FORMAL POWER SERIES WITH CYCLICALLY ORDERED EXPONENTS

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Abstract. We define and study a notion of ring of formal power series with exponents in a cyclically ordered group. Such a ring is a quotient of various subrings of classical formal power series rings. It carries a two variable valuation function. In the particular case where the cyclically ordered group is actually totally ordered, our notion of formal power series is equivalent to the classical one in a language enriched with a predicate interpreted by the set of all monomials.

1. Introduction

Throughout this paper, k will denote a commutative ring with unity. Further, C will denote a *cyclically ordered* abelian group, that is, an abelian group equipped with a ternary relation $(\cdot,\cdot,\cdot)$ which has the following properties:
(1) $\forall a,b,c,\ (a,b,c) \Rightarrow a \neq b \neq c \neq a$
(2) $\forall a,b,c,\ (a,b,c) \Rightarrow (b,c,a)$
(3) $\forall c \in C$, $(c,\cdot,\cdot)$ is a strict total order on $C \setminus \{c\}$.
(4) $(\cdot,\cdot,\cdot)$ is compatible, i.e., $\forall a,b,c,d,\ (a,b,c) \Rightarrow (a+d,b+d,c+d)$.
For all $c \in C$, we will denote by $\leq_c$ the associated order on C with first element c. For $\emptyset \neq X \subset C$, $\min_c X$ will denote the minimum of $(X,\leq_c)$, if it exists.

For instance, any totally ordered group is cyclically ordered with respect to the following ternary relation: (a,b,c) iff $a<b<c$ or $b<c<a$ or $c<a<b$. Such a cyclically ordered group is called a *linear cyclically ordered group*.

Let us review quickly two basic facts about cyclically ordered groups.

A theorem of Rieger (see [**?**], IV, 6, th. 21), shows that there exist a totally ordered abelian group G and a positive element z in G, such that the subgroup $\mathbb{Z}z$ generated by z is cofinal in G and $C \simeq G/\mathbb{Z}z$, cyclically ordered in the following way: for all $a,b,c \in G$, $(a+\mathbb{Z}z, b+\mathbb{Z}z, c+\mathbb{Z}z)$ holds if and only if there exist $a' \in a+\mathbb{Z}z$, $b' \in b+\mathbb{Z}z$, $c' \in c+\mathbb{Z}z$ such that one of $0 \leq a' < b' < c' < z$ or $0 \leq b' < c' < a' < z$ or $0 \leq c' < a' < b' < z$ holds.

Let $l(G)$ be the maximal convex subgroup of G not containing z; i.e., $l(G)$ is the largest proper convex subgroup of G. Note that $l(G) \subseteq \,]-z,z[\,$. Then the restriction of the canonical epimorphism $p: G \to C$ to $l(G)$ is a monomorphism of totally ordered groups. Its image $l(C)$ is called the *linear part of* C because it is the largest totally ordered subgroup of C with respect to the ordering $\leq_\epsilon$ that we will define in Section **??**.

By Swirczkowski's theorem (see [**?**]), $C/l(C)$ embeds in the multiplicative group of all complex numbers of absolute value 1.

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In Section **??** we define $k[[C]]$, the ring of formal power series with coefficients in k and exponents in C, and $k[C]$, its ring of polynomials, in such a way that they coincide with the classical notions when C is a linear cyclically ordered group.

In Section **??**, we prove the following:

Proposition 1. *There exists a subring A of $k[[G]]$ such that $A/(1-X^z)A \simeq k[[C]]$.*

For such a ring A, let p_A be the canonical mapping from A onto $A/(1-X^z)A$.

In Section **??**, we give some examples of units and zero divisors in $k[[C]]$. Then, we prove:

Proposition 2. *(a) The semi-group of all zero divisors of $k[[C]]$ is the union of the following two subsets:*
- the p_A-image of the semi-group of all zero-divisors of A,
- the p_A-image of the set of all elements σ such that $\sigma A \cap (1-X^z)A \neq \sigma(1-X^z)A$.
(b) The group of all units of $k[[C]]$ is the p_A-image of the semi-group of all elements σ such that $\sigma A + (1-X^z)A = A$.

Theorem 3. *If $k[[C]]$ is a field, then k is a field, C is torsion free and $C/l(C)$ embeds in the group of all roots of unity. Conversely, if k is a field, C is torsion free and $C/l(C)$ is finite, then $k[[C]]$ is a field.*

Open Problem: Suppose that $k[[C]]$ is a field. Does this imply that $C/l(C)$ must be finite?

Theorem 4. *$k[C]$ is an integral domain if and only if k is an integral domain and C is torsion free.*

Open Problem: Is it true that $k[[C]]$ is an integral domain if and ony if C is torsion free and k is an integral domain?

In Section **??** we define the *cyclic valuation* on $k[[C]]$ as a mapping v from $C \times k[[C]]$ onto $C_\infty := C \cup \{\infty\}$ (with $a <_b \infty$ for all $a, b \in C$). This extends the classical notion of a valuation in a natural way. Indeed, if C is a linear cyclically ordered group, then the usual valuation can be defined first order by means of the $v(a, \cdot)$, $a \in C$.

The first order properties of the cyclic valuations can be stated in several languages which we will compare, and which we will use to define the first order notion of a cyclically valued ring. Then, we prove:

Theorem 5. *A domain R is a subring of a ring $k[[C]]$, for some domain k, if and only if the following conditions hold:*

(a) C is isomorphic to a subgroup C' of the group of units of R,
(b) there is a mapping $v : C \times R \to C \cup \{\infty\}$ such that (R, v) is a cyclically valued ring,
(c) for all σ in R, the set $\{v(a, \sigma) \mid a \in C\}$ is well-ordered.

Throughout this paper, $\mathbb{N}$ will denote the semi-group of all non-negative integers, and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ will denote the set all positive integers.

2. Definition of formal power series

A subset S of C will be called *well-ordered* if the totally ordered set $(S, \leq_0)$ is well-ordered. It is not difficult to check that S is well-ordered if and only if, for all (or for some) $c \in C$, the totally ordered set $(S, \leq_c)$ is well-ordered.

