we introduce the following valuation $\tilde{\nu}$ of K^*

$$\tilde{\nu}(f) = \frac{\nu^*(f)}{\nu^*(x)}$$

for all $f \in K^*$. Then $\tilde{\nu}(x) = 1$ and we use the construction above with ν^* replaced by $\tilde{\nu}$. This procedure is equivalent to normalizing the value group Γ^* so that $\nu^*(x) = 1$.

We will see in Section 5 that the sequence of jumping polynomials $\{T_i\}_{i \geq 0}$ in S is well defined (Corollary 5.10). The next goal is to show that it forms a generating sequence of ν^*.

4.1. **Discrete case.** We suppose that the value group of ν^* is isomorphic to $\mathbb{Z}$. After performing a sequence of quadratic transforms along ν^* and normalizing Γ^* we may suppose that S has a system of regular parameters (x, y) such that $\nu^*(x) = 1$ generates Γ^*. Then by [10] (p.154) we have that a set $\{Q_i\}_{i \geq 0} \subset S$ is a generating sequence of

ν^* provided $\nu^*(Q_0) = 1$, each Q_i is a regular parameter of S such that (Q_0, Q_i) form a system of regular parameters, and $\lim_{i \to \infty} \nu^*(Q_i) = \infty$.

In particular, there are no minimal generating sequences in S. Any infinite subsequence of a generating sequence which contains Q_0 is a generating sequence itself.

Theorem 4.5. *The above $\{T_i\}_{i \geq 0} \subset S$ form a generating sequence of ν^*.*

Proof. Since $\nu^*(x)$ generates Γ^* we see that $q_i = 1$ for all $i \geq 1$. Thus $T_1 = y$ and $T_{i+1} = y - \lambda_1 x^{n_{1,0}} - \lambda_2 x^{n_{2,0}} - \cdots - \lambda_i x^{n_{i,0}}$ are linear in y for all $i \geq 1$. In particular T_i is a regular parameter of S and (x, T_i) form a system of regular parameters in S. Notice also that $\beta_i = \nu^*(T_i) \in \mathbb{Z}$ and $\beta_{i+1} > \beta_i$. This implies that $\lim_{i \to \infty} \nu^*(T_i) = \infty$. $\square$

4.2. **Non-discrete case.** We assume now that the value group of ν^* is a non-discrete subgroup of $\mathbb{Q}$.

Theorem 4.6. *With notations as above, $\{T_i\}_{i \geq 0} \subset S$ form a generating sequence of ν^*. Furthermore, if $p_1 > 1$ then $\{H_l\}_{l \geq 0} \subset S$ form a minimal generating sequence of ν^*. If $p_1 = 1$ then $\{H_l\}_{l \geq 1} \subset S$ form a minimal generating sequence of ν^*.*

Proof. This is shown in [10], in Chapter 2 of [7], and in [2]. $\square$

We will give an alternative proof of the above theorem in Section 7.1. Namely, we will show that the set $\{H_l\}_{l \geq 0} \subset S$ satisfies the sufficient condition for a sequence of elements of S to be a generating sequence of ν^* given in [5].

Remark 4.7. The two preceding theorems imply that the values of jumping polynomials $\{\nu^*(T_i)\}_{i \geq 0}$ generate the value group Γ^*.

5. ARITHMETICS

In this section we prove several properties of the numbers q_i and β_i defined in Section 4.

Definition 5.1. Given two rational numbers a and b we say that a is $\mathbb{Z}$-*divisible* by b, or equivalently b $\mathbb{Z}$-*divides* a, and write $b|a$, if a is an integer multiple of b, that is $a \in b\mathbb{Z}$ or $a = nb$ for some $n \in \mathbb{Z}$. The *greatest common divisor* of a and b, denoted by (a, b), is as usual the greatest rational number g such that $g|a$ and $g|b$.

Proposition 5.2. *Let p, q, t be nonzero integers with $(p, q) = 1$. Then $(\frac{1}{t}, \frac{p}{tq}) = \frac{1}{tq}$.*

Proof. If $g = (\frac{1}{t}, \frac{p}{tq})$ then $\frac{1}{tq}|g$. On the other hand

$$\frac{1}{tq} = \frac{1}{tq}(\alpha p + \beta q) = \alpha \frac{p}{tq} + \beta \frac{1}{t}$$

for some integers α and β since $1 = (p, q)$. Thus $g|\frac{1}{tq}$. $\square$

Proposition 5.3. *For $k \geq 1$, we have $(\beta_0, \beta_1, \ldots, \beta_k) = \frac{1}{Q_k}$.*

Proof. We use induction on k. For $k = 1$, clearly $(\beta_0, \beta_1) = (1, \frac{p_1}{q_1}) = \frac{1}{q_1}$. Assume now that $(\beta_0, \beta_1, \ldots, \beta_{k-1}) = \frac{1}{Q_{k-1}}$. Then

$$(\beta_0, \beta_1, \ldots, \beta_k) = ((\beta_0, \beta_1, \ldots, \beta_{k-1}), \beta_k - q_{k-1}\beta_{k-1}) = (\frac{1}{Q_{k-1}}, \frac{p_k}{Q_{k-1}q_k}) = \frac{1}{Q_k}.$$

$\square$

Corollary 5.4. *For $k \geq 0$, let $\Gamma_k = \langle \beta_0, \beta_1, \ldots, \beta_k \rangle$. Then, $\Gamma_k = \frac{1}{Q_k}\mathbb{Z}$ for all $k \geq 1$. (That is, the group generated by the values of the first $k + 1$ jumping polynomials is isomorphic to $\frac{1}{Q_k}\mathbb{Z}$.)*

Proposition 5.5. *For $k \geq 1$, we have $q_k\beta_k \in \Gamma_{k-1}$. Moreover, if $q_k > 1$ then $q_k\beta_k$ has order q_k in $\frac{\Gamma_{k-1}}{q_k\Gamma_{k-1}}$.*

Proof. We have $q_k\beta_k = q_k q_{k-1}\beta_{k-1} + p_k\frac{1}{Q_{k-1}}$ is $\mathbb{Z}$-divisible by $\frac{1}{Q_{k-1}}$. Thus, $q_k\beta_k \in \Gamma_{k-1}$.

Moreover, if $q_k > 1$ then $G = \frac{\Gamma_{k-1}}{q_k\Gamma_{k-1}} \cong \mathbb{Z}_{q_k}$ is not trivial and $\mathrm{ord}_G(q_k\beta_k) = \mathrm{ord}_G(p_k\frac{1}{Q_{k-1}}) = \mathrm{ord}_{\mathbb{Z}_{q_k}}(p_k) = q_k$ since $(p_k, q_k) = 1$. $\square$

Corollary 5.6. *For $k \geq 1$, we have $(\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_k) = \frac{1}{Q_k}$ and $\bar{\Gamma}_k = \langle \bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_k \rangle = \frac{1}{Q_k}\mathbb{Z}$. Also, $\bar{q}_k\bar{\beta}_k \in \bar{\Gamma}_{k-1}$ and $\bar{q}_k\bar{\beta}_k$ has order $\bar{q}_k$ in $\frac{\bar{\Gamma}_{k-1}}{\bar{q}_k\bar{\Gamma}_{k-1}}$.*

Remark 5.7. *With notations as above, we have $\bar{\Gamma}_k = \Gamma_{i_k}$ and $\bigcup_{k \geq 0} \bar{\Gamma}_k = \bigcup_{k \geq 0} \Gamma_k$.*

Proposition 5.8. *If $x \in \Gamma_k$ and $x \geq q_k\beta_k$ then there exists a unique representation*

$$x = \sum_{j=0}^{k} a_j\beta_j \tag{5.1}$$

with integer coefficients $0 \leq a_j < q_j$.

Proof. We first show existence of the presentation (5.1). We use induction on k. The claim is trivial for $k = 0$. Let $x \in \Gamma_1$ and $x \geq p_1$, that is $x = y + \frac{p}{q_1}$ for some $p, y \in \mathbb{Z}$ such that $y \geq p_1$ and $0 \leq p < q_1$. Since $(p_1, q_1) = 1$ there exists an integer $0 \leq a_1 < q_1$ such that $a_1 p_1 = p + t q_1$ for some $t \in \mathbb{Z}$. Notice that $a_1 p_1 < q_1 p_1$ and therefore $t < p_1$. So $x = (y - t)\beta_0 + a_1\beta_1$ is the required presentation.

Now assume that $k \geq 2$ and that a presentation (5.1) exists for $k - 1$. Let $x \in \Gamma_k$ and $x \geq q_k\beta_k$, then $x = \frac{y}{Q_{k-1}} + \frac{p}{Q_{k-1}q_k}$ for some $p, y \in \mathbb{Z}$ such that $0 \leq p < q_k$. Since $(p_k, q_k) = 1$ there exists an integer $0 \leq a_k < q_k$ such that $a_k p_k = p + t q_k$ for some $t \in \mathbb{Z}$. Then $x = \frac{y}{Q_{k-1}} + a_k(\beta_k - q_{k-1}\beta_{k-1}) - \frac{t}{Q_{k-1}}$, so that $x - a_k\beta_k = \frac{y-t}{Q_{k-1}} - a_k q_{k-1}\beta_{k-1} \in \Gamma_{k-1}$ and $x - a_k\beta_k \geq (q_k - a_k)\beta_k \geq \beta_k > q_{k-1}\beta_{k-1}$. Thus, by the inductive assumption we have that $x - a_k\beta_k = \sum_{j=0}^{k-1} a_j\beta_j$ with integer coefficients $0 \leq a_j < q_j$.