Lemma 2.1. *The sum of any two well-odered subsets of C is well-ordered.*

Proof. Let S_1 and S_2 be two well-ordered subsets and $S = S_1 + S_2 = \{c_1 + c_2 \mid c_1 \in S_1,\ c_2 \in S_2\}$. The ordered set $(C, \leq_0)$ is isomorphic to the subset $[0, z[$ of G. S_1 and S_2 are isomorphic to well-ordered subsets $\tilde{S}_1$ and $\tilde{S}_2$ of $[0, z[$, and $\tilde{S}_1 + \tilde{S}_2$ is a well-ordered subset of $[0, 2z[$. $S_1 + S_2$ is isomorphic to $((\tilde{S}_1 + \tilde{S}_2) \cap [0, z[) \cup (((\tilde{S}_1 + \tilde{S}_2) \cap [z, 2z[) - z)$, and this set is well-ordered. Hence $S_1 + S_2$ is well-ordered. □

For all $\sigma \in k^C$, the set $\mathrm{Supp}\,(\sigma) := \{c \in C \mid \sigma(c) \neq 0\}$ is called the *support* of σ. We denote by $k[[C]]$ the subset of k^C of all elements with well-ordered support. An element σ of $k[[C]]$ will be written in the form $\sigma = \Sigma_{c \in S} \sigma_c X^c$, where S is well-ordered, and for all c in S, $\sigma_c = \sigma(c)$. We call σ a *cyclic formal power series*. Note that if $\sigma_c = 0$ for all $c \notin S \cap S'$, then $\Sigma_{c \in S} \sigma_c X^c = \Sigma_{c \in S'} \sigma_c X^c$.

We let $k[C]$ be the subset of $k[[C]]$ of elements with finite support. An element of $k[C]$ will be called a *polynomial*.

We define addition on $k[[C]]$ by:

$$\sigma + \sigma' = \Sigma_{c \in S \cup S'} (\sigma_c + \sigma'_c) X^c,$$

and multiplication by:

$$\sigma \cdot \sigma' = \Sigma_{c \in S+S'} \left(\sum_{d \in S,\, d' \in S',\, d+d'=c} \sigma_d \cdot \sigma'_{d'} \right) X^c \ .$$

The small Σ is the symbol for a series, the large $\sum$ represents the sum in the ring. Indeed, like in classical power series, one can prove that if S and S' are well-ordered, then for all $c \in S + S'$, there is only a finite number of $(d, d') \in S \times S'$ such that $d + d' = c$.

It is routine to prove that $(k[[C]], +, \cdot)$ is a ring, and $(k[C], +, \cdot)$ is a subring of $(k[[C]], +, \cdot)$.

Note that k is embedded in $k[C]$ in a natural way. Hence we will consider it as a subring of $k[C]$, the elements of k being the functions with support $\{0\}$. Also note that any ring of formal power series is a ring of cyclic formal power series.

3. Cyclic versus classical formal power series rings

In this section, we focus on connections between rings of formal power series with exponents in a cyclically ordered group, and rings of formal power series with exponents in a totally ordered group. We set some notations which we will use throughout this paper:

- $G_+ := \{g \in G \mid g \geq 0\}$.
- p is the canonical map from G onto C.
- $k[G]$ is the ring of polynomials with (possibly negative) exponents in G and coefficients in k.
- If $k[[C]]$ or $k[C]$ is an integral domain, then its fraction field is denoted $k((C))$ or $k(C)$, respectively.

Note that, if k is a field, then the fraction field $k((G))$ of $k[[G_+]]$ is equal to $k[[G]]$ because G is totally ordered. It is also equal to $\bigcup_{g\in G_+} X^{-g}k[[G_+]]$. Nothing of this kind holds for $k[[C]]$ (see Section **??**), which need not even be an integral domain.

In order to prove that $k[[C]]$ is isomorphic to quotients of classical power series rings, we define the following subsets of $k[[G]]$:
$A_M := \{\sigma \in k[[G]] \mid \forall c \in C,\ p^{-1}(\{c\})\cap \mathrm{Supp}(\sigma)$ is finite and $p(\mathrm{Supp}(\sigma))$ is well-ordered$\}$.
$A_{M+} := \{\sigma \in A_M \mid \mathrm{Supp}(\sigma) \geq 0\}$.
$A_m := \{\sigma \in A_M \mid \mathrm{Supp}(\sigma)$ is bounded$\}$.
$A_{m+} := A_{M+} \cap A_m$.
$A_0 := \{\sigma \in A_M \mid \mathrm{Supp}(\sigma) \subset [0, z[\}$.
A_0 is an additive subgroup of $k[[G]]$ but, in general, not a subring.

Observe that we can't drop the condition "$p(\mathrm{Supp}(\sigma))$ is well-ordered" in the definition of the ring A_M. Indeed, we can find a subset S of G which is well-ordered although $p(S)$ is not. Take $G = \mathbb{Q}$ (the group of rational numbers) and $z = 1$. Then $S = \{n + \frac{1}{n} \mid n \in \mathbb{N}^*\}$ is well-ordered, while $p(S) = \{\frac{1}{n} \mid n \in \mathbb{N}^*\}$ is not well-ordered.

Lemma 3.1. *The sets A_M and A_m are subrings of $k[[G]]$, A_{M+} and A_{m+} are subrings of $k[[G_+]]$.*

Proof. Note that these sets are non-empty because they contain 0 and $1 = X^0$.

We prove here that A_M is a subring of $k((G))$, the other proofs being similar. Take $\sigma, \sigma' \in A_M$. Then $\mathrm{Supp}(-\sigma) = \mathrm{Supp}(\sigma)$, hence $-\sigma \in A_M$. Further, $p(\mathrm{Supp}(\sigma + \sigma')) \subset p(\mathrm{Supp}(\sigma) \cup \mathrm{Supp}(\sigma')) = p(\mathrm{Supp}(\sigma)) \cup p(\mathrm{Supp}(\sigma'))$ is well-ordered. For all $c \in C$, $p^{-1}(c) \cap (\mathrm{Supp}(\sigma) \cup \mathrm{Supp}(\sigma')) = (p^{-1}(c) \cap \mathrm{Supp}(\sigma)) \cup (p^{-1}(c) \cap \mathrm{Supp}(\sigma'))$ is finite. Hence, $\sigma + \sigma' \in A_M$.