To prove uniqueness of the presentation (5.1) it suffices to show that if $\sum_{j=0}^{k} c_j\beta_j = 0$ for some integer coefficients $-q_j < c_j < q_j$ then $c_j = 0$ for all j. We again use induction on k. The claim is trivial for $k = 0$. Assume that the claim is true for

$k-1$ and suppose that $\sum_{j=0}^{k} c_j \beta_j = 0$ for some integer coefficients $-q_j < c_j < q_j$. We only need to show that $c_k = 0$. Since $c_k \beta_k = -\sum_{j=0}^{k-1} c_j \beta_j$, $c_k \beta_k \in \Gamma_{k-1}$ and therefore $c_k \frac{p_k}{Q_{k-1}q_k} \in \Gamma_{k-1}$. Thus $\frac{c_k p_k}{Q_{k-1}q_k}$ is $\mathbb{Z}$-divisible by $\frac{1}{Q_{k-1}}$, that is $q_k | c_k p_k$. This implies that $q_k | c_k$, since $(p_k, q_k) = 1$, and therefore $c_k = 0$ due to the inequality $-q_k < c_k < q_k$. $\square$

Corollary 5.9. *For all $k > 0$ there exists a unique representation $q_k \beta_k = \sum_{j=0}^{k-1} n_{k,j} \beta_j$ with integer coefficients $0 \le n_{k,j} < q_j$.*

Proof. The statement is clear for $k = 1$, since $q_1 \beta_1 = p_1 \beta_0$. For $k > 1$ the conclusion follows from Proposition 5.8, since $q_k \beta_k \in \Gamma_{k-1}$ and $q_k \beta_k > q_{k-1} \beta_{k-1}$. $\square$

Corollary 5.10. *In the notations of Section 4, the sequence of jumping polynomials $\{T_i\}_{i \ge 0}$ in S is well-defined.*

Proof. The statement follows immediately from Corollary 5.9 and the assumption that $V^*/m_{V^*} = \mathbf{k}$. $\square$

We now recall some well-known facts about continued fractions. Let p and q be positive integers such that $(p, q) = 1$. Consider the Euclidian algorithm for finding the greatest common divisor of p and q:

$$r_0 = f_1 r_1 + r_2$$
$$r_1 = f_2 r_2 + r_3$$
$$\cdots$$
$$r_{N-2} = f_{N-1} r_{N-1} + 1$$
$$r_{N-1} = f_N \cdot 1,$$

where $r_0 = p$, $r_1 = q$ and $r_1 > r_2 > \cdots > r_{N-1} > r_N = 1$. Denote by $N = N(p, q)$ the number of divisions in the Euclidian algorithm for p and q and by $f_1, f_2, \ldots, f_N$ the coefficients in the Euclidian algorithm for p and q. Define $F_i = f_1 + \cdots + f_i$ and $\epsilon(p, q) = f_1 + \cdots + f_N = F_N$, $f_1(p, q) = f_1 = \left[\frac{p}{q}\right]$. Let a and b be integers such that $0 < a \le p$, $0 \le b < q$, and $aq - bp = 1$.

Remark 5.11. With notations as above, $\dfrac{p}{q} = f_1 + \cfrac{1}{f_2 + \cdots + \cfrac{1}{f_N}}$.

Let $\{P_k(z_1, \ldots, z_k)\}_{k \in \mathbb{N}_0}$ be a sequence of polynomials as in [10]. So $P_k(z_1, \ldots, z_k) \in \mathbb{N}_0[z_1, \ldots, z_k]$ is a polynomial in k variables with nonnegative integer coefficients such that for any set of numbers $c_1, \ldots, c_K$ we have

$$c_1 + \cfrac{1}{c_2 + \cdots + \cfrac{1}{c_K}} = \frac{P_K(c_1, \ldots, c_K)}{P_{K-1}(c_2, \ldots, c_K)}.$$

We also assume that $P_0 = 1$ and set $P_{-1} = 0$.

Then it follows from properties (1.2)-(1.6) in [10] that

$$p = P_N(f_1, \ldots, f_N),$$
$$q = P_{N-1}(f_2, \ldots, f_N),$$
$$a = P_{N-1}(f_1, \ldots, f_{N-1}), \qquad b = P_{N-2}(f_2, \ldots, f_{N-1}), \qquad \text{if } N \text{ is odd},$$
$$a = p - P_{N-1}(f_1, \ldots, f_{N-1}), \quad b = q - P_{N-2}(f_2, \ldots, f_{N-1}), \quad \text{if } N \text{ is even}.$$

We also recall property (1.5) from [10] here since it will be used in the sequel

$$P_k(f_1, \ldots, f_k) = f_k P_{k-1}(f_1, \ldots, f_{k-1}) + P_{k-2}(f_1, \ldots, f_{k-2}),$$
$$P_{k-1}(f_2, \ldots, f_k) = f_k P_{k-2}(f_2, \ldots, f_{k-1}) + P_{k-3}(f_2, \ldots, f_{k-2}).$$

6. Sequences of quadratic transforms

We will now consider a sequence

$$S = S_0 \to S_1 \to S_2 \to \ldots \to S_i \to \ldots$$

of quadratic transforms along ν^*. Suppose that E is a nonsingular irreducible curve on $\operatorname{Spec} S$. Denote by π_i the map $\operatorname{Spec} S_i \to \operatorname{Spec} S$ and by E_i the reduced simple normal crossing divisor $\pi_i^{-1}(E)_{red}$. We say that S_i is *free* if E_i has exactly one irreducible component. For a free ring S_i and a regular parameter $x_i \in S_i$ we will say that x_i is an *exceptional parameter* if x_i is supported on E_i. A system of parameters (x_i, y_i) of a free ring S_i is called *permissible* if x_i is an exceptional parameter.

If S_i has regular parameters (x_i, y_i) then we can choose regular parameters (x_{i+1}, y_{i+1}) in S_{i+1} as follows

a) if $\nu^*(x_i) < \nu^*(y_i)$ then $x_{i+1} = x_i$ and $y_{i+1} = \frac{y_i}{x_i}$,
b) if $\nu^*(x_i) > \nu^*(y_i)$ then $x_{i+1} = \frac{x_i}{y_i}$ and $y_{i+1} = y_i$,
c) if $\nu^*(x_i) = \nu^*(y_i)$ then $x_{i+1} = x_i$ and $y_{i+1} = \frac{y_i}{x_i} - c$, where $c \in \mathbf{k}$ is the residue of $\frac{y_i}{x_i}$.

Our goal is to describe explicitly the sequence of quadratic transforms of S along ν^*. Assume that (x, y) is a permissible system of parameters in S. Let p and q be positive coprime integers such that $\frac{\nu^*(y)}{\nu^*(x)} = \frac{p}{q}$. We denote by μ the value $\nu^*(x)$. Let $N = N(p, q)$, $f_1, \ldots, f_n$ and $F_1, \ldots, F_N$ be defined by the Euclidian algorithm for p and q as in Section 5. Let a and b be integers such that $0 < a \le p$, $0 \le b < q$ and $aq - bp = 1$. We will investigate the following sequence of quadratic transforms along ν^*

$$S = S_0 \to S_1 \to \ldots \to S_{F_1} \to \ldots \to S_{F_j} \to \ldots \to S_{F_N}.$$

If $N > 1$ then for all $0 \le j \le F_1$, the ring S_j is free and has a permissible system of parameters $(x, \frac{y}{x^j})$. In particular,

$$(X_1, Y_1) = \left(x, \frac{y}{x^{f_1}} \right) = \left(\frac{x^{P_0}}{y^{P_{-1}}}, \frac{y^{P_0}}{x^{P_1(f_1)}} \right)$$

is a permissible system of regular parameters in S_{F_1} with $\nu^*(X_1) = \mu$ and $\nu^*(Y_1) = \frac{r_2}{q}\mu$. If $N = 1$ then $S = S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_{F_N}$ is a sequence of free rings and S_{F_N} has a permissible system of parameters

$$(X_N, Y_N) = (X_1, Y_1) = \left(x, \frac{y}{x^{f_1}} - c\right) = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c\right),$$

where $c \in \mathbf{k}$ is the residue of $\frac{y^q}{x^p}$. Notice also that $\nu^*(X_N) = \mu = (\nu^*(x), \nu^*(y))$.

If $N > 2$ then for all $0 < j \leq f_2$, the ring S_{F_1+j} is not free and has a system of regular parameters $(\frac{X_1}{Y_1^j}, Y_1)$. In particular,

$$(X_2, Y_2) = \left(\frac{X_1}{Y_1^{f_2}}, Y_1\right) = \left(\frac{x^{P_0+f_2P_1(f_1)}}{y^{f_2}}, \frac{y^{P_0}}{x^{P_1(f_1)}}\right) = \left(\frac{x^{P_2(f_1,f_2)}}{y^{P_1(f_2)}}, \frac{y^{P_0}}{x^{P_1(f_1)}}\right)$$

are regular parameters in S_{F_2} and $\nu^*(X_2) = \frac{r_3}{q}\mu$, $\nu^*(Y_2) = \frac{r_2}{q}\mu$.

In general, for all $1 < k < N$ and $0 < j \leq f_k$, the ring $S_{F_{k-1}+j}$ is not free and has a system of regular parameters $(\frac{X_{k-1}}{Y_{k-1}^j}, Y_{k-1})$ if k is even or $(X_{k-1}, \frac{Y_{k-1}}{X_{k-1}^j})$ if k is odd. In particular, if k is even then S_{F_k} has a system of regular parameters

$$(X_k, Y_k) = \left(\frac{X_{k-1}}{Y_{k-1}^{f_k}}, Y_{k-1}\right)$$

where $\nu^*(X_k) = \frac{r_{k+1}}{q}\mu$ and $\nu^*(Y_k) = \frac{r_k}{q}\mu$. We also notice that since

$$\frac{X_{k-1}}{Y_{k-1}^{f_k}} = \frac{x^{P_{k-2}(f_1,\ldots,f_{k-2})+f_kP_{k-1}(f_1,\ldots,f_{k-1})}}{y^{P_{k-3}(f_2,\ldots,f_{k-2})+f_kP_{k-2}(f_2,\ldots,f_{k-1})}} = \frac{x^{P_k(f_1,\ldots,f_k)}}{y^{P_{k-1}(f_2,\ldots,f_k)}}$$

the regular parameters (X_k, Y_k) satisfy the equality

$$(X_k, Y_k) = \left(\frac{x^{P_k(f_1,\ldots,f_k)}}{y^{P_{k-1}(f_2,\ldots,f_k)}}, \frac{y^{P_{k-2}(f_2,\ldots,f_{k-1})}}{x^{P_{k-1}(f_1,\ldots,f_{k-1})}}\right).$$

If k is odd then S_{F_k} has a system of regular parameters

$$(X_k, Y_k) = \left(X_{k-1}, \frac{Y_{k-1}}{X_{k-1}^{f_k}}\right)$$

where $\nu^*(X_k) = \frac{r_k}{q}\mu$ and $\nu^*(Y_k) = \frac{r_{k+1}}{q}\mu$. We notice that since

$$\frac{Y_{k-1}}{X_{k-1}^{f_k}} = \frac{y^{P_{k-3}(f_2,\ldots,f_{k-2})+f_kP_{k-2}(f_2,\ldots,f_{k-1})}}{x^{P_{k-2}(f_1,\ldots,f_{k-2})+f_kP_{k-1}(f_1,\ldots,f_{k-1})}} = \frac{y^{P_{k-1}(f_2,\ldots,f_k)}}{x^{P_k(f_1,\ldots,f_k)}}$$

the regular parameters (X_k, Y_k) satisfy the equality

$$(X_k, Y_k) = \left(\frac{x^{P_{k-1}(f_1,\ldots,f_{k-1})}}{y^{P_{k-2}(f_2,\ldots,f_{k-1})}}, \frac{y^{P_{k-1}(f_2,\ldots,f_k)}}{x^{P_k(f_1,\ldots,f_k)}}\right).$$

Finally, if $N > 1$ is odd then for all $0 < j < f_N$ the ring $S_{F_{N-1}+j}$ is not free and has a system of regular parameters $(X_{N-1}, \frac{Y_{N-1}}{X_{N-1}^j})$. Moreover, S_{F_N} is the first free ring after a sequence of non-free rings $S_{F_1+1} \to \ldots \to S_{F_N-1}$. If $c \in \mathbf{k}$ is the residue of $\frac{y^q}{x^p}$ then

$$(X_N, Y_N) = \left(X_{N-1}, \frac{Y_{N-1}}{X_{N-1}^{f_N}} - c \right) =$$
$$= \left(\frac{x^{P_{N-1}(f_1,\ldots,f_{N-1})}}{y^{P_{N-2}(f_2,\ldots,f_{N-1})}}, \frac{y^{P_{N-1}(f_2,\ldots,f_N)}}{x^{P_N(f_1,\ldots,f_N)}} - c \right) = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right)$$

form a permissible system of parameters in S_{F_N} with $\nu^*(X_N) = \frac{r_N}{q}\mu = \frac{1}{q}\mu = (\nu^*(x), \nu^*(y))$.

If $N > 1$ is even then for all $0 < j < f_N$ the ring $S_{F_{N-1}+j}$ is not free and has a system of regular parameters $(\frac{X_{N-1}}{Y_{N-1}^j}, Y_{N-1})$. Moreover, S_{F_N} is the first free ring after a sequence of non-free rings $S_{F_1+1} \to \ldots \to S_{F_N-1}$. If $c \in \mathbf{k}$ is the residue of $\frac{y^q}{x^p}$ then

$$(X_N, Y_N) = \left(\frac{X_{N-1}}{Y_{N-1}^{f_N-1}}, \frac{Y_{N-1}^{f_N}}{X_{N-1}} - c \right) = \left(\frac{X_{N-1}}{Y_{N-1}^{f_N}} \cdot Y_{N-1}, \frac{Y_{N-1}^{f_N}}{X_{N-1}} - c \right) =$$
$$= \left(\frac{x^{P_N(f_1,\ldots,f_N)-P_{N-1}(f_1,\ldots,f_{N-1})}}{y^{P_{N-1}(f_2,\ldots,f_N)-P_{N-2}(f_2,\ldots,f_{N-1})}}, \frac{y^{P_{N-1}(f_2,\ldots,f_N)}}{x^{P_N(f_1,\ldots,f_N)}} - c \right) = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right)$$

form a permissible system of parameters in S_{F_N} with $\nu^*(X_N) = \frac{r_N}{q}\mu = \frac{1}{q}\mu = (\nu^*(x), \nu^*(y))$.

The following lemma summarizes the above discussion. We will often refer to it in the rest of the paper.

Lemma 6.1. *Suppose that S is a free ring and (x, y) is a permissible system of parameters in S such that $\dfrac{\nu^*(y)}{\nu^*(x)} = \dfrac{p}{q}$ for some coprime integers p and q. Let $k = \epsilon(p, q)$, $f_1 = f_1(p, q) = \left\lceil \dfrac{p}{q} \right\rceil$ and let a and b be nonnegative integers such that $a \leq p$, $b < q$, and $aq - bp = 1$. Then the sequence of quadratic transforms along ν^**

$$S = S_0 \to S_1 \to \ldots \to S_{f_1} \to S_{f_1+1} \to \ldots \to S_{k-1} \to S_k \qquad (6.1)$$

has the following properties:

1) *$S_0, S_1, \ldots, S_{f_1}$ and S_k are free rings.*
2) *Non-free rings appear in (6.1) if and only if $k > f_1$, that is if $q > 1$. In this case $S_{f_1+1}, \ldots, S_{k-1}$ are non-free.*

3) S_k *has a permissible system of coordinates* $(X, Y) = \left(\dfrac{x^a}{y^b}, \dfrac{y^q}{x^p} - c \right)$, *where*

$c \in \mathbf{k}$ *is the residue of* $\dfrac{y^q}{x^p}$. *Moreover,* $\nu^*(X) = (\nu^*(x), \nu^*(y)) = \dfrac{1}{q} \nu^*(x)$ *and*

$x = X^q(Y + c)^b$, $y = X^p(Y + c)^a$.

Proof. We only check that $X^q(Y + c)^b = \dfrac{x^{aq}}{y^{bq}} \cdot \dfrac{y^{qb}}{x^{pb}} = x^{aq-bp} = x$ and $X^p(Y + c)^a =$

$\dfrac{x^{ap}}{y^{bp}} \cdot \dfrac{y^{qa}}{x^{pa}} = y^{aq-bp} = y$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7. Properties of jumping polynomials

In this section assumptions and notations are as in Section 4. We fix regular parameters (x, y) of S and we further assume that (x, y) is a permissible system of parameters in S by setting E to be the curve on $\operatorname{Spec} S$ defined by $x = 0$.

For all $k > 0$ let $d_k = (p_1, p_2, \ldots, p_k)$. We will use this notation often in the rest of the paper.

Theorem 7.1. *Suppose that R is a regular local ring dominated by S and (u, v) are regular parameters of R such that*

$$u = x^t$$
$$v = y,$$

where t is a positive integer.

If $t | d_k$ for some $k > 0$ then $\{u, \{T_i\}_{i=1}^{k+1}\} = \{T_i'\}_{i=0}^{k+1}$ is the beginning of a sequence of jumping polynomials in R. Moreover, for all $1 \le i \le k$ the pair of coprime integers defined in the construction of jumping polynomials $\{T_i'\}_{i \ge 0}$ in R is $(p_i', q_i') = (\dfrac{p_i}{t}, q_i)$.

Proof. Since $\nu^*(u) = t$ in order to construct the sequence of jumping polynomials $\{T_i'\}_{i \ge 0}$ in R we use the following valuation $\tilde{\nu}$ of K^*:

$$\tilde{\nu}(f) = \dfrac{\nu^*(f)}{t} \text{ for all } f \in K^*.$$

We have $T_1' = v = y = T_1$ and the coprime integers p_1' and q_1' are such that $\dfrac{p_1'}{q_1'} = \tilde{\nu}(y) = \dfrac{p_1}{tq_1}$. Assume $t | p_1$. Since $(p_1, q_1) = 1$ we get $p_1' = \dfrac{p_1}{t}$ and $q_1' = q_1$. Then $T_2' = v^{q_1} - \lambda_1' u^{p_1'} = y^{q_1} - \lambda_1' x^{p_1}$, where λ_1' is the residue of $\dfrac{v^{q_1}}{u^{p_1'}} = \dfrac{y^{q_1}}{x^{p_1}}$, that is $\lambda_1' = \lambda_1$ and $T_2' = T_2$. The statement is proved for $k = 1$.