Set $S := \mathrm{Supp}(\sigma)$ and $S' := \mathrm{Supp}(\sigma')$. We have that $\mathrm{Supp}(\sigma\sigma') \subset S + S'$. Furthermore, the mapping $p : G \twoheadrightarrow C$ is a group homomorphism, hence $p(S + S') = p(S) + p(S')$ is well-ordered by Lemma **??**. It follows that $p(\mathrm{Supp}(\sigma\sigma'))$ is well-ordered. Now, take any $c \in C$ and assume that $p^{-1}(\{c\}) \cap \mathrm{Supp}(\sigma\sigma')$ is infinite. Set $H := \{h \in S \mid \exists h' \in S',\ p(h + h') = c\}$, $H' := \{h' \in S' \mid \exists h \in S,\ p(h + h') = c\}$. Then one of H and H' is infinite. But, since each element in $p(S)$ has finitely many pre-images, H is finite if and only if $p(H)$ is, and the same holds for H' and $p(H')$, where $p(H)$ and $p(H')$ are both well-ordered in C_0. Assume for instance, that $p(H)$ is infinite, hence it contains an infinite increasing sequence, then $p(H')$ contains an infinite decreasing sequence, a contradiction. Consequently, $\sigma\sigma' \in A_M$. □

Proposition **??** can be rephrased as follows:

Proposition 3.2. *Let A stand for any of the rings A_{M+}, A_M, A_{m+}, A_m, then $k[[C]] \simeq A/(1 - X^z)A$.*

Proof. We first prove this proposition for $A = A_M$. We define $p'_A : A_M \twoheadrightarrow k[[C]]$. For an element $\Sigma_{g\in S}\sigma_g X^g$ in A_M, we set

$$p'_A(\Sigma_{g\in S}\sigma_g X^g) = \Sigma_{c\in p(S)} \left(\sum_{h\in S\cap p^{-1}(\{c\})} \sigma_h \right) X^c \in k[[C]].$$

This definition makes sense because for each c in $p(S)$, $S \cap p^{-1}(\{c\})$ is a finite set. It is routine to check that the map p'_A is a ring epimorphism.

Take

$$\sigma = \Sigma_{g\in S}\sigma_g X^g \in A_M$$

and assume that S is the support of σ. We have $X^{-g_0}\sigma \in A_+$ for $g_0 = \min(\mathrm{Supp}(\sigma))$. Therefore, we may assume that $\sigma \in A_{M+}$. We have

$$p'_A(\sigma) = \Sigma_{c\in p(S)} \left(\sum_{h\in S\cap p^{-1}(\{c\})} \sigma_h \right) X^c \ .$$

Hence,

$$p'_A(\sigma) = 0 \text{ if and only if, for all } c \in p(S), \sum_{h\in S\cap p^{-1}(\{c\})} \sigma_h = 0 \ .$$

We assume $p'_A(\sigma) = 0$ and prove that $\sigma = (1 - X^z)\sigma'$ for some $\sigma' \in A_M$. For $c \in C$, we will denote by $\tilde{c}$ the unique element of $[0, z[$ such that $p(\tilde{c}) = c$. Set $\tilde{S} := \{\tilde{c} \mid c \in p(S)\} = [0, z[\cap(p^{-1}(p(S)))$. Since $p(S)$ is well-ordered and isomorphic to $\tilde{S}$, $\tilde{S} + \mathbb{N}z$ is well-ordered too.

For $c \in p(S)$, $p^{-1}(\{c\}) = \tilde{c} + \mathbb{Z}z$, and $S \cap p^{-1}(\{c\})$ is finite. Let m_c and n_c in $\mathbb{N}$ be such that $\sigma_{\tilde{c}+m_c\cdot z} \neq 0$, $\sigma_{\tilde{c}+n_c\cdot z} \neq 0$, and for $n > n_c$ or $n < m_c$, $\sigma_{\tilde{c}+n\cdot z} = 0$. It follows that

$$\sigma_{\tilde{c}+n_c\cdot z} = -\sigma_{\tilde{c}+m_c\cdot z} - \sigma_{\tilde{c}+(m_c+1)\cdot z} - \cdots - \sigma_{\tilde{c}+(n_c-1)\cdot z} \ .$$

For all $c \in p(S)$ and $m_c \leq i < n_c$, set $\sigma'_{\tilde{c}+i\cdot z} := \sigma_{\tilde{c}+m_c\cdot z} + \sigma_{\tilde{c}+(m_c+1)\cdot z} + \cdots + \sigma_{\tilde{c}+i\cdot z}$; otherwise set $\sigma'_g := 0$. Further, set

$$\sigma' = \Sigma_{g\in \tilde{S}+\mathbb{N}z}\sigma'_g X^g \ .$$

This is an element of A_{M+} since $\tilde{S} + \mathbb{N}z$ is well-ordered, and for all $c \in C$, $p^{-1}(\{c\})\cap \mathrm{Supp}(\sigma')$ is either empty or bounded above by $\tilde{c} + n_c z$ showing that its cardinality is at most n_c. Write

$$(1 - X^z)\sigma' \ = \ \Sigma_{g\in G}\sigma''_g X^g \in k[[G]] \ .$$

If $g \notin \mathrm{Supp}(\sigma') \cup (z + \mathrm{Supp}(\sigma'))$, then $\sigma''_g = 0$. If $g \in \mathrm{Supp}(\sigma') \cup (z + \mathrm{Supp}(\sigma'))$, then there are $c \in p(S)$ and $m_c \leq i \leq n_c$ with $g = \tilde{c} + i \cdot z$. We have:

$$\sigma''_g = \begin{cases} \sigma'_{\tilde{c}+m_c\cdot z} = \sigma_{\tilde{c}+m_c\cdot z} & \text{if } i = m_c \\ \sigma'_{\tilde{c}+i\cdot z} - \sigma'_{\tilde{c}+(i-1)\cdot z} = \sigma_{\tilde{c}+i\cdot z} & \text{if } m_c < i < n_c \\ -\sigma'_{\tilde{c}+(n_c-1)\cdot z} = -\sigma_{\tilde{c}+m_c\cdot z} - \sigma_{\tilde{c}+(m_c+1)\cdot z} - \cdots - \sigma_{\tilde{c}+(n_c-1)\cdot z} = \sigma_{\tilde{c}+n_c\cdot z} & \text{if } i = n_c \ . \end{cases}$$

It follows that $(1 - X^z)\sigma' = \sigma$, that is, $\sigma \in (1 - X^z)A_M$.