By induction on k it suffices to show that the statement holds for k provided it holds for $k - 1$. Then since $t | d_k$ and $d_k | d_{k-1}$ by the inductive assumption we have

$T_i' = T_i$ for all $1 \leq i \leq k$ and $(p_i', q_i') = (\frac{p_i}{t}, q_i)$ for all $1 \leq i \leq k - 1$. The coprime integers p_k' and q_k' satisfy the following equality

$$\frac{p_k'}{q_k'} = Q_{k-1}(\tilde{\nu}(T_k) - q_{k-1}\tilde{\nu}(T_{k-1})) = \frac{1}{t}Q_{k-1}(\nu^*(T_k) - q_{k-1}\nu^*(T_{k-1})) = \frac{1}{t} \cdot \frac{p_k}{q_k}.$$

Since $t | p_k$ and $(p_k, q_k) = 1$ we get $p_k' = \frac{p_k}{t}$ and $q_k' = q_k$. Then

$$T_{k+1}' = (T_k')^{q_k'} - \lambda_k' \prod_{i=0}^{k-1}(T_i')^{n_{k,i}'} = T_k^{q_k} - \lambda_k' u^{n_{k,0}'} \prod_{i=1}^{k-1} T_i^{n_{k,i}'},$$

where λ_k' is the residue of $\dfrac{T_k^{q_k}}{u^{n_{k,0}'} \prod_{i=1}^{k-1} T_i^{n_{k,i}'}}$ and $q_k\tilde{\nu}(T_k) = n_{k,0}' + \sum_{i=1}^{k-1} n_{k,i}'\tilde{\nu}(T_i)$ with $n_{k,i}' < q_i$ for all $1 \leq i \leq k - 1$. We notice that the last equality is equivalent to

$$q_k\nu^*(T_k) = n_{k,0}'t + \sum_{i=1}^{k-1} n_{k,i}'\nu^*(T_i).$$

We also have

$$q_k\nu^*(T_k) = n_{k,0} + \sum_{i=1}^{k-1} n_{k,i}\nu^*(T_i) \tag{7.1}$$

from the construction of jumping polynomials in S. Thus from the uniqueness of presentation (7.1) we obtain $n_{k,i}' = n_{k,i}$ for all $1 \leq i \leq k - 1$ and $n_{k,0}' = \dfrac{n_{k,0}}{t}$. Thus, $u^{n_{k,0}'} \prod_{i=1}^{k-1} T_i^{n_{k,i}'} = \prod_{i=0}^{k-1} T_i^{n_{k,i}}$, $\lambda_k' = \lambda_k$, the residue of $\dfrac{T_k^{q_k}}{\prod_{i=0}^{k-1} T_i^{n_{k,i}}}$, and $T_{k+1}' = T_{k+1}$. This completes the proof. $\qquad \square$

Corollary 7.2. *For all $k > 0$ all powers of x that appear in $T_2, T_3, \ldots, T_{k+1}$ are multiples of d_k.*

Proof. Notice that Theorem 7.1 in particular shows that if $t | d_k$, then all powers of x that appear in $T_2, T_3, \ldots, T_{k+1}$ are multiples of t. $\qquad \square$

Our next goal is to describe the images of jumping polynomials under blowups of S along ν^*.

Before we state the next results we notice that if $S \subset \bar{S}$ is any subring of the m_S-adic completion $\hat{S}$ of S we can extend the valuation ν^* to a valuation of $\bar{K} = QF(\bar{S})$ centered in $\bar{S}$. We first consider the unique extension $\hat{\nu}$ of ν^* to $\hat{K} = QF(\hat{S})$ centered in $\hat{S}$, then we restrict $\hat{\nu}$ to $\bar{K}$. By abuse of notations we will say that ν^* is also a valuation of $\bar{K}$.

We will be mostly interested in the case where $\bar{S}$ is an étale extension of S. If n is a maximal ideal of $\bar{S}$, we will say that the map $S \rightarrow \bar{S}_n$ is *local étale*. Most of the times we will have $n = m_{V^*} \cap \bar{S}$, the center of the valuation.

We will first consider a sequence of ring extensions

$$S = S_0' \to \bar{S}_0' \to S_1' \to \ldots \to \bar{S}_{i-1}' \to S_i' \to \ldots$$

such that for all $i \geq 0$ $\bar{S}_i' \to S_{i+1}'$ is a quadratic transform along ν^* and $S_i' \to \bar{S}_i'$ is a local étale extension. As before let E be a nonsingular irreducible curve on $\operatorname{Spec} S$, denote by π_i' the map $\operatorname{Spec} S_i' \to \operatorname{Spec} S$ and by E_i' the reduced simple normal crossing divisor $\pi_i'^{-1}(E)_{red}$.

In what follows, for all $i > 0$ let a_i, b_i be nonnegative integers such that $a_i q_i - b_i p_i = 1$ and $a_i \leq p_i$, $b_i < q_i$. The existence of a_i and b_i is due to the Euclidean division algorithm. Let $k_0 = 0$ and $k_i = k_{i-1} + \epsilon(p_i, q_i)$ ($\epsilon(p_i, q_i)$ was defined in Section 5).

Lemma 7.3. *There exists a sequence of ring extensions*

$$S = S_0' \to S_1' \to \ldots \to S_{k_1}' \to \bar{S}_{k_1}' \to S_{k_1+1}' \to \ldots$$
$$\ldots \to S_{k_2}' \to \bar{S}_{k_2}' \to \ldots \to S_{k_i}' \to \bar{S}_{k_i}' \to \ldots$$

such that for all $i > 0$ and $j \neq k_i$, $S_j' \to S_{j+1}'$ and $\bar{S}_{k_i}' \to S_{k_i+1}'$ are quadratic transforms along ν^*, $\bar{S}_{k_i}' = S_{k_i}'[\alpha_i]_{m_{V^*} \cap S_{k_i}'[\alpha_i]}$ are local étale extensions and the following hold:

1) *$\alpha_i^{p_{i+1}} \in S_{k_i}'$ is a unit.*
2) *S_{k_i}' is free and has a permissible system of parameters (z_i, w_i) such that z_i is an exceptional parameter and $w_i = \dfrac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$ is the strict transform of T_{i+1} in S_{k_i}'.*
3) *$\nu^*(z_i) = \dfrac{1}{Q_i}$ and $\nu^*(w_i) = \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$.*
4) *For all $0 \leq j \leq i$, $T_j = z_i^{Q_i \beta_i} \tau_{j,i}$, where $\tau_{j,i} \in S_{k_i}'$ is a unit.*

Proof. We apply induction on i. For $i = 1$, by Lemma 6.1 the ring S_{k_1}' is free and has a system of regular parameters (z_1, w_1), where

$$z_1 = \frac{x^{a_1}}{y^{b_1}} \qquad\qquad \nu^*(z_1) = (1, \frac{p_1}{q_1}) = \frac{1}{q_1}$$
$$w_1 = \frac{y^{q_1} - \lambda_1 x^{p_1}}{x^{p_1}} = \frac{T_2}{T_0^{n_{1,0}}} \qquad \nu^*(w_1) = \beta_2 - p_1 = \frac{1}{q_1} \cdot \frac{p_2}{q_2}.$$

We also have $T_0 = x = z_1^{q_1}(w_1 + \lambda_1)^{b_1} = z_1^{q_1}\tau_{0,1}$ and $T_1 = y = z_1^{p_1}(w_1 + \lambda_1)^{a_1} = z_1^{p_1}\tau_{1,1}$, where $\tau_{0,1}$ and $\tau_{1,1}$ are units in S_{k_1}'.

Now assume that the lemma is true for $i - 1$. We set $\alpha_{i-1} = \left(\dfrac{\prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}q_i}} \right)^{\frac{1}{p_i}}$

and $\bar{S}_{k_{i-1}}' = S_{k_{i-1}}'[\alpha_{i-1}]_{m_{V^*} \cap S_{k_{i-1}}'[\alpha_{i-1}]}$. Then $\alpha_{i-1}^{p_i} \in S_{k_{i-1}}'$ and $\bar{S}_{k_{i-1}}'$ is a local étale extension of $S_{k_{i-1}}'$. Let $\bar{z}_{i-1} = z_{i-1}\alpha_{i-1}$, then $(\bar{z}_{i-1}, w_{i-1})$ is a permissible system of parameters in $\bar{S}_{k_{i-1}}'$, $\nu^*(\bar{z}_{i-1}) = \nu(z_{i-1}) = \dfrac{1}{Q_{i-1}}$ and $\nu^*(w_{i-1}) = \dfrac{1}{Q_{i-1}} \cdot \dfrac{p_i}{q_i}$. Recall

that $k_i = k_{i-1} + \epsilon(p_i, q_i)$. Therefore, by Lemma 6.1 the ring S'_{k_i} is free and has a permissible system of parameters (z_i, w_i) such that

$$z_i = \frac{\bar{z}_{i-1}^{a_i}}{w_{i-1}^{b_i}} \quad \text{is an exceptional parameter with } \nu^*(z_i) = \left(\frac{1}{Q_{i-1}}, \frac{1}{Q_{i-1}} \cdot \frac{p_i}{q_i}\right) = \frac{1}{Q_i},$$

$$w_i = \frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}} - c_i, \quad \text{where } c_i \in \mathbf{k} \text{ is the residue of } \frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}}.$$