Since p'_A is a ring homomorphism and $p'_A(1 - X^z) = 0$, we have shown that $\ker p_A = (1 - X^z)A_M$, and therefore, $k[[C]] \simeq A_M/(1 - X^z)A_M$. Observe that p'_A is indeed the canonical mapping p_A.

If A stands for A_{M+}, A_m or A_{m+}, then $\ker p'_A\lceil A = \ker p'_A \cap A = (1 - X^z)A_M \cap A = (1 - X^z)A$. □

Since $A_0 \cup \{X^z\}$ generates A_{m+}, we have:

Remark 3.3. The family of all subrings A of A_M containing X^z and such that $p_A(A) = k[[C]]$ can be ordered by inclusion. Its smallest element is A_{m+}. Indeed, any element σ of A_M is equivalent modulo $(1 - X^z)A_M$ to an element of A_0, namely $(p_A\lceil A_0)^{-1}(p_A(\sigma))$.

4. Units and zero divisors

It is not difficult to see that $k[[C]]$ may contain zero divisors. For example, let $m > 1$ be an integer and suppose that there is some $g \in G$ such that $mg = z$. Then

$$(1 - X^g)(1 + X^g + X^{2g} + \cdots + X^{(m-1)g}) = 1 - X^z \in p_A^{-1}(\{0\}) .$$

Since $C \simeq G/\mathbb{Z}z$ where G is torsion free, such m and g exist if and only if C is not torsion free. Therefore, we have shown:

Lemma 4.1. *If $k[[C]]$ is an integral domain, then C is torsion free.*

Trivial examples of units are the elements X^c, with $c \in C$. The following properties of cyclically ordered groups will provide less trivial examples.

Definition 4.2. We set $P := \{c \in C \mid (0, c, -c)\}$. From the general properties of cyclically ordered groups, we know that there exists at most one element of order 2, which we shall denote by ϵ. Hence, C is a disjoint union $C = P \cup -P \cup \{c \mid c = -c\}$, where $\{c \mid c = -c\}$ has one or two elements. If there is such an ϵ in C, then we have $b \leq_0 \epsilon \leq_0 -a$ for all $(a, b) \in P \times P$. If not, then ϵ will denote the element of the Dedekind completion of $(C, \leq_0)$ such that for any $a, b \in P$, $b \leq_0 \epsilon \leq_0 -a$. In both cases, we define $\leq_\epsilon$ by: $a \leq_\epsilon b$ if and only if either $(a, b) \in -P \times (P \cup \{0\})$, or $(a, b) \in (-P \times (-P \cup \{0\})) \cup ((P \cup \{0\}) \times P)$ and $a \leq_0 b$.

The linear part $l(C)$ of C defined in the introduction is the largest totally ordered subgroup of $(C, \leq_\epsilon)$. (C is a linear cyclically ordered group if and only if $C = l(C)$.) As mentioned, $l(C)$ is the image of the maximal convex subgroup $l(G)$ of G not containing z under the canonical epimorphism $p : G \to C$. Its restriction to $l(G)$ is a monomorphism of totally ordered groups. We lift this monomorphism to $k[[l(G)]]$ by setting $p_l(\lambda X^g) = \lambda X^{p_l(g)}$ for $\lambda \in k$ and $g \in l(G)$, and then using additivity and multiplicativity, such that p_l becomes a field embedding. Via this embedding, we can assume $k[[l(G)]] \subset k[[C]]$.

If k is a field, then every element of $k[[l(G)]] \setminus \{0\}$ and every non-zero monomial is a unit in $k[[C]]$. But $k[[C]]$ may also contain units which are not monomials and do not belong to $k[[l(G)]]$. Indeed, let $m > 1$ be an integer. If λ_1, λ_2 are elements of k such that $\lambda_1^m - \lambda_2^m = 1$ and if there exists g such that $mg = z$, then

$$(\lambda_1 - \lambda_2 X^g)(\lambda_1^{m-1} + \lambda_1^{m-2}\lambda_2 X^g + \cdots + \lambda_2^{m-1} X^{(m-1)g}) = (\lambda_1^m - \lambda_2^m X^z) \in p_A^{-1}(\{1\})$$

For example, if $2g = z$, $(\sqrt{2} + X^g)(\sqrt{2} - X^g) = 2 - X^z \in p_A^{-1}(\{1\})$.

Let us give yet another example of a unit. Let q be a positive integer and G be the subgroup of $\mathbb{Q}\overrightarrow{\times}\mathbb{Q}$ generated by $(1, 0) = z$, $(0, 1)$ and $(\frac{1}{q}, \frac{1}{q})$. Now let $C := G/(\mathbb{Z}z)$, $k := \mathbb{Q}$, $c := p(\frac{1}{q}, \frac{1}{q}) \in C$, $S_i := \{p(\frac{i}{q}, n + \frac{i}{q}) \mid n \in \mathbb{N}\} = ic + p(\{0\} \times \mathbb{N})$ for $0 \leq i \leq q - 1$, $S := \bigcup_{j=0}^{q-1} S_j$, and $\sigma := \Sigma_{s \in S} X^s = \Sigma_{n \in \mathbb{N}} X^{p(0,n)}(1 + X^c + \cdots + X^{(q-1)c})$. Then $\sigma \in k[[C]]$ because S is a well-ordered subset of C. Since $X^{qc} = X^{p(0,1)}$, we have:

$$\begin{aligned}(1 - X^c)\sigma &= (1 - X^c)\Sigma_{n \in \mathbb{N}} X^{p(0,n)}(1 + X^c + \cdots + X^{(q-1)c}) \\ &= (1 - X^{qc})\Sigma_{n \in \mathbb{N}} X^{p(0,n)} = \Sigma_{n \in \mathbb{N}} X^{p(0,n)} - \Sigma_{n \in \mathbb{N}^*} X^{p(0,n)} = 1.\end{aligned}$$

Hence, $1 - X^c$ and σ are units.

Proof of Proposition ??.

(a) First we note that $1 - X^z$ is not a zero divisor. Indeed, 1 is a regular element, hence for all σ in A, the lowest element of the support of σ remains the lowest element of the support of $(1 - X^z)\sigma$. It follows that if $\sigma_1 \notin (1 - X^z)A$ and there exists σ_2 such that $\sigma_1\sigma_2 = 0$, then we can assume that $\sigma_2 \notin (1-X^z)A$. Consequently, $p_A(\sigma_1)$ is a zero divisor in $k[[C]]$.

Assume that σ_1 is a regular element. Then $p_A(\sigma_1)$ is a non-trivial zero divisor if and only if $p_A(\sigma_1) \neq 0$ and there exists $\sigma_2 \in A$ such that $p_A(\sigma_2) \neq 0$ and $p_A(\sigma_1)p_A(\sigma_2) = 0$, or equivalently:

(*) $\sigma_1 \notin (1 - X^z)A$ and there exist $\sigma_2 \notin (1 - X^z)A$, $\tau \in A$ such that $\sigma_1\sigma_2 = (1 - X^z)\tau$.

Now consider the condition $\sigma_1 A \cap (1 - X^z)A \neq \sigma_1(1 - X^z)A$. This is equivalent to $\exists\sigma_2 \in A$, $\exists\tau \in A$, $\sigma_1\sigma_2 = (1 - X^z)\tau$ and $\sigma_1\sigma_2 \notin \sigma_1(1 - X^z)A$ (because $\sigma_1(1 - X^z)A \subset \sigma_1 A \cap (1 - X^z)A$). Since σ_1 is a regular element, this condition is equivalent to (*).

(b) For all $\sigma \in A$,

$$\begin{aligned} p_A(\sigma) \text{ is a unit} &\Leftrightarrow \exists\sigma',\ p_A(\sigma\sigma') - 1 = 0 \\ &\Leftrightarrow \exists\sigma',\ \sigma\sigma' - 1 \in (1 - X^z)A \\ &\Leftrightarrow \exists\sigma',\ \exists\tau,\ \sigma\sigma' - 1 = (1 - X^z)\tau \\ &\Leftrightarrow \exists\sigma',\ \exists\tau,\ \sigma\sigma' + (1 - X^z)\tau = 1. \end{aligned}$$

□

Proof of Theorem ??.
First assume that $k[[C]]$ is a field. Take $\sigma \in k \setminus \{0\}$, that is, $\mathrm{Supp}(\sigma) = \{0\}$, and let σ^{-1} be the inverse of σ in $k[[C]]$. $\{0\} = \mathrm{Supp}(\{1\}) = \mathrm{Supp}(\sigma\sigma^{-1}) = \mathrm{Supp}(\sigma^{-1})$ (because $\sigma \in k$), hence $\sigma^{-1} \in k$. Therefore, k is a field. From Lemma **??**, it follows that C is torsion free.

We know that $C/l(C)$ embeds in the multiplicative group of all complex numbers of absolute value 1. We prove that, under our present hypothesis, every element of $C/l(C)$ must be a root of unity. In order to deduce a contradiction, assume that there exists $g \in G$ such that $l(G) < g < z + l(G)$ and that the class $\bar{g}$ of g modulo $l(G)$ and the class $\bar{z}$ of z modulo $l(G)$ are rationally independent. Set $c := p(g) \in C$, $c \neq 0$, and $\sigma := (1-X^c)$, and let $\sigma^{-1} := \Sigma_{s\in C}\lambda_s X^s$ be the inverse of $(1 - X^c)$ in $k[[C]]$. Then $1 = \Sigma_{s\in C}(\lambda_s - \lambda_{s-c})X^s$. Hence, for all $s \in C$,

$$\lambda_s - \lambda_{s-c} = \begin{cases} 0 & \text{if} \quad s \neq 0 \\ 1 & \text{if} \quad s = 0 \end{cases}$$

If $\lambda_0 \neq 0$, then $\lambda_{nc} = \lambda_0 \neq 0$ for all $n > 0$. If $\lambda_0 = 0$, then $\lambda_{-c} = 1 \neq 0$, and $\lambda_{-nc} = \lambda_{-c} = 1 \neq 0$ for all $n > 0$. Assume $\lambda_{nc} \neq 0$ for all $n > 0$ (the case of $\forall n > 0$, $\lambda_{nc} \neq 0$ for all $n > 0$ is similar, because $0 \neq -c \in C$).

The quotient group $\Gamma = G/l(G)$ is archimedean because $\mathbb{Z}z$ is cofinal in G. In the archimedean group Γ, $\bar{z}$ and $\bar{c}$ are rationally independent, hence the image of $\mathbb{Z}\bar{c}$ in the quotient group $\Gamma/(\mathbb{Z}\bar{z})$ is dense. In particular, it is not well-ordered. This implies that $\{nc \mid n > 1\}$ is a subset of $\mathrm{Supp}(\sigma^{-1})$ which is not well-ordered: a contradiction.

Now assume that C is torsion free and $C/l(C)$ is finite. Let n be the cardinality of $C/l(C)$; then $C/l(C)$ is isomorphic to the group of all n^{th} roots of unity. By hypothesis, there exist g in C and l in $l(C)$ such that $ng = l$ and C is the disjoint union $l(C) \cup$

$(g + l(C)) \cup \cdots \cup ((n-1)g + l(C))$. Note that, for every element c of C, nc belongs to $l(C)$. For c_1 and c_2 in C we set $c_1 <' c_2 \Leftrightarrow nc_1 <_\epsilon nc_2$. Then $(C, <')$ is a totally ordered group. Let $S \subset C$. Then there exist subsets $S_0,\ S_1, \ldots,\ S_{n-1}$ of $l(C)$ such that $S = S_0 \cup (g + S_1) \cup \cdots \cup ((n-1)g + S_{n-1})$. It follows that S is well-ordered in $(C, <_\epsilon)$ if and only if $S_0,\ S_1, \ldots,\ S_{n-1}$ are well-ordered, if and only if S is well-ordered in $(C, <')$. Hence, $k[[C]]$ is isomorphic to $k[[l(C)]][Y]$, with $Y^n = X^l \in k[[l(C)]]$. In particular, $k[[C]]$ is a field. □

¿From Lemma **??**, we see that a necessary condition for $k[[C]]$ to be an integral domain is that C is torsion free. Another condition is that k be an integral domain. We shall prove that for the ring of polynomials $k[C]$, these conditions are also sufficient. For this purpose, we need the following well known lemma.