We notice that for all $0 \leq l \leq i - 1$ the following equalities hold

$$\prod_{j=0}^{l} T_j^{n_{l+1,j}} = \prod_{j=0}^{l} z_{i-1}^{Q_{i-1}\beta_j n_{l+1,j}} \tau_{j,i-1}^{n_{l+1,j}} = z_{i-1}^{Q_{i-1} \sum_{j=0}^{l} n_{l+1,j}\beta_j} \prod_{j=0}^{l} \tau_{j,i-1}^{n_{l+1,j}}$$

$$= z_{i-1}^{Q_{i-1}q_{l+1}\beta_{l+1}} \prod_{j=0}^{l} \tau_{j,i-1}^{n_{l+1,j}} = \bar{z}_{i-1}^{Q_{i-1}q_{l+1}\beta_{l+1}} \alpha_{i-1}^{-Q_{i-1}q_{l+1}\beta_{l+1}} \prod_{j=0}^{l} \tau_{j,i-1}^{n_{l+1,j}}.$$

Hence,

$$\prod_{j=0}^{i-1} T_j^{n_{i,j}} = \bar{z}_{i-1}^{Q_i\beta_i} \alpha_{i-1}^{-Q_i\beta_i} \prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}, \qquad \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = \bar{z}_{i-1}^{Q_{i-1}q_{i-1}\beta_{i-1}} \alpha_{i-1}^{-Q_{i-1}q_{i-1}\beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}}$$

and

$$\frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}} = \frac{T_i^{q_i}}{\bar{z}_{i-1}^{p_i} \prod_{j=0}^{i-2} T_j^{n_{i-1,j}q_i}} = \frac{T_i^{q_i}}{\bar{z}_{i-1}^{Q_i q_{i-1}\beta_{i-1}+p_i} \alpha_{i-1}^{-Q_i q_{i-1}\beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}q_i}}$$

$$= \frac{T_i^{q_i}}{\bar{z}_{i-1}^{Q_i\beta_i} \alpha_{i-1}^{-Q_i\beta_i+p_i} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}q_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \cdot \frac{\prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}q_i}} \alpha_{i-1}^{-p_i}$$

$$= \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}. \tag{7.2}$$

Therefore, c_i coincides with λ_i, and we get $w_i = \dfrac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$ with $\nu^*(w_i) = \beta_{i+1} - q_i\beta_i = \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$.

Finally, in view of Lemma 6.1 we have $z_{i-1} = z_i^{q_i}(w_i + \lambda_i)^{b_i}\alpha_{i-1}^{-1}$ and $w_{i-1} = z_i^{p_i}(w_i + \lambda_i)^{a_i}$. Notice also that $Q_{i-1}\beta_j$ is an integer for all $0 \leq j \leq i - 1$. Thus, for all $0 \leq j \leq i - 1$ we get

$$T_j = z_{i-1}^{Q_{i-1}\beta_j} \tau_{j,i-1} = z_i^{Q_i\beta_j}((w_i + \lambda_i)^{b_i}\alpha_{i-1}^{-1})^{Q_{i-1}\beta_j} \tau_{j,i-1} = z_i^{Q_i\beta_j} \tau_{j,i},$$

where $\tau_{j,i}$ is a unit in S'_{k_i}, and

$$T_i = w_{i-1} \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = w_{i-1} z_{i-1}^{Q_{i-1} q_{i-1} \beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}} =$$

$$= z_i^{Q_i q_{i-1} \beta_{i-1} + p_i} (w_i + \lambda_i)^{a_i} ((w_i + \lambda_i)^{b_i} \alpha_{i-1}^{-1})^{Q_{i-1} q_{i-1} \beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}} = z_i^{Q_i \beta_i} \tau_{i,i},$$

where $\tau_{i,i}$ is a unit in S'_{k_i}. This completes the proof of the lemma. $\qquad\square$

Remark 7.4. In our set-up, assume that $\bar{S}$ is a local étale extension of S, S' is a quadratic transform of S along ν^*, and $\bar{S}'$ is a quadratic transform of $\bar{S}$ along ν^*. Without loss of generality, assume that S has regular parameters (x, y) with $\nu^*(y) \geq \nu^*(x)$. Then $S' = S[x, \frac{y}{x}]_{(x, \frac{y}{x} - \beta)}$, for some $\beta \in k$. Since $m_S \bar{S} = m_{\bar{S}}$, we have that $\bar{S}$ has regular parameters (x, y) and so $\bar{S}' = \bar{S}[x, \frac{y}{x}]_{(x, \frac{y}{x} - \beta')}$, for some $\beta' \in k$. Since the quadratic transforms are along ν^*, it follows that $\beta = \beta'$. Since $\bar{S}'$ is essentially of finite type over S' and $m_{S'} \bar{S}' = m_{\bar{S}'}$, we have that $\bar{S}'$ is a local étale extension of S'. Furthermore, if $\bar{S} = S[\alpha^{1/n}]_{m_{V^*} \cap S[\alpha^{1/n}]}$, where $\alpha \in S$, we have that $\bar{S}' = S'[\alpha^{1/n}]_{m_{V^*} \cap S'[\alpha^{1/n}]}$.

Theorem 7.5. *There exists a sequence of quadratic transforms along ν^**

$$S = S_0 \to S_1 \to \ldots \to S_{k_1 - 1} \to S_{k_1} \to S_{k_1 + 1} \to \ldots \to S_{k_i} \to \ldots$$

such that for all $i > 0$, S_{k_i} is free and has a system of regular parameters (x_i, y_i) such that x_i is an exceptional parameter, $T_j = x_i^{Q_i \beta_j} \gamma_{j,i}$ for $0 \leq j \leq i$, where $\gamma_{j,i} \in S_{k_i}$ is a unit, $\nu^*(x_i) = \dfrac{1}{Q_i}$, $y_i = \dfrac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$ is the strict transform of T_{i+1} in S_{k_i} and $\nu^*(y_i) = \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$.

Proof. We shall construct the required sequence from the sequence

$$S = S'_0 \to S'_1 \to \ldots \to S'_{k_1} \to \bar{S}'_{k_1} \to S'_{k_1 + 1} \to \ldots S'_{k_2} \to \bar{S}'_{k_2} \to \ldots \to S'_{k_i} \to \bar{S}'_{k_i} \to \ldots$$

of Lemma 7.3. It suffices to show, by induction on i, that we can construct a sequence

$$S = S_0 \to S_1 \to \ldots \to S_{k_1} \to S_{k_1 + 1} \to \ldots \to S_{k_i} \qquad (7.3)$$

with the required properties and such that $\bar{S}'_{k_i} = S_{k_i}[\alpha_1, \ldots, \alpha_i]_{m_{V^*} \cap S_{k_i}[\alpha_1, \ldots, \alpha_i]}$ is a local étale extension of S_{k_i}, where $\alpha_j^{p_{j+1}} \in S_{k_i}$ is a unit for $j = 1, \ldots, i$.

For $i = 1$, the sequence (7.3) is given by taking $S_j = S'_j$ for any $0 \leq j \leq k_1$ and setting $(x_1, y_1) = (z_1, w_1)$. We also notice that $\bar{S}'_{k_1} = S_{k_1}[\alpha_1]_{m_{V^*} \cap S_{k_1}[\alpha_1]}$, where $\alpha_1^{p_2} \in S_{k_1}$ is a unit. In general, suppose that the sequence (7.3) has been constructed for $i - 1$, i.e., we have a sequence of ring extensions

$$S = S_0 \to \cdots \to S_{k_1} \to \cdots \to S_{k_{i-1}} \to \bar{S}'_{k_{i-1}} \to S'_{k_{i-1}+1} \to \cdots \to S'_{k_i} \to \bar{S}'_{k_i} \to \ldots$$
$$(7.4)$$

where $S_{j-1} \to S_j$ is a quadratic transform for $1 \leq j \leq k_{i-1}$,

$$\bar{S}'_{k_{i-1}} = S_{k_{i-1}}[\alpha_1, \ldots, \alpha_{i-1}]_{m_{V^*} \cap S_{k_{i-1}}[\alpha_1, \ldots, \alpha_{i-1}]}$$

is a local étale extension (here, $\alpha_j^{p_j+1} \in S_{k_{i-1}}$ is a unit for $j = 1, \ldots, i-1$) and $\bar{S}'_{k_{i-1}} \to S'_{k_{i-1}+1}$, $S'_{j-1} \to S'_j$ is a quadratic transform for $k_{i-1} + 2 \leq j \leq k_i$. By applying Remark 7.4 to the subsequence

$$\bar{S}'_{k_{i-1}} \to S'_{k_{i-1}+1} \to \ldots \to S'_{k_i}$$

of the sequence (7.4) we obtain a new sequence of ring extensions

$$S = S_0 \to \cdots \to S_{k_{i-1}} \to S_{k_{i-1}+1} \to \cdots \to S_{k_i} \to \bar{S}'_{k_i} \to \ldots$$

where $S_{j-1} \to S_j$ is a quadratic transform for all $1 \leq j \leq k_i$ and

$$\bar{S}'_{k_i} = S_{k_i}[\alpha_1, \ldots, \alpha_i]_{m_{V^*} \cap S_{k_i}[\alpha_1, \ldots, \alpha_i]}$$

is a local étale extension (here, $\alpha_j^{p_j+1} \in S_{k_i}$ is a unit for $j = 1, \ldots, i$).

Let (x_{i-1}, y_{i-1}) be a system of parameters in $S_{k_{i-1}}$ satisfying the required properties. Then by Lemma 6.1 we get that $x_i = \dfrac{x_{i-1}^{a_i}}{y_{i-1}^{b_i}}$ is an exceptional parameter of S_{k_i} with $\nu^*(x_i) = \dfrac{1}{Q_{i-1}}$ and $\delta = \dfrac{y_{i-1}^{q_i}}{x_{i-1}^{p_i}}$ is a unit in S_{k_i}. Moreover, by computations similar to (7.2), we have

$$\frac{y_{i-1}^{q_i}}{x_{i-1}^{p_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \cdot \frac{\prod_{j=0}^{i-1} \gamma_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \gamma_{j,i-1}^{n_{i-1,j}q_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \theta,$$

where $\theta \in S_{k_i}$ is a unit.