Lemma 4.3. *Let k be a field containing a primitive n-th root of unity ζ, where $n \geq 2$. Then $1 - X^n = \prod_{i=0}^{n-1}(\zeta^i - X)$. Let $P(X) = 1 + \cdots$ and $Q(X) = 1 + \cdots$ be two polynomials of $k[X]$ such that $P(X)Q(X) = 1 - a^n X^n$, with $a \in k$. Then, there exists a partition $\{I, J\}$ of $\{0, \ldots, n-1\}$ such that $P(X) = (\prod_{i\in J} \zeta^i) \prod_{i\in I}(\zeta^i - aX)$ and $Q(X) = (\prod_{i\in I} \zeta^i) \prod_{i\in J}(\zeta^i - aX)$. In particular, $P(X)$ and $Q(X)$ belong to $\mathbb{Z}(\zeta)[aX]$.*

Proposition 4.4. *Assume that k is an integral domain. Then $1 - X^z$ is prime in $k[G_+]$ if and only if z is not divisible by any natural number bigger than 1 in G.*

Proof. First assume that there are two polynomials P and Q in $k[G_+]$ such that $P(X)Q(X) = 1 - X^z$. After multiplying with a suitable constant, we can assume that $P(X) = 1 + \lambda_1 X^{a_1} + \cdots$, $Q(X) = 1 + \lambda'_1 X^{b_1} + \cdots$. Then $P(X)Q(X) = 1 + \lambda_1 X^{a_1} + \lambda'_1 X^{b_1} + \lambda_1\lambda'_1 X^{a_1+b_1} + T(X)$, where all terms in $T(X)$ are of degree $> \min(a_1, b_1) > 0$. It follows that $\lambda_1 X^{a_1} + \lambda'_1 X^{b_1} = 0$. In particular, $a_1 = b_1$ and therefore, $\mathrm{Supp}(P) \cap \mathrm{Supp}(Q)$ contains non-zero elements.

Since k is assumed to be an integral domain, we may replace it by its fraction field. So we may assume k is a field. For further purposes, we also assume that k is closed under roots of unity.

Let $D(G)$ be the divisible hull of G. The number of exponents of the polynomials is finite. So we can find rationally independent exponents $\alpha_1, \ldots, \alpha_n$, such that z and the exponents of P and Q are all in $\mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_n$ (the existence of the α_i's can be proved by an iterative process like the Algorithm of Perron quoted in [**?**]). Set $X_1 = X^{\alpha_1}$, ... , $X_n = X^{\alpha_n}$.

Let $z = c_1\alpha_1 + \cdots + c_n\alpha_n$, with $c_i \in \mathbb{N}$, so

$$1 - X^z \ = \ 1 - X_1^{c_1} \cdots X_n^{c_n} \ .$$

We have $P(X_1, \ldots, X_n)Q(X_1, \ldots, X_n) = 1 - X_1^{c_1} \cdots X_n^{c_n}$. But $\mathrm{Supp}(P) \cap \mathrm{Supp}(Q)$ contains non-zero elements. Hence there exists i, say $i = n$, such that $\deg_{X_n}(P) \geq 1$ and $\deg_{X_n}(Q) \geq 1$ (hence $c_n = \deg_{X_n}(P) + \deg_{X_n}(Q) \geq 2$). Therefore, P and Q are in $k(X_1^{1/c_n}, \ldots, X_{n-1}^{1/c_n})[X_n]$.

Let $\zeta \in k$ be an n-th root of unity, and set $Y := X_1^{c_1/c_n} \cdots X_{n-1}^{c_{n-1}/c_n} X_n$. Then, by Lemma **??**, P and Q are elements of $\mathbb{Z}(\zeta)[Y]$. Furthermore, since $\deg_{X_n}(P) \geq 1$ and $\deg_{X_n}(Q) \geq 1$, we have $\deg_Y(P) \geq 1$ and $\deg_Y(Q) \geq 1$. Set $d := \deg_Y(P)$. Then there exists $r_d \in \mathbb{Z}(\zeta) \setminus \{0\}$ such that one of the monomials in P is $r_d X_1^{dc_1/c_n} \cdots X_n^{dc_{n-1}/c_n} X_n^d = r_d X^{(d/c_n)(c_1\alpha_1+\cdots+c_n\alpha_n)} = r_d X^{(d/c_n)z}$. Hence $(d/c_n)z \in G$.

Set $\frac{d}{c_n} = \frac{u}{v}$ with u, v in $\mathbb{N}^*$ and $\gcd(u, v) = 1$, and let u', v' in $\mathbb{Z}$ such that $uu' + vv' = 1$. Then $\frac{d}{c_n}z = \frac{u}{v}z$, hence $v'z + u'\frac{u}{v}z = \frac{vv'+uu'}{v}z = \frac{1}{v}z$ and $\frac{1}{v}z \in G$. But $d < c_n$, therefore $v > 1$. Hence z has a proper divisor in G.

If $1 - X^z$ divides $P(X)Q(X)$, then we can assume that $P(X)$, $Q(X)$, and X^z are polynomials (in the usual sense) with a finite number of variables, and they belong to a factorial ring (by embedding k in its fraction field). Therefore, there exist $P_1(X)$ and $Q_1(X)$ such that $P_1(X)Q_1(X) = 1 - X^z$, and as in the previous case, we see that z has a proper divisor. □

Proof of Theorem ??.
The condition "z is not divisible by an integer bigger than 1" is equivalent to "C is torsion free". Now, Theorem **??** follows from Proposition **??**. □

5. Cyclic valuations

In this section, k is an integral domain. If C contains a non-trivial 2-torsion element, then we let ϵ be this element; otherwise, $<_\epsilon$ has been defined in Definition **??**. We set $C_\epsilon = C \cup \{\epsilon\}$. We define a mapping v from $C_\epsilon \times k[[C]]$ onto C_∞ by setting $v(a, \sigma) = \min_a \mathrm{Supp}(\sigma)$ if $\sigma \neq 0$ and $v(a, 0) = \infty$.

It is routine to check that $(k[[C]], +, v(a, \cdot))$ is a valued group, for every $a \in C_\epsilon$. In particular, for all σ_1 and σ_2 in $k[[C]]$, $v(a, \sigma_1 - \sigma_2) \geq_a \min_a(v(a, \sigma_1), v(a, \sigma_2))$, and if $v(a, \sigma_1) \neq v(a, \sigma_2)$, then $v(a, \sigma_1 - \sigma_2) = \min_a(v(a, \sigma_1), v(a, \sigma_2))$.

The behaviour of the $v(a, \cdot)$'s with respect to multiplication is not trivial. However, observe that for all a, b in C and σ in $k[[C]]$, $v(a, \sigma) = v(b, X^{b-a}\sigma) + a - b$. If τ is a monomial of degree $b - a$, then $v(a, \sigma) = v(b, \tau\sigma) + a - b$.

If C is a linear cyclically ordered group, then $k[[C]]$ is the usual ring of formal power series, and we know that there is a canonical valuation defined on $k[[C]]$. This valuation is $v(\epsilon, \cdot)$ but is not among the $v(a, \cdot)$'s, $a \in C$. Indeed, for all $a \in C$, we can find some $\sigma \in k[C]$ (for instance, a sum of two monomials) such that $v(\epsilon, \sigma) \neq v(a, \sigma)$.

If C is not a linear cyclically ordered group, then there exist a and b in P with $a + b \in -P$, with P as defined in Definition **??**. Therefore, $v(\epsilon, 1 + X^a) = v(\epsilon, 1 + X^b) = 0$, $v(\epsilon, (1 + X^a)(1 + X^b)) = a + b \neq 0 = v(\epsilon, 1 + X^a) + v(\epsilon, 1 + X^b)$. Hence, $v(\epsilon, \cdot)$ is not a valuation in the usual sense.

We consider $k[[C]] \cup C_\epsilon$ as a structure of a two-sorted language $\mathcal{L}$ containing the usual function, relation and constant symbols for the ring and the value group sort. Then $\mathcal{L} \cup \{v\}$ will be the language $\mathcal{L}$ together with a function symbol interpreted by the mapping v.

The support mapping is interpretable in the language $\mathcal{L} \cup \{v\}$. Indeed, let $\sigma \in k[[C]]$. Then the support of σ is the set $\mathrm{Supp}(\sigma) = \{v(a, \sigma) \mid a \in C\} \setminus \{\infty\}$. It follows that the set of all monomials and the degree map of monomials are definable in the language $\mathcal{L} \cup \{v\}$. In the same way, the set of all constant polynomials is definable in the language $\mathcal{L} \cup \{v\}$.

Observe that we can also characterize the monomials by other means. Indeed, $\sigma \in k[[C]] \setminus \{0\}$ is a monomial if and only if $\forall \tau \in k[[C]]$, $v(v(0,\sigma),\tau\sigma) = v(0,\sigma) + v(0,\tau)$. If σ is a monomial, its degree is $v(a,\sigma) = v(0,\sigma)$ for all $a \in C$.

Another characterization of the constant polynomials follows from:

Remark 5.1. Let $\sigma \in k[[C]] \setminus \{0\}$, then σ is a constant if and only if one of the following equivalent formulas is satisfied in M.
(a) $\forall a \in C_\epsilon$, $\forall \tau \in k[[C]]$, $v(a,\sigma\tau) = v(a,\tau)$.
(b) $\exists a \in C$, $\forall \tau \in k[[C]]$, $v(a,\sigma\tau) = v(a,\tau)$.

Proof. If σ is a constant, then (a) holds. (a) $\Longrightarrow$ (b) is trivial.

Assume that σ is not a constant and take $a \in C$. If $v(a,\sigma) \neq 0$ and if $\tau = 1$, then $v(a,\sigma\tau) \neq v(a,\tau) = 0$. So we suppose that $v(a,\sigma) = 0$. Then there exist λ, $\lambda_1 \in k$, $g_1 \in C$, with $\lambda_1 \neq 0 \neq \lambda$, and $0 <_0 g_1 <_0 a$, such that $\sigma = \lambda + \lambda_1 X^{g_1} + \cdots$, (necessarily, all the elements of the support are in $[0,a[$, that is, greater than or equal to 0 with respect to $\leq_a$). Then set $\tau = X^{a-g_1}$, $v(a,\sigma\tau) = a \neq v(a,\tau)$. This proves that (b) implies that σ is a constant. □

Before we define structures in the language $\mathcal{L} \cup \{v\}$, we focus on two languages for the rings $k[[C]]$. First, we let M be a unary predicate interpreted by: $M(\sigma) \Leftrightarrow \sigma$ is a monomial. Then we have the following

Proposition 5.2. *In our structure $k[[C]] \cup C_\infty$, the languages $\mathcal{L} \cup \{v\}$ and $\mathcal{L} \cup \{v(0,\cdot),\, M\}$ are bi-interpretable.*

Proof. The set of all monomials is definable in $\mathcal{L} \cup \{v\}$. Hence, $\mathcal{L} \cup \{v(0,\cdot),\, M\}$ is interpretable in $\mathcal{L} \cup \{v\}$. Now, let $a \in C$ and $\sigma \in k[[C]]$ and take τ an invertible monomial such that $v(0,\tau) = a$. Then, $v(a,\sigma) = v(0,\tau^{-1}\sigma) + a$. Hence, $\mathcal{L} \cup \{v\}$ is interpretable in $\mathcal{L} \cup \{v(0,\cdot),\, M\}$. □

In the ring $k[[C]]$, the multiplicative subgroup $\{X^c \mid c \in C\}$ is isomorphic to the group C. We can introduce a cyclic order on $\{X^c \mid c \in C\}$ which is isomorphic to the cyclic order of C. We can also define a valuation in a different way: set $v'(X^a,\sigma) := X^b$ if and only if $v(a,\sigma) = b$, and for all $a \in C$, set $v'(X^a,0) := 0$ (the 0 of the ring replaces the ∞ of C). Now, we consider $k[[C]]$ as a structure of a language $\mathcal{L}'$ consisting of the language of rings together with the predicate v, a predicate interpreted by the multiplicative group $\{X^c \mid c \in C\}$, and a predicate for the cyclic order on $\{X^c \mid c \in C\}$, with, for all a, b in C, $(X^a, X^b, 0)$.