This implies that $\dfrac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \in S_{k_i}$ and, therefore, $\dfrac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} - \lambda_i = \dfrac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} = w_i \in S_{k_i}$. Since (z_i, w_i) form a permissible system of parameters in $\bar{S}'_{k_i}$, by replacing the exceptional parameter z_i by x_i we get a permissible system of parameters (x_i, w_i) in S_{k_i}. So we will choose a system of regular parameters (x_i, y_i) in S_{k_i} by letting $(x_i, y_i) = \left(\dfrac{x_{i-1}^{a_i}}{y_{i-1}^{b_i}}, w_i \right)$. Notice that $\nu^*(y_i) = \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$.

We further have $x_{i-1} = x_i^{q_i} \delta^{b_i}$ and $y_{i-1} = x_i^{p_i} \delta^{a_i}$. Thus, recalling that $Q_{i-1}\beta_j$ is an integer for all $0 \leq j \leq i-1$, we obtain

$$T_j = x_{i-1}^{Q_{i-1}\beta_j} \gamma_{j,i-1} = x_i^{Q_i\beta_j} \delta^{b_i Q_{i-1}\beta_j} \gamma_{j,i-1} = x_i^{Q_i\beta_j} \gamma_{j,i},$$

where $\gamma_{j,i} \in S_{k_i}$ is a unit, and for $j = i$ we have

$$T_i = y_{i-1} \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = x_i^{p_i} \delta^{a_i} \left(\prod_{j=0}^{i-2} x_i^{n_{i-1,j} Q_i \beta_j} \gamma_{j,i}^{n_{i-1,j}} \right) =$$

$$= x_i^{p_i + Q_i \sum_{j=0}^{i-2} n_{i-1,j} \beta_j} \left(\delta^{a_i} \prod_{j=0}^{i-2} \gamma_{j,i}^{n_{i-1,j}} \right) = x_i^{p_i + Q_i q_{i-1} \beta_{i-1}} \gamma_{i,i} = x_i^{Q_i \beta_i} \gamma_{i,i}$$

where $\gamma_{i,i} \in S_{k_i}$ is a unit.

Hence, the sequence of ring extensions $S = S_0 \to S_1 \to \cdots \to S_{k_i} \to \bar{S}'_{k_i}$ has required properties. The result is proved. $\qquad\square$

7.1. **Remarks on generating sequences.** Assume that the value group of ν^* is a non-discrete subgroup of $\mathbb{Q}$. Suppose that

$$S = S_0 \to S_1 \to S_2 \to \ldots \to S_j \to \ldots. \tag{7.5}$$

is a sequence of quadratic transforms along ν^* and E_j is the exceptional divisor on S_j for all $j \geq 0$. Then a generating sequence of ν^* can be constructed as in [5] (see also [10], p. 150).

Definition 7.6. Set $s'_1 = \bar{s}_0 = 0$. For all $i > 0$ let $(s'_{i+1}, \bar{s}_i)$ be the pair of integers with the following properties:

1) $\bar{s}_i$ is the biggest integer $s \geq s'_i$ such that $S_{s'}$ is free for all s' with $s'_i \leq s' \leq s$;
2) s'_{i+1} is the smallest integer $s > \bar{s}_i$ such that S_s is free.

We notice here that the set of free S_j in (7.5) is infinite, as it follows from Theorem 7.5. Thus the sequences of integers $\{s'_i\}_{i>0}$ and $\{\bar{s}_i\}_{i\geq 0}$ are well defined.

Remark 7.7. Let $\{Q_i\}_{i\geq 0}$ be a sequence of elements in S such that Q_0 is an exceptional parameter in S, (Q_0, Q_1) form a system of parameters in S and the strict transform of $\operatorname{div}(Q_1)$ in $\operatorname{Spec} S_{\bar{s}_1}$ is not empty. For each $i \geq 2$ let $\operatorname{div}(Q_i)$ be an analytically irreducible curve in $\operatorname{Spec} S$ such that the strict transform of $\operatorname{div}(Q_i)$ in $\operatorname{Spec} S_{\bar{s}_i}$ is smooth and transversal to $E_{\bar{s}_i}$. Then $\{Q_i\}_{i\geq 0}$ is a generating sequence of ν^* [10].

We show that the set of all independent jumping polynomials in S satisfies Remark 7.7. Therefore we have an alternative argument that such set forms a generating sequence of ν^*. We will need to use the irreducibility criterion of Cossart and Moreno-Socías [2, Theorem 6.2] in the form of Remark 7.17 of [5]. Before we state this irreducibility criterion we recall the notations of Section 4.

Suppose that (x, y) are permissible parameters in S and the value group Γ^* of ν^* is normalized so that $\nu^*(x) = 1$. Let $\{T_i\}_{i\geq 0}$ denote the sequence of jumping polynomials in S, $\{H_l\}_{l\geq 0}$ denote the sequence of independent jumping polynomials in S and $\{i_l\}_{l\geq 0}$ denote the sequence of indexes such that $H_l = T_{i_l}$. Then for all $l \geq 1$ we have

$$H_{l+1} = H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l,i_j}} - \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'} \prod_{j=0}^{l} H_j^{n_{i',i_j}}, \tag{7.6}$$

where $0 \leq n_{i_l,i_j}, n_{i',i_j} < \bar{q}_j$ for all $i_l < i' < i_{l+1}$ and all $0 \leq j \leq l$.

Theorem 7.8. (*Remark 7.17,* [5])

Given a sequence of Weierstrass polynomials $\{H_l\}_{l\geq 1}$ satisfying (7.6) for all $l \geq 1$, set $\bar{\gamma}_0 = 1$ and define by induction on l the values

$$\bar{\gamma}_l = \frac{1}{\bar{q}_l} \sum_{j=0}^{l-1} n_{i_l,i_j} \bar{\gamma}_j.$$

Let $\Gamma_l = < \bar{\gamma}_0, \bar{\gamma}_1, \ldots, \bar{\gamma}_l >$. Then $\{H_l\}_{l\geq 0}$ is a generating sequence of a (uniquely determined) valuation ring $\bar{V}$ of $\hat{S} = \mathbf{k}[[x,y]]$, whose value group is a non-discrete subgroup of $\mathbb{Q}$ if for $l > 0$ the $\bar{\gamma}_l$'s satisfy the following three properties:

1) *$\bar{q}_l \bar{\gamma}_l$ has order precisely $\bar{q}_l$ in $\frac{\Gamma_{l-1}}{\bar{q}_l \Gamma_{l-1}}$,*
2) *$\bar{\gamma}_{l+1} > \bar{q}_l \bar{\gamma}_l$,*
3) *$\sum_{j=0}^{l} n_{i',i_j} \bar{\gamma}_j > \bar{q}_l \bar{\gamma}_l$ for all $i_l < i' < i_{l+1}$.*

It follows from the construction of jumping polynomials that the values $\bar{\gamma}_l$, defined for the sequence of independent jumping polynomials, coincide with $\bar{\beta}_l$ for all $l \geq 0$. Combining Remark 4.3 and Corollary 5.6 we see that independent jumping polynomials satisfy the conditions of Theorem 7.8. Thus the sequence of independent jumping polynomials $\{H_l\}_{l\geq 0}$ is a generating sequence for some valuation $\bar{\nu}$ of S. In particular, every element $H_l \in S$ is analytically irreducible in S.

Let the sequence of quadratic transform of S

$$S = S_0 \to S_1 \to S_2 \to \ldots \to S_{k_1} \to \ldots S_{k_2} \to \ldots \to S_{k_i} \to \ldots$$

be as in Theorem 7.5. Suppose that (x_{i-1}, y_{i-1}) are permissible regular parameters of $S_{k_{i-1}}$, such that $\nu^*(x_{i-1}) = \frac{1}{Q_{i-1}}$ and $\nu^*(y_{i-1}) = \frac{1}{Q_{i-1}} \cdot \frac{p_i}{q_i}$. Lemma 6.1 shows that non-free rings will appear in the subsequence $S_{k_{i-1}} \to \ldots \to S_{k_i}$ if and only if $q_i > 1$, that is if β_i is an independent j-value and $i = i_l$ for some $l > 0$. In this case the last free ring in this subsequence is $S_{k_{i-1}+\left[\frac{p_i}{q_i}\right]}$ and S_{k_i} is the first free ring following $S_{k_{i-1}+\left[\frac{p_i}{q_i}\right]}$. Thus for all $l > 0$, $\bar{s}_l = k_{i_l-1} + \left[\frac{p_{i_l}}{q_{i_l}}\right]$ and $s'_{l+1} = k_{i_l}$.

Since $H_0 = x$ and $H_1 = y - \sum_{j=1}^{i_1-1} \lambda_j x^{\beta_j}$, (H_0, H_1) form a permissible system of parameters in S.

By Theorem 7.5 there exists a permissible system of parameters (x_{i_l-1}, y_{i_l-1}) in $S_{k_{i_l-1}}$ such that y_{i_l-1} is the strict transform of H_l in $S_{k_{i_l-1}}$. If $f_1 = \left[\frac{p_{i_l}}{q_{i_l}}\right]$ then

$S_{k_{i_l-1}+f_1} = S_{\bar{s}_l}$ and

$$(X_l, Y_l) = \left(x_{i_l-1}, \frac{y_{i_l-1}}{x_{i_l-1}^{f_1}}\right)$$

form a permissible system of parameters in $S_{\bar{s}_l}$. Thus the strict transform of H_l in $S_{\bar{s}_l}$ is Y_l. In particular, the strict transform of $\mathrm{div}(H_1)$ in $\mathrm{Spec}\, S_{\bar{s}_1}$ is not empty, and for $l \geq 2$, the strict transform of $\mathrm{div}(H_l)$ in $\mathrm{Spec}\, S_{\bar{s}_l}$ is smooth and transversal to $E_{\bar{s}_l}$. Therefore $\{H_l\}_{l \geq 0} \subset S$ form a generating sequence of ν^*.

Remark 7.9. We have that $\{H_l\}_{l \geq 0}$ is a minimal generating sequence of ν^* if $p_1 > 1$. If $p_1 = 1$ then $\{H_l\}_{l > 0}$ is a minimal generating sequence of ν^*.

8. MONOMIALIZATION OF GENERATING SEQUENCES

The goal of this section is to prove the following theorem.

Theorem 8.1. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic 0, and let* K^*/K *be a finite extension of algebraic function fields of transcendence degree 2 over* $\mathbf{k}$. *Let* ν^* *be a* $\mathbf{k}$-*valuation of* K^*, *with valuation ring* V^* *and value group* Γ^*, *and let* ν *be the restriction of* ν^* *to* K, *with valuation ring* V *and value group* Γ. *Suppose that* $R \to S$ *is an extension of algebraic regular local rings with quotient fields* K *and* K^* *respectively, such that* V^* *dominates* S *and* S *dominates* R. *Then there exist sequences of quadratic transforms* $R \to \bar{R}$ *and* $S \to \bar{S}$ *along* ν^* *such that* $\bar{S}$ *dominates* $\bar{R}$ *and the map between generating sequences of* ν *and* ν^* *in* $\bar{R}$ *and* $\bar{S}$ *respectively, has a toroidal structure.*

The lemma below is crucial in the proof of the theorem.

Lemma 8.2. *In the set up of Theorem 8.1, assume that* Γ^* *is a subgroup of* $\mathbb{Q}$ *and* $V^*/m_{V^*} = \mathbf{k}$. *Suppose that* R *has regular parameters* (u, v) *and* S *has regular parameters* (x, y) *such that*

$$\begin{aligned} u &= x^t \delta \\ v &= y, \end{aligned} \tag{8.1}$$

where t *is a positive integer and* δ *is a unit in* S. *Let* p *and* q *be positive coprime integers such that* $\frac{\nu^*(y)}{\nu^*(x)} = \frac{p}{q}$ *and let* $k = \epsilon(p, q)$. *Let* $\bar{p}$ *and* $\bar{q}$ *be positive coprime integers such that* $\frac{\nu(v)}{\nu(u)} = \frac{\bar{p}}{\bar{q}}$ *and let* $\bar{k} = \epsilon(\bar{p}, \bar{q})$. *Let* g *be the greatest common divisor of* t *and* p.

Then the sequences of quadratic transforms $R = R_0 \to R_1 \to \ldots \to R_{\bar{k}}$ *and* $S = S_0 \to S_1 \to \ldots \to S_k$ *along* ν^* *satisfy the following property:* $R_{\bar{k}}$ *and* S_k *are free rings and there exist permissible systems of regular parameters* (U, V) *in* $R_{\bar{k}}$ *and* (X, Y) *in* S_k *such that*

$$\begin{aligned} U &= X^g \Delta \\ V &= Y \end{aligned} \tag{8.2}$$

for some unit $\Delta \in S_k$.

Proof. We notice first that $\frac{\nu(v)}{\nu(u)} = \frac{\nu^*(y)}{t\nu^*(x)} = \frac{p}{tq}$. Writing $t = gt'$ and $p = gp'$, where $(t', p') = 1$, gives $\bar{p} = p'$ and $\bar{q} = qt'$. Also after possibly multiplying u by a constant we may assume that $\delta = 1 + w$ for some $w \in m_S$.

Let a and b be nonnegative integers such that $a \leq p$, $b < q$ and $aq - bp = 1$. Let $\bar{a}$ and $\bar{b}$ be nonnegative integers such that $\bar{a} \leq \bar{p}$, $\bar{b} < \bar{q}$ and $\bar{a}\bar{q} - \bar{b}\bar{p} = 1$.

By Lemma 6.1 applied to S and R respectively, we get that S_k has a permissible system of parameters $(X, Y') = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right)$, where $c \in \mathbf{k}$ is the residue of $\frac{y^q}{x^p}$, and $R_{\bar{k}}$ has a permissible system of parameters $(U, V) = \left(\frac{u^{\bar{a}}}{v^{\bar{b}}}, \frac{v^{\bar{q}}}{u^{\bar{p}}} - \bar{c} \right)$, where $\bar{c} \in \mathbf{k}$ is the residue of $\frac{v^{\bar{q}}}{u^{\bar{p}}}$. Moreover, $x = X^q(Y' + c)^b$, $y = X^p(Y' + c)^a$ and $u = U^{\bar{q}}(V + \bar{c})^{\bar{b}}$, $v = U^{\bar{p}}(V + \bar{c})^{\bar{a}}$.

Now

$$U = \frac{u^{\bar{a}}}{v^{\bar{b}}} = \frac{x^{t\bar{a}}\delta^{\bar{a}}}{y^{\bar{b}}} = \frac{[X^q(Y' + c)^b]^{t\bar{a}}\delta^{\bar{a}}}{[X^p(Y' + c)^a]^{\bar{b}}} = \frac{X^{qt\bar{a}}}{X^{p\bar{b}}}\delta^{\bar{a}}(Y' + c)^{bt\bar{a} - a\bar{b}} = X^g\Delta,$$

where $\Delta = \delta^{\bar{a}}(Y' + c)^{bt\bar{a} - a\bar{b}}$. Notice that the last equality holds since $qt\bar{a} - p\bar{b} = qgt'\bar{a} - gp'\bar{b} = g(qt'\bar{a} - p'\bar{b}) = g(\bar{q}\bar{a} - \bar{p}\bar{b}) = g$.

Furthermore notice that $\frac{v^{\bar{q}}}{u^{\bar{p}}} = \frac{y^{\bar{q}}}{x^{t\bar{p}}}\delta^{-\bar{p}} = \frac{y^{qt'}}{x^{gt'p'}}\delta^{-\bar{p}} = \left(\frac{y^q}{x^p} \right)^{t'}\delta^{-\bar{p}} = \left(\frac{y^q}{x^p} \right)^{t'}(1 + w)^{-\bar{p}}$. Therefore $\bar{c} = c^{t'}$, and

$$V = \frac{v^{\bar{q}}}{u^{\bar{p}}} - \bar{c} = \left(\frac{y^q}{x^p} \right)^{t'}\delta^{-\bar{p}} - c^{t'} = (Y' + c)^{t'}(1 + w)^{-\bar{p}} - c^{t'} = (Y' + c)^{t'} - c^{t'} + W,$$

where $W \in wS \subset m_S$. Since $m_S \subset (X)S_k$ we have that $V = Y'\delta_2 + XZ$ for some $Z \in S_k$ and unit $\delta_2 = \frac{\left(\frac{y^q}{x^p} \right)^{t'} - c^{t'}}{\left(\frac{y^q}{x^p} - c \right)} \in S_k$. Thus (X, V) form a permissible system of parameters in S_k. We set $Y = V$ to complete the proof. $\square$

Lemma 8.3. *In the set up of Theorem 8.1, assume that Γ^* is a subgroup of $\mathbb{Q}$ and $V^*/m_{V^*} = \mathbf{k}$. Suppose that R has regular parameters (u, v) and S has regular parameters (x, y) such that*

$$\begin{aligned} u &= x^t\delta \\ v &= y, \end{aligned} \tag{8.3}$$

where t is a positive integer and δ is a unit in S.

Let $\tilde{S} = S[\delta^{1/t}]_{m_{V^*} \cap S[\delta^{1/t}]}$ and let $\{T_i\}_{i \geq 0}$ be the sequence of jumping polynomials in $\tilde{S}$ such that $T_0 = \tilde{x} = x\delta^{1/t}$ and $T_1 = y$. Then either $\{u, \{T_i\}_{i>0}\}$ is a sequence of jumping polynomials in R or there exist sequences of quadratic transforms $R \to R'$ and $S \to S'$ such that R' has a system of regular parameters (U, V), S' has a system

of regular parameters (X, Y) and

$$U = X^g \Delta$$
$$V = Y,$$

where $g < t$ is a positive integer and Δ is a unit in S'.

Proof. Without loss of generality we may assume that Γ^* is normalized so that $\nu^*(x) = 1$. Then $\nu^*(\tilde{x}) = 1$ and $\nu(u) = t$.

For all $i > 0$ let p_i and q_i be coprime integers defined in the construction of jumping polynomials $\{T_i\}_{i \geq 0}$ in $\tilde{S}$. Denote by $M = \min\{i > 0 |\ t \nmid p_i\}$. We assume first that $M = \infty$, that is p_i is multiple of t for every i. Then since $u = \tilde{x}^t$, by Theorem 7.1 we get that $\{u, \{T_i\}_{i>0}\}$ is a sequence of jumping polynomials in R.