Proposition 5.3. *The subgroup $\{X^c \mid c \in C\}$ is not definable in the language $\mathcal{L} \cup \{v\}$.*

Proof. Assume that k is algebraically closed, and that C contains a torsion free divisible subgroup C_1, such that C_1 is a direct summand, $C = C_1 \oplus C_2$. We know that C_1 is a $\mathbb{Q}$-vector space, hence it admits a basis. For all element c in this basis, we let α_c be a constant, with $0 \neq \alpha_c \neq 1$. For $c \in C_2$, we set $\alpha_c = 1$, and we extend $c \mapsto \alpha_c$ as an isomorphism from the group C into $(k,\cdot)$. Now, we set $\forall c \in C$, $\varphi(X^c) := \alpha_c X^c$, and we extend φ to a homomorphism from $k[[C]]$ onto $k[[C]]$, such that the restriction of φ to k is the identity. Then φ induces an $\mathcal{L}' \cup \{v'(X^0,\cdot)\}$-isomorphism. We have constructed

two distinct $\mathcal{L}' \cup \{v'(X^0, \cdot)\}$-structures on $k[[C]]$. But in both cases, the induced $\mathcal{L} \cup \{v\}$-structure is the same. It follows that the subgroup $\{X^c \mid c \in C\}$ is not definable in the language $\mathcal{L} \cup \{v\}$. However, it is interpretable because it is isomorphic to the quotient of the group of invertible monomials by the group of invertible constants. □

The remainder of this section is devoted to the definition of $\mathcal{L} \cup \{v\}$-structures involving the rings $k[[C]]$. We will specify which of them can be embedded in a ring of formal power series with cyclically ordered exponents.

Definition 5.4. Let $(R, +, \cdot)$ be a domain, v a mapping from $C \times R$ onto $C \cup \{\infty\}$, and $\sigma \in R$. The *support* of σ is the set $\mathrm{Supp}(\sigma) := \{v(a, \sigma) \mid a \in C\}$; σ is a *monomial* if the support of σ is a singleton. σ is a *constant* if $\sigma = 0$ or $\mathrm{Supp}(\sigma) = \{0\}$. We will say that (R, v) is a *cyclically valued ring* if the following conditions hold.
(1) For all $a \in C$, $(R, +, v(a, \cdot))$ is a valued group.
(2) For all $\sigma \in R$ and $a \in C$, if $v(a, \sigma) = a$, then there exists a unique monomial $\mu_{a,\sigma}$ such that $v(a, \sigma - \mu_{a,\sigma}) \neq a$.
(3) For all $\sigma \in R$ and $a \in C$, $\min_a(\mathrm{Supp}(\sigma))$ exists and is equal to $v(a, \sigma)$.
(4) For all σ and σ' in R, $\mathrm{Supp}(\sigma\sigma') \subset \mathrm{Supp}(\sigma) + \mathrm{Supp}(\sigma')$.
(5) For all $n \in \mathbb{N}^*$, $a \in C$, $\sigma \in R$ and $\sigma' \in R$, if $\mathrm{Supp}(\sigma) \cap (a - \mathrm{Supp}(\sigma')) = \{a_1, \ldots, a_n\}$, then $\mu_{a,\sigma\sigma'} = \mu_{a_1,\sigma}\mu_{a-a_1,\sigma'} + \cdots + \mu_{a_n,\sigma}\mu_{a-a_n,\sigma'}$.

Clearly, in the language $\mathcal{L} \cup \{v\}$, the ring $k[[C]]$ is a cyclically valued ring.

Observe that if (R, v) is a cyclically valued ring, then the set of all constants of R is a subring of R. Indeed, let σ_1 and σ_2 be constants. By condition (4), $\sigma_1\sigma_2$ is a constant. Let $a \in \mathrm{Supp}(\sigma_1 + \sigma_2)$. Then $a = v(a, \sigma_1 + \sigma_2) \geq_a \min_a(v(a, \sigma_1), v(a, \sigma_2)) = 0$. Hence $a = 0$. Theorem **??** now follows from:

Theorem 5.5. *Let (R, v) be a cyclically valued ring with ring of constants k. Assume that the support of every element of R is well-ordered, and that there is an isomorphism $c \mapsto m_c$ from $(C, +)$ to $(R, \cdot)$ such that for all $c \in C$, m_c is a monomial of degree c. Then the cyclically valued ring (R, v) is isomorphic to a subring of $k[[C]]$.*

Proof. Let $\sigma \in R$ and $a \in C$. If $v(a, \sigma) \neq a$, we set $\sigma_a := 0$; if $v(a, \sigma) = a$, we set $\sigma_a := \mu_{a,\sigma} m_a^{-1}$. In any case, σ_a is a constant. We have $\Sigma_{a \in C} \sigma_a X^a \in k[[C]]$, because the support of σ is well-ordered. Now, we set $\varphi(\sigma) := \Sigma_{a \in C} \sigma_a X^a$. From (3), we deduce $\forall a \in C$, $\forall \sigma \in R$, $v(a, \varphi(\sigma)) = v(a, \sigma)$.

Take $\sigma, \sigma' \in R$. For all $a \in C$, $(\sigma + \sigma')_a = \sigma_a + \sigma'_a$, hence $\varphi(\sigma + \sigma') = \varphi(\sigma) + \varphi(\sigma')$. By (5), we have $\varphi(\sigma\sigma') = \varphi(\sigma)\varphi(\sigma')$. We have proved that φ is a homomorphism.

Assume that $\sigma \neq \sigma'$, and let $a \in \mathrm{Supp}(\sigma - \sigma')$. Then $v(a, \sigma - \sigma') = a$, and

$$\begin{array}{rcl} v(a, (\sigma - \sigma') - (\sigma_a m_a - \sigma'_a m_a)) & = & v(a, (\sigma - \sigma_a m_a) - (\sigma' - \sigma'_a m_a)) \\ & \geq_a & \min_a(v(a, \sigma - \sigma_a m_a), v(a, \sigma' - \sigma'_a m_a)) \\ & >_a & a. \end{array}$$

Hence, $v(a, \sigma_a m_a - \sigma'_a m_a) = a$. In particular, $\sigma_a \neq \sigma'_a$. It follows that φ is one-to-one. □

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