Assume now that $M < \infty$. Since $u = \tilde{x}^t$ and $t|p_i$ for all $i < M$, by Theorem 7.1 we get that $\{u, \{T_i\}_{i=1}^M\} = \{T_i'\}_{i=0}^M$ is the beginning of a sequence of jumping polynomials in R and for all $i < M$ the pairs of coprime integers (p_i', q_i') defined in the construction of the sequence $\{T_i'\}_{i \geq 0}$ are $(p_i', q_i') = (\frac{p_i}{t}, q_i)$.

Recall that the integers k_i are defined as $k_0 = 0$ and $k_i = k_{i-1} + \epsilon(p_i, q_i)$ if $i > 0$. Let $k_0' = 0$ and $k_i' = k_{i-1}' + \epsilon(p_i', q_i')$ for all $i > 0$. We will show first that the sequences of quadratic transforms

$$S = S_0 \to \ldots \to S_{k_1} \to \ldots \to S_{k_{M-1}}$$

and

$$R = R_0 \to \ldots \to R_{k_1'} \to \ldots \to R_{k_{M-1}'}$$

have the following property: for all $0 \leq i \leq M - 1$ the rings $R_{k_i'}$ and S_{k_i} are free, there exist permissible systems of parameters (u_i, v_i) in $R_{k_i'}$ and (x_i, y_i) in S_{k_i} and a unit $\delta_i \in S_{k_i}$ such that

$$u_i = x_i^t \delta_i, \qquad v_i = y_i$$

and

$$\nu^*(x_i) = \frac{1}{Q_i}, \qquad \nu^*(y_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

The statement is trivial for $i = 0$. Assume that $i > 0$ and that the statement holds for $i - 1$. Then Lemma 8.2 applies to $R_{k_{i-1}'} \subset S_{k_{i-1}}$. We notice that $\frac{\nu^*(y_{i-1})}{\nu^*(x_{i-1})} = \frac{p_i}{q_i}$ and $\frac{\nu(v_{i-1})}{\nu(u_{i-1})} = \frac{p_i}{tq_i} = \frac{p_i'}{q_i'}$, and therefore $k = \epsilon(p_i, q_i)$ and $\bar{k} = \epsilon(p_i', q_i')$. Thus $R_{k_i'}$ and S_{k_i} are free rings and there exist permissible systems of regular parameters (u_i, w_i) in $R_{k_i'}$ and (x_i, z_i) in S_{k_i} such that $u_i = x_i^t \delta_i$ and $w_i = z_i$ for some unit $\delta_i \in S_{k_i}$.

Now by Theorem 7.5 applied to R with ν replaced by $\tilde{\nu} = \frac{1}{t}\nu$ we get that $R_{k_i'}$ has a system of regular parameters (h_i, v_i) such that h_i is an exceptional parameter,

$$\nu(h_i) = t\tilde{\nu}(h_i) = t\frac{1}{q_1' \cdots q_i'} = \frac{t}{q_1 \cdots q_i} = \frac{t}{Q_i}$$

and

$$\nu(v_i) = t\tilde{\nu}(v_i) = t\frac{1}{q_1' \cdots q_i'} \cdot \frac{p_{i+1}'}{q_{i+1}'} = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Since u_i is also an exceptional parameter in $R_{k_i'}$ we have $u_i = h_i\gamma$ for some unit $\gamma \in R_{k_i'}$. Therefore (u_i, v_i) form a permissible system of parameters in $R_{k_i'}$ and $\nu(u_i) = \dfrac{t}{Q_i}$. Notice also that $v_i = \alpha u_i + \beta w_i$, where $\alpha, \beta \in S_{k_i}$. Moreover, β is a unit in $R_{k_i'}$, since the image of v_i is a regular parameter in $R_{k_i'}/(u_i)$. This implies that $v_i = \alpha x^t \delta_i + \beta z_i$ is also a regular parameter in S_{k_i} and (x_i, v_i) form a permissible system of parameters in S_{k_i}. We set $y_i = v_i$ and observe that $\nu^*(x_i) = \dfrac{1}{t}\nu(u_i) = \dfrac{1}{Q_i}$ and $\nu^*(y_i) = \nu(v_i) = \dfrac{1}{Q_i} \cdot \dfrac{p_{i+1}}{q_{i+1}}$.

To finish the proof of the lemma we apply Lemma 8.2 to $R_{k_{M-1}'} \subset S_{k_{M-1}}$. We have $p = p_M$, $q = q_M$ and $\bar{p} = p_M'$, $\bar{q} = q_M'$. Thus $R' = R_{k_M'}$ has regular parameters (U, V) and $S' = S_{k_M}$ has regular parameters (X, Y) such that

$$U = X^g \Delta$$
$$V = Y,$$

where Δ is a unit in S_{k_M} and $g = (p_M, t) < t$. $\qquad\square$

We are now ready to prove Theorem 8.1.

Proof. By the discussion of Section 3 we only need to consider the case when Γ^* is a subgroup of $\mathbb{Q}$ and $\mathrm{trdeg}_{\mathbf{k}}(V^*/m_{V^*}) = 0$. Then by the Strong Monomialization theorem we may assume that there exist regular parameters (u, v) in R and (x, y) in S such that $u = x^t\delta$ and $v = y$ for some unit $\delta \in S$. If $t = 1$ then $R = S$ and the conclusion of the theorem is trivial, so assume that $t > 1$.

We set $\tilde{S} = S[\delta^{1/t}]_{m_{V^*} \cap S[\delta^{1/t}]}$ and $\tilde{x} = x\delta^{1/t}$. Let $\{T_i\}_{i \geq 0}$ be a sequence of jumping polynomials in $\tilde{S}$ such that $T_0 = \tilde{x}$ and $T_1 = y$. For all $i > 0$ let the coprime integers p_i and q_i be defined as in the construction of jumping polynomials $\{T_i\}_{i \geq 0}$ in $\tilde{S}$.

First, let us assume that $t | p_i$ for all $i > 0$. Then by the proof of Lemma 8.3 we have that $\{u, \{T_i\}_{i>0}\}$ is a sequence of jumping polynomials in R. In particular, this implies that $T_i \in S$ for all $i > 0$.

If Γ^* is a discrete subgroup of $\mathbb{Q}$, after performing a sequence of quadratic transforms along ν^* and normalizing Γ^* we may assume that $\nu^*(x) = 1$ generates Γ^*. In this case Corollary 5.4 shows that $q_i = 1$ for all $i > 0$. Then $\nu^*(T_i) = \sum_{j=1}^{i} p_j$ is a multiple of t for all $i > 0$. Thus in view of Remark 4.7 we have that $\nu^*(u) = t$ generates Γ. By Theorem 4.5 the sequence $\{u, \{T_i\}_{i>0}\}$ form a generating sequence in R. Now since $T_1 = y$ and for all $i > 0$

$$T_{i+1} = v - \lambda_1 u^{n_1} - \lambda_2 u^{n_2} - \cdots - \lambda_i u^{n_i} = y - \lambda_1 u^{n_1} - \lambda_2 u^{n_2} - \cdots - \lambda_i u^{n_i}$$

is linear in y, repeating the proof of Theorem 4.5 we get that $\{x, \{T_i\}_{i>0}\}$ is a generating sequence in S.

If Γ^* is a non-discrete subgroup of $\mathbb{Q}$, let $\{H_i\}_{i\geq 0}$ be the sequence of independent jumping polynomials in $\tilde{S}$. Then $\{u, \{H_i\}_{i>0}\}$ is a sequence of independent jumping polynomials in R. By Theorem 4.6 we have that $\{H_i\}_{i\geq 0}$ is a generating sequence in $\tilde{S}$ and $\{u, \{H_i\}_{i>0}\}$ is a generating sequence in R. Moreover, $\{H_i\}_{i\geq 0}$ is a minimal generating sequence of ν^* since $p_1 > 1$ as a multiple of t. This implies that $\{x, \{H_i\}_{i>0}\}$ is a minimal generating sequence in S.

If $t \nmid p_i$ for some $i > 0$, let R', S' and g be as in the proof of Lemma 8.3. Notice that $g < t$, i.e., the exponent of Strong Monomialization has dropped. We now repeat the above argument starting with the rings R' and S' (instead of R and S). After a finite number of iterations we obtain the desired conclusion. $\square$

Remark 8.4. In the proof of Theorem 8.1 we have $\{u, \{H_i\}_{i>0}\}$ is a minimal generating sequence of ν in R if $p_1 \neq t$, otherwise $\{H_i\}_{i>0}$ is a minimal generating sequence of ν in R.

Remark 8.5. Assumptions and notations are as in the statement of Theorem 8.1. By [5, Theorem 6.1] there exist sequences of quadratic transforms $R \to \bar{R}$, $S \to \bar{S}$ along ν^* such that $\bar{R}$ has regular parameters (u, v), $\bar{S}$ has regular parameters (x, y) such that

$$
\begin{aligned}
u &= x^e \delta \\
v &= y
\end{aligned}
\tag{8.4}
$$

where δ is a unit in $\bar{S}$, and $e = [\Gamma^* : \Gamma]$ is the ramification index of ν^* relative to ν. The monomial form of (8.4) is preserved by the sequences of quadratic transforms of the proof of Theorem 8.1. Furthermore, the exponent e does not drop under such sequences of quadratic transforms (see the proof of [5, Theorem 6.1]). It follows from the proof of Theorem 8.1 that the map between generating sequences of ν and ν^* in $\bar{R}$ and $\bar{S}$ respectively, has the desired toroidal structure.

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